THE FUZZY GENERALIZED TAYLOR'S EXPANSION WITH APPLICATION IN FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, the generalized Taylor's expansion is presented for fuzzy-valued functions. To achieve this aim, fuzzy fractional mean value theorem for integral, and some properties of Caputo generalized Hukuhara derivative are necessary that we prove them in details. In application, the fractional Euler's method is derived for solving fuzzy fractional differential equations in the sense of Caputo differentiability. The effectiveness of the proposed method is verified by three examples.

1. Introduction

Fractional calculus have been the focus of many studies due to their frequent appearance in various applications in fluid flow, electrical networks, fractal theory, control theory, optics, biology, chemistry and etc [9]. The introduction of fuzzy sets in the model allows a better description of the uncertainty which appears in applications [25]. Therefore, fuzzy fractional differential equations have attracted lots of attention in mathematics and engineering researches. First work devoted to the subject of fuzzy fractional differential equations is the paper by Agarwal et al. [1]. They have defined the Riemann-Liouville differentiability concept under the Hukuhara differentiability to solve fuzzy fractional differential equations. Specifically, in [10], the existence and uniqueness are proved for solutions of fuzzy fractional differential equations under Riemann-Liouville differentiability. Ref.[29] deals with the solutions of fuzzy fractional differential equations under Riemann-Liouville H-differentiability by fuzzy Laplace transforms. Moreover, the explicit solutions of uncertain fractional differential equations under Riemann-Liouville H-differentiability using Mittag-Leffler functions are obtained in [7]. Authors in [3] proved the existence and uniqueness results for fuzzy fractional integral and integro-differential equations involving Riemann-Liouville differential operators. Moreover, the existence, uniqueness and approximate solutions of fuzzy fractional differential equations (FDEs) under Caputo's H-differentiability are studied in [28] and [2]. Based on generalized Hukuhara derivative [30], the concept of fractional Caputo derivative is introduced in [4], thence fuzzy fractional differential equations are investigated under this type of differentiability. Hereinafter, fuzzy fractional

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integro-differential equations defined in [5], and sufficient conditions which guarantee the existence and uniqueness of the solution are provided for this set of fractional equations (see e.g. [19, 20, 21, 22].

Up to now, all numerical methods for solving fuzzy fractional differential equations had been conducted based on converting fuzzy problems to two crisp corresponding problems, then find the approximate solutions of crisp problems and hence obtain an approximation of solutions of the fuzzy problems, see [2, 18, 24] for this scheme. The aim of this paper is to provide a method for solving fuzzy fractional differential equations, without converting it into two crisp equations. To achieve this, the mean value theorem for fuzzy Riemann-Liouville integral and several properties of Caputo generalized Hukuhara derivative are obtained, in order to obtain fuzzy generalized Taylor's formula. Then from it, we get the fractional Euler's method for various type of differentiability.
The paper is organized as follows: As preliminaries, we represent some basic results on fuzzy numbers and the differentiability and integrability properties for the fuzzy-valued functions. In Section 3, we give mean value theorem for fuzzy Riemann-Liouville integral. In Section 4, we present the Taylor's expansion for Caputo generalized Hukuhara derivatives, and also Section 5 is devoted to the generalized Taylor's expansion for fuzzy-valued functions. Consequently, Section 6 offers fuzzy fractional Euler's method for solving fuzzy differential equations of fractional orders. The approximate solution of three examples of fuzzy fractional differential equation is obtained in Section7. Conclusions are discussed in Section 8.

2. **Preliminaries**

A fuzzy subset of $\mathbb{R}$ is defined in terms of membership function such that a membership function $u : \mathbb{R} \to [0,\ 1]$ is assigned to each point $x \in \mathbb{R}$ a grade of membership in the fuzzy set. Let $\mathbb{R}_{\mathcal{F}}$ denote the fuzzy subsets of real axis ($i.e.$ $u : \mathbb{R} \to [0,\ 1]$) which satisfies

- **(i):** u is fuzzy convex;
- **(ii):** u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;
- **(iii):** u is upper semi-continuous;
- **(iv):** closure of $\{x \in \mathbb{R} | u(x) > 0\}$ is compact.

Then $\mathbb{R}_{\mathcal{F}}$ is called the space of fuzzy numbers(see e.g. [15]).
For each $0 < r \leq 1$ the r-level set $[u]^r$ of a fuzzy number u is the subset of points $x \in \mathbb{R}$ with membership grade $u(x)$ of at least r, that is $[u]^r = \left\{x \in \mathbb{R}\Big|u(x) \geq r\right\} = [u_-(r),\ u_+(r)]$ and $[u]^0 = cl\left\{x \in \mathbb{R}\Big|u(t) > 0\right\}$. Then it is well-known that for each $r \in [0,\ 1]$, $[u]^r$ is a bounded closed interval. Also the addition and scalar multiplication of fuzzy numbers in $\mathbb{R}_{\mathcal{F}}$ will be defined level set wise, that is, $\forall u,\ v \in \mathbb{R}_{\mathcal{F}}$ and $\forall k \in \mathbb{R}$, $[u \oplus v]^r = [u]^r + [v]^r$, $[k \odot\ u]^r = k\ [u]^r$, $\forall r \in [0,\ 1]$.

Definition 2.1. A fuzzy number u is determined by a pair (u_-, u_+) of functions $u_-, u_+ : [0,1] \to \mathbb{R}$, the end-points of the r-level sets, which satisfy the following requirements:

(i) $u_-(r)$ is a bounded non-decreasing left continuous function in $]0,\ 1]$, and right continuous at 0,

(ii) $u_+(r)$ is a bounded non-increasing left continuous function in $]0,\ 1]$, and right continuous at 0,

(iii) $u_-(r) \leq u_+(r)$, $0 \leq r \leq 1$.

A crisp number u is represented by $u_-(r) = u_+(r)$, $0 \leq r \leq 1$. Also, we recall a triangular fuzzy number is defined as a fuzzy set in $\mathbb{R}_{\mathcal{F}}$, that is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$ with $a \leq b \leq c$ such that $u_-(r) = a + (b - a)r$ and $u_+(r) = c - (c - b)r$ are the endpoints of r-level sets for each $r \in [0, 1]$.
The Hausdorff distance between fuzzy numbers is given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}^+ \cup \{0\}$ as in [26]

$$D(u,\ v) = \sup_{0<r\leq 1} \max\{|u_-(r) - v_-(r)|, |u_+(r) - v_+(r)|\},$$

where $[u]^r = [u_-(r),\ u_+(r)]$, $[v]^r = [v_-(r),\ v_+(r)]$. Then D is a metric in $\mathbb{R}_{\mathcal{F}}$ and has the following properties:

$D(u \oplus w,\ v \oplus w) = D(u,\ v),\ \forall u,\ v,\ w \in \mathbb{R}_{\mathcal{F}};$
$D(\lambda u,\ \lambda v) = |\lambda| D(u,\ v),\ \forall \lambda \in \mathbb{R}, \forall\ u,\ v \in \mathbb{R}_{\mathcal{F}};$
$D(u \oplus v,\ w \oplus z) \leq D(u,\ w) + D(v,\ z)\ \forall u,\ v,\ w,\ z \in \mathbb{R}_{\mathcal{F}};$

and $(D, \mathbb{R}_{\mathcal{F}})$ is a complete metric space.

Definition 2.2. ([11]). Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v \oplus w$, then w is called the H-difference of u and v, and it is denoted by $w = u \ominus v$.

Definition 2.3. ([12]). Given two fuzzy numbers u , $v \in \mathbb{R}_{\mathcal{F}}$, the generalized Hukuhara difference (gH-difference for short) is the fuzzy number w, if it exists, such that

$$u \ominus_{gH} v = w \Longleftrightarrow \begin{cases} (i\ ) & u = v \oplus w\ , \\ or\ (ii) & v = u \oplus (-1)w\ . \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number.
In terms of r-levels we have

$$[u \ominus_{gH} v]^r = [\min\{u_-(r) - v_-(r), u_+(r) - v_+(r)\}, \max\{u_-(r) - v_-(r), u_+(r) - v_+(r)\}]$$

and the conditions for the existence of $w = u \ominus_{gH} v \in \mathbb{R}_{\mathcal{F}}$ are

$$case(i) \begin{cases} w_-(r) = u_-(r) - v_-(r) \text{ and } w_+(r) = u_+(r) - v_+(r)\ \forall r \in [0,1], \\ \text{with } w_-(r) \text{ increasing, } w_+(r) \text{ decreasing, } w_-(r) \leq w_+(r). \end{cases}$$

$$case(ii) \begin{cases} w_-(r) = u_+(r) - v_+(r) \text{ and } w_+(r) = u_-(r) - v_-(r)\ \forall r \in [0,1], \\ \text{with } w_-(r) \text{ increasing, } w_+(r) \text{ decreasing, } w_-(r) \leq w_+(r). \end{cases}$$

If the gH-difference $u \ominus_{gH} v$ do not define a fuzzy number, the nested property of r-levels can be used and obtained a fuzzy number by

$$[u \ominus_g v]^r \quad = \quad cl \bigcup_{r_0 \geq r} ([u]^{r_0} \ominus_{gH} [v]^{r_0}), \text{ for } r \in [0,1],$$

where $u \ominus_g v$ define generalized difference of two fuzzy number u , $v \in \mathbb{R}_{\mathcal{F}}$ [13], which has been extended and studied in [12].
Note that the function $f : [a,\ b] \to \mathbb{R}_{\mathcal{F}}$, $[a,\ b] \subset \mathbb{R}$ is called the fuzzy-valued function. The r-level representation of fuzzy-valued function f is expressed by $f(x;r) = [f_-(x;r),\ f_+(x;r)]$, $\forall x \in [a,\ b]$, $\forall r \in [0,\ 1]$.

Definition 2.4. ([30]). The generalized Hukuhara derivative of a fuzzy-valued function $f : (a,\ b) \to \mathbb{R}_{\mathcal{F}}$ at x_0 is defined as

$$f'_{gH}(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) \ominus_{gH} f(x_0)}{h}. \tag{1}$$

If $f'_{gH}(x_0) \in \mathbb{R}_{\mathcal{F}}$ satisfying (1) exists, we say that f is generalized Hukuhara differentiable (gH-differentiable for short) at x_0.
Also we say that f is (i)-gH-differentiable at x_0 if

$$(i)\ f'_{gH}(x_0;r) = [f'_-(x_0;r),\ f'_+(x_0;r)],\ 0 \leq r \leq 1, \tag{2}$$

and f is (ii)-gH-differentiable at x_0 if

$$(ii)\ f'_{gH}(x_0;r) = [f'_+(x_0;r),\ f'_-(x_0;r)],\ 0 \leq r \leq 1. \tag{3}$$

Definition 2.5. ([30]) We say that a point $\xi_0 \in (a,\ b)$ is a switching point for the differentiability of $f(x)$, if in any neighborhood V of ξ_0 there exist points $x_1 < \xi_0 < x_2$ such that
(type I) at x_1 (2) holds while (3) does not hold and at x_2 (3) holds and (2) does not hold, or
(type II) at x_1 (3) holds while (2) does not hold and at x_2 (2) holds and (3) does not hold.

Definition 2.6. ([4]). Let $f : (a,\ b) \to \mathbb{R}_{\mathcal{F}}$. We say that $f(x)$ is n-th order gH-differentiable at x_0 whenever the function $f(x)$ is gH-differentiable of the order $i,\ i = 0, 1, ..., n-1$, at x_0, and if there exist $f^{(n)}_{gH}(x_0) \in \mathbb{R}_{\mathcal{F}}$ such that

$$f^{(n)}_{gH}(x_0) = \lim_{h \to 0} \frac{f^{(n-1)}_{gH}(x_0 + h) \ominus_{gH} f^{(n-1)}_{gH}(x_0)}{h}.$$

Definition 2.7. ([23]). A fuzzy-valued function $f : [a,\ b] \to \mathbb{R}_{\mathcal{F}}$ is said to be absolutely continuous if, for each $\epsilon > 0$, there exists $\delta > 0$ such that, for each family $\{(s_k, x_k) | k = 1, 2, ..., n\}$ of disjoint open intervals in $[a, b]$ with $\sum_{k=1}^{n}(x_k - s_k) < \delta$, we have $\sum_{k=1}^{n} D(f(x_k), f(s_k)) < \epsilon$.

Definition 2.8. ([8]). A fuzzy valued function $f : [a,\ b] \to \mathbb{R}_{\mathcal{F}}$ is said to be continuous at $x_0 \in [a,\ b]$ if for each $\epsilon > 0$ there is $\delta > 0$ such that $D(f(x), f(x_0)) < \epsilon$, whenever $x \in [a, b]$ and $|x - x_0| < \delta$. We say that f is fuzzy continuous on $[a, b]$ if f is continuous at each $x_0 \in [a,\ b]$ such that the continuity is one-sided at endpoints $a,\ b$.

Lemma 2.9. *([16]). Let $f : [a,\ b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then $\int_a^b f(x)dx$ exists and belongs to $\mathbb{R}_{\mathcal{F}}$, furthermore it holds*

$$\int_a^b f(x;r)dx = \left[\int_a^b f_-(x;\ r)dx,\ \int_a^b f_+(x;\ r)dx \right],\ 0 \le r \le 1.$$

Throughout this paper, let $A_{\mathbb{F}}^n[a,b]$ denote the space of fuzzy-valued functions from $[a,b]$ into $\mathbb{R}_{\mathcal{F}}$ with $n-1$ gH-derivative absolutely continuous function on $[a,b]$. Also, the space of all fuzzy-valued functions that have n-th continuous gH-derivatives on $[a,\ b]$ is denoted by $C_{\mathbb{F}}^n[a,b]$.

Definition 2.10. ([27]). Consider $f : [a,b] \to \mathbb{R}$, the fractional derivative of $f(t)$ in the Caputo sense is defined as

$(D_*^\alpha f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{(m-\alpha-1)} f^{(m)}(t)dt,\ m-1 < \alpha \le m\ ,\ m \in \mathbb{N}\ ,\ x > a.$

Definition 2.11. ([29]). Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$; the fuzzy fractional Riemann-Liouville integral of fuzzy-valued function f is defined as follows

$$(J_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt,$$

for $a \le x$, and $0 < \alpha \le 1$.

Definition 2.12. ([9]). Let $f \in A_{\mathbb{F}}^m[a,b]$. The Caputo generalized Hukuhara differentiability of fuzzy-valued function $f(\ ^{cf}[gH]-$differentiability for short) is defined as following:

$$(\ _{gH}D_*^\alpha f)(x) = J_a^{m-\alpha} f_{gH}^{(m)}(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{(m-\alpha-1)} (f_{gH}^{(m)})(t)\mathrm{d}t$$

where $m-1 < \alpha \le m\ ,\ m \in \mathbb{N}\ ,\ x > a$.
Also we say that f is $^{cf}[(i)-gH]$-differentiable at x_0 if

$$(i).\ (_{gH}D_*^\alpha f)(x_0;r) = [(D_*^\alpha f_-)(x_0;r), (D_*^\alpha f_+)(x_0;r)],\ 0 \le r \le 1, \tag{4}$$

and that f is $^{cf}[(ii)-gH]$-differentiable at x_0 if

$$(ii).\ (_{gH}D_*^\alpha f)(x_0;r) = [(D_*^\alpha f_+)(x_0;r), (D_*^\alpha f_-)(x_0;r)],\ 0 \le r \le 1, \tag{5}$$

where $(D_*^\alpha f_-)$ and $(D_*^\alpha f_+)$ are defined in definition 2.10.

Definition 2.13. ([4]). We say that a point $\xi_0 \in (a,b)$ is a switching point for the differentiability of f, if in any neighborhood V of ξ_0 there exist points $x_1 < \xi_0 < x_2$ such that
(type I) at x_1 (4) hold while (5) does not hold and at x_2 (5) holds and (4) does not hold, or
(type II) at x_1 (5) hold while (4) does not hold and at x_2 (4) holds and (5) does not hold.

3. Fuzzy Mean Value theorem for Riemann-Liouville Integral

Mean value theorem for integral in fractional calculus was introduced in crisp context [14] that is as

$$J_a^\alpha f(x)g(x) = f(c)J_a^\alpha g(x)$$

with some $c \in [a,b]$, and $f : [a,b] \to \mathbb{R}$ be continuous function and g is an integrable real function on $[a,b]$, and that g does not change its sign in this interval. Here, we introduce it by using fuzzy Riemann-Liouville integral. To this, a ranking concept must be exerted that we consider a partial ordering as follows (see [17]):

Definition 3.1. Let the partial ordering $\preceq$ in $\mathbb{R}_\mathcal{F}$ by

$$u \preceq v \text{ if and only if } u_-(r) \le v_-(r) \text{ and } u_+(r) \le v_+(r),\ \forall r \in [0,1],$$

and the strict inequality $\prec$ in $\mathbb{R}_\mathcal{F}$ is defined by

$$u \prec v \text{ if and only if } u_-(r) < v_-(r) \text{ and } u_+(r) < v_+(r),\ \forall r \in [0,1],$$

where $[u]^r = [u_-(r), u_+(r)], [v]^r = [v_-(r), v_+(r)]$.

Hereunder, some properties of considered ranking have been demonstrated.

proposition 3.2. *If $u \preceq v$ then $-v \preceq -u$.*

Proof. see [6]. □

proposition 3.3. *If $u \preceq v$ and $v \preceq u$ then $u = v$.*

Proof. see [6]. □

proposition 3.4. *Let $f, g : [a,b] \to \mathbb{R}_\mathcal{F}$ are two continuous fuzzy-valued function. If $f(x) \preceq g(x)$ for all $x \in [a,b]$ then $\int_a^b f(x)dx \preceq \int_a^b g(x)dx$.*

Proof. Using definition 3.1, for assumption gets

$$f_-(x;r) \le g_-(x;r),\ f_+(x,r) \le g_+(x;r),\ \forall x \in [a,b],\ \forall r \in [0,1].$$

By monotonicity of the integral we have

$$\int_a^b f_-(x;r)dx \le \int_a^b g_-(x;r)dx,\ \int_a^b f_+(x,r)dx \le \int_a^b g_+(x;r)dx,\ \forall r \in [0,1].$$

So the required result is obtained. □

To prove fuzzy mean value theorems for Riemann-Liouville integral, the fuzzy intermediate value theorem and fuzzy mean value theorem for integrals are needed, so we first prove these theorems.

Theorem 3.5. *If $f(x)$ is continuous fuzzy-valued function on $[a,b]$, and there exists a fuzzy number γ such that $f(a) \preceq \gamma \preceq f(b)$, then there exists at least $c \in [a,b]$, such that $f(c) = \gamma$.*

Proof. Define

$$\mathcal{S} = \{x | x \in [a, b], f(x) \preceq \gamma\}.$$

Since $\mathcal{S}$ is nonempty set and is bounded above by b, $c = \sup \mathcal{S}$ exists as a real number.
First suppose that $f(c) \succ \gamma$. By definition 3.1, for every $r \in [0, 1]$ we have $f_-(c; r) > \gamma_-(r), f_+(c; r) > \gamma_+(r)$.
Moreover, since f is continuous we have

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x\big(\mid x - c \mid < \delta \Rightarrow D(f(x), f(c)) < \epsilon\big). \tag{6}$$

For any fixed $r \in [0, 1]$, let $\epsilon = \min\{f_-(c; r) - \gamma_-(r), f_+(c; r) - \gamma_+(r)\}$, so from definition of distance D find that

$$\mid f_-(x; r) - f_-(c; r) \mid < \epsilon \Rightarrow f_-(x; r) > f_-(c; r) - \epsilon \geq \gamma_-(r) \tag{7}$$

and

$$\mid f_+(x; r) - f_+(c; r) \mid < \epsilon \Rightarrow f_+(x; r) > f_+(c; r) - \epsilon \geq \gamma_+(r) \tag{8}$$

From equations (7) and (8), for all $x \in (c - \delta, c + \delta)$, we deduce $f_-(x; r) > \gamma_-(r),\ f_+(x; r) > \gamma_+(r)$. This requires that $c - \delta$ be an upper bounded for $\mathcal{S}$, which is a contradiction, because no point in the interval $(c - \delta, c]$ for $f_-(x; r) > \gamma_-(r),\ f_+(x; r) > \gamma_+(r)$, can be belonged to $\mathcal{S}$ and c is considered as the supremum for $\mathcal{S}$. We then conclude that $f_-(c; r) \leq \gamma_-(r),\ f_+(c; r) \leq \gamma_+(r)$ for every $r \in [0, 1]$. So, according to definition 3.1, $f(c) \preceq \gamma$.
Now suppose that $f(c) \prec \gamma$. Again, by continuity and properties of D, for every $r \in [0, 1]$ in equation (6) we have

$$\mid f_-(x; r) - f_-(c; r) \mid < \epsilon \Rightarrow f_-(x; r) < f_-(c; r) + \epsilon \leq \gamma_-(r) \tag{9}$$

and

$$\mid f_+(x; r) - f_+(c; r) \mid < \epsilon \Rightarrow f_+(x; r) < f_+(c; r) + \epsilon \leq \gamma_+(r) \tag{10}$$

From equation (9) and (10), for all $x \in (c - \delta, c + \delta)$, $f_-(x; r) < \gamma_-(r),\ f_+(x; r) < \gamma_+(r)$, thus $c + \frac{\delta}{2} \in \mathcal{S}$, which is a contradiction. Hence, for every $r \in [0, 1]$, $f_-(c; r) \geq \gamma_-(r),\ f_+(c; r) \geq \gamma_+(r)$, and it means that $f(c) \succeq \gamma$. Therefore, by proposition 3.3 gets $f(c) = \gamma$. □

Theorem 3.6. *Let $f : [a, b] \to \mathbb{R}_{\mathcal{F}}$ be continuous fuzzy-valued function such that $f(a) \preccurlyeq f(x) \preccurlyeq f(b)$ for all $x \in [a, b]$, $g : [a, b] \to \mathbb{R}$ is an integrable real function on the interval $]a, b[$, then there exists at least one number $c \in]a, b[$ such that*

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx$$

Proof. First we suppose that $g(x) > 0$ then by proposition 3.4 and assumption, we have

$$\int_a^b f(a)g(x)\mathrm{d}x \preccurlyeq \int_a^b f(x)g(x)\mathrm{d}x \preccurlyeq \int_a^b f(b)g(x)\mathrm{d}x$$

Therefore,

$$f(a) \preccurlyeq \frac{\int_a^b f(x)g(x)\mathrm{d}x}{\int_a^b g(x)\mathrm{d}x} \preccurlyeq f(b)$$

By fuzzy intermediate value theorem, there is at least one number $c \in [a,b]$, such that

$$f(c) = \frac{\int_a^b f(x)g(x)\mathrm{d}x}{\int_a^b g(x)\mathrm{d}x}$$

The proof of $g(x) \leq 0$ by considering proposition 3.2 is similar to proof of $g(x) > 0$, which proves the theorem. □

Now we present fuzzy mean value theorem for Riemann-Liouville integral as following:

Theorem 3.7. *Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be continuous fuzzy-valued function such that $f(a) \preccurlyeq f(x) \preccurlyeq f(b)$ for all $x \in [a,b]$, g is a continuous and integrable real function on the interval $]a,b[$, such that g does not change sign in this interval, then there exists at least one number $c \in]a,b[$ such that*

$$J_a^\alpha f(x)g(x) = f(c)J_a^\alpha g(x)$$

Proof. By definition of fuzzy Riemann-Liouville integral 2.5, we have

$$J_a^\alpha f(x)g(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)g(t)}{(x-t)^{1-\alpha}}\mathrm{d}t$$

Setting $\tilde{g}(t) = \frac{g(t)}{\Gamma(\alpha)(x-t)^{1-\alpha}}$, so the assumption of theorem implies that $\tilde{g}$ is continuous and integrable in $]a,b[$ and it does not change sign in this interval. Hence, from theorem 3.6, we can write

$$J_a^\alpha f(x)g(x) = \int_a^x f(t)\tilde{g}(t)\mathrm{d}t = f(c)\int_a^x \tilde{g}(t)\mathrm{d}t = f(c)J_a^\alpha g(x)$$

□

4. Taylor's Expansion for Caputo gH-Derivative

Theorem 4.1. *Let f is belongs to $C_{\mathbb{F}}^n[a,b]$, $1 \leq k \leq n$.*

- **(i):** *If all order derivatives of f are (i)-gH-differentiable, then $f_{gH}^{(k-1)}(x) = f_{gH}^{(k-1)}(a) \oplus \int_a^x f_{gH}^{(k)}(t)dt$.*
- **(ii):** *If all order derivatives of f are (ii)-gH-differentiable, then $f_{gH}^{(k-1)}(x) = f_{gH}^{(k-1)}(a) \oplus \int_a^x f_{gH}^{(k)}(t)dt$.*
- **(iii):** *If odd-order derivatives of f are (ii)-gH-differentiable and even-order derivatives of f are (i)-gH-differentiable or vice versa, then $f_{gH}^{(k-1)}(x) = f_{gH}^{(k-1)}(a) \ominus (-1)\int_a^x f_{gH}^{(k)}(t)dt$.*

Proof. Assuming that $f \in C^n_{\mathbb{F}}[a,b]$ implies that $f^{(k)}_{gH}$, $k = 0,1,...,n$ are integrable on $[a,\ b]$.
(i). Let $f^{(k)}_{gH}(x)$, for $k = 1,2,...,n$ are differentiable on $[a,b]$ as in definition 2.4(i).

$$\begin{aligned}\int_a^x f^{(k)}_{gH}(t)dt &= [\int_a^x f^{(k)}_{-}(t)dt, \int_a^x f^{(k)}_{+}(t)dt] \\ &= [f^{(k-1)}_{-}(x) - f^{(k-1)}_{-}(a), f^{(k-1)}_{+}(x) - f^{(k-1)}_{+}(a)] \\ &= [f^{(k-1)}_{-}(x), f^{(k-1)}_{+}(x)] \ominus [f^{(k-1)}_{-}(a), f^{(k-1)}_{+}(a)]] \\ &= f^{(k-1)}_{gH}(x) \ominus f^{(k-1)}_{gH}(a)\end{aligned}$$

(ii). Let $f^{(k)}_{gH}(x)$, for $k = 1,2,...,n$ are differentiable on $[a,b]$ as in definition 2.4(ii).

$$\begin{aligned}\int_a^x f^{(k)}_{gH}(t)dt &= [\int_a^x f^{(k)}_{+}(t)dt, \int_a^x f^{(k)}_{-}(t)dt] \\ &= [f^{(k-1)}_{+}(x) - f^{(k-1)}_{+}(a), f^{(k-1)}_{-}(x) - f^{(k-1)}_{-}(a)] \\ &= [f^{(k-1)}_{+}(x), f^{(k-1)}_{-}(x)] \ominus [f^{(k-1)}_{+}(a), f^{(k-1)}_{-}(a)] \\ &= f^{(k-1)}_{gH}(x) \ominus f^{(k-1)}_{gH}(a)\end{aligned}$$

(iii). Suppose that $f^{(2k)}_{gH}(x)$ is differentiable as in definition 2.4 (i), and $f^{(2k-1)}_{gH}(x)$ is differentiable as in definition 2.4(ii).

$$\begin{aligned}\int_a^x f^{(2k+1)}_{gH}(t)dt &= [\int_a^x f^{(2k+1)}_{-}(t)dt, \int_a^x f^{(2k+1)}_{+}(t)dt] \\ &= [f^{(2k)}_{-}(x) - f^{(2k)}_{-}(a), f^{(2k)}_{+}(x) - f^{(2k)}_{+}(a)] \\ &= [f^{(2k)}_{-}(x), f^{(2k)}_{+}(x)] \ominus [f^{(2k)}_{-}(a), f^{(2k)}_{+}(a)]] \\ &= (-1)f^{(2k)}_{gH}(a) \ominus (-1)f^{(2k)}_{gH}(x)\end{aligned}$$

□

Lemma 4.2. *([4]). Suppose that $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy-valued function and $f \in A_{\mathbb{F}}[a,b]$. Then*

$$J^{\alpha}_{a}(\ _{gH}D^{\alpha}_{*}f)(x) = f(x) \ominus_{gH} f(a),\ 0 < \alpha \le 1.$$

Now, we present fuzzy Taylor's expansion for Caputo derivative.

Theorem 4.3. *Assume that $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = [\alpha] + 1$, and $f^{(m)}_{gH} \in C_{\mathbb{F}}[a,b]$.*

(i): *If f and it's high order derivatives are differentiable on $[a,\ b]$ according to definition 2.4(i), then*

$$f(x) = f(a) \oplus \sum_{k=1}^{m-1} \frac{(x-a)^k}{k!} \odot f^{(k)}_{gH}(a) \oplus J^{\alpha}_{a}(\ _{gH}D^{\alpha}_{*}f)(x)$$

(ii): *If f and it's high order derivatives are differentiable on $[a,\ b]$ according to definition 2.4(ii), then*

$$f(x) = f(a) \ominus (-1) \sum_{k=1}^{m-1} \frac{(x-a)^k}{k!} \odot f_{gH}^{(k)}(a) \ominus (-1) J_a^{\alpha}(\ {}_{gH}D_*^{\alpha} f)(x)$$

(iii): *If f is differentiable on $[a,b]$ according to definition 2.4(i), and it's high order derivatives, decussate change from (i) to (ii) differentiability, then*

$$\begin{aligned} f(x) = f(a) \quad &\ominus \quad (-1) \sum_{\substack{k=1 \\ k \text{ is odd}}}^{m-1} \frac{(x-a)^k}{k!} \odot f_{gH}^{(k)}(a) \\ &\oplus \quad \sum_{\substack{k=1 \\ k \text{ is even}}}^{m-1} \frac{(x-a)^k}{k!} \odot f_{gH}^{(k)}(a) \ominus (-1) J_a^{\alpha}({}_{gH}D_*^{\alpha} f)(x) \end{aligned}$$

(iv): *If f has a switching point at $\varepsilon \in [a,\ b]$ of type II and it's derivative has a switching point of type I at $\delta \in [\varepsilon, b]$. If $f_{gH}^{(k)}$, $k = 2, 3, ..., m$ are differentiable on $[a,\ b]$ according to definition 2.4(i), then*

$$f(x) = \begin{cases} f(a) \oplus \sum_{k=1}^{m-1} \frac{(x-a)^k}{k!} \odot f_{gH}^{(k)}(a) \oplus J_a^{\alpha}(\ {}_{gH}D_*^{\alpha} f)(x) & a \le x \le \varepsilon \\ f(a) \ominus (-1) \sum_{k=1}^{m-1} \frac{(x-a)^k}{k!} \odot f_{gH}^{(k)}(a) \ominus (-1) J_a^{\alpha}(\ {}_{gH}D_*^{\alpha} f)(x) & \varepsilon \le x \le \delta \\ f(a) \oplus (x-a) f'_{gH}(a) \ominus (-1) \sum_{k=2}^{m-1} \frac{(x-a)^k}{k!} \odot f_{gH}^{(k)}(a) \ominus (-1) J_a^{\alpha}(\ {}_{gH}D_*^{\alpha} f)(x) & \delta \le x \le b \end{cases}$$

(v): *If f has a switching point at $\varepsilon \in [a,\ b]$ of type I and it's derivative has a switching point of type II at $\delta \in [\varepsilon, b]$. If $f_{gH}^{(k)}$, $k = 2, 3, ..., m$ are differentiable on $[a,\ b]$ according to definition 2.4(ii), then*

$$f(x) = \begin{cases} f(a) \ominus (-1) \sum_{k=1}^{m-1} \frac{(x-a)^k}{k!} \odot f_{gH}^{(k)}(a) \ominus (-1) J_a^{\alpha}(\ {}_{gH}D_*^{\alpha} f)(x) & a \le x \le \varepsilon \\ f(a) \oplus \sum_{k=1}^{m-1} \frac{(x-a)^k}{k!} \odot f_{gH}^{(k)}(a) \oplus J_a^{\alpha}(\ {}_{gH}D_*^{\alpha} f)(x) & \varepsilon \le x \le \delta \\ f(a) \ominus (-1)(x-a) f'_{gH}(a) \oplus \sum_{k=2}^{m-1} \frac{(x-a)^k}{k!} \odot f_{gH}^{(k)}(a) \oplus J_a^{\alpha}(\ {}_{gH}D_*^{\alpha} f)(x) & \delta \le x \le b \end{cases}$$

Proof. Since $f_{gH}^{(m)} \in C_{\mathbb{F}}[a,b]$, according to definition 2.12, f is ${}^{cf}[gH]$-differentiable of order α. Then

$$J_a^{\alpha}(\ {}_{gH}D_*^{\alpha} f)(x) = J_a^{\alpha}(J_a^{m-\alpha}(f_{gH}^{(m)})(x)) = J_a^{m}(f_{gH}^{(m)})(x)$$

By theorem 4.1 and Lemma 4.2, we obtain
(i). Suppose that $f_{gH}^{(k)}$, for $k = 0, 1, ..., n$ are differentiable as in definition 2.4 (i).

$$\begin{aligned}
& J_a^\alpha({}_{gH}D_*^\alpha f)(x) = J_a^m(f_{gH}^{(m)})(x) \\
= \; & J_a^{m-1}(f_{gH}^{(m-1)})(x) \ominus J_a^{m-1}(f_{gH}^{(m-1)})(a) \\
= \; & J_a^{m-2}(f_{gH}^{(m-2)})(x) \ominus J_a^{m-2}(f_{gH}^{(m-2)})(a) \ominus J_a^{m-1}(f_{gH}^{(m-1)})(a) \\
\vdots \; & \\
= \; & J_a(f'_{gH})(x) \ominus J_a^2(f''_{gH})(a) \ominus \ldots \ominus J_a^{m-1}(f_{gH}^{(m-1)})(a) \\
= \; & f(x) \ominus f(a) \ominus J_a(f'_{gH})(a) \ominus J_a^2(f''_{gH})(a) \ominus \ldots \ominus J_a^{m-1}(f_{gH}^{(m-1)})(a)
\end{aligned}$$

Since

$$J_a^k = \int_a^x \int_a^x \cdots \int_a^x \underbrace{dx\ dx\ \cdots\ dx}_{k\ times} = \frac{(x-a)^k}{k!}$$

So, we find that

$$J_a^\alpha({}_{gH}D_*^\alpha f)(x) = f(x) \ominus f(a) \ominus (x-a)f'_{gH}(a) \ominus \tfrac{(x-a)^2}{2!} f''_{gH}(a) \ominus \ldots \ominus \tfrac{(x-a)^{m-1}}{(m-1)!} f_{gH}^{(m-1)}(a)$$

And thus gives

$$f(x) = f(a) \oplus \sum_{k=1}^{m-1} \frac{(x-a)^k}{k!} \odot f_{gH}^{(k)}(a) \oplus J_a^\alpha({}_{gH}D_*^\alpha f)(x)$$

(ii). Suppose that $f_{gH}^{(k)}$, for $k = 0, 1, ..., n$ are differentiable as in definition 2.4 (ii).

$$\begin{aligned}
& J_a^\alpha({}_{gH}D_*^\alpha f)(x) = J_a^m(f_{gH}^{(m)})(x) \\
= \; & J_a^{m-1}(f_{gH}^{(m-1)})(x) \ominus J_a^{m-1}(f_{gH}^{(m-1)})(a) \\
= \; & J_a^{m-2}(f_{gH}^{(m-2)})(x) \ominus J_a^{m-2}(f_{gH}^{(m-2)})(a) \ominus J_a^{m-1}(f_{gH}^{(m-1)})(a) \\
\vdots \; & \\
= \; & J_a(f'_{gH})(x) \ominus J_a^2(f''_{gH})(a) \ominus \ldots \ominus J_a^{m-1}(f_{gH}^{(m-1)})(a) \\
= \; & (-1)f(a) \ominus (-1)f(x) \ominus J_a(f'_{gH})(a) \ominus J_a^2(f''_{gH})(a) \ominus \ldots \ominus J_a^{m-1}(f_{gH}^{(m-1)})(a)
\end{aligned}$$

similar to proof of (i), we get

$$f(x) = f(a) \ominus (-1) \sum_{k=1}^{m-1} \frac{(x-a)^k}{k!} \odot f_{gH}^{(k)}(a) \ominus (-1) J_a^\alpha({}_{gH}D_*^\alpha f)(x)$$

(iii). Suppose that $f_{gH}^{(2k)}(x)$, for $k = 0, 1, ..., \frac{m}{2}$ are differentiable according to definition 2.4 (i) and $f_{gH}^{(2k-1)}(x)$, $k = 1, ..., \frac{m}{2}$ are differentiable as in definition 2.4

(ii).

$$\begin{aligned}
& J_a^{\alpha}(\ _{gH}D_*^{\alpha} f)(x) = J_a^m(f_{gH}^{(m)})(x) \\
= & -J_a^{m-1}(f_{gH}^{(m-1)})(a) \ominus (-1) J_a^{m-1}(f_{gH}^{(m-1)})(x) \\
= & -J_a^{m-1}(f_{gH}^{(m-1)})(a) \ominus (-1)\Big(- J_a^{m-2}(f_{gH}^{(m-2)})(a) \ominus (-1) J_a^{m-2}(f_{gH}^{(m-2)})(x)\Big) \\
& \vdots \\
= & -J_a^{m-1}(f_{gH}^{(m-1)})(a) \ominus J_a^{m-2}(f_{gH}^{(m-2)})(a) \oplus \cdots \oplus f(x) \ominus f(a) \\
= & \frac{-(x-a)^{m-1}}{(m-1)!} \odot f_{gH}^{(m-1)}(a) \ominus \frac{(x-a)^{m-2}}{(m-2)!} \odot f_{gH}^{(m-2)}(a) \oplus \cdots \oplus f(x) \ominus f(a)
\end{aligned}$$

Therefore

$$f(x) = f(a) \ominus (-1) \sum_{\substack{k=1 \\ k \text{ is odd}}}^{m-1} \tfrac{(x-a)^k}{k!} \odot f_{gH}^{(k)}(a) \oplus \sum_{\substack{k=1 \\ k \text{ is even}}}^{m-1} \tfrac{(x-a)^k}{k!} \odot f_{gH}^{(k)}(a) \oplus J_a^{\alpha}(_{gH}D_*^{\alpha} f)(x)$$

(iv). Suppose that $f_{gH}^{(k)}$, for $k = 2, 3, ..., n$ are differentiable as in definition 2.4 (i).

$$\begin{aligned}
J_a^{\alpha}(\ _{gH}D_*^{\alpha} f)(x) &= J_a^m(f_{gH}^{(m)})(x) = J_a^{m-1}(f_{gH}^{(m-1)})(x) \ominus J_a^{m-1}(f_{gH}^{(m-1)})(a) \\
&= J_a^2(f''_{gH})(x) \ominus J_a^2(f''_{gH})(a) \ominus \cdots \ominus J_a^{m-1}(f_{gH}^{(m-1)})(a) \quad (11)
\end{aligned}$$

Since $f'_{gH}(x)$ is differentiable on $[a,\ \delta]$ as in definition 2.4 (i) and on $[\delta,\ b]$ as in definition 2.4 (ii), we have

$$J_a^2(f''_{gH})(x) = \begin{cases} J_a(f'_{gH})(x) \ominus J_a(f'_{gH})(x) & a \le x \le \delta; \\ -J_a(f'_{gH})(a) \ominus (-1) J_a(f'_{gH})(x) & \delta \le x \le b. \end{cases} \quad (12)$$

And, Also $f(x)$ is differentiable on $[a,\ \varepsilon]$ according to definition 2.4 (ii) and on $[\varepsilon,\ b]$ as in definition 2.4 (i), so we have

$$J_a(f'_{gH})(x) = \begin{cases} -f(a) \ominus (-1) f(x) & a \le x \le \varepsilon; \\ f(a) \ominus\ f(a) & \varepsilon \le x \le \delta; \\ -f(a) \ominus (-1) f(x) & \delta \le x \le b. \end{cases} \quad (13)$$

Now, if we substitute (12) and (13) into (11), the proof of (iv) is obtained, and the proof of (v) is similar to (iv), which proves the theorem. □

5. **Fuzzy Generalized Taylor's Expansion**

In this section we are going to obtain a generalized Taylor's expansion for fuzzy-valued functions by using the concept of Caputo generalized Hukuhara derivative. To this end, we need to following theorems and Lemmas.

Lemma 5.1. *Let* $_{gH}D_*^{k\alpha} f\ :]a, b[\to \mathbb{R}_{\mathcal{F}}$ *be a continuous fuzzy function for* $k = 0, 1, ..., n$ *where* $0 < \alpha \le 1$.

(i): *If* $({}_{gH}D_*^{(n-1)\alpha} f)(x)$, *and* $({}_{gH}D_*^{n\alpha} f)(x)$ *for* $n \geq 1$ *are* ${}^{cf}[gH]$*-differentiable of order* α *in same type of differentiability concept, then*

$$J_a^{\alpha}({}_{gH}D_*^{(n+1)\alpha} f)(x) \quad = \quad ({}_{gH}D_*^{n\alpha} f)(x) \ominus ({}_{gH}D_*^{n\alpha} f)(a)$$

(ii): *If* $({}_{gH}D_*^{(n-1)\alpha} f)(x)$, *and* $({}_{gH}D_*^{n\alpha} f)(x)$ *for* $n \geq 1$ *are* ${}^{cf}[gH]$*-differentiable of order* α *in different type of differentiability concept, then*

$$J_a^{\alpha}({}_{gH}D_*^{(n+1)\alpha} f)(x) \quad = \quad (-1)({}_{gH}D_*^{n\alpha} f)(a) \ominus (-1)({}_{gH}D_*^{n\alpha} f)(x)$$

Proof. It is immediate by theorem 4.1 and Lemma 4.2. □

Lemma 5.2. *Let* ${}_{gH}D_*^{k\alpha} f :]a, b[\to \mathbb{R}_{\mathcal{F}}$ *be a continuous fuzzy function for* $k = 0, 1, ..., n$ *where* $0 < \alpha \leq 1$.

(i): *If* $({}_{gH}D_*^{(n-1)\alpha} f)(x)$, *and* $({}_{gH}D_*^{n\alpha} f)(x)$ *for* $n \geq 1$ *are* ${}^{cf}[gH]$*-differentiable in same sense of definition 2.12 of order* α*, then*

$$J_a^{n\alpha}({}_{gH}D_*^{n\alpha} f)(x) \ominus J_a^{(n+1)\alpha}({}_{gH}D_*^{(n+1)\alpha} f)(x) \quad = \quad \frac{(x-a)^{n\alpha}}{\Gamma(n\alpha+1)}({}_{gH}D_*^{n\alpha} f)(a)$$

(ii): *If* $({}_{gH}D_*^{(n-1)\alpha} f)(x)$, *and* $({}_{gH}D_*^{n\alpha} f)(x)$ *for* $n \geq 1$ *are* ${}^{cf}[gH]$*-differentiable in different sense of definition 2.12 of order* α*, then*

$$J_a^{n\alpha}({}_{gH}D_*^{n\alpha} f)(x) \oplus (-1) J_a^{(n+1)\alpha}({}_{gH}D_*^{(n+1)\alpha} f)(x) \quad = \quad \frac{(x-a)^{n\alpha}}{\Gamma(n\alpha+1)}({}_{gH}D_*^{n\alpha} f)(a)$$

Proof. (i). Using properties of Riemann-Liouville integral and generalized Hukuhara Caputo derivative along with Lemma 5.1 implies that

$$\begin{aligned} & J_a^{n\alpha}({}_{gH}D_*^{n\alpha} f)(x) \ominus J_a^{(n+1)\alpha}({}_{gH}D_*^{(n+1)\alpha} f)(x) \\ = \quad & J_a^{n\alpha}\Big(D_*^{n\alpha} f)(x) \ominus J_a^{\alpha}({}_{gH}D_*^{(n+1)\alpha} f)(x) \Big) \\ = \quad & J_a^{n\alpha}\Big(D_*^{n\alpha} f)(x) \ominus (({}_{gH}D_*^{n\alpha} f)(x) \ominus ({}_{gH}D_*^{n\alpha} f)(a) \Big) \\ = \quad & J_a^{n\alpha}({}_{gH}D_*^{n\alpha} f)(a) = \frac{(x-a)^{n\alpha}}{\Gamma(n\alpha+1)} \odot ({}_{gH}D_*^{n\alpha} f)(a) \end{aligned}$$

The proof of (ii) is similar to (i) then omitted here. □

Here, the generalized Taylor's formula is presented for Caputo ${}^{cf}[gH]$-differentiable functions.

Theorem 5.3. *Let* ${}_{gH}D_*^{k\alpha} f(x) \in C_{\mathbb{F}}[a, b]$ *for* $k = 0, 1, ..., n$ *where* $0 < \alpha \leq 1$.

(i): *If* $f(x)$ *is Caputo differentiable in sense of definition 2.12 (i) of order* $k\alpha$ *for* $k = 0, 1, ..., n$ *on* $[a, \ b]$*, then there exists* ξ *in* $]a, b[$ *such that*

$$f(x) = f(a) \oplus \sum_{k=1}^{n} \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \odot ({}_{gH}D_*^{k\alpha} f)(a) \oplus \frac{({}_{gH}D_*^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha}$$

(ii): *If $f(x)$ is Caputo differentiable in sense of definition 2.12 (ii) of order $k\alpha$ for $k = 0, 1, ..., n$ on $[a,\ b]$, then there exists ξ in $]a, b[$ such that*

$$f(x) = f(a) \ominus (-1) \sum_{k=1}^{n} \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \odot (\ {}_{gH}D_*^{k\alpha} f)(a) \ominus (-1) \frac{(\ {}_{gH}D_*^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-a)^{(n+1)\alpha}$$

(iii): *If $f(x)$ is Caputo differentiable in sense of definition 2.12 (i) of order $2k\alpha$ for $k = 0, 1, ..., \frac{n}{2}$, and also it is Caputo differentiable in sense of definition 2.12 (ii) of order $(2k-1)\alpha$ for $k = 1, ..., \frac{n}{2}$ on $[a,\ b]$, where n is an even number, then there exists ξ in $]a, b[$ such that*

$$\begin{aligned} f(x) &= f(a) \ominus (-1) \sum_{\substack{k=1 \\ k \text{ is odd}}}^{n} \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \odot ({}_{gH}D_*^{k\alpha} f)(a) \oplus \sum_{\substack{k=1 \\ k \text{ is even}}}^{n} \Big(\frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \\ &\odot \ ({}_{gH}D_*^{k\alpha} f)(a) \Big) \ominus (-1) \frac{({}_{gH}D_*^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-a)^{(n+1)\alpha} \end{aligned}$$

(iv): *If f has a switching point at $\varepsilon \in [a, b]$ of type II, the Caputo differentiability changes in sense of definition 2.12 (ii) to (i) at $t = \varepsilon$, and suppose that the type of differentiability for ${}_{gH}D_*^{k\alpha} f(x)$ for $k = 1, 2, ..., n$ are in sense of definition 2.12 (i) of order α on $[a, b]$. Then there exists ξ in $]a, b[$ such that*

$$f(x) = \begin{cases} f(a) \ominus (-1) \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} \odot (\ {}_{gH}D_*^{\alpha} f)(a) \\ \oplus \sum_{k=2}^{n} \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \odot (\ {}_{gH}D_*^{k\alpha} f)(a) \oplus \frac{(\ {}_{gH}D_*^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-a)^{(n+1)\alpha} & a \le x \le \epsilon; \\ \\ f(a) \oplus \sum_{k=1}^{n} \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \odot (\ {}_{gH}D_*^{k\alpha} f)(a) \oplus \frac{(\ {}_{gH}D_*^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-a)^{(n+1)\alpha} & \epsilon \le x \le b. \end{cases}$$

(v): *If f has a switching point at $\varepsilon \in [a, b]$ of type I, the Caputo differentiability changes from sense of definition 2.12 (i) to (ii) at $t = \varepsilon$, and suppose that the type of differentiability for ${}_{gH}D_*^{k\alpha} f(x)$ for $k = 1, 2, ..., n$ are in sense of definition 2.12 (ii) of order α on $[a, b]$. Then there exists ξ in $]a, b[$ such that*

$$f(x) = \begin{cases} f(a) \oplus \frac{(x-a)^{\alpha}}{\Gamma(\alpha+1)} \odot (\ {}_{gH}D_*^{\alpha} f)(a) \ominus (-1) \sum_{k=1}^{n} \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \odot (\ {}_{gH}D_*^{k\alpha} f)(a) \\ \ominus(-1) \frac{(\ {}_{gH}D_*^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-a)^{(n+1)\alpha}, & a \le x \le \epsilon; \\ \\ f(a) \ominus (-1) \sum_{k=1}^{n} \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)} \odot (\ {}_{gH}D_*^{k\alpha} f)(a) \\ \ominus(-1) \frac{(\ {}_{gH}D_*^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-a)^{(n+1)\alpha}, & \epsilon \le x \le b. \end{cases}$$

Proof. (i).Soppuse that ${}_{gH}D_*^{k\alpha} f(x)$ for $k = 0, 1, ..., n$ are ${}^{cf}[(i)-gH]$-differentiable, so

$$\begin{aligned} & \sum_{k=0}^{n} \big(J_a^{k\alpha}(\ {}_{gH}D_*^{k\alpha} f)(x) \ominus_{gH} J_a^{(k+1)\alpha}(\ {}_{gH}D_*^{(k+1)\alpha} f)(x) \big) \\ = \ & f(x) \ominus J_a^{\alpha}(\ {}_{gH}D_*^{\alpha} f)(x) \oplus J_a^{\alpha}(\ {}_{gH}D_*^{\alpha} f)(x) \ominus J_a^{2\alpha}(\ {}_{gH}D_*^{2\alpha} f)(x) \\ \oplus \ & ... \oplus J_a^{n\alpha}(\ {}_{gH}D_*^{n\alpha} f)(x) \ominus J_a^{(n+1)\alpha}(\ {}_{gH}D_*^{(n+1)\alpha} f)(x) \end{aligned}$$

Then by Lemma 5.1 and 5.2, we have

$$f(x) \ominus J_a^{(n+1)\alpha}({}_{gH}D_*^{(n+1)\alpha} f)(x) = f(a) \oplus \sum_{k=1}^{n} \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)}({}_{gH}D_*^{k\alpha} f)(a) \qquad (14)$$

Also, using fractional mean-valued theorem 3.7 gives

$$\begin{aligned} J_a^{(n+1)\alpha}({}_{gH}D_*^{(n+1)\alpha} f)(x) &= ({}_{gH}D_*^{(n+1)\alpha} f)(\xi) \int_a^x \frac{(x-t)^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)} dt \\ &= \frac{({}_{gH}D_*^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha} \qquad (15) \end{aligned}$$

Now, substituting (15) into (14), completes the proof (i).
(ii). Assuming that $({}_{gH}D_*^{k\alpha} f)(x)$ for $k = 0, 1, ..., n$ are ${}^{cf}[(ii) - gH]$-differentiable implies that

$$\begin{aligned} &\sum_{k=0}^{n} \left(J_a^{k\alpha}({}_{gH}D_*^{k\alpha} f)(x) \ominus_{gH} J_a^{(k+1)\alpha}({}_{gH}D_*^{(k+1)\alpha} f)(x)\right) \\ = \;& f(x) \oplus (-1)J_a^{\alpha}({}_{gH}D_*^{\alpha} f)(x) \ominus (-1)J_a^{\alpha}({}_{gH}D_*^{\alpha} f)(x) \oplus (-1)J_a^{2\alpha}({}_{gH}D_*^{2\alpha} f)(x) \\ \ominus \;& \cdots \ominus (-1)J_a^{n\alpha}({}_{gH}D_*^{n\alpha} f)(x) \oplus (-1)J_a^{(n+1)\alpha}({}_{gH}D_*^{(n+1)\alpha} f)(x) \end{aligned}$$

Using Lemmas 4.2 and 5.2 gets

$$f(x) \oplus (-1)J_a^{(n+1)\alpha}({}_{gH}D_*^{(n+1)\alpha} f)(x) = f(a) \ominus (-1) \sum_{k=1}^{n} \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)}({}_{gH}D_*^{k\alpha} f)(a)$$

The rest of the proof is similar to the proof of (i).

(iii). Let suppose that for $k = 0, 1, ..., n$, if k is even, then $({}_{gH}D_*^{k\alpha} f)(x)$ is ${}^{cf}[(i) - gH]$-differentiable and if k is odd, it is ${}^{cf}[(ii) - gH]$-differentiable. Therefore

$$\begin{aligned} &\sum_{k=0}^{n} J_a^{k\alpha}({}_{gH}D_*^{k\alpha} f)(x) \ominus_{gH} J_a^{(k+1)\alpha}({}_{gH}D_*^{(k+1)\alpha} f)(x) \\ = \;& f(x) \oplus (-1)J_a^{\alpha}({}_{gH}D_*^{\alpha} f)(x) \ominus (-1)J_a^{\alpha}({}_{gH}D_*^{\alpha} f)(x) \\ \ominus \;& J_a^{2\alpha}({}_{gH}D_*^{2\alpha} f)(x) \oplus \cdots \oplus J_a^{n\alpha}({}_{gH}D_*^{n\alpha} f)(x) \ominus (-1)J_a^{(n+1)\alpha}({}_{gH}D_*^{(n+1)\alpha} f)(x) \end{aligned}$$

By reasoning similar to those in (i), we get

$$\begin{aligned} & f(x) \oplus (-1)J_a^{(n+1)\alpha}({}_{gH}D_*^{(n+1)\alpha} f)(x) \\ = \;& f(a) \ominus (-1) \sum_{\substack{k=1 \\ k \text{ is odd}}}^{n} J_a^{k\alpha}({}_{gH}D_*^{k\alpha} f)(a) \oplus \sum_{\substack{k=1 \\ k \text{ is even}}}^{n} J_a^{k\alpha}({}_{gH}D_*^{k\alpha} f)(a) \\ = \;& f(a) \ominus (-1) \sum_{\substack{k=1 \\ k \text{ is odd}}}^{n} \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)}({}_{gH}D_*^{k\alpha} f)(a) \oplus \sum_{\substack{k=1 \\ kiseven}}^{n} \frac{(x-a)^{k\alpha}}{\Gamma(k\alpha+1)}({}_{gH}D_*^{k\alpha} f)(a) \end{aligned}$$

Since the proofs of (iv), (v) are similar to the proofs of (i), (ii), (iii), hence are omitted. □

6. **Fuzzy Fractional Euler's Method**

Consider a fractional differential equation with fuzzy initial condition

$$\begin{cases} ({}_{gH}D_*^{\alpha}y)(x) = f(x, y(x)) \ , \ x \in [a, b] \\ y(a) = \beta \in \mathbb{R}_{\mathcal{F}} \end{cases} \tag{16}$$

with $0 < \alpha \leq 1$, and $f : [a, b] \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ is supposed to be continuous. Also ${}_{gH}D_*^{\alpha}$ denote the generalized Hukuhara Caputo derivative of $y(t)$ which must be determined. See [4], for existence and uniqueness of solutions of problem (16).
This section is introduced the fractional Euler's method for solving fuzzy fractional initial value problem (16). Indeed, A sequence of approximations to the solution $y(x)$ will be obtained at several points, called gride points. To derive Euler's Method, the interval $[a, b]$ is divided to into N equal subintervals, each of length h, by the gride points $x_i = a + ih, i = 0, 1, 2, \cdots, N$. The distance between points, $h = \frac{b-a}{N}$ is called the grid size.
First let us suppose that the solution of problem (16) be continuous and ${}^{cf}[(i) - gH]-$differentiable on $[a, b]$. Consider the generalized Taylor's expansion (5.3) of $y(x)$ about x_k, for each $k = 0, 1, \cdots, N - 1$,

$$y(x_{k+1}) = y(x_k) \oplus \frac{(x_{k+1} - x_k)^{\alpha}}{\Gamma(\alpha + 1)} \odot ({}_{gH}D_*^{\alpha}y)(x_k) \oplus \frac{(x_{k+1} - x_k)^{2\alpha}}{\Gamma(2\alpha + 1)} \odot ({}_{gH}D_*^{2\alpha}y)(\xi_k)$$

and, also $h = x_{k+1} - x_k$,

$$y(x_{k+1}) = y(x_k) \oplus \frac{h^{\alpha}}{\Gamma(\alpha + 1)} \odot ({}_{gH}D_*^{\alpha}y)(x_k) \oplus \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)} \odot ({}_{gH}D_*^{2\alpha}y)(\xi_k)$$

If the gride size h is small chosen enough, the second-order term can be ignored, and also since $y(x)$ satisfies in problem (16), we obtain

$$y(x_{k+1}) = y(x_k) \oplus \frac{h^{\alpha}}{\Gamma(\alpha + 1)} \odot f(x_k, y(x_k))$$

Fractional Euler's method contracts $y_k = y(x_k)$, for each $k = 1, 2, ..., N$, so that initial condition (16) implies $y_0 = \beta$, and

$$y_{k+1} = y_k \oplus \frac{h^{\alpha}}{\Gamma(\alpha + 1)} \odot f(x_k, y_k), \ \ , k = 0, 1, ..., N - 1. \tag{17}$$

Now, consider the unique continuous solution of problem (16) is ${}^{cf}[(ii) - gH]-$differentiable on $[a, b]$. So the fractional Taylor's formula of $y(x)$ about the point x_k at x_{k+1} is as follows

$$y(x_{k+1}) = y(x_k) \ominus (-1)\tfrac{(x_{k+1}-x_k)^{\alpha}}{\Gamma(\alpha+1)} \odot ({}_{gH}D_*^{\alpha}y)(x_k) \ominus (-1)\tfrac{(x_{k+1}-x_k)^{2\alpha}}{\Gamma(2\alpha+1)} \odot ({}_{gH}D_*^{2\alpha}y)(\xi_k)$$

In similar way to previous case, the fractional Euler's method for ${}^{cf}[(ii) - gH]-$differentiability is

$$y_{k+1} = y_k \ominus (-1)\frac{h^{\alpha}}{\Gamma(\alpha + 1)} \odot f(x_k, y_k), \ \ , k = 0, 1, ..., N - 1.$$

Assuming that $y(x)$ has a switching point type I at $\xi \in [a,b]$, such that $x_0, x_1, ..., x_j, \xi, x_{j+1}, ..., x_N$. Thus the fractional Euler's method given by

$$\begin{cases} y_{k+1} = y_k \oplus \frac{h^\alpha}{\Gamma(\alpha+1)} \odot f(x_k, y_k), & k = 0,1,...,j \\ y_{k+1} = y_k \ominus (-1)\frac{h^\alpha}{\Gamma(\alpha+1)} \odot f(x_k, y_k), & k = j+1, j+2, ..., N-1, \end{cases}$$

and, if ξ be a switching point of type II, the Euler's method takes the form

$$\begin{cases} y_{k+1} = y_k \ominus (-1)\frac{h^\alpha}{\Gamma(\alpha+1)} \odot f(x_k, y_k), & k = 0,1,...,j \\ y_{k+1} = y_k \oplus \frac{h^\alpha}{\Gamma(\alpha+1)} \odot f(x_k, y_k), & k = j+1, j+2, ..., N-1. \end{cases}$$

7. Numerical Examples

In this section, the fuzzy fractional Euler's method is implemented for tackling problem (16). To illustrate the effectiveness of this novel method, we shall consider three examples in all type of differentiability.

Example 7.1. Suppose fractional Euler's method is used to obtain numerical solution of initial-value problem

$$({}_{gH}D_*^\alpha y)(x) = (0,1,1.5)\Gamma(\alpha+1), \; 0 \le x \le 1 \tag{18}$$

with $y(0) = 0$. From formula given in (17), we have

$$y_{k+1} = y_k \oplus h^\alpha \odot (0,1,1.5), \; , k = 0,1,...,N-1.$$

The exact ${}^{cf}[i-gH]$-differentiable solutions is $y(x) = (0,1,1.5)x^\alpha$. Table 1 shows the approximate solutions for equation (18) obtained for different values of α with garde size 0.1 and 0.01. The exact solution and it's Caputo gH-derivative for $\alpha = 0.5, 0.75, 1$ have been shown in Figures 1 - 3.

	α=0.5		α=0.75		$\alpha = 1$	
x	h=0.1	h=0.01	h=0.1	h=0.01	h=0.1	h=0.01
0.1	$(0, 0.31, 0.47)$	$(0, 0.1, 0.15)$	$(0, 0.17, 0.26)$	$(0, 0.03, 0.04)$	$(0, 0.1, 0.15)$	$(0, 0.1, 0.15)$
0.2	$(0, 0.63, 0.94)$	$(0, 0.2, 0.30)$	$(0, 0.35, 0.53)$	$(0, 0.06, 0.09)$	$(0, 0.2, 0.30)$	$(0, 0.2, 0.30)$
0.3	$(0, 0.94, 1.42)$	$(0, 0.3, 0.45)$	$(0, 0.53, 0.80)$	$(0, 0.09, 0.14)$	$(0, 0.3, 0.45)$	$(0, 0.3, 0.45)$
0.4	$(0, 1.26, 1.89)$	$(0, 0.4, 0.60)$	$(0, 0.71, 1.06)$	$(0, 0.12, 0.18)$	$(0, 0.4, 0.60)$	$(0, 0.4, 0.60)$
0.5	$(0, 1.58, 2.37)$	$(0, 0.5, 0.75)$	$(0, 0.88, 1.33)$	$(0, 0.15, 0.23)$	$(0, 0.5, 0.75)$	$(0, 0.5, 0.75)$
0.6	$(0, 1.89, 2.84)$	$(0, 0.6, 0.90)$	$(0, 1.06, 1.60)$	$(0, 0.18, 0.28)$	$(0, 0.6, 0.90)$	$(0, 0.6, 0.90)$
0.7	$(0, 2.21, 3.32)$	$(0, 0.7, 1.05)$	$(0, 1.24, 1.86)$	$(0, 0.22, 0.33)$	$(0, 0.7, 1.05)$	$(0, 0.7, 1.05)$
0.8	$(0, 2.52, 3.79)$	$(0, 0.8, 1.20)$	$(0, 1.42, 2.13)$	$(0, 0.25, 0.37)$	$(0, 0.8, 1.20)$	$(0, 0.8, 1.20)$
0.9	$(0, 2.84, 4.26)$	$(0, 0.9, 1.35)$	$(0, 1.60, 2.40)$	$(0, 0.28, 0.42)$	$(0, 0.9, 1.35)$	$(0, 0.9, 1.35)$
1.0	$(0, 3.16, 4.74)$	$(0, 1.0, 1.50)$	$(0, 1.77, 2.66)$	$(0, 0.31, 0.47)$	$(0, 1.0, 1.50)$	$(0, 1.0, 1.50)$

Table 1. The Numerical Solution of Example 7.1

Example 7.2. Suppose that we want to apply fuzzy fractional Euler's method to initial-value problem

$$\begin{cases} ({}_{gH}D_*^\alpha y)(x) = -y(x), \quad 0 \le x \le 1 \\ \\ y(0) = (0,1,2). \end{cases} \tag{19}$$

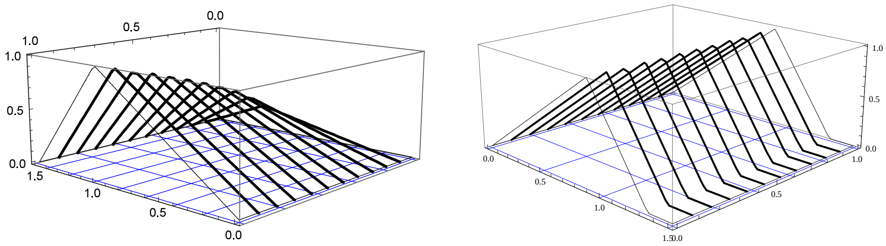

FIGURE 1. The solution of Example 7.1 for $\alpha = 0.5$ (Left), and it's Caputo gH-derivative (Right).

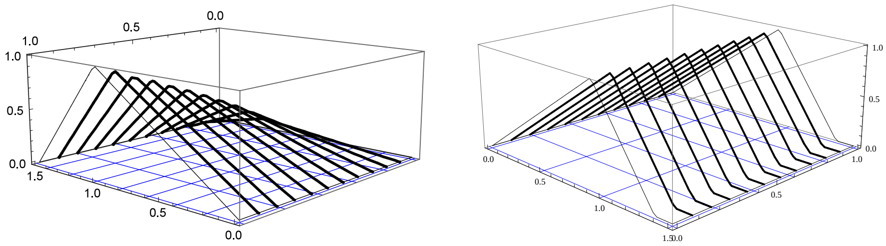

FIGURE 2. The solution of Example 7.1 for $\alpha = 0.75$ (Left), and it's Caputo gH-derivative (Right).

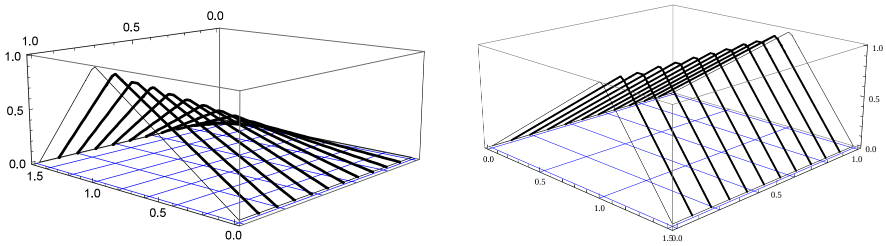

FIGURE 3. The solution of Example 7.1 for $\alpha = 1$ (Left), and it's Caputo gH-derivative (Right).

The exact ${}^{cf}[(ii) - gH]$-differentiable solution is $y(x) = (0, 1, 2)E_{\alpha}(-x^{\alpha})$. The fuzzy fractional Euler's method is, Consequently

$$\begin{cases} y_0 = (0, 1, 2), \\ y_{k+1} = y_k \ominus \frac{h^{\alpha}}{\Gamma(\alpha+1)} \odot y_k, \quad \text{k=0, 1, ..., N-1.} \end{cases}$$

Table 2 lists the numerical result of equation (19) for various fractional order with $h = 0.1$ and $h = 0.01$.

With different values of α the exact solution and it's Caputo generalized Hukuhara derivative are shown in Figures 4- 6.

	α=0.5		α=0.75		$\alpha=1$	
x	h=0.1	h=0.01	h=0.1	h=0.01	h=0.1	h=0.01
0.1	$(0, 0.64, 1.28)$	$(0, 0.3020, 0.6040)$	$(0, 0.80, 1.61)$	$(0, 0.70, 1.40)$	$(0, 0.90, 1.80)$	$(0, 0.90, 1.80)$
0.2	$(0, 0.41, 0.82)$	$(0, 0.0912, 0.1824)$	$(0, 0.65, 1.30)$	$(0, 0.49, 0.99)$	$(0, 0.81, 1.62)$	$(0, 0.81, 1.63)$
0.3	$(0, 0.26, 0.53)$	$(0, 0.0275, 0.0551)$	$(0, 0.52, 1.04)$	$(0, 0.34, 0.69)$	$(0, 0.72, 1.45)$	$(0, 0.73, 1.47)$
0.4	$(0, 0.26, 0.34)$	$(0, 0.0083, 0.0166)$	$(0, 0.42, 0.84)$	$(0, 0.24, 0.49)$	$(0, 0.65, 1.31)$	$(0, 0.66, 1.33)$
0.5	$(0, 0.17, 0.22)$	$(0, 0.0025, 0.0050)$	$(0, 0.34, 0.68)$	$(0, 0.17, 0.34)$	$(0, 0.59, 1.18)$	$(0, 0.60, 1.21)$
0.6	$(0, 0.11, 0.22)$	$(0, 0.0009, 0.0017)$	$(0, 0.27, 0.55)$	$(0, 0.12, 0.24)$	$(0, 0.53, 1.06)$	$(0, 0.54, 1.09)$
0.7	$(0, 0.07, 0.14)$	$(0, 0.0003, 0.0006)$	$(0, 0.22, 0.44)$	$(0, 0.08, 0.17)$	$(0, 0.47, 0.95)$	$(0, 0.49, 0.98)$
0.8	$(0, 0.04, 0.09)$	$(0, 0.0001, 0.0002)$	$(0, 0.17, 0.35)$	$(0, 0.05, 0.11)$	$(0, 0.43, 0.86)$	$(0, 0.44, 0.89)$
0.9	$(0, 0.02, 0.05)$	$(0, 0.0000, 0.0001)$	$(0, 0.14, 0.28)$	$(0, 0.04, 0.08)$	$(0, 0.38, 0.77)$	$(0, 0.40, 0.80)$
1.0	$(0, 0.01, 0.03)$	$(0, 0.0000, 0.0000)$	$(0, 0.11, 0.23)$	$(0, 0.03, 0.06)$	$(0, 0.34, 0.69)$	$(0, 0.36, 0.73)$

TABLE 2. The Numerical Solutions of Example 7.2

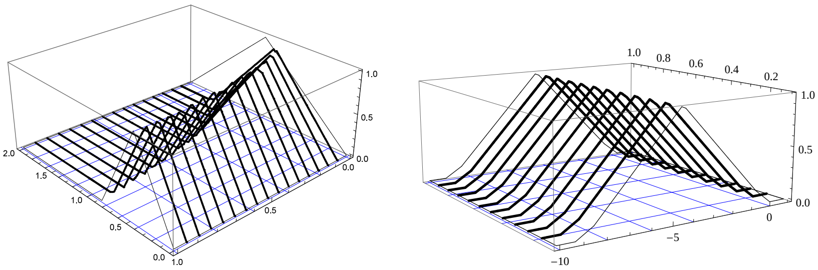

FIGURE 4. The solution of Example 7.2 for $\alpha = 0.5$ (Left), and it's Caputo gH-derivative (Right).

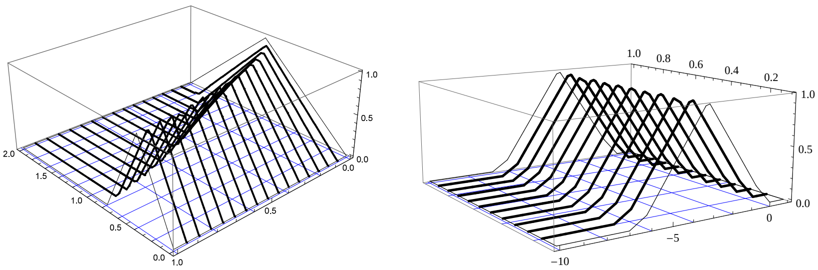

FIGURE 5. The solution of Example 7.2 for $\alpha = 0.75$ (Left), and it's Caputo gH-derivative (Right).

Example 7.3. Using fuzzy fractional Euler's method to obtain approximate to the solution of initial-value problem

$$\begin{cases} ({}_{gH}D_*^{\alpha}y)(x) = \frac{\pi x^{1-\alpha}\alpha}{\Gamma(2-\alpha)}(0, \frac{1}{2}, 1)\ {}_pF_q(1; [1-\frac{\alpha}{2}, \frac{3}{2}-\frac{\alpha}{2}]; -\frac{1}{4}\pi^2x^2\alpha^2), \quad 1 \leq x \leq 2 \\ y(1) = (0, 0.5, 1)\sin(\alpha\pi), \end{cases} \tag{20}$$

where ${}_pF_q(a; b; z)$ is the generalized hypergeometric function, with $\alpha = 0.5$ and various grade size gives the results listed in Table 3. The exact solution is $y(x) =$

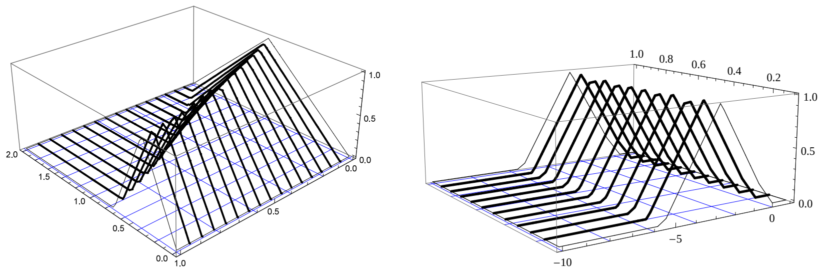

FIGURE 6. The solution of Example 7.2 for $\alpha = 1$ (Left), and it's Caputo gH-derivative (Right).

$(0, 0.5, 1)\sin(\pi\alpha x)$, that has a switching point of type I at $x = 1.463$ in sense of definition 2.13. Dividing the interval $[1, 2]$ to N subinterval $[x_k, x_{k+1}]$, for $k = 0, 1, ..., N-1$, and assuming the switching point belongs to $[x_j, x_{j+1}]$, the fuzzy fractional Euler's method for equation (20) is as follows

$$\begin{cases} y_{k+1} = y_k \oplus \frac{h^\alpha}{\Gamma(\alpha+1)} \odot (\frac{\pi x_k^{1-\alpha}\alpha}{\Gamma(2-\alpha)}(0, \frac{1}{2}, 1)\ {}_pF_q(a; b; zx_k^2)), & k = 0, 1, ..., j \\ y_{k+1} = y_k \ominus (-1)\frac{h^\alpha}{\Gamma(\alpha+1)} \odot (\frac{\pi x_k^{1-\alpha}\alpha}{\Gamma(2-\alpha)}(0, \frac{1}{2}, 1)\ {}_pF_q(a; b; zx_k^2)), & k = j+1, j+2, ..., N-1. \end{cases}$$

where $a = 1, b = [1 - \frac{\alpha}{2}, \frac{3}{2} - \frac{\alpha}{2}]$ and $z = -\frac{1}{4}\pi^2\alpha^2$.

	α=0.5		
t	h=0.1	h=0.01	h=0.001
1.1	$(0, 0.49, 0.98)$	$(0, 0.4936, 0.9875)$	$(0, 0.4942, 0.9823)$
1.2	$(0, 0.46, 0.93)$	$(0, 0.4745, 0.9514)$	$(0, 0.4747, 0.9512)$
1.3	$(0, 0.42, 0.86)$	$(0, 0.4468, 0.8923)$	$(0, 0.4452, 0.8920)$
1.4	$(0, 0.41, 0.80)$	$(0, 0.4024, 0.8092)$	$(0, 0.4023, 0.8031)$
1.5	$(0, 0.35, 0.71)$	$(0, 0.3526, 0.7072)$	$(0, 0.3523, 0.7062)$
1.6	$(0, 0.30, 0.59)$	$(0, 0.2945, 0.5835)$	$(0, 0.2923, 0.5832)$
1.7	$(0, 0.20, 0.43)$	$(0, 0.2248, 0.4583)$	$(0, 0.2246, 0.4545)$
1.8	$(0, 0.16, 0.31)$	$(0, 0.1536, 0.3148)$	$(0, 0.1536, 0.3903)$
1.9	$(0, 0.04, 0.13)$	$(0, 0.0264, 0.1265)$	$(0, 0.0011, 0.0234)$
2.0	$(0, -0.00, -0.00)$	$(0, -0.018, -0.012)$	$(0, 0.0000, 0.0000)$

TABLE 3. The Numerical Solutions of Example 7.3

The exact solution and it's ${}^{cf}[gH]$-derivative with different values of α are shown graphically, see Figure 7.

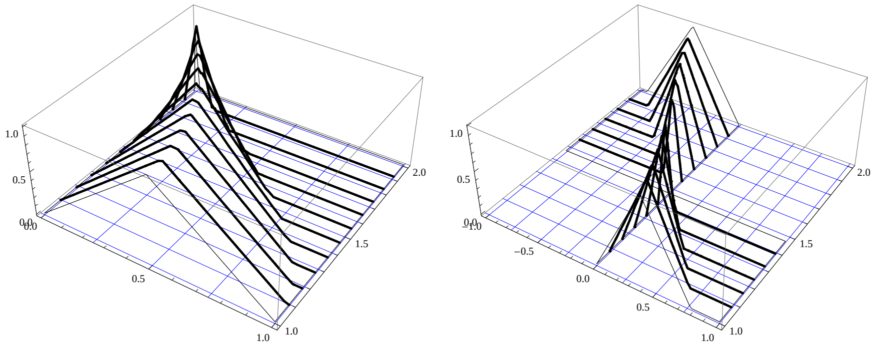

FIGURE 7. The solution of Example 7.3 for $\alpha = 0.5$ (Left), and it's Caputo gH-derivative (Right).

8. Conclusion

We have proved some main theorem in fuzzy fractional calculus that are called mean value theorem for fuzzy Riemann-Liouville integral, Taylor's expansion for Caputo gH-derivative, and fuzzy generalized Taylor's formula. As an application of this concepts, the fuzzy fractional Euler's method is presented under Caputo gH-differentiability. This method can be used for solving fuzzy fractional differential equations without embedding it to two crisp fractional differential equations. Solving three fuzzy fractional differential equations illustrated the method and its simplicity efficiency.

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