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RELATIONSHIPS BETWEEN COMPLETENESS OF FUZZY QUASI-UNIFORM SPACES

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Abstract. In this paper, we give a kind of Cauchy 1-completeness in probabilistic quasi-uniform spaces by using 1-filters. Utilizing the relationships among probabilistic quasi-uniformities, classical quasi-uniformities and Hutton $[0,1]$-quasi-uniformities, we show the relationships between their completeness. In fuzzy quasi-metric spaces, we establish the relationships between the completeness of induced probabilistic quasi-uniform spaces and both completeness of induced classical quasi-uniform spaces and induced Hutton $[0,1]$-quasi-uniform spaces.

1. Introduction

The theory of uniform structures is an important area of analysis and topology because it provides an appropriate context to link metrics with general topological structures. Quasi-uniformity is a uniformity structure which does not satisfy the symmetric condition. With the development of fuzzy topology, many mathematical structures have been generalized to the fuzzy case, such as fuzzy convergence structures [19, 20] and fuzzy convex structures [21, 22, 26, 27]. For uniformities, many researchers put forward various lattice-valued (quasi-)uniformities and obtain a series of interesting results: see e.g. Höhle's probabilistic (quasi-)uniformity [14], Lowen's (quasi-)uniformity [17], Hutton's L-(quasi-)uniformity [12, 31], Shi's pointwise (quasi-)uniformity [25, 31] and J. Gutiérrez García's L-uniformity [4]. In [4], J. Gutiérrez García studied the relationships between the different notions of fuzzy (quasi-)uniformities. It is worth mentioning that Zhang [32] studied a comparison of various types of uniformities in fuzzy topology and then analyzed the relationships between several notions of lattice-valued (quasi-)uniformities in [33].

The completeness discussed by means of filters theory is an important content in uniform spaces. Lowen in [17, 18] studied the completeness of fuzzy uniform spaces based on prefilters. Höhle studied $\top$-completeness of probabilistic uniform spaces based on $\top$-filters in [15]. J. Gutiérrez García and M. A. De Prada Vicente in [5] studied the completeness of Hutton $[0,1]$-quasi-uniform spaces based on tight and stratified L-filters. In this paper, with the help of the idea of J. L. Sieber and W. J. Pervin [23], we propose a kind of completeness of probabilistic quasi-uniform spaces, which is called Cauchy 1-completeness based on 1-filters in the unit interval

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$[0, 1]$. Inspired by the relationships between various types of lattice-valued (quasi-)uniformities, we want to discuss relationships between completeness of probabilistic quasi-uniformities and both completeness of classical quasi-uniformities and Hutton $[0, 1]$-quasi-uniformities.

Fuzzy (quasi-)metric spaces draw much attention in fuzzy mathematics. The usual concept of fuzzy (quasi-)metric spaces can date back to George and Veeramani [7, 8], which slightly modified the definition given by Kramosil and Michalek [16] who adapted the concept of probabilistic metrics to the fuzzy setting. Furthermore, the completeness of fuzzy (quasi-)metric spaces also has studied in [9, 10, 11]. What's more, many authors associated to each fuzzy (quasi-)metric space a lattice-valued (quasi-)uniform space (such as [28, 29]).

The paper is organized as follows. In section 2 we provide lattice theoretical environment and some concepts of lattice-valued quasi-uniformities used in this paper. Furthermore, we give a kind of Cauchy 1-completeness in probabilistic quasi-uniform spaces by using 1-filters. Section 3 and Section 4 are devoted to study the relationships between completeness of classical quasi-uniform spaces and probabilistic quasi-uniform spaces, and the relationships between completeness of probabilistic quasi-uniform spaces and Hutton $[0, 1]$-quasi-uniform spaces. Finally, in section 5, in the framework of fuzzy quasi-metric spaces, we establish the relationships between completeness of induced classical quasi-uniform spaces and induced probabilistic quasi-uniform spaces and the relationships between the completeness of induced probabilistic quasi-uniform spaces and induced Hutton $[0, 1]$-quasi-uniform spaces.

2. **Preliminaries**

In this paper, we use the unit interval $I = [0, 1]$ as the true table although most of the results are also valid in complete residuated lattices.

2.1. **Lattice Theoretical Preliminaries.**

Definition 2.1. [24] A binary operation $* : I \times I \to I$ is called a *(left-)continuous t-norm* if it satisfies the following conditions:

(1) $*$ is associative and commutative;
(2) 1 is the unit, i.e., $a * 1 = a$ for all $a \in I$;
(3) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$;
(4) $*$ is (left-)continuous.

In this paper, we always assume $*$ is a left-continuous t-norm on I. We say that the left-continuous t-norm $*$ does not have nontrivial zero divisors, if $\alpha * \beta \neq 0$ whenever $\alpha, \beta \neq 0$.

If $*$ is a left-continuous t-norm, since the map $\alpha * (-) : I \to I$ preserves arbitrary joins for each $\alpha \in I$, it has a right adjoint $\alpha \xrightarrow{*} (-) : I \to I$ determined by the adjoint property

$$\alpha * \beta \leqslant \gamma \Leftrightarrow \beta \leqslant \alpha \xrightarrow{*} \gamma, \quad \alpha, \beta, \gamma \in I.$$

Hence the implication $\to$ is the binary operation on I given by

$$\alpha \xrightarrow{*} \gamma = \bigvee \{\beta \in I \mid \alpha * \beta \leqslant \gamma\}, \quad \alpha, \gamma \in I.$$

Three distinguished examples of left-continuous t-norm are $\wedge$, $\cdot$ and $*_{Ł}$ (the Łukasiewicz t-norm) which are given as

$$\alpha \wedge \beta = min\{\alpha, \beta\}, \quad \alpha \cdot \beta = \alpha\beta \quad and \quad \alpha *_{Ł} \beta = max\{\alpha + \beta - 1, 0\};$$

$$\alpha \xrightarrow{\wedge} \beta = \begin{cases} 1 & \text{if } \alpha \leqslant \beta, \\ \beta & \text{if } \beta < \alpha, \end{cases} \alpha \xrightarrow{\cdot} \beta = \begin{cases} 1 & \text{if } \alpha \leqslant \beta \\ \dfrac{\beta}{\alpha} & \text{if } \beta < \alpha \end{cases} \text{and } \alpha \xrightarrow{Ł} \beta = min\{1 - \alpha + \beta, 1\},$$

for all $\alpha, \beta \in I$. It is well-known and easy that $* \leqslant \wedge$ for each left-continuous t-norm $*$.

For a set X, a binary map $\mathcal{S}_X(-,-) : I^X \times I^X \to I$ is defined by $\mathcal{S}_X(A, B) = \bigwedge_{x \in X} \big(A(x) \to B(x)\big)$ for each $A, B \in I^X$, where $\mathcal{S}_X(A, B)$ can be interpreted as the degree to which A is a subset of B. It is called *fuzzy inclusion order* or *subsethood degree* of I-subsets. $\mathcal{S}_X(-,-)$ is also denoted as $\mathcal{S}(-,-)$.

2.2. **Relationships Among Classical Filters, 1-filters and I-filters.**

Below we collect some definitions and results regarding classical filters, 1-filters and I-filters for the unit interval I, that will be needed later on.

Definition 2.2. [15] A nonempty subset $\mathbb{F}$ of I^X is called a 1-*filter* on X provided it satisfies the following properties:

(IF1) If $A \in I^X$ such that $\bigvee_{B \in \mathbb{F}} S(B, A) = 1$, then $A \in \mathbb{F}$;
(IF2) $A_1 \wedge A_2 \in \mathbb{F}$ for all $A_1, A_2 \in \mathbb{F}$;
(IF3) $\bigvee_{x \in X} A(x) = 1$ for all $A \in \mathbb{F}$.

The set of all 1-filters on X is denoted by $\mathbb{F}_I(X)$. For a 1-filter on X, the axiom $(IF1)$ means $B \in \mathbb{F}$ whenever $A \in \mathbb{F}$ with $A \leqslant B$. For each $x \in X$, let $[x] \subseteq I^X$ be $[x] = \{A \in I^X \mid A(x) = 1\}$. Then $[x]$ is a 1-filter and $[x]$ is usually called *the point* 1-*filter of* x.

Definition 2.3. [15] A nonempty subset $\mathbb{B} \subseteq I^X$ is called a 1-*filter base* on the set X if it satisfies the following conditions:

(B1) $\bigvee_{B \in \mathbb{B}} S(B, C \wedge D) = 1$ for all $C, D \in \mathbb{B}$;
(B2) $\bigvee_{x \in X} C(x) = 1$ for all $C \in \mathbb{B}$.

Definition 2.4. [13] Let X be a set. A map $\upsilon : I^X \to I$ is called an I-*filter* on X if it satisfies the following properties:

(F0) $\upsilon(1_X) = 1$;
(F1) If $f_1,\ f_2 \in I^X$, $f_1 \leqslant f_2$ then $\upsilon(f_1) \leqslant \upsilon(f_2)$;
(F2) $\upsilon(f_1) \wedge \upsilon(f_2) \leqslant \upsilon(f_1 \wedge f_2)$ for each $f_1,\ f_2 \in I^X$;
(F3) $\upsilon(1_\emptyset) = 0$.

An I-filter υ is said to be *stratified* if it satisfies the following additional axiom:

(F4) $\alpha * \upsilon(f) \leqslant \upsilon(\alpha * f)$ for all $\alpha \in I$ and $f \in I^X$.

An I-filter υ is said to be *tight* if it satisfies the following important axiom:

(F5) $\upsilon(\alpha * 1_X) = \alpha$ for all $\alpha \in I$.

From [30], we have the following results between classical filters and 1-filters.

Lemma 2.5. *Let $\mathcal{F}_I(X)$ be the set of all classical filter on X and $\mathbb{F}_I(X)$ be the set of all 1-filter on X. The following statements hold:*

(1)The order-preserving mapping $\omega : \mathcal{F}_I(X) \to \mathbb{F}_I(X)$ is given by

$$\omega(\mathcal{F}) = \{A \in I^X \mid \bigvee_{F \in \mathcal{F}} \bigwedge_{x \in F} A(x) = 1\},$$

for each $\mathcal{F} \in \mathcal{F}_I(X)$. Then $\omega(\mathcal{F})$ is a 1-filter.

(2)The order-preserving mappings $[\ ]$, $\iota : \mathbb{F}_I(X) \to \mathcal{F}_I(X)$ are respectively given by $[\mathbb{F}] = \{u \in 2^X \mid \chi_u \in \mathbb{F}\}$ and $\iota(\mathbb{F}) = \{\sigma_r(A) \mid A \in \mathbb{F}, r \in [0, 1)\}$, for each $\mathbb{F} \in \mathbb{F}_I(X)$, where $\sigma_r(A) = \{x \in X \mid A(x) > r\}$. Then $[\mathbb{F}]$ and $\iota(\mathbb{F})$ are classical filters.

Lemma 2.6. *Let $\mathcal{F}$ be a classical filter on X and $\mathbb{F}$ be a 1-filter. Then*

(1) $\iota(\omega(\mathcal{F})) = \mathcal{F}$;
(2) $\omega(\iota(\mathbb{F})) \supseteq \mathbb{F}$;
(3) $[\omega(\mathcal{F})] = \mathcal{F}$;
(4) $\omega([\mathbb{F}]) \subseteq \mathbb{F}$.

The lemma state that these adjoint connections hold, i.e., $\iota \dashv \omega \dashv [\ ]$.

Definition 2.7. $\mathbb{F}$ is a 1-filter. $\mathbb{F}$ is called an *induced 1-filter* if there exists a classical filter $\mathcal{F}$ such that $\mathbb{F} = \omega(\mathcal{F})$.

It is easy to check the following results.

Lemma 2.8. *$\mathbb{F}$ is an induced 1-filter if and only if $\iota(\mathbb{F}) = [\mathbb{F}]$.*

Furthermore, $\omega(\iota(\mathbb{F})) = \mathbb{F}$.

Next, we will mention the relationship between 1-filters and stratified and tight I-filters.

Lemma 2.9. [2] *Every 1-filter $\mathbb{F}$ on X induces a stratified and tight I-filter $\upsilon_{\mathbb{F}}$ by*

$$\upsilon_{\mathbb{F}}(f) = \bigvee\{\alpha \in I \mid \alpha \to f \in \mathbb{F}\}, \quad \forall f \in I^X.$$

Lemma 2.10. [2] *Every stratified and tight I-filter υ on X determines a 1-filter $\mathbb{F}_\upsilon$ by*

$$\mathbb{F}_\upsilon = \{f \in I^X \mid \upsilon(f) = 1\}.$$

J. Gutiérrez García established a relationship between 1-filters and I-filters by using some properties of the characteristic value in [3]. Here, we do not utilize the relationship to discuss the following problems and just hope that readers know it.

2.3. $(X,\mathbb{U})$**,** $(X,\mathcal{U})$ **and** $(X,\mathfrak{U})$**.**

In this part, we now describe briefly the definition and some properties about classical quasi-uniformity $\mathbb{U}$, probabilistic quasi-uniformity $\mathcal{U}$ and Hutton $[0,1]$-quasi-uniformity $\mathfrak{U}$.

Definition 2.11. [4, 15] A nonempty subset $\mathcal{U} \subseteq I^{X\times X}$ is called a *probabilistic quasi-uniformity* on X, if it is a 1-filter and still satisfies the following conditions:

(IU0) $U \in \mathcal{U}$ implies $U(x,x) = 1$ for all $x \in X$,
(IUC) $U \in \mathcal{U}$ implies that there exist $V \in \mathcal{U}$ such that $V \circ V \leqslant U$.

For a probabilistic quasi-uniform space $(X,\mathcal{U})$, $\mathbb{N}_x^{\mathcal{U}}$ is the 1-filter generated by the set $N_x^{\mathcal{U}} = \{U(-,x) \mid U \in \mathcal{U}\}$. Therefore,

$$\tau_{\mathcal{U}} = \{A \in I^X \mid \forall x \in X,\ A(x) \leqslant \bigvee_{U\in\mathcal{U}} S(U(-,x),A)\}$$

is generated I-topology by $\mathbb{N}_x^{\mathcal{U}}$.

Next, we will introduce the definition of Hutton $[0,1]$-quasi uniformities from J. Gutiérrez García [5].

Let X be a set and $(I,\leqslant)$ be a complete lattice. By $\mathcal{H}_I(X)$ we denote the collection of all maps $\phi : I^X \to I^X$ satisfying the following conditions:

(1) $\phi(a) \geqslant a$ for each $a \in I^X$ (Enlarging),
(2) $\phi(\bigvee_{j\in J} a_j) = \bigvee_{j\in J} \phi(a_j)$ for each $\{a_j\}_{j\in J} \subset I^X$ (Join-preserving) and $\phi(1_\emptyset) = 1_\emptyset$.

Remark 2.12. As pointed out in [5], each arbitrary join-preserving element $\phi \in (I^X)^{I^X}$ is completely determined by the collection of I-set

$$\{\phi(\alpha * 1_{\{x\}}) \mid \alpha \in (0,1],\ x \in X\}.$$

Definition 2.13. [5] Let X be a set and $([0,1],\leqslant)$ be a complete lattice. A Hutton $[0,1]$-quasi-uniformity on X is a nonempty subset $\mathfrak{U}$ of $\mathcal{H}_I(X)$ such that

(HU1) if $\phi_1 \in \mathfrak{U},\ \phi_1 \leqslant \phi_2$ and $\phi_2 \in \mathcal{H}_I(X)$ then $\phi_2 \in \mathfrak{U}$,
(HU2) if $\phi_1,\ \phi_2 \in \mathfrak{U}$, there exist $\phi \in \mathfrak{U}$ such that $\phi \leqslant \phi_1$ and $\phi \leqslant \phi_2$,
(HU3) if $\phi \in \mathfrak{U}$, there exist $\psi \in \mathfrak{U}$ such that $\psi \circ \psi \leqslant \phi$ (where $\circ$ denotes the usual composition of functions).

The pair $(X,\mathfrak{U})$ is called a *Hutton* $[0,1]$*-quasi-uniform space* such that X is a set and $\mathfrak{U}$ is a Hutton $[0,1]$-quasi-uniformity on X. A nonempty subset $\mathfrak{B}$ of $\mathfrak{U}$, is a base for $\mathfrak{U}$ if for each $\phi \in \mathfrak{U}$, there exists $\varphi \in \mathfrak{B}$ such that $\varphi \leqslant \phi$.

A Hutton $[0,1]$-quasi-uniformity $\mathfrak{U}$ is said to be *stratified* if it has a base $\mathfrak{B}$ which satisfies:

(HU4) if $\forall\varphi \in \mathfrak{B},\ \forall\alpha \in I,\ \forall x \in X,\ \alpha * \varphi(1_{\{x\}}) \leqslant \varphi(\alpha * 1_{\{x\}})$.

2.4. **Cauchy** 1**-completeness of Probabilistic Quasi-uniformity.**

In this part, we use the idea of Sieber and Pervin in [23] to give the Cauchy 1-completeness of probabilistic quasi-uniform spaces by 1-filters used in this paper.

Definition 2.14. Let $(X,\mathcal{U})$ be a probabilistic quasi-uniform space. A 1-filter $\mathbb{F}$ is called a *Cauchy* 1-*filter* if and only if for every $U \in \mathcal{U}$ there exists a point $x \in X$ such that $U(-,x) \in \mathbb{F}$.

If $\mathbb{F}$ is an induced 1-filter, then $\mathbb{F}$ is called an *induced Cauchy* 1-*filter* if and only if for every $U \in \mathcal{U}$ there exists a point $x \in X$ such that $U(-,x) \in \mathbb{F}$.

It is easy to know that $\mathbb{N}_x^{\mathcal{U}}$ is a Cauchy 1-filter and we call it *the neighborhood* 1-*filter of* x. If $\mathbb{N}_x^{\mathcal{U}} \subseteq \mathbb{F}$, we call $\mathbb{F}$ converges to x.

Definition 2.15. A probabilistic quasi-uniform space $(X,\mathcal{U})$ will be said to be *Cauchy* 1-*complete* if and only if each Cauchy 1-filter converges.

Furthermore, a probabilistic quasi-uniform space $(X,\mathcal{U})$ is said to be *induced Cauchy* 1-*complete* if each induced Cauchy 1-filter converges.

Now we recall the definition about the completeness of classical quasi-uniform space $(X,\mathbb{U})$ in [23].

Definition 2.16. [23] A filter $\mathcal{F}$ in a quasi-uniform space $(X,\mathbb{U})$ will be called a *Cauchy filter* if and only if for every $u \in \mathbb{U}$ there exists a point $z \in X$ such that $u[z] \in \mathcal{F}$, where $u[z] = \{y \in X \mid (y,z) \in u\}$.

From general topology, we obtain that for a quasi-uniform space $(X,\mathbb{U})$, $\mathbf{N}_z^{\mathbb{U}}$, the neighborhood of z, is generated by the set $N_z^{\mathbb{U}} = \{u[z] \mid u \in \mathbb{U}\}$. If $\mathbf{N}_z^{\mathbb{U}} \subseteq \mathcal{F}$, we call $\mathcal{F}$ converges to z.

Definition 2.17. [23] A quasi-uniform space $(X,\mathbb{U})$ will be said to be *complete* if and only if every Cauchy filter converges.

3. Relationship Between the Completeness of $(X,\mathbb{U})$ and $(X,\mathcal{U})$

In this section, we will discuss the relationships between the completeness of classical quasi-uniform spaces and Cauchy 1-completeness of probabilistic quasi-uniform spaces.

Next, we firstly introduce two functors Φ, Ψ about classical quasi-uniformities and probabilistic quasi-uniformities.

Proposition 3.1. [6, 33] *Let $(X,\mathbb{U})$ be a classical quasi-uniform space and $(X,\mathcal{U})$ be a probabilistic quasi-uniform space. Let $\Phi(\mathbb{U})$ be the probabilistic quasi-uniformity generated by*

$$\{1_u \mid u \in \mathbb{U}\}.$$

Let $\Psi(\mathcal{U})$ be the classical quasi-uniformity generated by

$$\{\Psi(U) \mid U \in \mathcal{U}\}$$

where $\Psi(U) = \{(x,y) \in X \times X \mid U(x,y) = 1\}$. Then:

(1) $\Psi(\Phi(\mathbb{U})) = \mathbb{U}$;
(2) $\Phi(\Psi(\mathcal{U})) \subseteq \mathcal{U}$;
(3) Ψ is a right adjoint of Φ.

Lemma 3.2. *Let $(X,\mathbb{U})$ be a classical quasi-uniform space, $\mathcal{F}$ be a classical filter on X and $x_0 \in X$. Then:*

(1) $\mathcal{F}$ *is a Cauchy filter in* $(X,\mathbb{U})$ *if and only if* $\omega(\mathcal{F})$ *is a Cauchy* 1*-filter in* $(X,\Phi(\mathbb{U}))$.

(2) $\mathcal{F}$ *converges to* x_0 *in* $(X,\mathbb{U})$ *if and only if* $\omega(\mathcal{F})$ *converges to* x_0 *in* $(X,\Phi(\mathbb{U}))$.

Proof. (1) Necessity: Let $u\in\mathbb{U}$ and $1_u\in\Phi(\mathbb{U})$. Since $\mathcal{F}$ is a Cauchy filter in $(X,\mathbb{U})$, there exists $x\in X$ such that $u[x]\in\mathcal{F}$. Hence,

$$\bigvee_{F\in\mathcal{F}}\bigwedge_{y\in F}1_u(y,x)\geqslant\bigwedge_{y\in u[x]}1_u(y,x)=1.$$

Therefore, $1_u(-,x)\in\omega(\mathcal{F})$.

Sufficiency: Let $u\in\mathbb{U}$. Then $1_u\in\Phi(\mathbb{U})$. Since $\omega(\mathcal{F})$ is a Cauchy 1-filter in $(X,\Phi(\mathbb{U}))$, there exists $x\in X$ such that $1_u(-,x)\in\omega(\mathcal{F})$. Then there is $\bigvee_{F\in\mathcal{F}}\bigwedge_{y\in F}1_u(y,x)=1$. So we can find $F_r\in\mathcal{F}$ satisfying $\bigwedge_{y\in F_r}1_u(y,x)>r$ for all $r\in(0,1)$. When $y\in F_r$, we have $(y,x)\in u$, namely, $y\ inu[x]$. Hence, $F_r\subseteq u[x]$. Therefore, $u[x]\in\mathcal{F}$.

(2) Necessity: Let $u\in\mathbb{U}$, $1_u\in\Phi(\mathbb{U})$ and $1_u(-,x_0)\in\mathbb{N}_{x_0}^{\Phi(\mathbb{U})}$. Since $\mathcal{F}$ converges to x_0 in $(X,\mathbb{U})$, we have $\mathbf{N}_{x_0}^{\mathbb{U}}\subseteq\mathcal{F}$. So, for $u[x_0]\in\mathbf{N}_{x_0}^{\mathbb{U}}$, there is $u[x_0]\in\mathcal{F}$. Hence,

$$\bigvee_{F\in\mathcal{F}}\bigwedge_{y\in F}1_u(y,x_0)\geqslant\bigwedge_{y\in u[x_0]}1_u(y,x_0)=1.$$

Therefore, $1_u(-,x_0)\in\omega(\mathcal{F})$.

Sufficiency: Let $u\in\mathbb{U}$. Then $1_u\in\Phi(\mathbb{U})$ and $u[x_0]\in\mathbf{N}_{x_0}^{\mathbb{U}}$. Since $\omega(\mathcal{F})$ converges to x_0 in $(X,\Phi(\mathbb{U}))$, we have $\mathbb{N}_{x_0}^{\Phi(\mathbb{U})}\subseteq\omega(\mathcal{F})$. Furthermore, when $1_u(-,x_0)\in\mathbb{N}_{x_0}^{\Phi(\mathbb{U})}$, we have $1_u(-,x_0)\in\omega(\mathcal{F})$. Then there is $\bigvee_{F\in\mathcal{F}}\bigwedge_{y\in F}1_u(y,x_0)=1$. So we can find $F_r\in\mathcal{F}$ satisfying $\bigwedge_{y\in F_r}1_u(y,x_0)>r$ for all $r\in(0,1)$. When $y\in F_r$, we have $(y,x_0)\in u$. Then there is $y\in u[x_0]$. Hence, $F_r\subseteq u[x_0]$. Therefore, $u[x_0]\in\mathcal{F}$. □

Lemma 3.3. *Let* $(X,\mathcal{U})$ *be a probabilistic quasi-uniform space,* $\mathbb{F}$ *be an induced* 1*-filter and* $x_0\in X$*. Then:*

(1) $\mathbb{F}$ *is a Cauchy* 1*-filter in* $(X,\mathcal{U})$ *if and only if* $\iota(\mathbb{F})$ *is a Cauchy filter in* $(X,\Psi(\mathcal{U}))$*;*

(2) $\mathbb{F}$ *converges to* x_0 *in* $(X,\mathcal{U})$ *if and only if* $\iota(\mathbb{F})$ *converges to* x_0 *in* $(X,\Psi(\mathcal{U}))$.

Proof. (1) Sufficiency: Let $U\in\mathcal{U}$. Then $\Psi(U)\in\Psi(\mathcal{U})$. Since $\iota(\mathbb{F})$ is a Cauchy filter in $(X,\Psi(\mathcal{U}))$, there exists $x\in X$ such that $\Psi(U)[x]\in\iota(\mathbb{F})=[\mathbb{F}]$. Let me denote $\Psi(U)[x]=A$. Furthermore, we have $\chi_A\in\mathbb{F}$. Hence,

$$\bigvee_{B\in\mathbb{F}}S(B,U(-,x))\geqslant\bigwedge_{y\in A}(\chi_A(y)\to U(y,x))=\bigwedge_{y\in A}U(y,x)=1.$$

Therefore, $U(-,x)\in\mathbb{F}$.

Necessity: Let $u\in\Psi(\mathcal{U})$. Then there exists $U\in\mathcal{U}$ such that $\Psi(U)\subseteq u$. Since $\mathbb{F}$ is a Cauchy 1-filter in $(X,\mathcal{U})$, there is some $x\in X$ satisfying $U(-,x)\in\mathbb{F}$. On account of $\omega(\iota(\mathbb{F}))=\mathbb{F}$, it follows that $U(-,x)\in\omega(\iota(\mathbb{F}))$. Then there is

$\bigvee_{G\in\iota(\mathbb{F})}\bigwedge_{y\in G} U(y,x)=1$. So we can find $G_r\in\iota(\mathbb{F})$ such that $\bigwedge_{y\in G_r} U(y,x)>r$ for any $r\in[0,1)$. When $y\in G_r$, we have $U(y,x)>r$ for any $r\in[0,1)$. Hence, $y\in\Psi(U)[x]$, where $\Psi(U)[x]=\{y\in X\mid U(y,x)=1\}$. Furthermore, there is $G_r\subseteq\Psi(U)[x]\subseteq u[x]$. Therefore, $u[x]\in\iota(\mathbb{F})$.

(2) Sufficiency: Let $U\in\mathcal{U}$, $\Psi(U)\in\Psi(\mathcal{U})$ and $U(-,x_0)\in\mathbb{N}^{\mathcal{U}}_{x_0}$. Since $\iota(\mathbb{F})$ converges to x_0 in $(X,\Psi(\mathcal{U}))$, we have $\mathbf{N}^{\Psi(\mathcal{U})}_{x_0}\subseteq\iota(\mathbb{F})$. Let me denote $\Psi(U)[x_0]=A$. For $A\in\mathbf{N}^{\Psi(\mathcal{U})}_{x_0}$, there is $A\in\iota(\mathbb{F})$. Furthermore, we obtain $\chi_A\in\mathbb{F}$. Hence,

$$\bigvee_{B\in\mathbb{F}} S(B,U(-,x_0))\geqslant\bigwedge_{y\in A}(\chi_A(y)\to U(y,x_0))=\bigwedge_{y\in A}U(y,x_0)=1.$$

Therefore, $U(-,x_0)\in\mathbb{F}$.

Necessity: Let $U\in\mathcal{U}$ and $\Psi(U)[x_0]\in\mathbf{N}^{\Psi(\mathcal{U})}_{x_0}$. Since $\mathbb{F}$ converges to x_0 in $(X,\mathcal{U})$, we have $\mathbb{N}^{\mathcal{U}}_{x_0}\subseteq\mathbb{F}$. For $U(-,x_0)\in\mathbb{N}^{\mathcal{U}}_{x_0}$, there is $U(-,x_0)\in\mathbb{F}$. On account of $\omega(\iota(\mathbb{F}))=\mathbb{F}$, it follows that $U(-,x_0)\in\omega(\iota(\mathbb{F}))$. Then there is $\bigvee_{G\in\iota(\mathbb{F})}\bigwedge_{y\in G}U(y,x_0)=1$. So we can find $G_r\in\iota(\mathbb{F})$ such that $\bigwedge_{y\in G_r}U(y,x_0)>r$ for any $r\in[0,1)$. When $y\in G_r$, we have $U(y,x_0)>r$ for any $r\in[0,1)$. Hence, $y\in\Psi(U)[x_0]$, where $\Psi(U)[x_0]=\{y\in X\mid U(y,x_0)=1\}$. Furthermore, there is $G_r\subseteq\Psi(U)[x_0]$. Therefore, $\Psi(U)[x_0]\in\iota(\mathbb{F})$. □

It is easy to check the following results by Lemma 3.2 and Lemma 3.3.

Theorem 3.4. *If $(X,\Phi(\mathbb{U}))$ is Cauchy 1-complete, then $(X,\mathbb{U})$ is complete.*

Corollary 3.5. *(1) $(X,\mathbb{U})$ is complete if and only if $(X,\Phi(\mathbb{U}))$ is induced Cauchy 1-complete;*

(2) if $(X,\Psi(\mathcal{U}))$ is complete, then $(X,\mathcal{U})$ is induced Cauchy 1-complete.

4. Relationship Between the Completeness of $(X,\mathcal{U})$ and $(X,\mathfrak{U})$

In this section, we will study the completeness of probabilistic quasi-uniformity $\mathcal{U}$ and Hutton $[0,1]$-quasi-uniformity $\mathfrak{U}$ through the following functors Λ and Υ. Gutiérrez García in [4, 6] has studied the functors and discussed the relationship between probabilistic quasi-uniformities and Hutton $[0,1]$-quasi-uniformities by use of them.

Proposition 4.1. [4, 6] *Let $(X,\mathcal{U})$ be a probabilistic quasi-uniform space and $(X,\mathfrak{U})$ be a Hutton $[0,1]$-quasi-uniform space. Let $\Lambda(\mathcal{U})$ be the Hutton $[0,1]$-quasi-uniformity generated by*

$$\{\Lambda(U)\mid U\in\mathcal{U}\},$$

where $[\Lambda(U)](a)(x)=\bigvee_{y\in X}U(x,y)*a(y)$ for each $U\in\mathcal{U}$, $a\in I^X$ and $x\in X$. Let $\Upsilon(\mathfrak{U})$ be the probabilistic quasi-uniformity generated by

$$\{\Upsilon(\phi)\mid\phi\in\mathfrak{U}\},$$

where $[\Upsilon(\phi)](x,y)=\bigwedge_{\alpha\in I}\alpha\to[\phi(\alpha*1_{\{y\}})](x)$ for each $\phi\in\mathfrak{U}$ and $x,y\in X$. Then:

(1) $\Upsilon(\Lambda(\mathcal{U})) = \mathcal{U}$*;*
(2) $\Lambda(\Upsilon(\mathfrak{U})) \subseteq \mathfrak{U}$*;*
(3) Υ *is a right adjoint of* Λ*.*

We observe that Λ preserves arbitrary sups since $*$ does it. Since 0 is the zero element with respect to $*$, it follows that for each $\alpha \in I$ and $x \in X$,

$$[\Lambda(U)](\alpha * 1_{\{x\}}) = U(-,x) * \alpha = [\Lambda(U)](1_{\{x\}}) * \alpha.$$

If Hutton $[0,1]$-quasi-uniformity $\mathfrak{U}$ is stratified, we can observe that $[\Upsilon(\phi)](-,x) = \phi(1_{\{x\}})$ for each $\phi \in \mathfrak{B}$ and $x \in X$, where $\mathfrak{B}$ is a base for $\mathfrak{U}$.

First of all, we recall the completeness of Hutton $[0,1]$-quasi-uniform space $(X,\mathfrak{U})$ in [5].

Lemma 4.2. [5] *Let* I*-filter* υ *and Hutton* $[0,1]$*-quasi-uniformity* $\mathfrak{U}$ *on* X *be stratified and* $p \in X$*. Then:*

(1) υ *converges to* p *if and only if* $\forall \varphi \in \mathfrak{B}$, $\upsilon(\varphi(1_{\{p\}})) = 1$*;*
(2) υ *is a Cauchy* I*-filter if and only if* $\forall \varphi \in \mathfrak{B}, \exists p_\varphi \in X, \upsilon(\varphi(1_{\{p_\varphi\}})) = 1$.

Where $\mathfrak{B}$ *is a base for* $\mathfrak{U}$*.*

Definition 4.3. [5] A Hutton $[0,1]$-quasi-uniform space $(X,\mathfrak{U})$ is said to be *complete* if any stratified and tight Cauchy I-filter on X is convergent.

Next, we will discuss the relationship between the completeness of probabilistic quasi-uniformities and Hutton [0,1]-quasi-uniformities.

Proposition 4.4. *Let* $(X,\mathcal{U})$ *be a probabilistic quasi-uniform space,* $(X,\mathfrak{U})$ *be a stratified Hutton* $[0,1]$*-quasi-uniform space,* $\mathbb{F}$ *be a* 1*-filter and* υ *be a stratified* I*-filter. Then:*

(1) $\mathbb{F}$ *is a Cauchy* 1*-filter in* $(X,\mathcal{U})$ *if and only if* $\upsilon_{\mathbb{F}}$ *is a stratified Cauchy* I*-filter in* $(X,\Lambda(\mathcal{U}))$*;*
(2) υ *is a stratified Cauchy* I*-filter in* $(X,\mathfrak{U})$ *if and only if* $\mathbb{F}_\upsilon$ *is a Cauchy* 1*-filter in* $(X,\Upsilon(\mathfrak{U}))$*.*

Proof. (1) Necessity: Let $U \in \mathcal{U}$ and $\Lambda(U)$ be a base element for $(X,\Lambda(\mathcal{U}))$. Since $\mathbb{F}$ is a Cauchy 1-filter in $(X,\mathcal{U})$, there exists $p_U \in X$ such that $U(-,p_U) \in \mathbb{F}$. Hence,

$$\upsilon_{\mathbb{F}}(\Lambda(U)(1_{\{p_U\}})) = \bigvee\{\alpha \in [0,1] \mid \alpha \to \Lambda(U)(1_{\{p_U\}}) \in \mathbb{F}\} =$$

$$= \bigvee\{\alpha \in [0,1] \mid \alpha \to U(-,p_U) \in \mathbb{F}\} \geqslant \{1 \in [0,1] \mid 1 \to U(-,p_U) \in \mathbb{F}\} = 1.$$

Therefore, $\upsilon_{\mathbb{F}}$ is a Cauchy I-filter on $(X,\Lambda(\mathcal{U}))$.

Sufficiency: Let $U \in \mathcal{U}$. Then $\Lambda(U)$ is a base element for $(X,\Lambda(\mathcal{U}))$. Since $\upsilon_{\mathbb{F}}$ is a stratified Cauchy I-filter in $(X,\Lambda(\mathcal{U}))$, there exists $p_U \in X$ such that $\upsilon_{\mathbb{F}}(\Lambda(U)(1_{\{p_U\}})) = 1$. Specifically,

$$\upsilon_{\mathbb{F}}(\Lambda(U)(1_{\{p_U\}})) = \bigvee\{\alpha \in [0,1] \mid \alpha \to \Lambda(U)(1_{\{p_U\}}) \in \mathbb{F}\} =$$

$$= \bigvee\{\alpha \in [0,1] \mid \alpha \to U(-,p_U) \in \mathbb{F}\} = 1.$$

So for any $\alpha \in [0,1]$, we can find $\beta_\alpha \in [0,1]$ satisfying $\beta_\alpha \to U(-,p_U) \in \mathbb{F}$ such that $\beta_\alpha \geqslant \alpha$. Hence,

$$\bigvee_{B\in\mathbb{F}} S(B,U(-,p_U)) \geqslant \bigvee_{\alpha\in[0,1]} S(\beta_\alpha \to U(-,p_U),U(-,p_U)) =$$

$$= \bigvee_{\alpha\in[0,1]} \bigwedge_{x\in X} ((\beta_\alpha \to U(x,p_U)) \to U(x,p_U)) \geqslant \bigvee_{\alpha\in[0,1]} \beta_\alpha \geqslant \bigvee_{\alpha\in[0,1]} \alpha = 1.$$

Therefore, $U(-,p_U) \in \mathbb{F}$.

(2) Necessity: Let $\phi \in \mathcal{B}$ and $\Upsilon(\phi) \in \Upsilon(\mathfrak{U})$, where $\mathfrak{B}$ is a base for $\mathfrak{U}$. Since υ is a stratified Cauchy I-filter in $(X,\mathfrak{U})$, there exists $p_\phi \in X$ such that $\upsilon(\phi(1_{\{p_\phi\}})) = 1$. An account of $\mathbb{F}_\upsilon = \{f \in I^X \mid \upsilon(f) = 1\}$, it follows that $\phi(1_{\{p_\phi\}}) \in \mathbb{F}_\upsilon$. Hence, $\Upsilon(\phi)(-,p_\phi) = \phi(1_{\{p_\phi\}}) \in \mathbb{F}_\upsilon$.

Sufficiency: Let $\phi \in \mathfrak{B}$, where $\mathfrak{B}$ is a base for $\mathfrak{U}$. Then $\Upsilon(\phi) \in \Upsilon(\mathfrak{U})$. Since $\mathbb{F}_\upsilon$ is a Cauchy 1-filter in $(X,\Upsilon(\mathfrak{U}))$, there exists $p \in X$ such that $\Upsilon(\phi)(-,p) \in \mathbb{F}_\upsilon$. Hence, $1 = \upsilon(\Upsilon(\phi)(-,p)) = \upsilon(\phi(1_{\{p\}}))$. □

Proposition 4.5. *Let $(X,\mathcal{U})$ be a probabilistic quasi-uniform space, $(X,\mathfrak{U})$ be a stratified Hutton $[0,1]$-quasi-uniform space, $\mathbb{F}$ be a 1-filter, υ be a stratified I-filter and $p \in X$. Then:*

(1) $\mathbb{F}$ converges to p in $(X,\mathcal{U})$ if and only if $\upsilon_\mathbb{F}$ converges to p in $(X,\Lambda(\mathcal{U}))$;

(2) υ converges to p in $(X,\mathfrak{U})$ if and only if $\mathbb{F}_\upsilon$ converges to p in $(X,\Upsilon(\mathfrak{U}))$.

Proof. (1) Necessity: Let $U \in \mathcal{U}$ and $\Lambda(U)$ be a base element for $(X,\Lambda(\mathcal{U}))$. Since $\mathbb{F}$ converges to p in $(X,\mathcal{U})$, we have $\mathbb{N}_p^{\mathcal{U}} \subseteq \mathbb{F}$. Then for any $U(-,p) \in \mathbb{N}_p^{\mathcal{U}}$, there is $U(-,p) \in \mathbb{F}$. Hence,

$$\upsilon_\mathbb{F}(\Lambda(U)(1_{\{p\}})) = \bigvee\{\alpha \in [0,1] \mid \alpha \to \Lambda(U)(1_{\{p\}}) \in \mathbb{F}\} =$$

$$= \bigvee\{\alpha \in [0,1] \mid \alpha \to U(-,p) \in \mathbb{F}\} \geqslant \{1 \in [0,1] \mid 1 \to U(-,p) \in \mathbb{F}\} = 1.$$

Therefore, $\upsilon_\mathbb{F}$ converges to p in $(X,\Lambda(\mathcal{U}))$.

Sufficiency: Let $U \in \mathcal{U}$ and $U(-,p) \in \mathbb{N}_p^{\mathcal{U}}$. Since $\upsilon_\mathbb{F}$ converges to p in $(X,\Lambda(\mathcal{U}))$, there exist $\Lambda(U)$ being a base element for $\Lambda(\mathcal{U})$ such that $\upsilon_\mathbb{F}(\Lambda(U)(1_{\{p\}})) = 1$. Specifically,

$$\upsilon_\mathbb{F}(\Lambda(U)(1_{\{p\}})) = \bigvee\{\alpha \in [0,1] \mid \alpha \to \Lambda(U)(1_{\{p\}}) \in \mathbb{F}\} =$$

$$= \bigvee\{\alpha \in [0,1] \mid \alpha \to U(-,p) \in \mathbb{F}\} = 1.$$

So for any $\alpha \in [0,1]$, we can find $\beta_\alpha \in [0,1]$ satisfying $\beta_\alpha \to U(-,p) \in \mathbb{F}$ such that $\beta_\alpha \geqslant \alpha$. Hence,

$$\bigvee_{B\in\mathbb{F}} S(B,U(-,p)) \geqslant \bigvee_{\alpha\in[0,1]} S(\beta_\alpha \to U(-,p),U(-,p)) =$$

$$= \bigvee_{\alpha\in[0,1]} \bigwedge_{x\in X} ((\beta_\alpha \to U(x,p)) \to U(x,p)) \geqslant \bigvee_{\alpha\in[0,1]} \beta_\alpha \geqslant \bigvee_{\alpha\in[0,1]} \alpha = 1.$$

Therefore, $U(-,p) \in \mathbb{F}$.

(2) Necessity: Let $\phi \in \mathcal{B}$ and $\Upsilon(\phi)(-,p) \in \mathbb{N}_p^{\Upsilon(\mathfrak{U})}$, where $\mathcal{B}$ is a base for $\mathfrak{U}$. Since υ converges to p in $(X,\mathfrak{U})$, we have $\upsilon(\phi(1_{\{p\}})) = 1$. Hence, $\upsilon(\Upsilon(\phi)(-,p)) = \upsilon(\phi(1_{\{p\}})) = 1$. Therefore, $\Upsilon(\phi)(-,p) \in \mathbb{F}_\upsilon$.

Sufficiency: Let $\phi \in \mathcal{B}$, where $\mathcal{B}$ is a base for $\mathfrak{U}$. Since $\mathbb{F}_\upsilon$ converges to p in $(X,\Upsilon(\mathfrak{U}))$, we have $\Upsilon(\phi)(-,p) \in \mathbb{N}_p^{\Upsilon(\mathfrak{U})} \subseteq \mathbb{F}_\upsilon$. Furthermore, we have $\upsilon(\Upsilon(\phi)(-,p)) = 1$. Therefore, $\upsilon(\phi(1_{\{p\}})) = \upsilon(\Upsilon(\phi)(-,p)) = 1$. □

Theorem 4.6. *Let $(X,\mathcal{U})$ be a probabilistic quasi-uniform space and $\Lambda(\mathcal{U})$ be Hutton [0,1]-quasi-uniformity induced by the mapping Λ. Then:*
$(X,\mathcal{U})$ is Cauchy 1-complete if and only if $(X,\Lambda(\mathcal{U}))$ is complete.

Proof. Sufficiency: Suppose that $(X,\Lambda(\mathcal{U}))$ is complete, then $(X,\mathcal{U})$ is Cauchy 1-complete by Proposition 4.4(1) and Proposition 4.5(1).

Necessity: Let υ be a Cauchy I-filter on $(X,\Lambda(\mathcal{U}))$. Since $\Upsilon(\Lambda(\mathcal{U})) = \mathcal{U}$, $\mathbb{F}_\upsilon$ is a Cauchy filter on $(X,\Upsilon(\Lambda(\mathcal{U})))$ by Proposition 4.4(2). Since $(X,\Upsilon(\Lambda(\mathcal{U})))$ is Cauchy 1-complete, then $\mathbb{F}_\upsilon$ converges to p in $(X,\Upsilon(\Lambda(\mathcal{U})))$. By Proposition 4.5(2), we have $\upsilon_{\mathbb{F}_\upsilon}$ converges to p in $(X,\Lambda(\mathcal{U}))$. On account of $\upsilon_{\mathbb{F}_\upsilon} \leqslant \upsilon$, it follows that υ converges to p in $(X,\Lambda(\mathcal{U}))$. Hence, $(X,\Lambda(\mathcal{U}))$ is complete. □

Theorem 4.7. *Let $(X,\mathfrak{U})$ be a Hutton $[0,1]$-quasi-uniform space and $\Upsilon(\mathfrak{U})$ be probabilistic quasi-uniformity induced by the mapping Υ. Then:*
If $(X,\Upsilon(\mathfrak{U}))$ is Cauchy 1-complete, then $(X,\mathfrak{U})$ is complete.

Proof. It is easy to check the result by Proposition 4.4(2) and Proposition 4.5(2). □

5. Completeness in Fuzzy Quasi-metric Spaces

In this section, we will discuss the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced classical quasi-uniform spaces and the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced Hutton $[0,1]$-quasi-uniform space in fuzzy quasi-metric spaces. Let $(X,M,*)$ be a fuzzy quasi-metric space. From [7], we can associate $(X,M,*)$ with a classical quasi-uniformity $\mathbb{U}_M$ induced by the base $\{U_{\varepsilon,t} \mid \varepsilon \in (0,1), t > 0\}$, where $U_{\varepsilon,t} = \{(x,y) \mid M(x,y,t) > 1-\varepsilon\}$. We can also associate $(X,M,*)$ with a probabilistic quasi-uniformity $\mathcal{U}_M$ induced by the base $\{M(-,-,t) \mid t > 0\}$, i.e., $\mathcal{U}_M = \{U \in [0,1]^{X\times X} \mid \bigvee_{t>0} S(M(-,-,t),U) = 1\}$. From [1], We can associate $(X,M,*)$ with a Hutton $[0,1]$-quasi-uniformity $\mathfrak{U}_M$ induced by the base $\{\phi^M_{\varepsilon,t} \mid \varepsilon \in (0,1),\quad t > 0\}$, where the mapping $\phi^M_{\varepsilon,t} : [0,1]^X \to [0,1]^X$ is defined by $\phi^M_{\varepsilon,t}(\alpha * 1_{\{x\}})(y) = \alpha * ((1-\varepsilon) \to M(x,y,t))$ for each $x,y \in X$ and $\alpha \in [0,1]$.

Lemma 5.1. *Let $(X,M,*)$ be a fuzzy quasi-metric space, $\mathcal{F}$ be a classical filter on X and $\mathbb{F}$ be a 1-filter. Then:*

(1) $\mathcal{F}$ is a Cauchy filter in $(X,\mathbb{U}_M)$ if and only if $\omega(\mathcal{F})$ is a Cauchy 1-filter in $(X,\mathcal{U}_M)$;
(2) If $\mathbb{F}$ is a Cauchy 1-filter in $(X,\mathcal{U}_M)$, then $\iota(\mathbb{F})$ is a Cauchy filter in $(X,\mathbb{U}_M)$.

Lemma 5.2. *Let $(X, M, *)$ be a fuzzy quasi-metric space, $\mathcal{F}$ be a classical filter on X, $\mathbb{F}$ be a* 1*-filter and $x_0 \in X$. Then:*

(1) $\mathcal{F}$ converges to x_0 in $(X, \mathbb{U}_M)$ if and only if $\omega(\mathcal{F})$ converges to x_0 in $(X, \mathcal{U}_M)$;

(2) If $\mathbb{F}$ converges to x_0 in $(X, \mathcal{U}_M)$, then $\iota(\mathbb{F})$ converges to x_0 in $(X, \mathbb{U}_M)$.

Corollary 5.3. *Let $(X, M, *)$ be a fuzzy quasi-metric space and $x_0 \in X$. If $\mathbb{F}$ is an induced* 1*-filter, Then:*

(1) $\mathbb{F}$ is a Cauchy 1*-filter in $(X, \mathcal{U}_M)$ if and only if $\iota(\mathbb{F})$ is a Cauchy filter in $(X, \mathbb{U}_M)$;*

(2) $\mathbb{F}$ converges to x_0 in $(X, \mathcal{U}_M)$ if and only if $\iota(\mathbb{F})$ converges to x_0 in $(X, \mathbb{U}_M)$.

The lemmas above can be similarly proved according to [30]. Through the lemmas above, we obtain the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced classical quasi-uniform spaces in fuzzy quasi-metric spaces.

Theorem 5.4. *If $(X, \mathcal{U}_M)$ is Cauchy* 1*-complete, then $(X, \mathbb{U}_M)$ is complete.*

Corollary 5.5. *$(X, \mathcal{U}_M)$ is induced Cauchy* 1*-complete if and only if $(X, \mathbb{U}_M)$ is complete.*

By the following proposition, we have the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced Hutton $[0, 1]$-quasi-uniform spaces. According to the idea of Gutiérrez García [1] Corollary 16, we have the following results.

Proposition 5.6. *Let $(X, M, *)$ be a fuzzy quasi-metric space. Then:*

(1) $\mathfrak{U}_M = \Lambda(\mathcal{U}_M)$;

(2) $\mathcal{U}_M = \Upsilon(\mathfrak{U}_M)$.

Proof. (1) We know that the collection $\{\Lambda((1-\varepsilon) \to M(-, -, t)) \mid \varepsilon \in (0, 1),\ t > 0\}$ is a basis for $\Lambda(\mathcal{U}_M)$. According to the definition of Λ, we have that $[\Lambda((1 - \varepsilon) \to M(-, -, t))](1_{\{y\}})(x) = (1 - \varepsilon) \to M(x, y, t)$ as desired.

(2) We know that the collection $\{\Upsilon(\phi^M_{\varepsilon,t}) \mid \varepsilon \in (0, 1),\ t > 0\}$ is a basis for $\Upsilon(\mathfrak{U}_M)$. According to the definition of Υ, we have that $\Upsilon(\phi^M_{\varepsilon,t})(x, y) = \phi^M_{\varepsilon,t}(1_{\{y\}})(x) = (1 - \varepsilon) \to M(x, y, t)$,for each $\varepsilon \in (0, 1)$ and $t > 0$ as desired. □

Theorem 5.7. *Let $(X, M, *)$ be a fuzzy quasi-metric space, $\mathcal{U}_M$ be an induced probabilistic quasi-uniformity and $\mathfrak{U}_M$ be an induced Hutton [0,1]-quasi-uniformity. Then:*

$(X, \mathcal{U}_M)$ is Cauchy 1*-complete if and only if $(X, \mathfrak{U}_M)$ is complete.*

6. **Conclusions**

In this paper, we use 1-filters to give a kind of Cauchy 1-completeness of probabilistic quasi-uniform spaces. We have established close relationships between the completeness of probabilistic quasi-uniform spaces and both completeness of classical quasi-uniform spaces and Hutton $[0, 1]$-quasi-uniform spaces. In the framework of fuzzy quasi-metric spaces, we establish the relationship between completeness of induced classical quasi-uniform spaces and induced probabilistic quasi-uniform spaces and the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced Hutton $[0,1]$-quasi-uniform spaces. These results illustrate the reasonableness of 1-filters in discussing the problem of completeness in probabilistic quasi-uniform spaces. Consequently, we have reasons to believe 1-filters are a good tool to study probabilistic quasi-uniform convergence spaces. In the future, we will investigate the relationship between the completeness of probabilistic quasi-uniform spaces and probabilistic quasi-uniform convergence spaces.

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