A DUALITY BETWEEN LM-FUZZY POSSIBILITY COMPUTATIONS AND THEIR LOGICAL SEMANTICS

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Abstract. Let X be a dcpo and let L be a complete lattice. The family $\sigma_L(X)$ of all Scott continuous mappings from X to L is a complete lattice under pointwise order, we call it the L-fuzzy Scott structure on X. Let E be a dcpo. A mapping $g : \sigma_L(E) \longrightarrow M$ is called an LM-fuzzy possibility valuation of E if it preserves arbitrary unions. Denote by $\pi_{LM}(E)$ the set of all LM-fuzzy possibility valuations of E. The denotational semantics assigning to an LM-fuzzy possibility computation from a dcpo D to another one E is a Scott continuous mapping from D to $\pi_{LM}(E)$, which is a model of non-determinism computation in Domain Theory. A healthy LM-fuzzy predicate transformer from D to E is a sup-preserving mapping from $\sigma_L(E)$ to $\sigma_M(D)$, which is always interpreted as the logical semantics from D to E. In this paper, we establish a duality between an LM-fuzzy possibility computation and its LM-fuzzy logical semantics.

1. Introduction

As has been described in [20], the semantics of programming languages has been intensively studied by both mathematicians and computer scientists. By proposing domain theory, in the late 1960s Dana S. Scott invented appropriate semantic domains in [16, 17]. In domain theory, computation in general involves two classes: determinism and non-determinism. Deterministic computation means that the computed results are deterministic by a given input. However, if a non-deterministic program runs several times with the same input, it may produce different outputs. As we know, an important problem in domain theory is the modelling of non-deterministic features of programming languages and of parallel features treated in a non-deterministic way. In order to describe this non-determinism, the concept of powerdomain [6, 11, 12, 19] is introduced. Another method for description of non-determinism consists in quantifying this non-determinism by means of probability or possibility. Probabilistic non-determinism has also been studied and has led to the probabilistic powerdomain as a model [8, 9, 13, 15]. Different runs of a probabilistic program with the same input may again result in different outputs. In this situation, we need to know how likely these outputs are. Thus, a probability distribution or continuous valuation on the domain of states is chosen to describe

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such a behaviour. Based on this idea, Chen and Wu [3] proposed the concept of possibility valuation.

In computer science, standard or classic predicates are subsets of states, or can equivalently be regarded as $\{0,1\}$-valued functions defined on states. Chen and Jung [2] generalized classic predicates to the fuzzy case in such a way that fuzzy predicates on a dcpo D are Scott-continuous functions from D to the unit interval $[0,1]$. Lu [10] generalized [0,1] to a complete residuated lattice and studied it as a model of fuzzy Scott topology. In [3], Chen and Wu introduced the notion of healthy fuzzy predicate transformers and gave the logical semantics of possibility computations. They successfully established the duality (an equivalence) between denotational and logical semantics of possibility computations. Shen et al. [18] generalized Chen-Wu's duality into quantale-valued fuzzy ordered setting.

In this paper, we will consider variable-basis type of non-deterministic computation with respect to two complete lattice L and M, called LM-fuzzy possibility computations. The LM-fuzzy possibility computations, in contrast with the possibility computations, consider the LM-fuzzy possibility degree (i.e., the results of possibility value [0,1] are replaced by two different complete lattice L, M) as a state that the domain of states lies in an $L(M)$-valued fuzzy Scott open set on dcpos, which is more extensive than the possibility computation in [3]. In other words, we will define an LM-fuzzy possibility distribution of states on the fuzzy Scott structure of dcpos by giving the LM-fuzzy possibility that a state belongs to a certain fuzzy Scott open set. The goal of this kind of non-deterministic computations is to give, for a given input, the LM-fuzzy possibility distributions of final states on fuzzy Scott topology on dcpos. Precisely, for a given dcpo, we will establish a one-to-one correspondence between an LM-fuzzy possibility computation and its LM-fuzzy logical semantics. Sometimes, we also say this correspondence a duality or an equivalence between an LM-fuzzy possibility computation and its LM-fuzzy logical semantics.

2. **Preliminaries**

We refer to [4] for lattice theory and domain theory, to [7] for fuzzy sets and fuzzy topology.

Let L be a complete lattice. The greatest element of L is denoted by 1 and the least element of L is denoted by 0. For $S \subseteq L$, write $\bigvee S$ for the least upper bound of S and $\bigwedge S$ for the greatest lower bound of S. In particular, $\bigvee \emptyset = 0$ and $\bigwedge \emptyset = 1$. If $S = \{s_i |\ i \in I\}$, then we write $\bigvee_i s_i$ and $\bigwedge_i s_i$ instead of $\bigvee S$ and $\bigwedge S$ respectively. For the two-element lattice $\{0,1\}$, we always write it as 2. Let L be a complete lattice and let X be set. For $a \in L$, a_X denotes the constant mapping from X to L with the value a.

Let L be a complete lattice and let $* : L \times L \longrightarrow L$ be a semigroup operation. If for every $a \in L$ and every directed subset $D \subseteq L$, it holds that

$$a * (\bigvee D) = \bigvee_{d \in D} (a * d) \text{ and } (\bigvee D) * a = \bigvee_{d \in D} (d * a),$$

then we call the pair $(L, *)$ a $*$-*continuous semigroup*.

A $*$-continuous semigroup $(L,*)$ is called a *quantale* [14] if $*$ is distributive over arbitrary joins. A quantale $(L,*)$ is called *integral* [14] if the top element 1 is the unit of $*$, i.e., $1*a=a=a*1$ holds for every $a\in L$. It is a routine to check that if the operation $*$ is commutative, then $(L,*)$ is an integral quantale iff it is a *complete residuated lattice* [5].

A *stratified* L*-topology* on a set X is a subfamily $\delta\subseteq L^X$ that satisfies:
(T0) $a_X\in\delta$ for every $a\in L$;
(T1) $\forall A,B\in\delta,\ A*B\in\delta$ (Here $A*B$ is defined pointwisely);
(T2) $\forall\{A_i|\ i\in I\}\subseteq\delta,\ \bigvee_i A_i\in\delta$.

A nonempty subset D of a poset X is called *directed* if for all $a,b\in D$ there exists an upper bound $c\in D$ of $\{a,b\}$. Denote by $\mathcal{D}(X)$ the family of all directed subset of X. A poset X is called *directed complete* (or a *dcpo*, in short) if each directed subset of X has a join.

A subset $U\subseteq X$ is called *Scott open* if U is an upper set and $\bigvee D\in U$ implies $D\cap U\neq\emptyset$ for all directed subset $D\subseteq X$. The family of all Scott open sets forms a topology on X, called the *Scott topology*, in symbols $\sigma(X)$. For every $x\in X$, $X\backslash{\downarrow}x$ always is a Scott open set of X.

A mapping $f:X\longrightarrow Y$ between two dcpos is called *Scott continuous* if f preserves joins of directed subsets, which is equivalently to say that, f is continuous with respect to the Scott topologies. For two dcpos D and E, we use the notation $[D\to E]_S$ to denote the set of all Scott continuous mappings from D to E.

3. Duality between LM-fuzzy possibility computations and their LM-fuzzy logical semantics

In this section, we will generalize the contents in [3] from [0,1] to two complete lattices L and M. Here different dcpos may have different valued sets.

For the ordinary Scott topology on a dcpo X, it is well-known that a subset U is Scott open iff $U:X\longrightarrow\{0,1\}$ is Scott continuous. It motivates us to define a fuzzy Scott topology on X.

Proposition 3.1. *Let X be a dcpo and let L be a complete lattice. The family $\sigma_L(X)$ of all Scott continuous mappings from X to L is a complete lattice under pointwise order.*

Proof. For any $a\in L$, $a_X:X\longrightarrow L$ is obviously a Scott continuous mapping. Thus $a_X\in\sigma_L(X)$. Suppose that $\{A_i|\ i\in I\}\subseteq\sigma_L(X)$. For each directed subset $S\subseteq X$, $\bigvee_i A_i(\bigvee S)=\bigvee_i\bigvee_{s\in S}A_i(d)=\bigvee_{s\in S}\bigvee_i A_i(s)$. Thus $\bigvee_i A_i\in\sigma_L(X)$. □

The family $\sigma_L(X)$ in Proposition 3.1 maybe is not an L-topology since we can not prove that it is closed under finite meets. In this case we call $\sigma_L(X)$ the *L-fuzzy Scott structure* on X. But for $L=[0,1]$ with $*=\wedge$, $\sigma_L(X)$ turns out to be the fuzzy Scott topology in [3]; for a complete residuated lattice L, $\sigma_L(X)$ turns out to be the fuzzy Scott topology in [10]; and for a $*$-continuous complete lattice L, $\sigma_L(X)$ turns out to be the fuzzy Scott topology in [21].

Definition 3.2. Let E be a dcpo. A mapping $g : \sigma_L(E) \longrightarrow M$ is called an *LM-fuzzy possibility valuation* of E if it preserves arbitrary unions, i.e., $g(\bigvee \mathcal{A}) = \bigvee g(\mathcal{A})$ for each $\mathcal{A} \subseteq \sigma_L(E)$. Denote by $\pi_{LM}(E)$ the set of all LM-fuzzy possibility valuations of E.

Definition 3.3. Let D and E be two dcpos. The *denotational semantics* assigns to an LM-fuzzy possibility computation F from D to E a Scott continuous function $h : D \longrightarrow \pi_{LM}(E)$. Denote by $[D \to \pi_{LM}(E)]_S$ the set of all LM-fuzzy possibility computation of F from D to E.

From this definition, we can see the result of a possibility computation at one input state. For example, the assignment F gives every element $d \in D$ a fuzzy possibility (a value in M) for itself being belonging to an L-fuzzy Scott open set $U \in \sigma_L(E)$.

Definition 3.4. An *LM-fuzzy predicate transformer* from D to E is a mapping $t : \sigma_L(E) \longrightarrow \sigma_M(D)$. An LM-fuzzy predicate transformer is called *healthy*, if it satisfies the following healthy conditions:

(H) Sup-preserving: for $\mathcal{A} \subseteq \sigma_L(E)$, $t(\bigvee \mathcal{A}) = \bigvee t(\mathcal{A})$.

Denote by $[\sigma_L(E) \to \sigma_M(D)]$ the set of all healthy L-fuzzy predicate transformers from D to E under the pointwise order.

Definition 3.5. Let D and E be two dcpos. The *logical semantics* assigns to an LM-fuzzy possibility computation F from D to E a healthy LM-fuzzy predicate transformer from D to E.

Next, we will show how to establish a duality between LM-fuzzy possibility computations and their logical semantics.

Proposition 3.6. *Define*

$$\alpha : [\sigma_L(E) \to \sigma_M(D)] \longrightarrow [D \to \pi_{LM}(E)]_S,$$

$$\alpha(t)(x)(U) = t(U)(x) \ (\forall t \in [\sigma_L(E) \to \sigma_M(D)], \forall x \in D, \forall U \in \sigma_L(E)).$$

Then α is a well-defined mapping.

Proof. Step 1. $\alpha(t)(x) \in \pi_{LM}(E)$ for every $x \in D$. In fact, for $\mathcal{A} \subseteq \sigma_L(E)$, we have

$$\begin{aligned}
[\alpha(t)(x)](\bigvee \mathcal{A}) &= t(\bigvee \mathcal{A})(x) \\
&= (\bigvee t(\mathcal{A}))(x) \\
&= [\bigvee_{U \in \mathcal{A}} t(U)](x) \\
&= \bigvee_{U \in \mathcal{A}} t(U)(x) \\
&= \bigvee_{U \in \mathcal{A}} \alpha(t)(x)(U) \\
&= \bigvee [\alpha(t)(x)](\mathcal{A}).
\end{aligned}$$

Step 2. $\alpha(t) \in [D \to \pi_{LM}(E)]_S$ for every $t \in [\sigma_L(E) \to \sigma_M(D)]$. In fact, for $S \in \mathcal{D}(D)$, for any $U \in \sigma_L(E)$,

$$\begin{array}{rcl} [\alpha(t)(\bigvee S)](U) & = & t(U)(\bigvee S) \\ & = & \bigvee t(U)(S) \\ & = & \bigvee\limits_{x\in S} t(U)(x) \\ & = & \bigvee\limits_{x\in S} \alpha(t)(x)(U) \\ & = & [\bigvee \alpha(t)(S)](U). \end{array}$$

□

Proposition 3.7. *Define*

$$\beta : [D \to \pi_{LM}(E)]_S \longrightarrow [\sigma_L(E) \to \sigma_M(D)]$$

$$\beta(h)(U)(x) = h(x)(U) \ (\forall h \in [D \to \pi_{LM}(E)]_S, \forall x \in D, \forall U \in \sigma_L(E)).$$

Then β is a well-defined mapping.

Proof. Step 1. $\beta(h)(U) \in \sigma_M(D)$ for every $U \in \sigma_L(E)$. In fact, for every $S \in \mathcal{D}(D)$, we have

$$\begin{array}{rcl} \beta(h)(U)(\bigvee S) & = & h(\bigvee S)(U) \\ & = & (\bigvee h(S))(U) \\ & = & (\bigvee\limits_{x\in S} h(x))(U) \\ & = & \bigvee\limits_{x\in S} h(x)(U) \\ & = & \bigvee\limits_{x\in S} \beta(h)(U)(x) \\ & = & \bigvee[\beta(h)(U)](S). \end{array}$$

Step 2. $\beta(h) \in [\sigma_L(E) \to \sigma_M(D)]$ for every $h \in [D \to \pi_{LM}(E)]_S$. Let $\mathcal{A} \subseteq \sigma_L(E)$ and $x \in D$. Then

$$\begin{array}{rcl} \beta(h)(\bigvee \mathcal{A})(x) & = & h(x)(\bigvee \mathcal{A}) \\ & = & \bigvee h(x)(\mathcal{A}) \\ & = & \bigvee\limits_{U\in\mathcal{A}} h(x)(U) \\ & = & \bigvee\limits_{U\in\mathcal{A}} \beta(h)(U)(x) \\ & = & [\bigvee\limits_{U\in\mathcal{A}} \beta(h)(U)](x) \\ & = & [\bigvee(\beta(h))(\mathcal{A})](x). \end{array}$$

□

Theorem 3.8. *Let D and E be two dcpos. Then*

$$[\sigma_L(E) \to \sigma_M(D)] \cong [D \to \pi_{LM}(E)]_S$$

via the pair of functions (α, β).

Proof. Step 1. For all $h \in [D \to \pi_{LM}(E)]_S,\ x \in D$ and $U \in \sigma_L(E))$, we have

$$(\alpha \circ \beta)[(h)(x)(U)] = [\alpha(\beta(h))](x)(U) = \beta(h)(U)(x) = h(x)(U).$$

That is, $\alpha \circ \beta = \mathrm{id}_{[D \to \pi_{LM}(E)]_S}$.

Step 2. For all $t \in [D \to \pi_{LM}(E)]_S,\ x \in D$ and $U \in \sigma_L(E))$, we have

$$(\beta \circ \alpha)[(t)(U)(x)] = [\beta(\alpha(t))](U)(x) = \alpha(t)(x)(U) = t(U)(x).$$

That is, $\beta \circ \alpha = \mathrm{id}_{[\sigma_L(E) \to \sigma_M(D)]}$. □

4. **Two restricted cases and their generalizations**

Case 1. $L = \{0, 1\}$**.**

Since $\pi_{2M}(E) = [\sigma(E) \longrightarrow M]$, by the duality in Theorem 3.8, we have

$$[\sigma(E) \to [D \to M]_S] \cong [D \to [\sigma(E) \longrightarrow M]]_S.$$

Since $\sigma(E)$ is a complete lattice, we can replace it by a complete lattice L, that is,

$$[L \to [D \to M]_S] \cong [D \to [L \to M]]_S,$$

We will prove it step by step.

Proposition 4.1. *Define*

$$\alpha : [L \to [D \to M]_S] \longrightarrow [D \to [L \to M]]_S,$$

$$\alpha(t)(x)(n) = t(n)(x)\ (\forall t \in [L \to [D \to M]_S], \forall x \in D, \forall n \in L).$$

Then α is a well-defined mapping.

Proof. Step 1. $\alpha(t)(x) \in [L \to M]$ for every $x \in D$. In fact, for $A \subseteq L$, we have

$$[\alpha(t)(x)](\bigvee A) = t(\bigvee A)(x) = \bigvee{}_{n\in A} t(n)(x) = \bigvee{}_{n\in A} \alpha(t)(x)(n) = \bigvee [\alpha(t)(x)](A).$$

Step 2. $\alpha(t) \in [D \to [L \to M]]_S$ for every $t \in [L \to [D \to M]_S]$. In fact, for $S \in \mathcal{D}(D)$, for any $n \in L$,

$$[\alpha(t)(\bigvee S)](n) = t(n)(\bigvee S) = \bigvee{}_{x\in S} t(n)(x) = \bigvee{}_{x\in S} \alpha(t)(x)(n) = [\bigvee \alpha(t)(S)](n).$$

□

Proposition 4.2. *Define*

$$\beta : [D \to [L \to M]]_S \longrightarrow [L \to [D \to M]_S]$$

$$\beta(h)(n)(x) = h(x)(n)\ (\forall h \in [D \to [L \to M]]_S, \forall x \in D, \forall n \in L).$$

Then β is a well-defined mapping.

Proof. Step 1. $\beta(h)(n) \in [D \to M]_S$ for every $n \in L$. In fact, for every $S \in \mathcal{D}(D)$, we have

$$\beta(h)(n)(\bigvee S) = h(\bigvee S)(n) = (\bigvee h(S))(n) = \bigvee{}_{x\in S} \beta(h)(n)(x) = \bigvee [\beta(h)(n)](S).$$

Step 2. $\beta(h) \in [L \to [D \to M]_S]$ for every $h \in [D \to [L \to M]]_S$. Let $A \subseteq L$ and $x \in D$. Then

$$\beta(h)(\bigvee A)(x) = h(x)(\bigvee A) = \bigvee{}_{n\in A} h(x)(U) = \bigvee{}_{n\in A} \beta(h)(n)(x) = [\bigvee (\beta(h))(A)](x).$$

□

Theorem 4.3. *Let D be a dcpo and let L, M be two complete lattices. Then*

$$[L \to [D \to M]_S] \cong [D \to [L \to M]]_S.$$

Proof. Almost the same as that of Theorem 3.8. □

Remark 4.4. In a monoidal closed category $\mathbf{C}$ [1], we know that for all objects A, B, C, it holds that

$$\hom_{\mathbf{C}}(A \otimes C, B) \cong \hom_{\mathbf{C}}(C, \hom_{\mathbf{C}}(A, B)).$$

Since $A \otimes C \cong C \otimes A$, we have

$$\hom_{\mathbf{C}}(A, \hom_{\mathbf{C}}(C, B)) \cong \hom_{\mathbf{C}}(C, \hom_{\mathbf{C}}(A, B)).$$

It is well-known that the category **Sup** of complete lattices with sup-preserving mappings is monoidal closed [1], and the category of **DCPO** of dcpos with Scott continuous mappings is Cartesian closed and so is monoidal closed [4]. Clearly, if A and B are two complete lattices, then $\hom_{\mathbf{Sup}}(A, B) = [A \to B]$; if A and B are two dcpos, then $\hom_{\mathbf{DCPO}}(A, B) = [A \to B]_S$. Then, if A, B, C are complete lattices, then

$$[C \to [A \to B]] \cong [A \to [C \to B]].$$

If A, B, C are dcpos, then

$$[C \to [A \to B]_S]_S \cong [A \to [C \to B]_S]_S.$$

Theorem 4.3 means that, in categorical terminology,

$$\hom_{\mathbf{Sup}}(L, \hom_{\mathbf{DCPO}}(D, M)) \cong \hom_{\mathbf{DCPO}}(D, \hom_{\mathbf{Sup}}(L, M)).$$

It is very interesting that this duality seems a mutually interaction between the monoidal closedness of **Sup** and **DCPO**, which is a new result in lattice theory.

Case 2. $M = \{0, 1\}$**.**

For a complete lattice L, it is easily proved that a member in $[L \to 2]$ just has the form of characterization functions of $L \backslash {\downarrow} n$ for a unique $n \in L$. So $[L \to 2]$ is order isomorphic to L^{op}. In this sense, $\pi_{L2}(E) = [\sigma_L(E) \to 2] \cong \sigma_L(E)^{op}$. So for $M = 2$, the duality in Theorem 3.8 becomes the case

$$[D \to \sigma_L(E)^{op}]_S \cong [\sigma_L(E) \to \sigma(D)].$$

Since $\sigma_L(E)$ is a complete lattice, replacing it by a complete lattice L, we have the following generalized duality:

$$[D \to L^{op}]_S \cong [L \to \sigma(D)].$$

We will prove it step by step.

Theorem 4.5. *Define $\alpha : [D \to L^{op}]_S \longrightarrow [L \to \sigma(D)]$ by*

$$\alpha(t)(n) = t^{-1}(L \backslash {\uparrow} n) \ (\forall t \in [D \to L^{op}]_S, \ \forall n \in L).$$

Then α is a well-defined mapping.

Proof. Step 1. The subset $L\backslash\uparrow n$ in L is the same as $L^{op}\backslash\downarrow n$ in L^{op}, which is a Scott open set of L^{op}. Since $t : D \to L^{op}$ is Scott continuous, we have

$$\alpha(t)(n) = t^{-1}(L\backslash\uparrow n) = t^{-1}(L^{op}\backslash\downarrow n) \in \sigma(D).$$

Step 2. Firstly, $\alpha(t)(0) = t^{-1}(\emptyset) = \emptyset$. Secondly, for $\{n_i |\ i \in I\} \subseteq L$, we have

$$\alpha(t)(\bigvee{}_i n_i) = t^{-1}(L\backslash\uparrow(\bigvee{}_i n_i)) = t^{-1}(\bigcup{}_i N\backslash\uparrow n_i) = \bigcup{}_i t^{-1}(L\backslash\uparrow n_i) = \bigcup{}_i \alpha(t)(n_i).$$

□

For every $h \in [L \to \sigma(D)]$, it has a right adjoint $h^* : \sigma(D) \longrightarrow L$ [4], which is given by

$$h^*(U) = \bigvee\{n \in L|\ h(n) \subseteq U\}.$$

By the Adjoint Theorem, it holds that $h(n) \subseteq U$ iff $n \leq h^*(U)$ for all $n \in L$ and $U \in \sigma(E)$.

Lemma 4.6. *Let $h \in [L \to \sigma(D)]$, $n \in L$ and $x \in D$. It holds that*

$$n \not\leq h^*(D\backslash\downarrow x) \textit{ iff } x \in h(n).$$

Proof. $n \not\leq h^*(D\backslash\downarrow x)$ iff $h(n) \not\subseteq D\backslash\downarrow x$ iff $h(n) \cap \downarrow x \neq \emptyset$ iff $x \in h(n)$, notice that $h(n)$ is an upper set. □

Theorem 4.7. *Define $\beta : [L \to \sigma(D)] \longrightarrow [D \to L^{op}]_S$ by*

$$\beta(h)(x) = h^*(D\backslash\downarrow x)\ (\forall h \in [L \to \sigma(D)],\ \forall x \in D).$$

Then β is a well-defined mapping.

Proof. We need to show that $\beta(h) : D \to L^{op}$ is a Scott continuous mapping. That is for every direct subset S in D, it holds that $\beta(h)(\bigvee S) = \bigwedge_{s\in S} \beta(h)(s)$ in L. In fact, by Lemma 4.6, we have $a \not\leq \beta(h)(x)$ iff $x \in h(a)$. Now we have

$$\begin{array}{rll} a \not\leq \beta(h)(\bigvee S) & \text{iff} & \bigvee S \in h(a) \\ & \text{iff} & \exists s \in S,\ s \in h(a) \\ & \text{iff} & \exists s \in S,\ a \not\leq \beta(h)(s) \\ & \text{iff} & a \not\leq \bigwedge_{s\in S} \beta(h)(s). \end{array}$$

Notice that $h(a)$ is a Scott open set and S is directed. □

Lemma 4.8. *For all $t \in [D \to L^{op}]_S$, $n \in L$ and $x \in D$, it holds that*

$$t^{-1}(L\backslash\uparrow n) \cap \downarrow x = \emptyset \textit{ iff } x \notin t^{-1}(L\backslash\uparrow n).$$

Proof. It is obvious since $t^{-1}(L\backslash\uparrow n) \in \sigma(D)$ which is an upper set. □

Theorem 4.9. *Let D be a dcpo and let L be a complete lattice. Then*

$$[D \to L^{op}]_S \cong [L \to \sigma(D)].$$

Proof. Step 1. For all $h \in [L \to \sigma(D)]$ and $a \in L$, we have

$$\begin{array}{rcl}(\alpha \circ \beta)(h)(a) & = & (\beta(h))^{-1}(L \backslash \uparrow a) \\ & = & \{x \in D |\ a \not\leq \beta(h)(x)\} \\ & = & \{x \in D |\ a \not\leq h^*(D \backslash \downarrow x)\} \\ & = & \{x \in D |\ x \in h(a)\} \\ & = & h(a).\end{array}$$

Hence $\alpha \circ \beta = \mathrm{id}_{[L \to \sigma(D)]}$.

Step 2. For all $t \in [D \to L^{op}]_S$ and $x \in D$, we have

$$\begin{array}{rcl}(\beta \circ \alpha)(t)(x) & = & (\alpha(t))^*(D \backslash \downarrow x) \\ & = & \bigvee\{n \in L |\ \alpha(t)(n) \subseteq D \backslash \downarrow x\} \\ & = & \bigvee\{n \in L |\ t^{-1}(L \backslash \uparrow n) \cap \downarrow x = \emptyset\} \\ & = & \bigvee\{n \in L |\ x \notin t^{-1}(L \backslash \uparrow n)\} \\ & = & \bigvee\{n \in L |\ n \leq t(x)\} \\ & = & t(x).\end{array}$$

Hence, $\beta \circ \alpha = \mathrm{id}_{[D \to L^{op}]_S}$. □

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