GENERALIZED STATES ON EQ-ALGEBRAS

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Abstract. In this paper, we introduce a notion of generalized states from an EQ-algebra $\mathcal{E}_1$ to another EQ-algebra $\mathcal{E}_2$, which is a generalization of internal states (or state operators) on an EQ-algebra $\mathcal{E}$. Also we give a type of special generalized state from an EQ-algebra $\mathcal{E}_1$ to $\mathcal{E}_1$, called generalized internal states (or GI-state). Then we give some examples and basic properties of generalized (internal) states on EQ-algebras. Moreover we discuss the relations between generalized states on EQ-algebras and internal states on other algebras, respectively. We obtain the following results: (1) Every state-morphism on a good EQ-algebra $\mathcal{E}$ is a G-state from $\mathcal{E}$ to the EQ-algebra $\mathcal{E}_0 = ([0,1], \wedge_0, \odot_0, \sim_0, 1)$. (2) Every state operator μ satisfying $\mu(x) \odot \mu(y) \in \mu(\mathcal{E})$ on a good EQ-algebra $\mathcal{E}$ is a GI-state on $\mathcal{E}$. (3) Every state operator τ on a residuated lattice $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ can be seen a GI-state on the EQ-algebra $(L, \wedge, \odot, \sim, 1)$, where $x \sim y := (x \rightarrow y) \wedge (y \rightarrow x)$. (4) Every GI-state σ on a good EQ-algebra $(L, \wedge, \odot, \sim, 1)$ is a internal state on equality algebra $(L, \wedge, \sim, 1)$. (5) Every GI-state σ on a good EQ-algebra $(L, \wedge, \odot, \sim, 1)$ is a left state operator on BCK-algebra $(L, \wedge, \rightarrow, 1)$, where $x \rightarrow y = x \sim x \wedge y$.

1. **Introduction**

Every many-valued logic is uniquely determined by the algebraic properties of the structure of its truth values. At present, it is generally accepted that in fuzzy logic, the algebraic structure should be a residuated lattice, possibly fulfilling some additional properties. BL-algebras, MTL-algebras, MV-algebras, etc., are the best known classes of residuated lattices [12, 16, 28]. Note that the typical operations on these algebras are multiplication $\odot$ and implication $\rightarrow$ which are closely tied by adjointness property.

Fuzzy type theory [25, 26] whose basic connective is a fuzzy equality was developed as a counterpart of the classical higher-order logic (type theory in which identity is a basic connective, see [1]). Since the algebra of truth values is no longer a residuated lattice, a specific algebra called an EQ-algebra [27] for fuzzy type theory was proposed by Novák and De Baets. EQ-algebras are interesting and important algebras from many points of view. First, the above residuated lattices based logical algebras are all particular cases of EQ-algebras. Second, the adjointness property which strictly couples $\odot$ and $\rightarrow$ on residuated lattices based logical algebras is relaxed. Indeed, in EQ-algebras, $\rightarrow$ is defined directly from fuzzy equality by the

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formula $x \to y = (x \wedge y) \sim x$. But the fuzzy equality $\sim$ cannot be reconstructed from the implication $\to$ in EQ-algebras in general. Third, EQ-algebras open a possibility to develop a fuzzy logic with a non-commutative conjunction but a single implication only [27]. From these points of view, it is meaningful to study EQ-algebras.

A state on MV-algebras ([24]) or BL-algebras ([29]) is an analogue of a probability measure, and it serves as an averaging process for formulas in Łukasiewicz logic or basic fuzzy logic, respectively. We note that states were introduced also in many different algebraic structures and they are intensively studied by many authors, e.g. [8, 15, 20, 4, 6, 22, 21] and others. Recently, Flaminio and Montagna [13, 14] have presented a new approach to states on MV-algebras: they added a unary operation τ to the language of MV-algebras as an internal state. It preserves the usual properties of states. It generalizes the state, as a function on the algebra taking values in the interval $[0, 1]$ with the addition property, as well as H*á*jek's approach to fuzzy logic with modality Pr (interpreted as probably). In the first case if s is a state and a an event, $s(a)$ denotes average of appearing the event a while in the second case, $Pr(a)$ is presented as truth value of appearing a. Consequently, the concepts of a state BL-algebra, a state equality algebra and a state BCK-algebra were introduced by Ciungu et al.[7, 5], by Borzooei et al. [2] respectively, as an extension of the concept of a state MV-algebra. Subsequently, the concept was extended by Dvurečenskij et al.[9] to *Rl-monoids* (not necessarily commutative). As a generalization of the notion of internal states for MV-algebras, BL- algebras and *Rl*-monoids, etc., the concept of a state residuated lattice was introduced by He et al.[17]. Especially the authors of this manuscript introduced the notions of states and internal states into hyper algebras, and established theories of states and internal states on hyper BCK-algebras[33, 32]. Recently Borzooei et al.[3] introduced the notion of state operators for EQ-algebras and studied some basic properties of it. But this state operators on EQ-algebras is not a generalization of the notion of internal states on residuated lattices.

In order to give a unified model of states and internal states of EQ-algebras, residuated lattices, equality algebras and BCK-algebras, we shall establish generalized state theory on EQ-algebras and discuss the relations between it and internal states of above-mentioned algebras in this paper.

This paper is organized as follows: In Section 2, we review some basic definitions and results about EQ-algebras, residuated lattices, equality algebras and BCK-algebras. In Section 3, we introduce the notion of generalized states on EQ-algebras and investigate some related properties of generalized states. In Section 4, we discuss the relation between generalized states and state-morphisms and internal states on EQ-algebras, respectively. In Section 5, we study the relation between generalized states on EQ-algebras and internal states on residuated lattices. In Section 6, we give the relations between generalized states on EQ-algebras and internal states on equality algebras and BCK-algebras.

2. **Preliminaries**

Definition 2.1. ([27]) An EQ-algebra is an algebra $\mathcal{E} := (E, \wedge, \odot, \sim, 1)$ of type (2,2,2,0) such that, for all $x, y, z, t \in E$:
(E1) $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1,
(E2) $(E, \odot, 1)$ is a commutative monoid and $\odot$ is isotone w.r.t. " $\leq$ " (where $x \leq y$ is defined as $x \wedge y = x$),
(E3) $x \sim x = 1$, (reflexivity axiom)
(E4) $((x \wedge y) \sim z) \odot (t \sim x) \leq z \sim (t \wedge y)$, (substitution axiom)
(E5) $(x \sim y) \odot (z \sim t) \leq (x \sim z) \sim (y \sim t)$, (congruence axiom)
(E6) $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$, (monotonicity axiom)
(E7) $x \odot y \leq x \sim y$. (boundedness axiom)

Definition 2.2. ([11]) Let $\mathcal{E}$ be an EQ-algebra. We say that it is
- semiseparated if for all $a, b \in E$, $a \sim 1 = 1$ implies $a = 1$.
- separated if for all $a, b \in \mathcal{E}$, $a \sim b = 1$ implies $a = b$.
- good if for all $a \in E$, $a \sim 1 = a$.
- residuated if for all $a, b, c \in E$,
$(a \odot b) \wedge c = a \odot b$ iff $a \wedge ((b \wedge c) \sim b) = a$. (1)
- idempotent it satisfies $x \odot x = x$ for all $x \in \mathcal{E}_1$.

Let $\mathcal{E}$ be an EQ-algebra and $x \to y := (x \wedge y) \sim x$. Then we have the following: for all $a, b, c \in E$,
(a) Symmetry: $a \sim b = b \sim a$,
(b) Transitivity: $(a \sim b) \odot (b \sim c) \leq a \sim c$,
(c) Transitivity of implication: $(a \to b) \odot (b \to c) \leq a \to c$.

Note that if the EQ-algebra is good then it is separated, but not vice-versa. Clearly, (1) can be written classically as $a \odot b \leq c$ iff $a \leq b \to c$.

Example 2.3. ([27]) Let $\mathcal{L} = (L, \vee, \wedge, \odot, \Rightarrow, 0, 1)$ be a residuated lattice and $f : L \to L$ be a $\wedge$-homomorphism such that $a \Leftrightarrow b \leq f(a) \Leftrightarrow f(b)$, holds for all $a, b \in L$, where $\Leftrightarrow$ is the biresidual operation defined by
$a \Leftrightarrow b = (a \Rightarrow b) \wedge (b \Rightarrow a)$. (2)
If we define the binary operation $\sim$ by $a \sim b = f(a) \Leftrightarrow f(b)$, then $(L, \wedge, \odot, \sim, 1)$ is an EQ-algebra.

Example 2.4. ([27]) Let $E = [0, 1], x \odot y = 0 \vee (x + y - 1)$ be the Łukasiewicz conjunction and define $f_k : [0, 1] \to [0, 1]$ by $f_k(x) = 1 \wedge (x + k)$, for some fixed $k \in [0, 1]$. It can be verified that f_k is a $\wedge$-homomorphism, i.e. $f_k(x \wedge y) = f_k(x) \wedge f_k(y)$. Moreover, $|f_k(x) - f_k(y)| \leq |x - y|$. Therefore, we may define
$x \sim y = 1 - |f_k(x) - f_k(y)|$, (3)
for all $x, y \in [0, 1]$. Since (3) is the biresidual operation (2) based on the Łukasiewicz implication, we conclude from Example 2.3 that $\mathcal{E} = ([0, 1], \wedge, \odot, \sim, 1)$ is an EQ-algebra in which $\hat{x} = f_k(x)$. This algebra is clearly not separated.

Especially taking $k = 0$, we define $x \sim_0 y = 1 - |x - y|$ for $x, y \in [0, 1]$. Then $([0, 1], \wedge, \odot, \sim_0, 1)$ is an EQ-algebra, which is denoted by $\mathcal{E}_0 = ([0, 1], \wedge_0, \odot_0, \sim_0, 1)$.

We list below some of the properties of EQ-algebras from [11] that will be used in the paper.

Proposition 2.5. *(*[27],[11]*) Let $\mathcal{E}$ be an EQ-algebra and $\overline{x} = x \sim 1$. Then the following properties hold, for all $x, y, z \in E$,*
(1) $x \odot y \leq x \wedge y \leq x, y$,
(2) $z \odot (x \wedge y) \leq (z \odot x) \wedge (z \odot y)$,
(3) $x \sim y \leq x \to y$,
(4) $x \to x = 1$,
(5) $x \leq \overline{x}$, $\overline{1} = 1$,
(6) $x \odot (x \sim y) \leq \overline{y}$,
(7) $(z \to (x \wedge y)) \odot (x \sim t) \leq z \to (t \wedge y)$,
(8) $(x \to y) \odot (y \to x) \leq x \sim y$,
(9) if $x \leq y \to z$, *then* $x \odot y \leq \overline{z}$,
(10) if $x \leq y \leq z$, *then* $z \sim x \leq z \sim y$ *and* $x \sim z \leq x \sim y$,
(11) $x \to y \leq (x \wedge z) \to y$,
(12) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,
(13) if $x \leq y$, *then* $x \to y = 1$ *and* $x \sim y = y \to x$,
(14) if $x \leq y$, *then* $z \to x \leq z \to y$ *and* $y \to z \leq x \to z$,
(15) $x \to y \leq (z \to x) \to (z \to y)$,
(16) $x \to y \leq (y \to z) \to (x \to z)$.

Proposition 2.6. *(*[27]*) (1) In every good EQ-algebra $\mathcal{E}$, the following inequality holds for all $a, b \in E$, $a \leq (a \sim b) \sim b$,*
(2) Each residuated EQ-algebra is good (and thus separated).

Proposition 2.7. *(*[11]*) Let $\mathcal{E}$ be a good EQ-algebra. Then the following hold:*
(1) $x = 1 \to x$,
(2) $x \leq y \to x = y \sim (x \wedge y)$,
(3) $x \leq (x \to y) \to y$,
(4) $x \to (y \to z) = y \to (x \to z)$.

Proposition 2.8. *Let $\mathcal{E}$ be a good EQ-algebra. Then the following hold for all $x, y, z \in E$:*
(1) $((x \sim x \wedge y) \sim y) \sim y = x \sim (x \wedge y)$,
(2) $x \to y = ((x \to y) \to y) \to y$,
(3) if $x \leq y$, *then* $x \leq x \sim y$,
(4) $x \sim y \leq (x \sim z) \sim (y \sim z)$,
(5) $x \leq (x \sim x \wedge y) \sim y$,
(6) $y \leq (x \sim x \wedge y) \sim y$,
(7) $x \odot (x \to x \odot y) \leq x \odot y$,
if $\mathcal{E}$ is a residuated EQ-algebra, then
(8) $(x \to y) \to ((x \odot z) \to (y \odot z)) = 1$,
(9) $(x \odot y) \to z = x \to (y \to z)$.

Proof. (1) Applying Proposition 2.6(1), we get $x \sim x \wedge y \leq ((x \sim x \wedge y) \sim y) \sim y$. From Proposition 2.6(1) we have $x \wedge y \leq x \leq (x \sim y) \sim y$ and applying Proposition 2.5(12) we get $x \wedge y \sim ((x \sim y) \sim y) \leq x \wedge y \sim x$, that is, $((x \sim y) \sim y) \sim x \wedge y \leq x \sim x \wedge y$. We conclude that $((x \sim x \wedge y) \sim y) \sim y = x \sim x \wedge y$.
(2) By Proposition 2.7(3), $x \to y \leq ((x \to y) \to y) \to y$. On the other hand, by

Proposition 2.7(3) again, we have $x \leq (x \to y) \to y$. Using Proposition 2.5(16) we get $x \to y \geq ((x \to y) \to y) \to y$. Therefore $x \to y = ((x \to y) \to y) \to y$.
(3) It follows Proposition 2.7(2) that $x \leq y \sim y \wedge x = y \sim x = x \sim y$.
(4) Taking $z = t$ in (E5), we get $(x \sim y) \odot 1 \leq (x \sim z) \sim (y \sim z)$. That is $x \sim y \leq (x \sim z) \sim (y \sim z)$.
(5) Applying (4), we have $x = x \sim 1 \leq (x \sim x \wedge y) \sim (1 \sim x \wedge y) = (x \sim x \wedge y) \sim x \wedge y$. By (2) we have $x \wedge y \leq y \leq x \sim x \wedge y$. Applying Proposition 2.5(12) we get $x \wedge y \sim (x \sim x \wedge y) \leq y \sim (x \sim x \wedge y)$, and so $x \leq (x \sim x \wedge y) \sim y$.
(6) By Proposition 2.7(2), we have $y \leq x \sim x \wedge y$. Using (3) we get $y \leq y \sim (x \sim x \wedge y) = (x \sim x \wedge y) \sim y$.
(7) Note that $x \odot (x \to x \odot y) = x \odot (x \sim (x \wedge x \odot y)) = x \odot (x \sim (x \odot y)) = (1 \sim x) \odot (x \sim (x \odot y)) \leq 1 \sim x \odot y = x \odot y$.
(8) Let $\mathcal{E}$ be a residuated EQ-algebra. Then $(x \to y) \to ((x \odot z) \to (y \odot z)) = 1$ iff $(x \to y) \leq ((x \odot z) \to (y \odot z))$ iff $(x \odot z) \odot (x \to y) \leq y \odot z$ iff $x \odot (x \to y) \leq y$ iff $x \to y \leq x \to y$.
(9) Let $\mathcal{E}$ be a residuated EQ-algebra. Then $x \odot ((x \odot y) \to z) \leq x \odot (y \to z) \leq y \to z$ and hence $(x \odot y) \to z \leq x \to (y \to z)$. Conversely $(x \to (y \to z)) \odot (x \odot y) = x \odot (x \to (y \to z)) \odot y \leq (y \to z) \odot y \leq z$, which implies $x \to (y \to z) \leq (x \odot y) \to z$. □

Definition 2.9. ([11]) Let $\mathcal{E}$ be an EQ-algebra and $\emptyset \neq F \subseteq E$. F is called a prefilter in $\mathcal{E}$ if it satisfies for any $x, y \in E$,
(F1) $1 \in F$,
(F2) If $x \in F$ and $x \to y \in F$, then $y \in F$.
A prefilter F is said to be a filter if it satisfies
(F3) If $x \to y \in F$, then $(x \odot z) \to (y \odot z) \in F$ for any $x, y, z \in E$.

Now we recall some notions and results of residuated lattices, equality algebras and BCK-algebras.

Definition 2.10. ([16], [31]) An algebraic structure $(L, \wedge, \vee, \odot, \to, 0, 1)$ of type (2, 2, 2, 2, 0, 0) is called a residuated lattice if it satisfies the following conditions:
(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(2) $(L, \odot, 1)$ is a commutative monoid,
(3) $x \odot y \leq z$ if and only if $x \leq y \to z$,
for all $x, y, z \in L$, where $\leq$ is the partial order of the lattice $(L, \wedge, \vee, 0, 1)$.

Proposition 2.11. *(*[30], [31], [34]*) In any residuated lattice* $(L, \wedge, \vee, \odot, \to, 0, 1)$, *the following properties hold:*
(1) $1 \to x = x$, $x \to 1 = 1$,
(2) $x \leq y$ *if and only if* $x \to y = 1$,
(3) $x \odot x^* = 0$, $x \odot y = 0$ *if and only if* $x \leq y^*$, *where* $x^* = x \to 0$,
(4) If $x \leq y$, *then* $y \to z \leq x \to z$, $z \to x \leq z \to y$ *and* $x \odot z \leq y \odot z$,
(5) $x \odot (x \to y) \leq y$,
(6) $x \odot y \leq x \wedge y$, $x \leq y \to x$,
(7) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$,
(8) $0^* = 1$, $1^* = 0$, $x \leq x^{**}$, $x^{***} = x^*$,
(9) $x \odot (y \to z) \leq y \to (x \odot z) \leq (x \odot y) \to (x \odot z)$,

(10) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
(11) $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$,
(12) $x \to (x \wedge y) = x \to y$,
(13) $x \odot y = x \odot (x \to x \odot y)$,
(14) $((x \to y) \to y) \to y = x \to y$,
(15) $x \vee y \leq (x \to y) \to y$,
for any $x, y, z \in L$.

Definition 2.12. ([19]) An equality algebra is an algebra $\mathcal{E} = (X, \sim, \wedge, 1)$ of type (2, 2, 0) such that the following axioms are fulfilled for all $x, y, z \in X$:
(e1) $(A, \wedge, 1)$ is a meet-semilattice with top element 1,
(e2) $x \sim y = y \sim x$,
(e3) $x \sim x = 1$,
(e4) $x \sim 1 = x$,
(e5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$,
(e6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,
(e7) $x \sim y \leq (x \sim z) \sim (y \sim z)$.

Definition 2.13. ([18]) A BCK-algebra is an algebra $B = (X, \to, 1)$ of type (2, 0) with the following axioms:
(BCK1) $(a \to b) \to ((b \to c) \to (a \to c)) = 1$,
(BCK2) $1 \to a = a$,
(BCK3) $a \to 1 = 1$,
(BCK4) $a \to b = 1$ and $b \to a = 1$ imply $a = b$.
For any BCK-algebra one can define a partial ordering relation (called the underlying partial order of B) by $a \leq b$ iff $a \to b = 1$.

Definition 2.14. ([19]) A $BCK-algebra$ with meet $B = (X, \to, \wedge, 1)$ is an algebra of type (2, 2, 0) such that $(X, \to, 1)$ is a BCK-algebra and the underlying partial order of $(X, \to, 1)$ is a $\wedge$-semilattice. The equivalence operation $\leftrightarrow$ of B is defined by $a \leftrightarrow b = (a \to b) \wedge (b \to a)$.

Definition 2.15. ([3]) A state EQ-algebra is a pair $(\mathcal{E}, \mu)$ such that $\mathcal{E}$ is an EQ-algebra and $\mu : E \to E$ is a unary operation on E satisfying the following conditions, for all $x, y \in E$:
(SO1) $\mu(x) \leq x$,
(SO2) $\mu(x) \leq \mu(\mu(x))$,
(SO3) $\mu(x \sim y) = \mu(x) \sim \mu(y)$,
(SO4) $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$,
(SO5) if $x \vee y$ and $\mu(x) \vee \mu(y)$ exist, then $\mu(x \vee y) \leq \mu(x) \vee \mu(y)$.
We call μ a state operator on $\mathcal{E}$.

Lemma 2.16. *(*[3]*) Let* $(\mathcal{E}, \mu)$ *be a state EQ-algebra. Then the following conditions hold: for all* $x, y \in \mathcal{E}$,
(1) $\mu(1) = 1$,
(2) $\mu(0) = 0$,
(3) if $x \leq y$, *then* $\mu(x) \leq \mu(y)$,
(4) $\mu(x \to y) = \mu(x) \to \mu(y)$.

Lemma 2.17. *([3]) Let $(\mathcal{E}, \mu)$ be a state EQ-algebra such that $\mathcal{E}$ is a good EQ-algebra. Then the following conditions hold: for all $x \in \mathcal{E}$,*
(i) $\mu(\mu(x)) = \mu(x)$,
(ii) $Im(\mu) = \{x \in \mathcal{E} | \mu(x) = x\}$.

Definition 2.18. ([2]) Let $(X, \to, 1)$ be a BCK-algebra. A map $\mu : X \to X$ is called a left (right) state operator on X if it satisfies the following conditions:
(S1) $x \to y = 1$ implies $\mu(x) \to \mu(y) = 1$,
(S2) $\mu(x \to y) = \mu((x \to y) \to y) \to \mu(y)$ $(\mu(x \to y) = \mu((y \to x) \to x) \to \mu(y))$,
(S3) $\mu(\mu(x) \to \mu(y)) = \mu(x) \to \mu(y)$.
A left (right) state BCK-algebra is a pair (X, μ), where X is a BCK-algebra and μ is a left (right) state operator on X.

3. Generalized states on EQ-algebras

In this section we introduce and study a notion of generalized states from an EQ-algebra X to an EQ-algebra Y, which are the generalization of states, and internal states of EQ-algebras. Also we give some types of special generalized states according to the structures of Y and discuss relations among them.

Definition 3.1. Let $(X, \sim_1, \wedge_1, \odot_1, 1_1)$ and $(Y, \sim_2, \wedge_2, \odot_2, 1_2)$ be two EQ-algebras. A map $\sigma : X \to Y$ is called a generalized state from X to Y, which is denoted simply G-state, if it is satisfies the following conditions:
for all $x, y, z \in X$,
(GS1) $x \leq_1 y$ implies $\sigma(x) \leq_2 \sigma(y)$,
(GS2) $\sigma(x \sim_1 x \wedge_1 y) = \sigma((x \sim_1 x \wedge_1 y) \sim_1 y) \sim_2 \sigma(y)$,
(GS3) $\sigma(x) \odot_2 \sigma(x \to_1 (x \odot_1 y)) \leq \sigma(x \odot_1 y)$,
(GS4) for all $x, y \in X$, $\sigma(x) \sim_2 \sigma(y) \in \sigma(X)$, where $\sigma(X) = \{\sigma(x) : x \in X\}$,
(GS5) for all $x, y \in X$, $\sigma(x) \wedge_2 \sigma(y) \in \sigma(X)$,
(GS6) for all $x, y \in X$, $\sigma(x) \odot_2 \sigma(y) \in \sigma(X)$.
Moreover, if $X = Y$ and $\sigma^2 = \sigma$, then σ is called a generalized internal state on X, simply GI-state on X.

Remark 3.2. (1) In the notion of internal states on residuated lattices (BL-algebras, BCK-algebras etc.), an internal state satisfies the condition $\sigma(\sigma(x) * \sigma(y)) = \sigma(x) * \sigma(y)$, where $* \in \{\odot, \to, \wedge, \vee\}$. But in case of EQ-algebras, for a map $\sigma : X \to Y$, $\sigma(\sigma(x) * \sigma(y))$ is not well-defined. Hence in Definition 3.1, we use (GS4),(GS5)and (GS6) to define G-states on EQ-algebras.
(2) The condition (GS2) of Definition 3.1 is inspired by the condition (S2) in the definition of state operators on BCK-algebras.

Example 3.3. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two EQ-algebras. Then the map $1_{\mathcal{E}}$, defined by $1_{\mathcal{E}}(x) = 1_2$ for all $x \in \mathcal{E}_1$, is a G-state from $\mathcal{E}_1$ to $\mathcal{E}_2$.

Example 3.4. If $\mathcal{E}$ is a good EQ-algebra, then the identity map $Id_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ is a GI-state on $\mathcal{E}$.

Proof. Indeed, it is obvious that (GS1), (GS4), (GS5) and (GS6) are verified. By Proposition 2.8(7), we have (GS3). By Proposition 2.7(1), we can prove (GS2). Moreover it is clear that $\sigma^2 = \sigma$. Therefore $Id_{\mathcal{E}}$ is a GI-state. □

Example 3.5. Let $\mathcal{E} = (E, \wedge, \odot, \sim, 1)$ be a linearly ordered good EQ-algebra. Then we can check that $\mathcal{E} \times \mathcal{E} = (E \times E, \wedge \times \wedge, \odot \times \odot, \sim \times \sim, 1 \times 1)$ is a good EQ-algebra, where the operations on $E \times E$ are defined according to the coordinate components. On $E \times E$ we define two operators, σ_1 and σ_2 as follows

$\sigma_1(a, b) = (a, a)$, $\sigma_2(a, b) = (b, b)$, $(a, b) \in E \times E$.

Then σ_1 and σ_2 are two GI-states on $E \times E$ that are also endomorphisms such that $\sigma_i^2 = \sigma_i, i = 1, 2$. Moreover, $(\mathcal{E} \times \mathcal{E}, \sigma_1)$ and $(\mathcal{E} \times \mathcal{E}, \sigma_2)$ are isomorphic state EQ-algebras under the isomorphism $(a, b) \mapsto (b, a)$.

Example 3.6. Let $\mathcal{E}_1 = \{0_1, a_1, b_1, 1_1\}$ and $\mathcal{E}_2 = \{0_2, a_2, b_2, 1_2\}$ be two chains with Cayley tables as follows respectively:

$\odot_1$	0_1	a_1	b_1	1_1
0_1	0_1	0_1	0_1	0_1
a_1	0_1	a_1	a_1	a_1
b_1	0_1	a_1	b_1	b_1
1_1	0_1	a_1	b_1	1_1

$\sim_1$	0_1	a_1	b_1	1_1
0_1	1_1	0_1	0_1	0_1
a_1	0_1	1_1	a_1	a_1
b_1	0_1	a_1	1_1	1_1
1_1	0_1	a_1	1_1	1_1

$\to_1$	0_1	a_1	b_1	1_1
0_1	1_1	1_1	1_1	1_1
a_1	0_1	1_1	1_1	1_1
b_1	0_1	a_1	1_1	1_1
1_1	0_1	a_1	1_1	1_1

$\odot_2$	0_2	a_2	b_2	1_2
0_2	0_2	0_2	0_2	0_2
a_2	0_2	0_2	0_2	a_2
b_2	0_2	0_2	0_2	b_2
1_2	0_2	a_2	b_2	1_2

$\sim_2$	0_2	a_2	b_2	1_2
0_2	1_2	a_2	0_2	0_2
a_2	a_2	1_2	a_2	a_2
b_2	0_2	a_2	1_2	b_2
1_2	0_2	a_2	b_2	1_2

$\to_2$	0_2	a_2	b_2	1_2
0_2	1_2	1_2	1_2	1_2
a_2	a_2	1_2	1_2	1_2
b_2	0_2	a_2	1_2	1_2
1_2	0_2	a_2	b_2	1_2

Then $(\mathcal{E}_1, \wedge_1, \odot_1, \sim_1, 1_1)$ is an EQ-algebra but is not a good EQ-algebra, and $(\mathcal{E}_2, \wedge_2, \odot_2, \sim_2, 1_2)$ is a good EQ-algebra ([23]).

One can easily check that the map $\sigma : \mathcal{E}_1 \to \mathcal{E}_2$, defined as follows: $\sigma(0_1) = 0_2$, $\sigma(a_1) = a_2$, $\sigma(b_1) = 1_2$, $\sigma(1_1) = 1_2$, is a G-state from $\mathcal{E}_1$ to $\mathcal{E}_2$.

Remark 3.7. (1) In Example 3.6, we can see that G-state σ can not be seen a state or an internal state on EQ-algebra $\mathcal{E}_1$, and hence it can not be seen a state or an internal state on a residuated lattice. It follows that the notion of G-state on EQ-algebras is a non-trivial generalization of one of states on residuated lattices.
(2) An internal state on a residuated lattice E can be seen a G-state from E to E (see Proposition 5.3).
(3) In general, for G-state σ of EQ-algebras $Ker(\sigma)$ is a prefilter, but it may not be a filter (see Example 3.5). As a result, it is different from the related result of residuated lattice. However, we can prove that whenever $\sigma : \varepsilon_1 \to \varepsilon_2$ is a G-sate and ε_1 is residuated, then $Ker(\sigma)$ is a filter(3.18).

Proposition 3.8. *Let σ be a G-state from a good EQ-algebra $\mathcal{E}_1$ to an EQ-algebra $\mathcal{E}_2$. Then*
(1) $\sigma(1_1) = 1_2$,
(2) $\sigma(x \to_1 y) = \sigma((x \to_1 y) \to_1 y) \to_2 \sigma(y)$,
(3) $\sigma(\mathcal{E}_1)$ is a subalgebra of $\mathcal{E}_2$,
(4) $\sigma(x \to_1 y) \leq_2 \sigma(x) \to_2 \sigma(y)$,

(5) if $\mathcal{E}_2$ *is good,* $Ker(\sigma)$ *is a prefilter of* $\mathcal{E}_1$*, where* $Ker(\sigma) = \{x \in \mathcal{E}_1 | \sigma(x) = 1_2\}$*,*
(6) $\sigma(\overline{x}) = \overline{\sigma(x)}$*.*

Proof. (1) Consider (GS2) for $y = x$, we have $\sigma(1_1) = \sigma(1_1 \sim x) \sim \sigma(x) = \sigma(x) \sim \sigma(x) = 1_2$.

(2) By (GS2), $\sigma(x \rightarrow_1 y) = \sigma((x \rightarrow_1 y) \sim_1 y) \sim_2 \sigma(y)$. By Proposition 2.5(15) and Proposition 2.8(2), we have $(x \rightarrow_1 y) \sim_1 y = (x \rightarrow_1 y) \rightarrow_1 y$ and hence $\sigma(x \rightarrow_1 y) = \sigma((x \rightarrow_1 y) \rightarrow_1 y) \sim_2 \sigma(y)$. By Proposition 2.7(2), $\sigma(y) \leq_2 \sigma((x \rightarrow_1 y) \rightarrow_1 y)$. It follows that $\sigma(x \rightarrow_1 y) = \sigma((x \rightarrow_1 y) \rightarrow_1 y) \rightarrow_2 \sigma(y)$.

(3) It follows from (1), and (GS4), (GS5) and (GS6).

(4) By (5) and (6) of Proposition 2.8, we have $x \leq_1 (x \sim_1 x \wedge_1 y) \sim_1 y$ and $y \leq_1 (x \sim_1 x \wedge_1 y) \sim_1 y$. Then by (GS2), we have $\sigma(x \rightarrow_1 y) = \sigma(x \sim_1 x \wedge_1 y) = \sigma((x \sim_1 x \wedge_1 y) \sim_1 y) \sim_2 \sigma(y)$. Using (15)and (16) of Proposition 2.5, we get $\sigma((x \sim_1 x \wedge_1 y) \sim_1 y) \sim_2 \sigma(y) \leq_2 \sigma((x \sim_1 x \wedge_1 y) \sim_1 y) \rightarrow_2 \sigma(y) \leq_2 \sigma(x) \rightarrow_2 \sigma(y)$.

(5) By (1), we have $1 \in Ker(\sigma)$. Let $a \in Ker(\sigma)$ and $a \rightarrow_1 b \in Ker(\sigma)$. Then $\sigma(a) = 1_2$ and $1_2 = \sigma(a \rightarrow_1 b) \leq \sigma(a) \rightarrow_2 \sigma(b)$ by (4). Thus $1_2 = \sigma(a) \rightarrow_2 \sigma(b) = 1 \rightarrow_2 \sigma(b) = \sigma(b)$. Therefore $b \in Ker(\sigma)$.

(6) By (1) and (GS2), we have $\sigma(\overline{x}) = \sigma(x \sim_1 1_1) = \sigma(1_1 \sim_1 (1_1 \wedge_1 x)) = \sigma((1_1 \sim_1 (1_1 \wedge_1 x)) \sim_1 x) \sim_2 \sigma(x) = \sigma(x \sim_1 x) \sim_2 \sigma(x) = 1_2 \sim_2 \sigma(x) = \overline{\sigma(x)}$. ☐

Proposition 3.9. *Let* σ *be a* GI*-state on an* EQ*-algebra* $\mathcal{E}$*. Then*
(1) $\sigma(\sigma(x) \sim \sigma(y)) = \sigma(x) \sim \sigma(y)$*,*
(2) $\sigma(\sigma(x) \wedge \sigma(y)) = \sigma(x) \wedge \sigma(y)$*,*
(3) $\sigma(\sigma(x) \odot \sigma(y)) = \sigma(x) \odot \sigma(y)$*,*
(4) $\sigma(\sigma(x) \rightarrow \sigma(y)) = \sigma(x) \rightarrow \sigma(y)$*,*
(5) $\sigma(\mathcal{E}) = \{x \in \mathcal{E} | x = \sigma(x)\}$*.*
Moreover if $\mathcal{E}$ *is good, we have the following:*
(6) $\sigma(1) = 1$*,*
(7) $\sigma(\mathcal{E})$ *is a subalgebra of* $\mathcal{E}$*.*

Proof. (1) By (GS4), there is $a \in E$ such that $\sigma(x) \sim \sigma(y) = \sigma(a)$. Hence $\sigma(\sigma(x) \sim \sigma(y)) = \sigma(\sigma(a)) = \sigma(a) = \sigma(x) \sim \sigma(y)$ since σ is a GI-state.

(2)-(4) They are similar to (1).

(5) Clearly, $\{x \in E | x = \sigma(x)\} \subseteq \sigma(E)$. Let $x \in \sigma(E)$, that is there exists $x_1 \in E$ such that $x = \sigma(x_1)$. Since $\sigma^2 = \sigma$, we have $x = \sigma(x_1) = \sigma(\sigma(x_1)) = \sigma(x)$, that is, $x \in \{x \in E | x = \sigma(x)\}$. Thus $\sigma(E) \subseteq \{x \in E | x = \sigma(x)\}$ and we conclude that $\sigma(E) = \{x \in E | x = \sigma(x)\}$.

(6) It follows from Proposition 3.8(1).

(7) It follows from Proposition 3.8(3). ☐

Proposition 3.10. *Let* σ *be a* G*-state from a good* EQ*-algebra* $\mathcal{E}_1$ *to a good* EQ*-algebra* $\mathcal{E}_2$ *and* $x, y \in \mathcal{E}_1$ *such that* $y \leq_1 x$*. Then the following hold:*
(1) $\sigma(x \sim_1 y) \leq \sigma(x) \sim_2 \sigma(y)$*,*
(2) $\sigma(x) \sim_2 \sigma(y) = \sigma(x) \rightarrow_2 \sigma(y)$*.*

Proof. (1) By Proposition 2.8(5), we have $y \leq_1 x \leq_1 (x \sim_1 x \wedge_1 y) \sim_1 y$. It follows that $\sigma(y) \leq_2 \sigma(x) \leq_2 \sigma((x \sim_1 x \wedge_1 y) \sim_1 y)$. Applying Proposition 2.5(10) we have $\sigma(y) \sim_2 \sigma((x \sim_1 x \wedge_1 y) \sim_1 y) \leq_2 \sigma(y) \sim_2 \sigma(x)$, and so by (GS2) we get $\sigma(x \sim_1 y) \leq_2 \sigma(x) \sim_2 \sigma(y)$.

(2) Since $y \leq_1 x$, we have $\sigma(y) \leq_2 \sigma(x)$ and $\sigma(x) \rightarrow_2 \sigma(y) = \sigma(x) \sim_2 \sigma(x) \wedge_2 \sigma(y) = \sigma(x) \sim_2 \sigma(y)$. □

Proposition 3.11. *Let σ be a G-state from a residuated EQ-algebra $\mathcal{E}_1$ to an EQ-algebra $\mathcal{E}_2$. Then for all $x, y \in E$, $\sigma(x) \odot_2 \sigma(y) \leq_2 \sigma(x \odot_1 y)$.*

Proof. Since $\mathcal{E}_1$ is a residuated EQ-algebra and $x \odot_1 y \leq_1 x \odot_1 y$, we have $y \leq_1 x \rightarrow_1 x \odot_1 y$. By (GS1), we have $\sigma(y) \leq_2 \sigma(x \rightarrow_1 (x \odot_1 y))$. Applying (GS3) and (E2), we get $\sigma(x \odot_1 y) \geq_2 \sigma(x) \odot_2 \sigma(x \rightarrow_1 (x \odot_1 y)) \geq_2 \sigma(x) \odot_2 \sigma(y)$. □

Remark 3.12. (1) Prefilters and filters coincide in residuated EQ-algebras([23]).
(2) Then, from logical point of view, the prefilters in EQ-algebras correspond to provable formulas, so that it is necessary to study it. However, from prefilters we can not induce congruences on EQ-algebras (see [10]). In order to induce congruences on EQ-algebras one needs to introduce the notion of filters on EQ-algebras.

We will denote by $F(\mathcal{E})$ the set of all filters of $\mathcal{E}$. Clearly, $\mathcal{E} \in F(\mathcal{E})$ and $F(\mathcal{E})$ is closed under arbitrary intersections. As a consequence, $(F(\mathcal{E}), \subseteq)$ is a lattice.

Lemma 3.13. *([11]) Let F be a prefilter of a separated EQ-algebra $\mathcal{E}$. For all $a, b \in E$, it holds that*
(1) if $a \in F$ and $a \leq b$, then $b \in F$,
(2) if $a, a \sim b \in F$, then $b \in F$,
(3) if $a, b \in F$, then $a \wedge b \in F$,
(4) if $a \sim b \in F$ and $b \sim c \in F$, then $a \sim c \in F$,
(5) $1 \sim b \in F$ iff $b \in F$,
(6) $F = \{b \in E | b \sim 1 \in F\}$.

Lemma 3.14. *([11]) Let F be a prefilter of a separated EQ-algebra $\mathcal{E}$, $a \sim b \in F$ and $a' \sim b' \in F$. Then the following hold:*
(1) $(a \wedge a') \sim (b \wedge b') \in F$,
(2) $(a \sim a') \sim (b \sim b') \in F$,
(3) $(a \rightarrow a') \sim (b \rightarrow b') \in F$.

Lemma 3.15. *([11]) Let F be a filter of a separated EQ-algebra $\mathcal{E}$. For all $a, b \in E$,*
(1) if $a, b \in F$, then $a \odot b \in F$,
(2) if $a \sim b \in F$, then $(a \odot c) \sim (b \odot c) \in F$ for all $c \in E$.

Given a prefilter $F \subseteq E$, usually, the following relation on $\mathcal{E}$ is an equivalence relation but not a congruence:
$a \approx_F b$ iff $a \sim b \in F$.
We shall denote by $[a]_F$ the equivalence class of $a \in E$ with respect to $\approx_F$ and by E/F the quotient set associated with $\approx_F$.

By Lemma 3.14 and 3.15, we can get the following proposition.

Proposition 3.16. *([27]) Let F be a filter of a separated EQ-algebra $\mathcal{E}$. Then the relation $\approx_F$ is a congruence relation on $\mathcal{E}$.*

Proposition 3.17. *Every filter of a good EQ-algebra $\mathcal{E}$ is a subalgebra of $\mathcal{E}$.*

Proof. Let $\mathcal{E}$ be a good EQ-algebra and F be a filter of $\mathcal{E}$. By (F1), $1 \in F$. Consider $x, y \in F$. By Proposition 2.6(1), $x \leq (x \sim y) \sim y$. It follows from Lemma 3.13(1) that $(x \sim y) \sim y \in F$. So $y \sim (x \sim y) \in F$. From Lemma 3.13(2) and $y, y \sim (x \sim y) \in F$ we get $x \sim y \in F$. Since a good EQ-algebra is separated, so $x \wedge y \in F$ and $x \odot y \in F$ by Lemma 3.13(3) and Lemma 3.15(1). Thus F is a subalgebra of $\mathcal{E}$. $\square$

Proposition 3.18. *Let σ be a G-state from a residuated EQ-algebra $\mathcal{E}_1$ to a good EQ-algebra $\mathcal{E}_2$. Then the following hold:*
(1) $Ker(\sigma) \in F(\mathcal{E}_1)$,
(2) $Ker(\sigma)$ is a subalgebra of $\mathcal{E}_1$.

Proof. (1) It follows from Proposition 3.8(5) and Lemmas 3.12.
(2) It is a corollary of (1) and Proposition 3.17.

$\square$

Proposition 3.19. *Let σ be a G-state from a residuated EQ-algebra $\mathcal{E}_1$ to a good EQ-algebra $\mathcal{E}_2$. Denote $K = Ker(\sigma)$.*
(1) $\approx_K$ is a congruence on $\mathcal{E}_1$,
(2) $\mathcal{E}_1/\approx_K$ is a good EQ-algebra,
(3) $[x]_K \leq [y]_K$ iff $x \to y \in K$.

Proof. (1) It follows from Propositions 3.18 and 3.16.
(2) It follows from (1) that the quotient algebra $\mathcal{E}_1/\approx_K$ is an EQ-algebra. Moreover for all $x \in \mathcal{E}_1$, note that $[1]_K \sim [x]_K = [1_1 \sim_1 x]_K$. Since $\mathcal{E}_1$ is residuated, and thus it is good, we get $1_1 \sim_1 x = x$. Therefore $[1]_K \sim [x]_K = [1_1 \sim_1 x]_K = [x]_K$. This shows that $\mathcal{E}_1/\approx_K$ is a good EQ-algebra.
(3) Note that $[x]_K \leq [y]_K$ iff $[x]_K \wedge [y]_K = [x]_K$ iff $[x \wedge_1 y]_K = [x]_K$ iff $x \wedge_1 y \sim_1 x \in K$ iff $x \to_1 y \in K$. $\square$

Definition 3.20. Let σ be a G-state from an EQ-algebra $\mathcal{E}_1$ to an EQ-algebra $\mathcal{E}_2$.
(1) σ is called strong if it satisfies $\sigma(x \sim_1 y) = \sigma(x) \sim_2 \sigma(y)$ for all $x, y \in \mathcal{E}_1$,
(2) σ is called residuated if it satisfies $\sigma((x \odot_1 y) \wedge_1 z) = \sigma(x \odot_1 y)$ iff $\sigma(x \wedge_1 ((y \wedge_1 z) \sim_1 y) = \sigma(x)$, for all $x, y, z \in \mathcal{E}_1$,
(3) σ is called divisible if it satisfies $\sigma(x \wedge x) = \sigma(x) \odot (\sigma(x) \to \sigma(y))$ for all $x, y \in \mathcal{E}_1$
(4) σ is called idempotent it satisfies $\sigma(x \odot x) = \sigma(x)$ for all $x \in \mathcal{E}_1$.

Remark 3.21. (1) Recall a strong internal state σ of BL-algebra L which is internal state of L satisfying the condition $(3')_{BL} : \sigma(x \odot y) = \sigma(x) \odot \sigma(x^- \vee y)$ (see [7]). It is easy to check that when an internal state σ of BL-algebra L satisfying $\sigma(x \to y) = \sigma(x) \to \sigma(y)$ for all $x, y \in L$, it is a strong internal state. Hence the notion of strong G-sates on EQ-algebras is a generalization of strong internal state on BL-algebras.
(2) If an internal state σ of BL-algebra L satisfies $\sigma(x \to y) = \sigma(x) \to \sigma(y)$ for all $x, y \in L$, then it is an endomorphism of L (see [7]). In the following we will get some corresponding result for strong G-states on EQ-algebras.

Proposition 3.22. *Let σ be a strong G-state from a residuated EQ-algebra $\mathcal{E}_1$ to a good EQ-algebra $\mathcal{E}_2$ and denote $K = Ker(\sigma)$. Then the following hold.*
(1) $\mathcal{E}_1/\approx_K$ is residuated if and only if σ is residuated,
(2) $\mathcal{E}_1/\approx_K$ is idempotent if and only if σ is idempotent.

Proof. (1) Let σ is residuated. Then we have $([x]_K \odot [y]_K) \wedge [z]_K = [x]_K \odot [y]_K$ iff $[(x \odot_1 y]) \wedge_1 z]_K = [x \odot_1 y]_K$ iff $(x \odot_1 y) \wedge_1 z \sim_1 x \odot_1 y \in K$ iff $(x \odot_1 y) \to_1 (x \odot_1 y) \wedge z \in K$ iff $\sigma((x \odot_1 y) \to_1 (x \odot_1 y) \wedge z) = 1_2$. Since σ is strong, we have that $\sigma((x \odot_1 y) \to_1 (x \odot_1 y) \wedge z) = 1_2$ iff $\sigma(x \odot_1 y) \to_2 \sigma((x \odot_1 y) \wedge_1 z) = 1_2$ iff $\sigma(((x \odot_1 y]) \wedge_1 z)) = \sigma(x \odot_1 y)$. Since σ is residuated, we have that $\sigma(((x \odot_1 y]) \wedge_1 z)) = \sigma(x \odot_1 y)$ iff $\sigma(x \wedge_1 ((y \wedge_1 z) \sim_1 y) = \sigma(x)$ iff $\sigma(x) \to_2 \sigma(x \wedge_1 ((y \wedge_1 z) \sim_1 y) = 1_2$ iff $\sigma(x \to_1 (x \wedge_1 (y \wedge_1 z) \sim_1 y)) = 1_2$ iff $x \to_1 (x \wedge_1 (y \wedge_1 z) \sim_1 y) \in K$ iff $x \sim_1 (x \wedge_1 (y \wedge_1 z) \sim_1 y) \in K$ iff $[x]_K = [(x \wedge_1 (y \wedge_1 z) \sim_1 y)]_K$ iff $[x]_K = [x]_K \wedge ([y]_K \wedge [z]_K \sim [y]_K)$. Similarly we can prove that if $\mathcal{E}_1/\approx_K$ is residuated, then σ is residuated.
(2) The proof is similar to one of (1). □

Now we introduce a notion of divisible EQ-algebras.

Definition 3.23. An EQ-algebra $\mathcal{E}$ is called divisible if it satisfies the following condition: for all $x, y \in \mathcal{E}$, $x \wedge y = x \odot (x \to y)$.

Proposition 3.24. *Let σ be a strong G-state from a residuated EQ-algebra $\mathcal{E}_1$ to a residuated EQ-algebra $\mathcal{E}_2$. Then we have the following:*
(1) if σ is divisible and $\mathcal{E}_2$ is divisible, $\sigma(x \wedge_1 y) = \sigma(x) \wedge_2 \sigma(y)$ for all $x, y \in \mathcal{E}_1$,
(2) $\sigma(x \to_1 y) = \sigma(x) \to_2 \sigma(y)$ for all $x, y \in \mathcal{E}_1$,
(3) $\sigma(x \odot_1 y) = \sigma(x) \odot_2 \sigma(y)$ for all $x, y \in \mathcal{E}_1$.

Proof. (1) Since σ is divisible and $\mathcal{E}_2$ is a divisible EQ-algebra, we have $\sigma(x \wedge_1 y) = \sigma(x) \odot_2 (\sigma(x) \to_2 \sigma(y)) = \sigma(x) \wedge_2 \sigma(y)$.
(2) Since σ is strong, we have $\sigma(x \to_1 y) = \sigma(x \sim_1 (x \wedge_1 y)) = \sigma(x) \sim_2 \sigma(x \wedge_1 y)$. Since $\mathcal{E}_2$ is a divisible EQ-algebra and by (1), we have $\sigma(x \to_1 y) = \sigma(x) \sim_2 (\sigma(x) \wedge_2 \sigma(y)) = \sigma(x) \to_2 \sigma(y)$.
(3) Note that
$\sigma(x \odot_1 y) \to_2 \sigma(z)$
$= \sigma((x \odot_1 y) \to_1 z)$ (by (2))
$= \sigma(x \to_1 (y \to_1 z))$ (by (9) in Proposition 2.8)
$= \sigma(x) \to_2 (\sigma(x) \to_2 \sigma(y))$ (by (2))
$= (\sigma(x) \odot_2 \sigma(y)) \to_2 \sigma(z)$. (by (9) in Proposition 2.8)
On the other hand, by (GS6) of Definition 3.1, $\sigma(x) \odot_2 \sigma(y) = \sigma(z)$ for some $z \in \mathcal{E}_1$. Hence $\sigma(x \odot_1 y) \to_2 (\sigma(x) \to_2 \sigma(y)) = \sigma(x \odot_1 y) \to_2 \sigma(z) = (\sigma(x) \odot_2 \sigma(y)) \to_2 \sigma(z) = (\sigma(x) \odot_2 \sigma(y)) \to_2 (\sigma(x) \odot_2 \sigma(y)) = 1$. This shows that $\sigma(x \odot_1 y) \leq \sigma(x) \odot_2 \sigma(y)$. Combining Proposition 3.11, we have $\sigma(x \odot_1 y) = \sigma(x) \odot_2 \sigma(y)$. □

Theorem 3.25. *Let σ be a strong divisible G-state from a residuated EQ-algebra $\mathcal{E}_1$ to a divisible residuated EQ-algebra $\mathcal{E}_2$.*
(1) If the map $\nu : \mathcal{E}_1 \to \mathcal{E}_1/ker(\sigma)$ is defined by $\nu(x) = x/ker(\sigma)$, then there is an unique map $\eta : \mathcal{E}_1/ker(\sigma) \to \mathcal{E}_2$ such that the following figure is commutative,

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[r, "\sigma"] & \mathcal{E}_2 \\
\mathcal{E}_1/ker(\sigma) \arrow[u, "\nu"] \arrow[ur, "\eta"'] &
\end{tikzcd}$$

(2) $\mathcal{E}_1/ker(\sigma) \cong \sigma(\mathcal{E}_2)$.

Proof. (1) Define a map $\eta : \mathcal{E}_1/ker(\sigma) \to \mathcal{E}_2$ by $\eta(x/ker(\sigma)) = \sigma(x)$. Then we can check that η is well-defined. Indeed, let $x/ker(\sigma) = y/ker(\sigma)$, then $x \to_1 y, y \to_1 x \in kae(\sigma)$. Hence $\sigma(x \to_1 y) = 1, \sigma(y \to_1 x) = 1$. Since σ is strong, then $\sigma(x) \to_2 \sigma(y)) = 1, \sigma(y) \to_2 \sigma(x)) = 1$, that is $\sigma(x) = \sigma(y)$. Now we prove $\sigma = \eta\nu$. For any $x \in \mathcal{E}_1$, $(\eta\nu)(x) = \eta(\nu(x)) = \eta(x/ker(\sigma)) = \sigma(x)$, that is, $\sigma = \eta\nu$. Then we shows that η is unique. Assume there is $\xi : \mathcal{E}_1/ker(\sigma) \to \mathcal{E}_2$ such that $\xi\nu = \sigma$. Then for any $x/ker(\sigma) \in \mathcal{E}_1/ker(\sigma)$, we have $\xi(x/ker(\sigma)) = \xi(\nu(x)) = (\xi\nu)(x) = (\eta\nu)(x) = \eta(\nu(x)) = \eta(x/ker(\sigma))$.
(2) We only need to prove that η is an EQ-isomorphism from $\mathcal{E}_1/ker(\sigma)$ to $\sigma(\mathcal{E}_2)$. From Proposition 3.24, we can check that η is an EQ-homomorphism from $\mathcal{E}_1/ker(\sigma)$ to $\sigma(\mathcal{E}_2)$. Now we prove that η is injective. Let $x/ker(\sigma) \neq y/ker(\sigma)$. If $\eta(x/ker(\sigma)) = \eta(y/ker(\sigma))$, then $\sigma(x) = \sigma(y)$ and so $\sigma(x \to y) = \sigma(x) \to \sigma(y) = 1$ and $\sigma(y \to x) = \sigma(y) \to \sigma(x) = 1$. This shows that $x \to y, y \to x \in ker(\sigma)$, and hence $x/ker(\sigma) = y/ker(\sigma)$. This contradicts to $x/ker(\sigma) \neq y/ker(\sigma)$. Clearly η is surjective. □

4. Generalized states and (internal) states on EQ-algebras

In [3], R.A. Borzooei introduced (internal) states on EQ-algebras. Now we discuss the relations between generalized states and (internal) states on EQ-algebras. In this section, we suppose that EQ-algebras are with a bottom element 0.

Definition 4.1. ([3]) Bosbach state on an EQ-algebra $\mathcal{E}$ is a function $s : \mathcal{E} \to [0, 1]$ such that the following conditions hold:
(BS1) $s(x) + s(x \to y) = s(y) + s(y \to x)$, for all $x, y \in \mathcal{E}$,
(BS2) $s(0) = 0$ and $s(1) = 1$.

Proposition 4.2. *([3]) Let s be a Bosbach state on an EQ-algebra $\mathcal{E}$. Then the following properties hold, for all $x, y, z \in \mathcal{E}$:*
(1) $x \leq y$ implies $s(x) \leq s(y)$,
(2) $x \leq y$ implies $s(y \to x) = 1 - s(y) + s(x) = s(x \sim y)$.

Proposition 4.3. *Let σ be a strong G-state from an EQ-algebra $\mathcal{E}_1$ to an EQ-algebra $\mathcal{E}_2$. If s is a Bosbach state on $\mathcal{E}_2$ then $s\sigma$ is a Bosbach state on $\mathcal{E}_1$.*

Proof. First we prove (BS1). For $x, y \in \mathcal{E}_1$, $(s\sigma)(x) + (s\sigma)(x \to_1 y) = s(\sigma(x)) + s(\sigma(x \to_1 y)) = s(\sigma(x)) + s(\sigma(x) \to_2 \sigma(y))) = s(\sigma(y)) + s(\sigma(x) \to_2 \sigma(y)) = s(\sigma(y)) + s(\sigma(x \to_1 y)) = (s\sigma)(y) + (s\sigma)(x \to_1 y))$. □

Definition 4.4. ([3]) A state-morphism on an EQ-algebra $\mathcal{E}$ is a function $m : E \to [0,1]$ such that:
(SM1) $m(0) = 0$,
(SM2) $m(x \to y) = min\{1, 1 - m(x) + m(y)\}$.

Proposition 4.5. ([3]) *Every state-morphism on an EQ-algebra $\mathcal{E}$ is a Bosbach state.*

Note that if m is a state-morphism on an EQ-algebra $\mathcal{E}$. Then $m(x^*) = 1-m(x)$, and $m(x^{**}) = m(x)$ where $x^* = x \to 0$.

Proposition 4.6. *Every state-morphism on a good EQ-algebra $\mathcal{E}$ is a G-state from $\mathcal{E}$ to the EQ-algebra $\mathcal{E}_0 = ([0,1], \wedge_0, \odot_0, \sim_0, 1)$ given in Example 2.4.*

Proof. Assume m is a state-morphism on $\mathcal{E}$. By Propositions 4.2(1) and 4.5, (GS1) holds.

Now we check (GS2). Let $x, y \in X$. By Proposition 2.8(6), $y \leq (x \sim x \wedge y) \sim y$. By Proposition 4.2(2), we have
$m((x \sim x \wedge y) \sim y) \sim_0 m(y)$
$= 1 - (m((x \sim x \wedge y) \sim y) - m(y))$
$= 1 + m(y) - m((x \sim x \wedge y) \sim y)$. (*)
By Proposition 2.8(2), $y \leq x \sim x \wedge y$ and hence
$m((x \sim x \wedge y) \sim y) = m((x \sim x \wedge y) \to y) = 1 + m(y) - m(x \sim x \wedge y)$. (**)
Taking (**) into (*) we can get $m((x \sim x \wedge y) \sim y) \sim_0 m(y) = m(x \sim x \wedge y)$. This shows that (GS2) holds.

For (GS3), we have
$m(x) \odot_0 m(x \to x \odot y)$
$= 0 \vee (m(x) + m(x \to x \odot y) - 1)$
$= 0 \vee (m(x) + (1 - m(x) + m(x \odot y)) - 1)$
$= 0 \vee m(x \odot y)$
$= m(x \odot y)$.

Let $x, y \in \mathcal{E}$ and $m(x) \leq m(y)$. Then $m(x) \sim_0 m(y) = 1 - |m(x) - m(y)| = 1 - m(y) + m(x) = min\{1, 1 - m(y) + m(x)\} = m(y \to x)$ and hence $m(x) \sim_0 m(y) \in m(\mathcal{E})$. This shows that (GS4) holds.

(GS5) is obviously true.

For (GS6), we have
$m((x^{**} \to y^*)^*) = 1-m(x^{**} \to y^*) = 1-(1-m(x^{**})+m(y^*)) = m(x)+m(y)-1$.
So $m(x) \odot_0 m(y) = 0 \vee (m(x) + m(y) - 1) = 0$ or $m((x^{**} \to y^*)^*)$.
It follows that $m(x) \odot_0 m(y) \in m(E)$, that is (GS6). □

Example 4.7. Consider the EQ-algebra $\mathcal{E}_2$ given in Example 3.6 and define a map $\sigma : \mathcal{E}_2 \to \mathcal{E}_2$ by $\sigma(0_2) = 0_2, \sigma(a_2) = a_2, \sigma(b_2) = 1_2, \sigma(1_2) = 1_2$. Then we can see that σ is a GI-state but it is not a state operator. This shows that a GI-state need not be a state operator on an EQ-algebra.

Now we discuss the relations between G-states and state operators on EQ-algebras.

Example 4.8. Consider the EQ-algebra $\mathcal{E}_1$ given in Example 3.6 and the identity map $Id_{\mathcal{E}_1}$. Obviously $Id_{\mathcal{E}_1}$ is a state operator of $\mathcal{E}_1$ but it is not a GI-state of $\mathcal{E}_1$ since $1_1 \odot_1 (1_1 \to_1 1_1 \odot_1 b) = 1 \nleq 1_1 \odot_1 b = b$, that is, (GS3) does not hold. This shows that a state operator need not be a GI-state on an EQ-algebra.

Proposition 4.9. *Let $\mathcal{E}$ be a good EQ-algebra and μ be a state operator on $\mathcal{E}$. Then μ is a strong GI-state if and only if it satisfies (GS6).*

Proof. The necessity is obvious. We prove the sufficiency. Let μ satisfy (GS6).
By Lemma 2.16(iii), (GS1) holds.
By (SO3), (GS4) holds.
By (SO4), (GS5) holds.
For (GS3), using Proposition 2.5(5), we have
$\mu(x) \odot \mu(x \to (x \odot y))$
$= \mu(x) \odot (\mu(x) \sim (\mu(x) \wedge \mu(x \odot y)))$
$= \mu(x) \odot (\mu(x) \sim \mu(x \odot y))$
$= (1 \sim \mu(x)) \odot (\mu(x) \sim \mu(x \odot y))$
$\leq 1 \sim \mu(x \odot y)$
$= \mu(x \odot y)$.
This shows that (GS3) holds.
To prove (GS2), let us consider the following. By Lemma 2.16(iv) and Proposition 2.7(3), $\mu((x \sim x \wedge y) \sim y) \sim \mu(y) = \mu((x \to y) \to y) \to \mu(y) = ((\mu(x) \to \mu(y)) \to \mu(y)) \to \mu(y) = \mu(x) \to \mu(y) = \mu(x \to y) = \mu(x \sim x \wedge y)$. This shows that (GS2) holds. Moreover by Lemma 2.17 $\mu^2 = \mu$. Therefore μ is a GI-state. By (SO3), μ is strong. □

Proposition 4.10. *Let σ be a G-state from a residuated EQ-algebra $\mathcal{E}_1$ to a residuated EQ-algebra $\mathcal{E}_2$.*
(1) If μ is a state operator on $\mathcal{E}_1$ satisfying (GS6), then $\sigma\mu$ is a G-state from $\mathcal{E}_1$ to $\mathcal{E}_2$.
(2) If μ is a state operator on $\mathcal{E}_1$ satisfying (GS6) and σ is strong, then $\sigma\mu$ is a strong G-state from $\mathcal{E}_1$ to $\mathcal{E}_2$.

Proof. (1) It is clear that $\sigma\mu$ satisfies (GS1). For (GS2), we have
$(\sigma\mu)(x \sim_1 (x \wedge_1 y))$
$= \sigma(\mu(x \sim_1 (x \wedge_1 y))$
$= \sigma(\mu(x \to_1 y)) =$
$= \sigma(\mu(x) \to_1 \mu(y))$ (by Lemma 2.16(4))
$= \sigma(\mu(x) \sim_1 (\mu(x) \wedge \mu(y)))$
$= \sigma(\mu(x) \sim_1 (\mu(x) \wedge \mu(y)) \sim_1 \mu(y)) \sim_2 \sigma(\mu(y))$ (σ satisfies (GS2))
$= \sigma((\mu(x) \to_1 \mu(y)) \to_1 \mu(y)) \sim_2 \sigma(\mu(y))$
$= (\sigma\mu)(x \to_1 y) \to_1 y) \sim_2 (\sigma\mu)(y)$
$= (\sigma\mu)(x \sim_1 (x \wedge y)) \sim_1 y) \sim_2 (\sigma\mu)(y)$.
Now we check (GS3). Since μ satisfies (GS6), by Proposition 4.9, μ is a strong GI-state on $\mathcal{E}_1$. By proposition 3.24(3), we have $\mu(x \odot_1 y) = \mu(x) \odot_1 \mu(y)$. Hence we have
$(\sigma\mu)(x) \odot_2 (\sigma\mu)(x \to_1 (x \odot_1 y))$

$= \sigma(\mu(x)) \odot_2 \sigma(\mu(x \rightarrow_1 (x \odot_1 y)))$
$= \sigma(\mu(x)) \odot_2 \sigma(\mu(x) \rightarrow_1 \mu(x \odot_1 y))$ (since μ is strong)
$= \sigma(\mu(x)) \odot_2 \sigma(\mu(x) \rightarrow_1 (\mu(x) \odot_1 \mu(y)))$
$\leq \sigma(\mu(x) \odot_1 \mu(y))$ (since σ satisfies (GS3))
$= \sigma(\mu(x \odot_1 y))$
$= (\sigma\mu)(x \odot_1 y)$.
It is easy to check that $\sigma\mu$ satisfies (GS4),(GS5) and (GS6). Hence $\sigma\mu$ is a G-state from $\mathcal{E}_1$ to $\mathcal{E}_2$.
(2) Straightforward. □

5. Generalized states on EQ-algebras and internal states on residuated lattices

In [17], the internal states on residuated lattices are introduced and studied. In this section we discuss the relations between generalized states on EQ-algebras and internal states on residuated lattices.

Definition 5.1. ([17]) Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a residuated lattice. A mapping $\tau : L \rightarrow L$ is called a state operator on L if it satisfies the following conditions:
(L1) $\tau(0) = 0$,
(L2) $x \rightarrow y = 1$ implies $\tau(x) \rightarrow \tau(y) = 1$,
(L3) $\tau(x \rightarrow y) = \tau(x) \rightarrow \tau(x \wedge y)$,
(L4) $\tau(x \odot y) = \tau(x) \odot \tau(x \rightarrow (x \odot y))$,
(L5) $\tau(\tau(x) \odot \tau(y)) = \tau(x) \odot \tau(y)$,
(L6) $\tau(\tau(x) \rightarrow \tau(y)) = \tau(x) \rightarrow \tau(y)$,
(L7) $\tau(\tau(x) \vee \tau(y)) = \tau(x) \vee \tau(y)$,
(L8) $\tau(\tau(x) \wedge \tau(y)) = \tau(x) \wedge \tau(y)$,
for any $x, y \in L$.

The pair (L, τ) is said to be a state residuated lattice, or more precisely, a residuated lattice with internal state.

Proposition 5.2. *([17]) Let (L, τ) be a state residuated lattice, then for any $x, y \in L$ we have:*
(1) $\tau(1) = 1$,
(2) $x \leq y$ implies $\tau(x) \leq \tau(y)$,
(3) $\tau^2(x) = \tau(x)$,
(4) $\tau(x \odot y) \geq \tau(x) \odot (y)$ and if $x \odot y = 0$, then $\tau(x \odot y) = \tau(x) \odot \tau(y)$,
(5) $\tau(x \odot y^*) \geq \tau(x) \odot (\tau(y))^*$ and if $x \leq y$, then $\tau(x \odot y^*) = \tau(x) \odot (\tau(y))^*$,
(6) $\tau(x \rightarrow y) \leq \tau(x) \rightarrow \tau(y)$. In particular, if x, y are comparable, then $\tau(x \rightarrow y) = \tau(x) \rightarrow \tau(y)$.

The following proposition shows that the notion of state EQ-algebras is a generalization of the notion of state residuated lattices.

Proposition 5.3. *Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a residuated lattice and (L, τ) be a state residuated lattice. Then we have the following.*
(1) $\mathcal{E} = (L, \wedge, \odot, \sim, 1)$ is an EQ-algebra, where the operator $\sim$ is defined by

$x \sim y := (x \to y) \wedge (y \to x)$,
(2) τ *is a* GI*-state on the* EQ*-algebra* $\mathcal{E}$.

Proof. (1) It follows from [27].
(2) It only need to prove that (GS2) holds. Note that $\tau(x \sim x \wedge y) = \tau((x \to x \wedge y) \wedge (x \wedge y \to x)) = \tau((x \to x \wedge y) \wedge 1) = \tau(x \to x \wedge y) = \tau(x \to y)$ by Proposition 2.11(12). On the other hand, $\tau((x \sim x \wedge y) \sim y) \sim \tau(y) = \tau((x \to y) \sim y) \sim \tau(y) = \tau((x \to y) \to y) \sim \tau(y)$. Since $y \leq (x \to y) \to y$, we have $\tau((x \to y) \to y) \sim \tau(y) = \tau((x \to y) \to y) \to \tau(y) = \tau(((x \to y) \to y) \to y) = \tau(x \to y)$ by Proposition 2.5(15) and Proposition 5.2(6). This shows that (GS2) holds. $\square$

Example 5.4. Let $L = [0,1]$ be the real unit interval. For all $x, y \in L$, we define $x \odot y = min\{x, y\}$ and $x \to y = \begin{cases} 1, & x \leq y \\ y, & otherwise \end{cases}$, then $(L, min, max, \odot, \to, 0, 1)$ becomes a residuated lattice, which is called Gödel structure. Now, for any $a \in L$, we define a map τ_a on L as follows:

$$\tau_a(x) = \begin{cases} x, & x \leq a \\ 1, & otherwise \end{cases}.$$

One can check that τ_a is a state operator on L. Therefore, (L, τ_a) is a state residuated lattice. Denote $\mathcal{E} = (L, min, \odot, \sim, 1)$, where

$$x \sim y := min\{x \to y, y \to x\} = \begin{cases} x, & x < y, \\ 1, & x = y,. \\ y, & y < x, \end{cases}$$

By Proposition 5.3, we have that τ_a is a GI-state on the EQ-algebra $\mathcal{E}$.

6. CONCLUSIONS

In this paper, we introduced a notion of generalized states on EQ-algebras. We gave a unified model of states and internal states on some important logic algebras. By the arguments in the paper we can see that generalized states on an EQ-algebra not only are generalization of states and internal states on EQ-algebras, but also are generalization of internal states on residuated lattices. Moreover generalized states on an EQ-algebra are state operators on their reductions, as equality algebras and BCK-algebras. Indeed let $(\mathcal{E}, \wedge, \odot, \sim, 1)$ be a good EQ-algebra and σ be a GI-state on $\mathcal{E}$. Then we can get the following results: (1) $(\mathcal{E}, \wedge, \sim, 1)$ is an equality algebra and (2) $(\mathcal{E}, \wedge, \sim, \sigma, 1)$ is a state equality algebra. At the same time let $(\mathcal{E}, \wedge, \odot, \sim, 1)$ be a good EQ-algebra and σ be a GI-state on $\mathcal{E}$, then we have the following: (3) $(\mathcal{E}, \wedge, \to, 1)$ is a BCK-algebra with meet, where $x \to y = x \sim (x \wedge y)$, and (4) $(\mathcal{E}, \wedge, \to, \sigma, 1)$ is a left state BCK-algebra.

In the next work, it is worthy to portray some types of logic algebras and corresponding logics by use of generalized states.

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