TOPOLOGICAL CHARACTERIZATION FOR FUZZY REGULAR LANGUAGES

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Abstract. We present a topological characterization for fuzzy regular languages: we show that there is a bijective correspondence between fuzzy regular languages and the set of all clopen fuzzy subsets with finite image in the induced fuzzy topological space of Stone space (Profinite space), and then we give a representation of closed fuzzy subsets in the induced fuzzy topological space via fuzzy regular languages. Moreover, we prove that the induced fuzzy topological space has a basis consisting of leveled characteristic functions of the closure of cut languages of fuzzy regular languages.

1. **Introduction**

Fuzzy finite automata were introduced by Santos [25, 26], Lee and Zadeh [15] in the late 1960s. Since finite automata constitute a discrete model that recognizes formal languages in computability theory, fuzzy finite automata are studied as an extended model that reduces the gap between precision and vagueness of computer languages. Algebraic properties of fuzzy languages had been studied, among others, by Mordeson and Malik [17] and other scholars [4, 9, 10, 11, 20, 22, 23, 24, 28, 29, 30]. In recent years, novel applications of fuzzy finite automata have emerged from numerous sciences, like biology, physics, cognitive sciences, control, and linguistics.

Hall [8] introduced profinite topologies on free groups and showed that every finitely generated subgroup of a free group is closed in the profinite topology. Profinite topology has been used as a powerful tool to study the classification of regular languages and construction of minimal finite automata. Pippenger [21], Pin [18], Gehrke [6, 7] have used Stone duality to investigate automata theory and provided a topological characterization for regular languages. Further, they have also established a bijective correspondence between Boolean algebra of regular languages and the set of all clopen subsets in Stone space (Profinite space). One of the benefits of using Stone duality in studies of automata theory is the decidable problems of regular languages, related to equivalence problems, reduce to profinite identities in the profinite topology, whose form is much simpler than the former. For more information on this topic we refer the reader to [1, 2, 19, 31].

Between ordinary topological spaces and fuzzy topological spaces, people considered a kind of special fuzzy topological spaces —-induced spaces, as a connection

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between these two categories [12]. The concept of induced fuzzy topological space, was first introduced in 1968 by Chang [3] and later extended by other authors. R. Lowen [14] studied this kind of fuzzy topological spaces with the notion of a lower semicontinuous function. Lower semicontinuous functions play an important role in defining the concept of basis for a fuzzy topology. The present study follows the approach of references [6, 12, 18, 21, 32] and provides a topological characterization for fuzzy regular languages. We also show the connections between some characteristic of the profinite topological space and its corresponding induced fuzzy topological space.

2. Preliminaries

This section summarizes some basic concepts and notations used in this paper. These concepts and notations are given in more detail in [17, 18, 27].

A metric space (X, d) is a set X equipped with a metric, i.e. a function $d : X \times X \to \mathbb{R}$, called the distance, such that for all $x, y, z \in X$:

(1) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$, and
(3) $d(x, y) \leq d(x, z) + d(y, z)$.

Every metric space is a topological space in a natural manner. Thus, all definitions and theorems of general topological spaces also apply to all metric spaces. In particular, for any $x \in X$, we define the open ball of radius $r > 0$ on x as the set $B(x; r) = \{y \in X \mid d(x, y) < r\}$. These open balls form the basis of a topology on X, which forms a topological space. A subset U of (X, d) is called *open* if there exists an $r > 0$ such that $B(x; r)$ is contained in U for every $x \in U$, and a subset $V \subseteq X$ is called *closed* if its complement $X \backslash V$ is open. A subset U is *clopen* if it is both open and closed.

Let (X, d) be a metric space. A sequence $\{x_n\}_{n \geq 0}$ in X is called a *Cauchy sequence* if for every $\epsilon > 0$, there exists a natural number N, such that $d(x_i, x_j) < \epsilon$ whenever $i \geq N$ and $j \geq N$.

A metric space is *complete* if its every Cauchy sequence is convergent. If metric space (X, d) is incomplete, it is always possible to construct a larger space which is complete and contains just enough points so that every Cauchy sequence in X has a limit in the larger space. Therefore, we need to adjoin new points to (X, d) and extend d to all these new points for the formerly nonconvergent Cauchy sequences to find limits among these new points. The new points are then the limits of the sequences in X. In fact, every metric space has a completion, which is unique up to an isomorphism. We refer to [27] for an detailed introduction of complete metric space.

In this section, Σ denotes a finite alphabet, let Σ^* denote the set of all words of finite length over Σ and let Λ denote the empty word. Then Σ^* is the free monoid generated by Σ with the operation of concatenation.

A *deterministic finite state automaton* (DFA, for short) is a tuple $W = (Q, \Sigma, \eta, q_0, F)$, where Q is a finite set of states, Σ is a finite alphabet, $\eta : Q \times \Sigma \longrightarrow Q$ is the next-state function, q_0 is the initial state, and F is the set of final states. W recognizes the input string $x \in \Sigma^*$ if the last state entered by W on application of x starting in state q_0 is a member of the set F. W recognizes the language L consisting of all such strings.

Let X be a nonempty finite set. Any mapping from X into $[0, 1]$ is called a *fuzzy subset* of X.

The concept of *fuzzy finite state automata* (FFA, for short) is a direct generalization of DFA (see [17]). Here, FFA are formally defined as a 5-tuple $\mathcal{W} = (Q, \Sigma, \eta, I, F)$, where Q and Σ are defined above, η is a fuzzy subset of $Q \times \Sigma \times Q$, which represents a fuzzy transition function, I and F denote the fuzzy initial state and the fuzzy final state as the fuzzy subsets of Q, respectively. We can extend η on Σ^*, denoted in sequel as $\eta^* : Q \times \Sigma^* \times Q \to [0, 1]$, as (1) $\eta^*(q, \Lambda, q) = 1$ and $\eta^*(q, \Lambda, p) = 0$ for any distinct elements q and p in Q; (2) for any input word $x \in \Sigma^*$, and input symbol $y \in \Sigma$, $\eta^*(q, xy, p) = \bigvee \big\{\eta^*(q, x, p') \wedge \eta(p', y, p) \mid p' \in Q\big\}$, where we use the symbol $\bigvee$ to represent the supremum of real numbers.

The fuzzy language recognized by an FFA is defined as a fuzzy subset of Σ^*, as denoted by $f : \Sigma^* \to [0, 1]$, for any input word $x \in \Sigma^*$, $f(x) = \bigvee \big\{I(q) \wedge \eta^*(q, x, p) \wedge F(p) \mid q, p \in Q\big\}$, and f is called a *fuzzy regular language* over Σ. The set of all fuzzy regular languages is denoted by $\mathrm{FReg}(\Sigma^*)$. Let 1_{Σ^*} and 0_{Σ^*} be constant fuzzy subsets that take the values of 1 and 0, respectively. Clearly, $1_{\Sigma^*}, 0_{\Sigma^*} \in \mathrm{FReg}(\Sigma^*)$. The set-theoretic operations, such as union, intersection and complement, can be defined by $(f \cup g)(x) = f(x) \vee g(x)$, $(f \cap g)(x) = f(x) \wedge g(x)$, and $f^c(x) = 1 - f(x)$, where $f, g \in \mathrm{FReg}(\Sigma^*)$, $x \in \Sigma^*$. The set of all fuzzy regular languages together with these operations forms a De Morgan algebra (see [17]).

3. The Fuzzy Topology Associated with Fuzzy Regular Languages

We recall the standard notion of separating words for a DFA, details of which could be found in [5, 7, 18, 21].

Definition 3.1. A DFA W *separates* two words x and y of Σ^* if W recognizes one of the words but not the other.

We denote by $|W|$ the number of states of this DFA. Now we let
$r(x, y) = \min\big\{|W| \mid W$ is a DFA that separates x and $y\big\}$, and
$d(x, y) = 2^{-r(x,y)}$.
With the usual conventions of $\min\emptyset = +\infty$, and $2^{-\infty} = 0$.

Similarly, a finite monoid M separates two words x and y of Σ^*, if there is a monoid morphism $\varphi : \Sigma^* \to M$ such that $\varphi(x) \neq \varphi(y)$. We also set $d'(x, y) = 2^{-r'(x,y)}$, where $r'(x, y) = \min\big\{|M| \mid M$ is a monoid that separates x and $y\big\}$.

Example 3.2. (1) Let $\varphi : \Sigma^* \to \mathbb{Z}/2\mathbb{Z}$ be the morphism defined by $\varphi(x) = |x| (\mathrm{mod}\ 2)$. Then $\varphi(ababab) = 0$ and $\varphi(baaabab) = 1$, and hence φ separates the words "*ababab*" and "*baaabab*". This shows that the words "*ababab*" and "*baaabab*" can be separated by a group of order 2.

(2) Let M be the monoid defined on the set $\{1, a, b\}$ by the operation $aa = ba = a$, $bb = ab = b$ and $1z = z1 = z$ for all $z \in \{1, a, b\}$. Let x and y be words of $\{a, b\}$. Then the words xa and yb can be separated by the morphism $\varphi : \Sigma^* \to M$ defined by $\varphi(a) = a$ and $\varphi(b) = b$ since $\varphi(ua) = a$ and $\varphi(ub) = b$.

It is trivial to check d is a metric on Σ^*. The phrase "(Σ^*, d) is a *metric space*" means that d is a metric on the set Σ^*. The metric space (Σ^*, d) in this paper is viewed as a topological space, as denoted by (Σ^*, Γ_d). Clearly, with this metric, (Σ^*, Γ_d) is incomplete, and its completion, denoted by $(\widehat{\Sigma^*}, \Gamma)$, is called a *profinite topological space*. The element of $\widehat{\Sigma^*}$ is called a *profinite word*.

Since Σ^* embeds naturally in $\widehat{\Sigma^*}$, every finite word is a profinite word. However, it is really difficult to give concrete examples of profinite words which are not words. A transparent example of such a sequence is provided by the Cauchy sequence $\{x^{n!}\}_{n\geq 0}$ in Σ^*, the profinite word which is the limit of this sequence is denoted by ν_x. The formal definition is $\nu_x = \lim\limits_{n\to\infty} x^{n!}$ (see [18]).

Lemma 3.3. [18, 21] *For the profinite topological space $(\widehat{\Sigma^*}, \Gamma)$, if $L \subseteq \Sigma^*$, then $L = \overline{L} \cap \Sigma^*$, where $\overline{L}$ is the closure of L (that is, the intersection of all closed subsets in $(\widehat{\Sigma^*}, \Gamma)$ that contain L). Furthermore, the following conditions are equivalent.*

(1) L is a regular language;

(2) $L = U \cap \Sigma^*$ for some clopen subset U of $(\widehat{\Sigma^*}, \Gamma)$;

(3) $\overline{L}$ is clopen in $(\widehat{\Sigma^*}, \Gamma)$.

Lemma 3.3 establishes a tight connection between Boolean algebra of regular languages and the set of all clopen subsets in the profinite topological space $(\widehat{\Sigma^*}, \Gamma)$. In the sequel, we show a correspondence between fuzzy regular languages and certain fuzzy topological spaces. That is, we shall give a topological characterization for fuzzy regular languages.

Definition 3.4. Let f be a mapping from the metric space (Σ^*, Γ_d) into the metric space $([0, 1], \rho)$ with the usual absolute value metric. Then

(1) f is *continuous* if and only if for every $x \in \Sigma^*$ and for every $\epsilon > 0$ there exists a $\delta > 0$, such that whenever $y \in \Sigma^*$ satisfies $d(x, y) < \delta$, then $\rho(f(x), f(y)) < \epsilon$.

(2) f is called *uniformly continuous* , if for every $\epsilon > 0$ there exists a $\delta > 0$, such that whenever $x, y \in \Sigma^*$ satisfy $d(x, y) < \delta$, then $\rho(f(x), f(y)) < \epsilon$.

Remark 3.5. With the metric d defined above, (Σ^*, Γ_d) is a discrete topological space, i.e., every word in Σ^* is isolated. Therefore, any function f from (Σ^*, Γ_d) into the metric space $([0, 1], \rho)$ is continuous.

Example 3.6. Consider the function which to every finite word assigns 0 and to every infinite profinite word assigns 1. It is not continuous, since an infinite profinite word is a limit of a sequence of finite words, over which the value is 0; however the value of the limit is 1. This shows that not every function $\widehat{f}$ from $(\widehat{\Sigma^*}, \Gamma)$ into $([0, 1], \rho)$ is continuous.

The concept of fuzzy topology was introduced by Chang [3] and the theory of fuzzy topologies was developed by many mathematicians [12, 13, 32]. Here, we focus on the set of $[0, 1]^{\widehat{\Sigma^*}} = \{\widehat{f} \mid \widehat{f} : \widehat{\Sigma^*} \to [0, 1]\}$. Chang states that a *fuzzy topological space* is a pair $\left([0, 1]^{\widehat{\Sigma^*}}, \tau\right)$, where τ is a *fuzzy topology* on $[0, 1]^{\widehat{\Sigma^*}}$, that is, a family of fuzzy subsets ($\tau \subset [0, 1]^{\widehat{\Sigma^*}}$) that satisfies the following three axioms,

(1) $1_{\widehat{\Sigma^*}} \in \tau$ and $0_{\widehat{\Sigma^*}} \in \tau$;
(2) if $\widehat{f}_1$ and $\widehat{f}_2$ belong to τ, then so does $\widehat{f}_1 \cap \widehat{f}_2$, and
(3) if $\widehat{f}_t$ belongs to τ for each $t \in T$, then so does $\bigcup_{t \in T} \widehat{f}_t$.

Members of τ are called *open fuzzy subsets*. Fuzzy subset that takes the form $1 - \widehat{f}$, where $\widehat{f}$ is an open fuzzy subset, is called a *closed fuzzy subset*. A fuzzy subset is a *clopen fuzzy subset* if it is both an open and closed fuzzy subset.

To emphasize this fact, we provide the following examples.

Example 3.7. (1) Take $\tau = \{a_{\widehat{\Sigma^*}} : a \in [0,1]\}$, where $a_{\widehat{\Sigma^*}}(\nu) = a$ for every $\nu \in \widehat{\Sigma^*}$. Then τ is a fuzzy topology on $[0,1]^{\widehat{\Sigma^*}}$.

(2) For the profinite topological space $(\widehat{\Sigma^*}, \Gamma)$, let $\mathcal{X}_U$ be the characteristic function of $U \subseteq \widehat{\Sigma^*}$, then $\tau = \{\mathcal{X}_U : U \in \Gamma\}$ is a fuzzy topology on $[0,1]^{\widehat{\Sigma^*}}$.

Definition 3.8. For the profinite space $(\widehat{\Sigma^*}, \Gamma)$, a mapping $\widehat{f} : \widehat{\Sigma^*} \to [0,1]$ is said to be *lower semicontinuous* if $\{\nu \in \widehat{\Sigma^*} \mid \widehat{f}(\nu) > \lambda\} \in \Gamma$ for every $\lambda \in [0,1]$.

Let $w(\Gamma)$ denote the family of all lower semicontinuous functions, that is,

$$w(\Gamma) = \big\{\widehat{f} : \widehat{\Sigma^*} \to [0,1] \mid \{\nu \in \widehat{\Sigma^*} \mid \widehat{f}(\nu) > \lambda\} \in \Gamma, \lambda \in [0,1]\big\}.$$

It is trivial to check that $w(\Gamma)$ is a fuzzy topology on $[0,1]^{\widehat{\Sigma^*}}$. In this case, $([0,1]^{\widehat{\Sigma^*}}, w(\Gamma))$ is called the *induced fuzzy topological space* of $(\widehat{\Sigma^*}, \Gamma)$. If $\widehat{f}$ is lower semicontinuous, then $\widehat{f}^{\,c} = 1 - \widehat{f}$ is called upper semicontinuous.

Lemma 3.9. [16] *For a fuzzy subset $f : \Sigma^* \to [0,1]$, given $r \in [0,1]$, we write* $\mathrm{Im}(f) = \{f(x) \mid x \in \Sigma^*\}$ *(the image set of f as a subset of [0,1]), $f_{[\lambda]} = \{x \in \Sigma^* \mid f(x) \geq \lambda\}$ (called the λ-cut of f), $f_{(\lambda)} = \{x \in \Sigma^* \mid f(x) > \lambda\}$ (called the strong λ-cut of f), and $f_{=\lambda} = \{x \in \Sigma^* \mid f(x) = \lambda\}$. The following statements are equivalent.*
(1) f is a fuzzy regular language;
(2) $\mathrm{Im}(f)$ *is finite, and $f_{[\lambda]}$ is a regular language for every $\lambda \in [0,1]$;*
(3) $\mathrm{Im}(f)$ *is finite, and $f_{(\lambda)}$ is a regular language for every $\lambda \in [0,1]$;*
(4) $\mathrm{Im}(f)$ *is finite, and $f_{=\lambda}$ is a regular language for every $\lambda \in [0,1]$.*

Theorem 3.10. *Let $f : \Sigma^* \to [0,1]$ be any mapping from the metric space (Σ^*, Γ_d) into the metric space $([0,1], \rho)$ with the usual absolute value metric. If f is a fuzzy regular language, then there exists a unique continuous mapping $\widehat{f}$ from $(\widehat{\Sigma^*}, \Gamma)$ into $([0,1], \rho)$ extending f. Moreover, for any $\lambda \in [0,1]$, we have*

$$\overline{f_{(\lambda)}} = \big\{\nu \in \widehat{\Sigma^*} \mid \widehat{f}(\nu) > \lambda\big\}.$$

Proof. **Step 1:** Define a function $\widehat{f} : \widehat{\Sigma^*} \to [0,1]$.

Since f is a fuzzy regular language, by Lemma 3.9, then for any $\lambda \in [0,1]$, $f_{[\lambda]}$ is a regular language, and $\overline{f_{[\lambda]}}$ is a clopen subset in $(\widehat{\Sigma^*}, \Gamma)$. According to the decomposition theorem of fuzzy subsets, we have $f = \bigcup_{\lambda \in \mathrm{Im}(f)} \lambda \mathcal{X}_{f_{[\lambda]}}$, where $\mathcal{X}_{f_{[\lambda]}}$ is the characteristic function of $f_{[\lambda]}$. Let $\widehat{f}:\widehat{\Sigma^*}\to[0,1]$ be the mapping defined as

$$\widehat{f}=\bigcup_{\lambda\in\mathrm{Im}(f)}\lambda\mathcal{X}_{\overline{f_{[\lambda]}}}.$$

It is easy to see that $\widehat{f}$ is well-defined.

Step 2: $\widehat{f}$ is an extension of f to $\widehat{\Sigma^*}$.

Since f has a finite range by Lemma 3.9, then $f_{(\lambda)}=f_{=\lambda_1}\cup f_{=\lambda_2}\cup\cdots\cup f_{=\lambda_n}$, and $f_{=\lambda_i}\cap f_{=\lambda_j}=\emptyset$ $(i\neq j)$, where $\lambda_1,\lambda_2,\cdots,\lambda_n$ are all the points of $\mathrm{Im}(f)$ that are greater than $\lambda\in[0,1]$. Without loss of generality, let $\lambda_1<\lambda_2<\cdots<\lambda_n$. Thus, we would rewrite $\widehat{f}$ as

$$\widehat{f}=\bigcup_{i\in\{1,2,\cdots,n-1\}}\lambda_i\mathcal{X}_{\left(\overline{f_{[\lambda_i]}}-\overline{f_{[\lambda_{i+1}]}}\right)}\cup\lambda_n\mathcal{X}_{\overline{f_{[\lambda_n]}}}.$$

Note that there is a natural embedding of Σ^*, which maps a word $x\in\Sigma^*$ to the equivalence class of the sequence which is constantly equal to x. Here and later, we may view Σ^* as a subset of $\widehat{\Sigma^*}$. Hence, for any $x\in\Sigma^*$, if $x\in\left(\overline{f_{[\lambda_i]}}-\overline{f_{[\lambda_{i+1}]}}\right)$, which implies that $\widehat{f}(x)=\lambda_i$. On the other hand, $\overline{f_{[\lambda_i]}}\cap\Sigma^*=f_{[\lambda_i]}$. Thus, we have $x\in f_{[\lambda_i]}$, but $x\notin f_{[\lambda_{i+1}]}$, that is, $x\in\{x\in\Sigma^*\mid\lambda_i\leq f(x)<\lambda_{i+1}\}$, i.e., $f(x)=\lambda_i$. In particular, if $x\in\overline{f_{[\lambda_n]}}$, then $\widehat{f}(x)=\lambda_n$. Similarly, we have $x\in f_{[\lambda_n]}$, then $f(x)=\lambda_n$. Therefore, for any $x\in\Sigma^*$, $\widehat{f}(x)=f(x)$, and thus $\widehat{f}$ extends f to $\widehat{\Sigma^*}$. In this case, f can be viewed as a mapping from the metric space (Σ^*,Γ_d) into $([0,1],\rho)$, and $\widehat{f}$ is a mapping from the profinite topological space $(\widehat{\Sigma^*},\Gamma)$ into $([0,1],\rho)$.

Step 3: $\widehat{f}$ is continuous.

According to the definition of $\widehat{f}$, for any $\mu\in[0,1]$, we have

$$\begin{aligned}\widehat{f}_{(\mu)}&=\{\nu\in\widehat{\Sigma^*}\mid\widehat{f}(\nu)>\mu\}\\&=\Big\{\nu\in\widehat{\Sigma^*}\mid\bigvee_{\lambda\in\mathrm{Im}(f)}\lambda\mathcal{X}_{\overline{f_{[\lambda]}}}(\nu)>\mu\Big\}\\&=\begin{cases}\emptyset, & \text{if }\lambda\leq\mu\text{ for every }\lambda\in\mathrm{Im}(f);\\\overline{f_{[\lambda]}}, & \text{if }\lambda>\mu\text{ for every }\lambda\in\mathrm{Im}(f).\end{cases}\end{aligned}$$

Clearly, $\emptyset$ is open (clopen) in $(\widehat{\Sigma^*},\Gamma)$, and for any $\lambda\in[0,1]$, $\overline{f_{[\lambda]}}$ is also open (clopen), then $\widehat{f}_{(\mu)}$ is open (clopen). This implies that $\widehat{f}\in w(\Gamma)$, i.e., $\widehat{f}$ is lower semicontinuous. Similarly, for any $\mu\in[0,1]$, $\widehat{f}_{[\mu]}=\{\nu\in\widehat{\Sigma^*}\mid\widehat{f}(\nu)\geq\mu\}=\big\{\nu\in\widehat{\Sigma^*}\mid\bigvee_{\lambda\in\mathrm{Im}(f)}\lambda\mathcal{X}_{\overline{f_{[\lambda]}}}(\nu)\geq\mu\big\}$. Thus, for every $\lambda\in\mathrm{Im}(f)$, if $\lambda<\mu$, then $\widehat{f}_{[\mu]}=\emptyset$, and $\widehat{f}_{[\mu]}=\overline{f_{[\lambda]}}$ otherwise, which implies that $\widehat{f}_{[\mu]}$ is closed (clopen) for any $\mu\in[0,1]$. That is, $\widehat{f}$ is a closed fuzzy subset in $([0,1]^{\widehat{\Sigma^*}},w(\Gamma))$, $\widehat{f}$ is upper semicontinuous.

Therefore, $\widehat{f}$ is a continuous function extending f.

Step 4: $\widehat{f}$ is unique.

Let $\widehat{f}$ and $\widehat{h}$ be continuous functions extending f to $\widehat{\Sigma^*}$. For any profinite word $\nu\in\widehat{\Sigma^*}$, there is a convergent Cauchy sequence $\{x_n\}_{n\geq0}$ in Σ^* which is convergent to ν. By continuity of $\widehat{f}$ and $\widehat{h}$,

$$\widehat{f}(\nu) = \lim_{n\to\infty} \widehat{f}(x_n) = \lim_{n\to\infty} \widehat{h}(x_n) = \widehat{h}(\nu).$$

Hence, $\widehat{f} = \widehat{h}$.

Step 5: Prove the equality.

For any $\nu \in \{\nu \in \widehat{\Sigma^*} \mid \widehat{f}(\nu) > \lambda\}$ and $\lambda \in [0,1]$, there exists a convergent Cauchy sequence $\{x_n\}_{n\geq 0}$ in Σ^* which is convergent to ν. Hence, the sequence $\widehat{f}(x_1), \widehat{f}(x_2), \cdots$ has limit $\widehat{f}(\nu)$ with respect to the continuous function $\widehat{f}$. That is, there exists a natural number N, such that whenever $n \geq N$, $\widehat{f}(x_n) = \widehat{f}(x_N)$, i.e., $f(x_n) = f(x_N)$. Thus, $\{x_n \mid n \geq N\} \subseteq f_{(\lambda)}$, which implies that $\overline{\{x_n \mid n \geq N\}} \subseteq \overline{f_{(\lambda)}}$, i.e., $\nu \in \overline{f_{(\lambda)}}$. Hence, $\overline{f_{(\lambda)}} \supseteq \{\nu \in \widehat{\Sigma^*} \mid \widehat{f}(\nu) > \lambda\}$.

The inclusion $\subseteq$ is obvious by the continuity of $\widehat{f}$. □

Remark 3.11. Since $(\widehat{\Sigma^*}, \Gamma)$ is compact (see [18]), every continuous function from $(\widehat{\Sigma^*}, \Gamma)$ into the metric space $([0,1], \rho)$ is automatically uniformly continuous. Therefore, Theorem 3.10 implies that every fuzzy regular language f is a uniformly continuous function from (Σ^*, Γ_d) into $([0,1], \rho)$ with respect to the underlying metric d.

From the above theorem we let Clopen(Γ) be the collection of all clopen subsets of $(\widehat{\Sigma^*}, \Gamma)$, and ω(Clopen(Γ)) be the family of all lower semicontinuous functions generated by Clopen(Γ), that is,

$$\omega(\text{Clopen}(\Gamma)) = \big\{\widehat{f} : \widehat{\Sigma^*} \to [0,1] | \{\nu \in \widehat{\Sigma^*} \mid \widehat{f}(\nu) > \lambda\} \in \text{Clopen}(\Gamma), \lambda \in [0,1]\big\}.$$

However, ω(Clopen(Γ)) does not form a fuzzy topology on $[0,1]^{\widehat{\Sigma^*}}$.

Theorem 3.12. *For any $\widehat{f} \in \omega\big(\text{Clopen}(\Gamma)\big)$, let f be the restriction of $\widehat{f}$ to Σ^*. Then $f_{(\lambda)}$ is a regular language for any $\lambda \in [0,1]$.*

Proof. For any $\widehat{f} \in \omega\big(\text{Clopen}(\Gamma)\big)$ and $\lambda \in [0,1]$, we have $\{\nu \in \widehat{\Sigma^*} | \widehat{f}(\nu) > \lambda\} \in$ Clopen(Γ). According to Lemma 3.3, $f_{(\lambda)} = \{\nu \in \widehat{\Sigma^*} \mid \widehat{f}(\nu) > \lambda\} \cap \Sigma^*$ is a regular language, where f is the restriction of $\widehat{f}$ to Σ^*. □

Theorem 3.13. *For any $\widehat{f} \in \omega\big(\text{Clopen}(\Gamma)\big)$, if $\text{Im}(\widehat{f})$ is finite, then*

(1) $\widehat{f}$ is a clopen fuzzy subset in $([0,1]^{\widehat{\Sigma^*}}, w(\Gamma))$;

(2) $\widehat{f}$ restricted to Σ^* is a fuzzy regular language.

Proof. (1) For any $\widehat{f} \in \omega\big(\text{Clopen}(\Gamma)\big)$, assume that $\text{Im}(\widehat{f})$ is finite. Let $\text{Im}(\widehat{f}) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. It is not difficult to show that $\widehat{f}_{=\lambda_i} \in$ Clopen(Γ) for every $\lambda_i \in \text{Im}(\widehat{f}), i = 1, 2, \cdots, n$. On the other hand, for any $\lambda \in [0,1]$, we have $\{\nu \in \widehat{\Sigma^*} \big| (1 - \widehat{f})(\nu) > \lambda\} = \{\nu \in \widehat{\Sigma^*} \big| \widehat{f}(\nu) < 1 - \lambda\} = \{\nu \in \widehat{\Sigma^*} \big| \widehat{f}(\nu) \leq 1 - \lambda\} \setminus \{\nu \in \widehat{\Sigma^*} \big| \widehat{f}(\nu) = 1 - \lambda\} = \{\nu \in \widehat{\Sigma^*} \big| \widehat{f}(\nu) \leq 1 - \lambda\} \cap \big\{\widehat{\Sigma^*} - \{\nu \in \widehat{\Sigma^*} \big| \widehat{f}(\nu) = 1 - \lambda\}\big\}$. Clearly, if $1 - \lambda = \lambda_i$, $i = 1, 2, \cdots, n$, then $\widehat{f}_{=(1-\lambda)} \in$ Clopen(Γ) and $\widehat{f}_{=(1-\lambda)} = \emptyset$ otherwise. Therefore, for any $\lambda \in [0,1]$, $\widehat{\Sigma^*} - \widehat{f}_{=(1-\lambda)} \in$ Clopen(Γ). Since $\{\nu \in \widehat{\Sigma^*} \big| \widehat{f}(\nu) \leq 1 - \lambda\} \in$ Clopen(Γ), then $\{\nu \in \widehat{\Sigma^*} \big| (1-\widehat{f})(\nu) > \lambda\} \in \text{Clopen}(\Gamma)$, i.e., $1-\widehat{f} \in \omega\big(\text{Clopen}(\Gamma)\big) \subseteq w(\Gamma)$. This shows that $\widehat{f}$ is clopen.

(2) As directly obtained by Lemma 3.9 and Theorem 3.12. □

Theorem 3.14. *Let* Clopen$(w(\Gamma))$ *be the set of all clopen fuzzy subsets in* $([0,1]^{\widehat{\Sigma^*}}, w(\Gamma))\}$, *i.e.,*

$$\text{Clopen}(w(\Gamma)) = \{\widehat{f} \in w(\Gamma) \mid \widehat{f} is a clopen fuzzy subset\}.$$

For any $\widehat{f} \in \text{Clopen}(w(\Gamma))$, *if* $\text{Im}(\widehat{f})$ *is finite, then* $\widehat{f}$ *restricted to* Σ^* *is a fuzzy regular language.*

Proof. For any $\widehat{f} \in \text{Clopen}(w(\Gamma))$ and $\text{Im}(\widehat{f})$ is finite, let $\text{Im}(\widehat{f}) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. We may assume, without loss of generality, that $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Since $\widehat{f}$ is both lower semicontinuous and upper semicontinuous, then for any $\lambda \in [0,1]$, $\widehat{f}_{(\lambda)}$ is an open subset in $(\widehat{\Sigma^*}, \Gamma)$, and $\widehat{f}_{[\lambda]}$ is closed. Thus, we get $\widehat{f}_{=\lambda_n} \in \text{Clopen}(\Gamma)$. From this, it is not difficult to deduce that $\widehat{f}_{=\lambda_i} \in \text{Clopen}(\Gamma)$ for every $\lambda_i \in \text{Im}(\widehat{f})$, $i = 1, 2, \cdots, n$. By Lemma 3.3, Lemma 3.9, and Theorem 3.13, $\widehat{f}$ restricted to Σ^* is a fuzzy regular language. □

Remark 3.15. In fact, for any $\widehat{f} \in \text{Clopen}(w(\Gamma))$, $\widehat{f}$ is therefore a continuous mapping from $(\widehat{\Sigma^*}, \Gamma)$ into the metric space $([0,1], \rho)$. That is, $\text{Clopen}(w(\Gamma)) = \{\widehat{f} \in w(\Gamma) \mid \widehat{f} \text{ is continuous}\}$. While not every $f \in \text{Clopen}(w(\Gamma))$ has a finite range, we shall now see that there exists a function over $\widehat{\Sigma^*}$ with an infinite range.

Example 3.16. Given some fixed word $x \in \Sigma^*$, consider the mapping $f_x : \Sigma^* \to [0,1]$ defined by

$$f_x(y) = d(x, y)$$

for any $y \in \Sigma^*$, and d is the above metric. Thus, for any Cauchy sequence $\{y_n\}_{n\geq 0}$ in Σ^*, the sequence $\{d(x, y_n)\}_{n\geq 0}$ is also a Cauchy sequence of $[0,1]$. A straightforward calculation shows that f_x is uniformly continuous. Then there exists a unique continuous function $\widehat{f_x}$ from $(\widehat{\Sigma^*}, \Gamma)$ into the metric space $([0,1], \rho)$ extending f_x. Hence, $\widehat{f_x} \in \text{Clopen}(w(\Gamma))$. However, it is not difficult to prove that $\text{Im}(f_x)$ is infinite, thus $\text{Im}(\widehat{f_x})$ is also infinite.

Corollary 3.17. *Let* $f : \Sigma^* \to [0,1]$ *be any mapping from the metric space* (Σ^*, Γ_d) *into the metric space* $([0,1], \rho)$ *with the usual absolute value metric.*

(1) If f *is uniformly continuous, then there exists a unique extension of* f *to a continuous mapping* $\widehat{f}$ *from* $(\widehat{\Sigma^*}, \Gamma)$ *into* $([0,1], \rho)$. *This means that* $\widehat{f} \in \text{Clopen}(w(\Gamma))$.

(2) If f *is a fuzzy regular language, and* $\widehat{f} : \widehat{\Sigma^*} \to [0,1]$ *is the continuous extension of* f, *then* $\widehat{f} \in \text{Clopen}(w(\Gamma))$, *and* $\text{Im}(\widehat{f})$ *is finite.*

Theorem 3.18. *Assume that a family* $\mathbf{F}\text{Clopen}(w(\Gamma))$ *is defined as*

$$\mathbf{F}\text{Clopen}(w(\Gamma)) = \{\widehat{f} \in \text{Clopen}(w(\Gamma)) \mid \text{Im}(\widehat{f}) is finite\}.$$

Then the maps $f \mapsto \widehat{f}$ *and* $\widehat{f} \mapsto \widehat{f}|_{\Sigma^*}$ *define mutually inverse isomorphism between De Morgan algebra* $\mathrm{FReg}(\Sigma^*)$ *and* $\mathbf{F}\mathrm{Clopen}(w(\Gamma))$*, where* $\widehat{f}$ *is the continuous extension of* f *to* $\widehat{\Sigma^*}$ *as depicted in Theorem 3.10, and* $\widehat{f}|_{\Sigma^*}$ *is the restriction of* $\widehat{f}$ *to* Σ^**.*

Proof. This follows immediately from Theorem 3.10, Corollary 3.17 and Theorem 3.14. □

Note that from this theorem it follows that there is a bijective correspondence between $\mathrm{FReg}(\Sigma^*)$ and the set of all clopen fuzzy subsets with finite image in the induced fuzzy topological space $([0,1]^{\widehat{\Sigma^*}}, w(\Gamma))$.

Theorem 3.19. *Let* f *be any function from the metric space* (Σ^*, Γ_d) *into the metric space* $([0,1], \rho)$ *with the usual absolute value metric. Then* f *is a fuzzy regular language if and only if* $\mathrm{Im}(f)$ *is finite and* $\mathcal{X}_{\overline{f_{(\lambda)}}} \in \mathbf{F}\mathrm{Clopen}(w(\Gamma))$ *for any* $\lambda \in [0,1]$*, where* $\mathcal{X}_{\overline{f_{(\lambda)}}}$ *is the characteristic function of* $\overline{f_{(\lambda)}}$*.*

Proof. Let $\mu \in [0,1]$ for the case of $\mu = 1$. We obtain $\big\{\nu \in \widehat{\Sigma^*} \mid \mathcal{X}_{\overline{f_{(\lambda)}}}(\nu) > \mu\big\} = \emptyset$. Moreover, for any $\mu \in [0,1)$, and the following equality applies: $\big\{\nu \in \widehat{\Sigma^*} \mid \mathcal{X}_{\overline{f_{(\lambda)}}}(\nu) > \mu\big\} = \overline{f_{(\lambda)}}$. Therefore, $\overline{f_{(\lambda)}} \in \mathrm{Clopen}(\Gamma)$ if and only if $\mathcal{X}_{\overline{f_{(\lambda)}}} \in w(\mathrm{Clopen}(\Gamma))$ for any $\lambda \in [0,1]$. Similarly, for any $\lambda \in [0,1]$, we have $\overline{f_{(\lambda)}} \in \mathrm{Clopen}(\Gamma)$ if and only if $\big\{\nu \in \widehat{\Sigma^*} \mid \mathcal{X}_{\overline{f_{(\lambda)}}}(\nu) \geq s\big\} \in w(\mathrm{Clopen}(\Gamma))$ for any $s \in [0,1]$. According to Lemma 3.3 and Lemma 3.9, we deduce that f is a fuzzy regular language if and only if $\overline{f_{(\lambda)}} \in \mathrm{Clopen}(\Gamma)$ and $\mathrm{Im}(f)$ is finite. □

The following theorem characterizes lower semicontinuous functions in terms of convergent Cauchy sequences.

Theorem 3.20. *Let* $\widehat{f}$ *be a mapping from the profinite topological space* $(\widehat{\Sigma^*}, \Gamma)$ *into the metric space* $([0,1], \rho)$ *with the usual absolute value metric. If* $\{x_n\}_{n \geq 0}$ *being a convergent Cauchy sequence in* Σ^* *with a limit in* $\widehat{\Sigma^*}$ *implies that*

$$\widehat{f}(\lim_{n\to\infty} x_n) \leq \lim_{n\to\infty} \inf\, f(x_n),$$

where f *is the restriction of* $\widehat{f}$ *to* Σ^**. Then* $\widehat{f} \in w(\Gamma)$*. In particular, if the restriction* f *is a fuzzy regular language over* Σ*, then* $\widehat{f} \in \omega\big(\mathrm{Clopen}(\Gamma)\big)$*.*

Proof. For any $\lambda \in [0,1]$, we let $V = \widehat{f}^{-1}\big([0,\lambda]\big)$. Thus, for $\nu \in \overline{V}$, then there exists a Cauchy sequence $\{x_n\}_{n\geq 0}$ in $V \cap \Sigma^*$ with $x_n \to \nu (n \to \infty)$. As $x_n \to \nu (n \to \infty)$, by hypothesis, $\widehat{f}(\lim_{n\to\infty} x_n) = \widehat{f}(\nu) \leq \lim_{n\to\infty} \inf\, f(x_n)$. Since both $\widehat{f}$ and f coincide over Σ^*, by definition of V we have $f(x_i) \leq \lambda$ for any $x_i \in V \cap \Sigma^*$. Therefore, $\widehat{f}(\nu) \leq \lambda$. This implies that $\nu \in V$, which means that V is a closed subset in $(\widehat{\Sigma^*}, \Gamma)$. Then $\widehat{f}^{-1}\big((\lambda, 1]\big)$ is open, that is, $\{\nu \in \widehat{\Sigma^*} \mid \widehat{f}(\nu) > \lambda\} \in \Gamma$. Hence, $\widehat{f} \in w(\Gamma)$.

Assume that f is fuzzy regular, by Lemma 3.9, the set $f^{[\lambda]} = \{x \in \Sigma^* \mid f(x) \leq \lambda\}$ is regular language for any $\lambda \in [0,1]$. Clearly, $\overline{f^{[\lambda]}} \in \mathrm{Clopen}(\Gamma)$. Since $\widehat{f}^{-1}\big([0,\lambda]\big) \cap$

$\Sigma^* = f^{[\lambda]}$, and $\widehat{f}^{-1}([0,\lambda])$ is closed for any $\lambda \in [0,1]$, then $\widehat{f}^{-1}([0,\lambda]) = \overline{f^{[\lambda]}}$. This means that $\widehat{f}^{-1}([0,\lambda]) \in \mathrm{Clopen}(\Gamma)$. Thus, $\widehat{f}^{-1}((\lambda,1]) \in \mathrm{Clopen}(\Gamma)$, that is, $\{\nu \in \widehat{\Sigma^*} \mid \widehat{f}(\nu) > \lambda\} \in \mathrm{Clopen}(\Gamma)$. Hence, $\widehat{f} \in \omega(\mathrm{Clopen}(\Gamma))$. □

4. Closure and Basis Associated with Fuzzy Regular Languages

Induced fuzzy topological spaces have very strong properties. Using the induced fuzzy topological space $([0,1]^{\widehat{\Sigma^*}}, w(\Gamma))$, we shall provide some characterizations for the closed fuzzy subsets in terms of fuzzy regular languages.

Definition 4.1. [12, 32] Let $([0,1]^{\widehat{\Sigma^*}}, w(\Gamma))$ be the induced fuzzy topological space of $(\widehat{\Sigma^*}, \Gamma)$, for any mapping $\widehat{f} : \widehat{\Sigma^*} \to [0,1]$, the *closure* of $\widehat{f}$ is the intersection of all closed fuzzy subsets in $([0,1]^{\widehat{\Sigma^*}}, w(\Gamma))$ that contain $\widehat{f}$, denoted by $\overline{\widehat{f}}$.

Given $r \in [0,1]$, $U \subseteq \widehat{\Sigma^*}$, $r\mathcal{X}_U$ denotes the fuzzy subset which takes value r at $\nu \in U$ and 0 at $\nu \notin U$, such a fuzzy subset is called a *leveled characteristic function*.

Theorem 4.2. *Let $f : \Sigma^* \to [0,1]$ be any function from the metric space (Σ^*, Γ_d) into the metric space $([0,1], \rho)$ with the usual absolute value metric. If f is a fuzzy regular language, we let $\widehat{f}$ denote any extension of f to $\widehat{\Sigma^*}$, then*

$$\overline{\widehat{f}} = \bigcup_{r\in[0,1]} r\mathcal{X}_{\overline{f_{(r)}}} = \bigcup_{r\in[0,1]} r\mathcal{X}_{\overline{f_{[r]}}}.$$

Proof. For any $f \in \mathrm{FReg}(\Sigma^*)$, suppose $\widehat{f}$ is an extension of f to $\widehat{\Sigma^*}$. It is easy to check that $\widehat{f}$ is a closed fuzzy subset in $([0,1]^{\widehat{\Sigma^*}}, w(\Gamma))$ if and only if $\widehat{f}_{[r]}$ is a closed subset in $(\widehat{\Sigma^*}, \Gamma)$ for any $r \in [0,1]$. We let

$$\widehat{g} = \bigcup_{r\in[0,1]} r\mathcal{X}_{\overline{f_{(r)}}},\ \widehat{h} = \bigcup_{r\in[0,1]} r\mathcal{X}_{\overline{f_{[r]}}}.$$

Since $\overline{\widehat{f}}$ is a closed fuzzy subset in $([0,1]^{\widehat{\Sigma^*}}, w(\Gamma))$, then $\big(\overline{\widehat{f}}\,\big)_{[r]}$ is closed in $(\widehat{\Sigma^*}, \Gamma)$. Note that f is a fuzzy regular language, by Lemma 3.3 and Lemma 3.9, then $\overline{f_{(r)}} \in \mathrm{Clopen}(\Gamma)$ for any $r \in [0,1]$, and $\overline{f_{(r)}} \cap \Sigma^* = \widehat{f}_{(r)} \cap \Sigma^*$. This implies that $\widehat{f}_{(r)} \subseteq \overline{f_{(r)}}$. Since $f_{(r)} \subseteq f_{[r]} \subseteq \big(\overline{\widehat{f}}\,\big)_{[r]}$, it follows that $\widehat{f}_{(r)} \subseteq \overline{f_{(r)}} \subseteq \overline{f_{[r]}} \subseteq \big(\overline{\widehat{f}}\,\big)_{[r]}$. By the decomposition theorem of fuzzy subsets, we get

$$\widehat{f} \le \widehat{g} \le \widehat{h} \le \bigcup_{r\in[0,1]} r\mathcal{X}_{(\overline{\widehat{f}}\,)_{[r]}} = \overline{\widehat{f}}.$$

It remains to show that $\widehat{g}$ is a closed fuzzy subset in $([0,1]^{\widehat{\Sigma^*}}, w(\Gamma))$. To show $\widehat{g}$ is closed in $([0,1]^{\widehat{\Sigma^*}}, w(\Gamma))$, it is sufficient to show that $\widehat{g}_{[r]}$ is closed in $(\widehat{\Sigma^*}, \Gamma)$ for any $r \in [0,1]$. In fact, for any $r \in [0,1]$ and any $\lambda \in [0,r]$, if $\nu \in \overline{f_{(\lambda)}}$, then $\widehat{g}(\nu) \ge \lambda\mathcal{X}_{\overline{f_{(\lambda)}}}(\nu) = \lambda$, which implies that $\widehat{g}(\nu) \ge r$, i.e., $\nu \in \widehat{g}_{[r]}$. On the other hand, if there exists a $\lambda_0 \in [0, r]$ such that $\nu \notin \overline{f_{(\lambda_0)}}$, for any $u \geq \lambda_0$, we get $\nu \notin \overline{f_{(u)}}$, that is to say, $u\mathcal{X}_{\overline{f_{(u)}}} = 0$. Therefore,

$$\widehat{g}(\nu) = \Big(\bigcup_{k<\lambda_0} k\mathcal{X}_{\overline{f_{(k)}}} \Big)(\nu) \leq \bigvee \{k \mid k < \lambda_0\} < \lambda_0 \leq r.$$

We then deduce that $\nu \in \widehat{g}_{[r]}$ if and only if for any $\lambda \in [0, r]$, $\nu \in \overline{f_{(\lambda)}}$. This means that $\widehat{g}_{[r]} = \bigcap_{\lambda \in [0,r]} \overline{f_{(\lambda)}}$. Clearly, $\widehat{g}_{[r]}$ is closed and so the proof is completed. □

Applying Theorem 4.2, Corollary 3.17, and Theorem 3.10, we obtain the following corollary.

Corollary 4.3. *Let $f : \Sigma^* \to [0, 1]$ be any function from the metric space (Σ^*, Γ_d) into the metric space $([0, 1], \rho)$ with the usual absolute value metric, and let f be a fuzzy regular language. Then $\widehat{f}$ is the continuous extension of f to $\widehat{\Sigma^*}$ if and only if*

$$\widehat{f} = \bigcup_{r\in[0,1]} r\mathcal{X}_{\overline{f_{(r)}}} = \bigcup_{r\in[0,1]} r\mathcal{X}_{\overline{f_{[r]}}}.$$

The concept of a basis is useful because many properties of fuzzy topologies can be reduced to statements about the basis that generates the corresponding topology, and many fuzzy topologies can be easily defined in terms of their basis.

Definition 4.4. [12] A *basis* for a fuzzy topological space $\big([0, 1]^{\widehat{\Sigma^*}}, \tau\big)$ is a subcollection $\mathcal{B}$ of τ, such that each member $\widehat{f}$ of τ can be written as $\widehat{f} = \bigcup_{t\in T} \widehat{f}_t$, where each $\widehat{f}_t$ belongs to $\mathcal{B}$.

Lemma 4.5. [18] *If L_1, L_2 are regular languages, then $\overline{L_1} \cap \overline{L_2} = \overline{L_1 \cap L_2}$.*

Next, we give a basis for the induced fuzzy topological space $([0, 1]^{\widehat{\Sigma^*}}, w(\Gamma))$. To do this, we need the following lemma.

Lemma 4.6. [32] *Let $([0, 1]^{\widehat{\Sigma^*}}, w(\Gamma))$ be the induced fuzzy topological space of $(\widehat{\Sigma^*}, \Gamma)$. For any $\widehat{g} \in [0, 1]^{\widehat{\Sigma^*}}$ and $\lambda \in [0, 1]$, if the characteristic function $\mathcal{X}_{\widehat{g}^{[\lambda]}}$ is a closed fuzzy subset of $([0, 1]^{\widehat{\Sigma^*}}, w(\Gamma))$, then $\widehat{g}$ is an open fuzzy subset in $([0, 1]^{\widehat{\Sigma^*}}, w(\Gamma))$, where $\widehat{g}^{[\lambda]} = \{\nu \in \widehat{\Sigma^*} \mid \widehat{g}(\nu) \leq \lambda\}$.*

The following theorem shows that the induced fuzzy topological space has a basis consisting of leveled characteristic functions of the closure of cut languages of fuzzy regular languages.

Theorem 4.7. *Let $([0, 1]^{\widehat{\Sigma^*}}, w(\Gamma))$ be the induced fuzzy topological space of $(\widehat{\Sigma^*}, \Gamma)$. Then the family*

$$\mathcal{B} = \big\{ r\mathcal{X}_{\overline{f_{[r]}}} \mid f \in \mathrm{FReg}(\Sigma^*), r \in [0, 1] \big\}$$

forms a basis for the fuzzy topology $w(\Gamma)$.

Proof. Since $1_{\Sigma^*} \in \mathrm{FReg}(\Sigma^*)$, for any $\lambda \in [0,1]$, then $(1_{\Sigma^*})_{[\lambda]} = \Sigma^*$, which implies that $1\mathcal{X}_{\overline{(1_{\Sigma^*})_{[\lambda]}}} = 1\mathcal{X}_{\widehat{\Sigma^*}} = 1_{\widehat{\Sigma^*}} \in \mathcal{B}$. That is, $\bigcup \mathcal{B} = 1_{\widehat{\Sigma^*}}$. Let

$$\mathbb{W} = \{\bigcup \mathcal{S} \mid \mathcal{S} \subset \mathcal{B}\}.$$

Thus, $1_{\widehat{\Sigma^*}} \in \mathbb{W}$ obtained by taking $1_{\widehat{\Sigma^*}} \in \mathcal{S}$. Taking $\mathcal{S} = \emptyset$, we then obtain $0_{\widehat{\Sigma^*}} \in \mathbb{W}$. Clearly, $\mathbb{W}$ is closed under arbitrary joins.

First we show that $\mathcal{B}$ has the finite intersection property (that is, a collection $\mathcal{A}$ of fuzzy subsets of $\mathcal{B}$ is said to have the intersection property if the intersection over any finite subcollection of $\mathcal{A}$ is nonempty). For any $r_1\mathcal{X}_{\overline{f_{[r_1]}}}$, $r_2\mathcal{X}_{\overline{h_{[r_2]}}} \in \mathcal{B}$, where $r_1, r_2 \in [0,1]$, $f, h \in \mathrm{FReg}(\Sigma^*)$, by Lemma 4.5, we have

$$r_1\mathcal{X}_{\overline{f_{[r_1]}}} \cap r_2\mathcal{X}_{\overline{h_{[r_2]}}} = (r_1 \wedge r_2)\mathcal{X}_{\overline{f_{[r_1]}} \cap \overline{h_{[r_2]}}} = r\mathcal{X}_{\overline{f_{[r_1]} \cap h_{[r_2]}}},$$

where $r = r_1 \wedge r_2$. Let $L = f_{[r_1]} \cap h_{[r_2]}$, then L is regular. Assume that $\mathfrak{x}$ is the characteristic function of L. Clearly, $\mathfrak{x}$ is fuzzy regular. Then we can write $r\mathcal{X}_{\overline{L}} = r\mathcal{X}_{\overline{\mathfrak{x}_{[r]}}}$ for any $r \in [0,1]$. By the definition of $\mathcal{B}$, we have $r\mathcal{X}_{\overline{\mathfrak{x}_{[r]}}} \in \mathcal{B}$. This implies that $r_1\mathcal{X}_{\overline{f_{[r_1]}}} \cap r_2\mathcal{X}_{\overline{h_{[r_2]}}} \in \mathcal{B}$.

Secondly, to show that $\mathbb{W}$ is a fuzzy topology on $[0,1]^{\widehat{\Sigma^*}}$, we need only show that $\widehat{g} \wedge \widehat{h} \in \mathbb{W}$ for any $\widehat{g}$, $\widehat{h} \in \mathbb{W}$. This topology can be obtained through the following computation.

Let $\mathcal{S}_1 = \{r_t\mathcal{X}_{\overline{f_{[r_t]}}} \mid t \in T_1\} \subseteq \mathcal{B}$ and $\mathcal{S}_2 = \{r_s\mathcal{X}_{\overline{f_{[r_s]}}} \mid s \in T_2\} \subseteq \mathcal{B}$ for some indexing sets T_1, T_2. Suppose $\widehat{g} = \bigcup \mathcal{S}_1$, $\widehat{h} = \bigcup \mathcal{S}_2$. Then

$$\widehat{g} \cap \widehat{h} = \Big(\bigcup_{t \in T_1} r_t\mathcal{X}_{\overline{f_{[r_t]}}}\Big) \cap \Big(\bigcup_{s \in T_2} r_s\mathcal{X}_{\overline{f_{[r_s]}}}\Big) = \bigcup_{t \in T_1, s \in T_2} (r_t\mathcal{X}_{\overline{f_{[r_t]}}} \cap r_s\mathcal{X}_{\overline{f_{[r_s]}}}).$$

Since $\mathcal{B}$ has the finite intersection property, then for any $t \in T_1, s \in T_2$, $r_t\mathcal{X}_{\overline{f_{[r_t]}}} \cap r_s\mathcal{X}_{\overline{h_{[r_s]}}} \in \mathcal{B}$. This implies that $\widehat{g} \cap \widehat{h} \in \mathbb{W}$. Hence, $\mathbb{W}$ forms a basis for a fuzzy topology on $[0,1]^{\widehat{\Sigma^*}}$.

In what follows we shall show that $\mathbb{W} = w(\Gamma)$. Let $\widehat{g} \in w(\Gamma)$, then for any $\lambda \in [0,1]$, $\widehat{g}^{[\lambda]}$ is a closed subset in $(\widehat{\Sigma^*}, \Gamma)$. By the definition of $\mathcal{B}$, $\mathcal{X}_{\widehat{g}^{[\lambda]}}$ is a closed fuzzy subset of $\mathbb{W}$. According to Lemma 4.6, we have $\widehat{g} \in \mathbb{W}$. Note that $\mathcal{B} \subseteq w(\Gamma)$, then $\mathbb{W} \subseteq w(\Gamma)$. Therefore, $\mathbb{W} = w(\Gamma)$. □

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