

STRATIFIED (L, M) -FUZZY DERIVED SPACES

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ABSTRACT. In this paper, the concepts of derived sets and derived operators are generalized to (L, M) -fuzzy topological spaces and their characterizations are given. What is more, it is shown that the category of stratified (L, M) -fuzzy topological spaces, the category of stratified (L, M) -fuzzy closure spaces and the category of stratified (L, M) -fuzzy quasi-neighborhood spaces are all isomorphic to the category of stratified (L, M) -fuzzy derived spaces.

1. Introduction

In point-set topology, derived set is an important and interesting topic. As we all know, neighborhood systems, interior operators and closure operators are good ways to characterize topology. Besides, there is a close relationship between derived sets and topologies in general topology [3, 7]. In [11], it has been shown that there exists a one-to-one correspondence between derived operators and topologies.

With the development of fuzzy mathematics [22], the notion of derived sets has been extended to fuzzy set theory. Many papers about accumulation points, derived sets, derived operators in L -topological spaces appeared, such as in [1, 4, 9, 10, 12, 16, 18]. In 1980, B.M. Pu and Y.M. Liu [10] proposed the notions of accumulation points and derived sets in L -topological spaces. In 1988, by using the concept of remote-neighborhood systems, G.J. Wang [18] introduced a different definition of accumulation points and derived sets in L -topological spaces. But this kind of definition about derived operator did not meet the property of finite union-preserving. In order to make up for this shortage, Y.C. Bai [1] proposed the concepts of N -derived sets and N -derived operators in L -topological spaces and studied their properties in 1990.

In 1991, M.S. Ying [20] gave a new approach to fuzzification of derived sets in a completely different direction and proposed the notion of $[0, 1]$ -fuzzifying (or fuzzifying, in short) derived set. But he did not give the axiomatic conditions of fuzzifying derived operators and did not discuss the relation between fuzzifying topologies and fuzzifying derived operators. In 1994, F.G. Shi [12] gave a definition of derived operators in L -topological spaces by using the notion of N -derived sets. Moreover, he also proved that there exists a one-to-one correspondence between his derived operators and L -topologies under some conditions. However, these satisfactory results were based on the binary logic, that is, for any fuzzy point x_λ

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and any fuzzy set A , x_λ is either an accumulation point of A or not, only one of the two is right. Now, we would give some degree to which a fuzzy point x_λ is an accumulation point of A in (L, M) -fuzzy topological spaces and generalize some results in L -topological spaces to (L, M) -fuzzy topological spaces.

The aims of this paper is to introduce the notions of (L, M) -fuzzy derived sets and (L, M) -fuzzy derived operators. Besides, the relationships among stratified (L, M) -fuzzy topological spaces, stratified (L, M) -fuzzy closure spaces, stratified (L, M) -fuzzy quasi-neighborhood spaces and stratified (L, M) -fuzzy derived spaces are also discussed from the perspective of category theory.

This paper is organized as follows. In Section 2, some preliminary notions and results are recalled. In Section 3, the notions of (L, M) -fuzzy derived sets and (L, M) -fuzzy derived operators are introduced and their characterizations are given. In Section 4, it is shown that the category $LM\text{-STop}$ of stratified (L, M) -fuzzy topological spaces and their continuous maps, the category $LM\text{-SCS}$ of stratified (L, M) -fuzzy closure spaces and their continuous maps, and the category $LM\text{-SQS}$ of stratified (L, M) -fuzzy quasi-neighborhood spaces and their continuous maps are all isomorphic to the category $LM\text{-SDS}$ of stratified (L, M) -fuzzy derived spaces and their continuous maps.

2. Preliminaries

Throughout this paper, both L and M denote completely distributive De Morgan algebras. A completely distributive De Morgan algebra is a completely distributive lattices with an order-reserving involution $'$. The smallest element and the largest element in L are denoted by \perp_L and \top_L , respectively. The smallest element and the largest element in M are denoted by \perp_M and \top_M , respectively.

Let X be a nonempty set. L^X denotes the set of all L -fuzzy subsets on X . L^X is also a completely distributive De Morgan algebra when it inherits the structure of the lattice L in a natural way, by defining \wedge, \vee, \leq and $'$ pointwisely. The smallest element and the largest element in L^X are denoted by \perp_{L^X} and \top_{L^X} , respectively. An element a in L is called a co-prime element if $b \vee c \geq a$ implies $b \geq a$ or $c \geq a$ for any $b, c \in L$ [6]. The set of all nonzero co-prime elements of L is denoted by $J(L)$ and the set of all nonzero co-prime elements of L^X is denoted by $J(L^X)$. It is easy to see that $J(L^X)$ is exactly the set of all fuzzy points x_λ ($\lambda \in J(L)$).

We say that a is wedge below b in L , in symbols, $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \geq b$ implies $a \leq d$ for some $d \in D$ [2]. We denote $\beta(a) = \{x \in L \mid x \prec a\}$ and $\beta^*(a) = \beta(a) \cap J(L)$. $a \prec^{op} b$ means that if for every subset $D \subseteq L$, $\bigwedge D \leq b$ implies $d \leq a$ for some $d \in D$. That is, $a \prec^{op} b \Leftrightarrow a' \prec b' \Leftrightarrow b \prec a$. We denote $\alpha(a) = \{x \in L \mid x \prec^{op} a\}$. A complete lattice L is completely distributive if and only if $a = \bigvee \beta(a) = \bigwedge \alpha(a)$ for each $a \in L$ [19]. The wedge below relation in a completely distributive lattice has the interpolation property, i.e., if $a \prec b$, then there exists $c \in L$ such that $a \prec c \prec b$. Moreover, it is easy to see that $a \prec \bigwedge_{i \in I} b_i$ implies $a \prec b_i$ for every $i \in I$, whereas $a \prec \bigvee_{i \in I} b_i$ implies $a \prec b_i$ for some $i \in I$ [9].

Lemma 2.1. [19] *Let L be a completely distributive lattice and let $\{a_i \mid i \in I\} \subseteq L$. Then*

- (1) $\alpha(\bigwedge_{i \in I} a_i) = \bigcup_{i \in I} \alpha(a_i)$, i.e., α is a \bigwedge - \bigcup mapping.
(2) $\beta(\bigvee_{i \in I} a_i) = \bigcup_{i \in I} \beta(a_i)$, i.e., β is a union-preserving mapping.

From Lemma 2.1, we know that α is an order-reversing mapping and β is an order-preserving mapping.

For any $A \in L^X$, we use the following notations [13]:

$$A_{[a]} = \{x \in X \mid A(x) \geq a\}, \quad A^{[a]} = \{x \in X \mid a \notin \alpha(A(x))\}.$$

Definition 2.2. [8, 17] An (L, M) -fuzzy topology on X is a mapping $\mathcal{T} : L^X \rightarrow M$ satisfying the following axioms:

- (LMT1) $\mathcal{T}(\perp_L) = \mathcal{T}(\top_L) = \top_M$;
(LMT2) $\forall A_1, A_2 \in L^X, \mathcal{T}(A_1) \wedge \mathcal{T}(A_2) \leq \mathcal{T}(A_1 \wedge A_2)$;
(LMT3) $\forall \{A_j \mid j \in J\} \subseteq L^X, \bigwedge_{j \in J} \mathcal{T}(A_j) \leq \mathcal{T}(\bigvee_{j \in J} A_j)$.

An (L, M) -fuzzy topology \mathcal{T} is called stratified if it satisfies

$$(LMT1)^* \forall s \in L, \mathcal{T}(\underline{s}) = \top_M, \text{ where } \underline{s} : X \rightarrow L \text{ is defined by } \underline{s}(x) = s.$$

Obviously, $(LMT1)^* \Rightarrow (LMT1)$. A set X equipped with an (L, M) -fuzzy topology \mathcal{T} , denoted by (X, \mathcal{T}) , is called an (L, M) -fuzzy topological space.

A continuous mapping from an (L, M) -fuzzy topological space (X, \mathcal{T}_X) to an (L, M) -fuzzy topological space (Y, \mathcal{T}_Y) is a mapping $f : X \rightarrow Y$ such that $\mathcal{T}_Y(B) \leq \mathcal{T}_X(f_L^{\leftarrow}(B))$ for all $B \in L^Y$, where $f_L^{\leftarrow}(B)(x) = B(f(x))$. The category of stratified (L, M) -fuzzy topological spaces and their continuous mappings is denoted by **LM-Stop**.

The following definitions were presented for (L, L) -fuzzy cases. They can easily be transformed to (L, M) -fuzzy cases as follows:

Definition 2.3. [5] An (L, M) -fuzzy quasi-neighborhood system on X is a set $\mathcal{Q} = \{\mathcal{Q}_{x_\lambda} \mid x_\lambda \in J(L^X)\}$ of mappings $\mathcal{Q}_{x_\lambda} : L^X \rightarrow M$ satisfying the following conditions:

- (LMQ1) $\mathcal{Q}_{x_\lambda}(\perp_L) = \perp_M, \mathcal{Q}_{x_\lambda}(\top_L) = \top_M$;
(LMQ2) $\mathcal{Q}_{x_\lambda}(A) \neq \perp_M \Rightarrow x_\lambda \not\leq A'$;
(LMQ3) $\mathcal{Q}_{x_\lambda}(A \wedge B) = \mathcal{Q}_{x_\lambda}(A) \wedge \mathcal{Q}_{x_\lambda}(B)$;
(LMQ4) $\mathcal{Q}_{x_\lambda}(A) = \bigvee_{x_\lambda \not\leq B \geq A'} \bigwedge_{y_\mu \not\leq B} \mathcal{Q}_{y_\mu}(B')$.

An (L, M) -fuzzy quasi-neighborhood system \mathcal{Q} is called stratified if it satisfies

$$(LMQ1)^* \mathcal{Q}_{x_\lambda}(\perp_L) = \perp_M; \forall x_\lambda \in J(L^X), \forall \lambda \not\leq s', \mathcal{Q}_{x_\lambda}(\underline{s}) = \top_M.$$

Obviously, $(LMQ1)^* \Rightarrow (LMQ1)$. A set X equipped with an (L, M) -fuzzy quasi-neighborhood system \mathcal{Q} , denoted by (X, \mathcal{Q}) , is called an (L, M) -fuzzy quasi-neighborhood space.

A continuous mapping from an (L, M) -fuzzy quasi-neighborhood space (X, \mathcal{Q}_X) to an (L, M) -fuzzy quasi-neighborhood space (Y, \mathcal{Q}_Y) is a mapping $f : X \rightarrow Y$ such that $(\mathcal{Q}_Y)_{f_L^{\rightarrow}(x_\lambda)}(B) \leq (\mathcal{Q}_X)_{x_\lambda}(f_L^{\leftarrow}(B))$ holds for all $x_\lambda \in J(L^X)$ and $B \in L^Y$, where $f_L^{\rightarrow}(x_\lambda) = f(x)_\lambda$. The category of stratified (L, M) -fuzzy quasi-neighborhood spaces and their continuous mappings is denoted by **LM-SQS**.

Definition 2.4. [15] An (L, M) -fuzzy closure operator on X is a mapping $Cl : L^X \rightarrow M^{J(L^X)}$ satisfying the following conditions:

- (LMC1) $\forall x_\lambda \in J(L^X), Cl(\underline{\perp}_L)(x_\lambda) = \perp_M$;
(LMC2) $\forall x_\lambda \leq A, Cl(A)(x_\lambda) = \top_M$;
(LMC3) $Cl(A \vee B) = Cl(A) \vee Cl(B)$;
(LMC4) $\forall x_\lambda \not\leq A, Cl(A)(x_\lambda) = \bigwedge_{x \not\leq B \geq A} \bigvee_{y_\mu \not\leq B} Cl(B)(y_\mu)$.

An (L, M) -fuzzy closure operator Cl is called stratified if it satisfies (LMC1)* $\forall x_\lambda \in J(L^X), \forall \lambda \not\leq s, Cl(\underline{s})(x_\lambda) = \perp_M$.

Obviously, (LMC1)* \Rightarrow (LMC1). A set X equipped with an (L, M) -fuzzy closure operator Cl , denoted by (X, Cl) , is called an (L, M) -fuzzy closure space.

A continuous mapping from an (L, M) -fuzzy closure space (X, Cl_X) to an (L, M) -fuzzy closure space (Y, Cl_Y) is a mapping $f : X \rightarrow Y$ such that $Cl_X(A)(x_\lambda) \leq Cl_Y(f_L^{\rightarrow}(A))(f_L^{\rightarrow}(x_\lambda))$ holds for all $x_\lambda \in J(L^X)$ and $A \in L^X$, where $f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$ for every $y \in Y$. The category of stratified (L, M) -fuzzy closure spaces and their continuous mappings is denoted by $LM\text{-SCS}$.

In what follows, we will recall some results about N -derived sets and derived operators in L -topological spaces.

Let (X, δ) be an L -topological space, where δ is an L -topology on X . For each $P \in L^X$, P is called a closed L -set if $P' \in \delta$. For any $x_\lambda \in J(L^X)$, a closed L -set P is called a closed remote-neighborhood of x_λ if $x_\lambda \not\leq P$. The set of all closed remote-neighborhoods of x_λ is denoted by $R(x_\lambda)$.

We next recall the definition of strong T_{-1} .

Definition 2.5. [12] Let (X, δ) be an L -topological space. Then (X, δ) is called strong T_{-1} if for any $x_\lambda \in J(L^X)$ and $a \in L$, there exists some $P \in R(x_\lambda)$ such that $x_a \leq P$ whenever $\lambda \not\leq a$.

Remark 2.6. Obviously, stratified L -topological spaces are strong T_{-1} .

In [1, 12], the notions of N -derived sets and L -derived operators were introduced as follows:

Definition 2.7. [1, 12] Let (X, δ) be an L -topological space and $A \in L^X$. $x_\lambda \in J(L^X)$ is called an N -accumulation point of A if $A \not\leq P \vee x_1$ for all $P \in R(x_\lambda)$. The supremum of all N -accumulation points of A is called the N -derived set of A , denoted by A^d .

In [12], F.G. Shi has proved that there exists a one-to-one correspondence between derived operators in the sense of Shi (or L -derived operators, for briefly) and L -topologies with the condition of strong T_{-1} .

Definition 2.8. [12] A mapping $d : L^X \rightarrow L^X$ is called an L -derived operator on X if it satisfies the following conditions:

- (LD1) $d(\underline{\perp}_L) = \underline{\perp}_L$;
(LD2) $\forall x_\lambda \in J(L^X), x_\lambda \not\leq d(x_1)$;
(LD3) $d(A \vee B) = d(A) \vee d(B)$;
(LD4) $d(d(A)) \leq A \vee d(A)$.

An L -derived operator d is called stratified if it satisfies the following condition: (LD1)* $\forall s \in L, d(\underline{s}) \leq s$. Obviously, (LD1)* \Rightarrow (LD1).

By Definition 2.7, we obtain a mapping $d^\delta : L^X \longrightarrow L^X$ defined by $A \longrightarrow A^d$. From [12], it is not difficult to see that if (X, δ) is stratified, then d^δ is a stratified L -derived operator on X . Conversely, if $d : L^X \longrightarrow L^X$ is a stratified L -derived operator on X , then $\delta^d = \{A \in L^X \mid d(A') \leq A'\}$ is a stratified L -topology on X . In addition, there exists a one-to-one correspondence between stratified L -derived operators and stratified L -topologies.

3. (L, M) -fuzzy Derived Spaces

In this section, we will introduce the notions of (L, M) -fuzzy derived sets and (L, M) -fuzzy derived operators. Then we will give their characterizations.

Now, we shall generalize the concept of N -accumulation points to (L, M) -fuzzy topological spaces as follows.

Definition 3.1. Let $\mathcal{T} : L^X \longrightarrow M$ be an (L, M) -fuzzy topology. Define a mapping $d^{\mathcal{T}} : L^X \longrightarrow M^{J(L^X)}$ by $\forall A \in L^X, \forall x_\lambda \in J(L^X)$,

$$d^{\mathcal{T}}(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq A - x_1} (\mathcal{T}(B'))',$$

where $A - x_1 \triangleq A \wedge (\text{supp}A - \{x\})$ and $\text{supp}A = \{x \in X \mid A(x) \neq \perp_L\}$.

$d^{\mathcal{T}}(A)(x_\lambda)$ is called the degree to which x_λ is an N -accumulation point of A with respect to \mathcal{T} . The mapping $d^{\mathcal{T}}(A) : J(L^X) \longrightarrow M$ is called the (L, M) -fuzzy derived set of A with respect to \mathcal{T} .

Remark 3.2. (1) If $M = 2$ in the above definition (i.e., if \mathcal{T} is an L -topology) and $d^{\mathcal{T}}(A)(x_\lambda) = \top_M$, then $B \geq A - x_1$ implies $B' \notin \mathcal{T}$, for all $B \in L^X$ with $x_\lambda \not\leq B$. That is, $B' \in \mathcal{T}$ implies $A - x_1 \not\leq B$ for all $B \in L^X$ with $x_\lambda \not\leq B$. It is exactly the definition of N -accumulation points in L -topological spaces.

(2) In [21], from another point of view, W. Yao generalized the concept of accumulation point of a filter in crisp topology to (L, M) -fuzzy cases and gave the definition of the degree to which a fuzzy point is an accumulation point of an (L, M) -fuzzy filter. However, the definition of accumulation points about (L, M) -fuzzy filters is different from the above definition.

Before studying the properties of (L, M) -fuzzy derived sets, we need the following lemma.

Lemma 3.3. Let $\mathcal{T} : L^X \longrightarrow M$ be a stratified (L, M) -fuzzy topology. Then for all $y_\mu \in J(L^X)$ and for any $B, D \in L^X$,

$$(1) \bigwedge_{y_\mu \not\leq D \geq B - y_1} (\mathcal{T}(D'))' = \bigwedge_{y_\mu \not\leq D \geq B} (\mathcal{T}(D'))' \text{ whenever } y_\mu \not\leq B.$$

$$(2) \bigvee_{y_\mu \not\leq B} \bigwedge_{y_\mu \not\leq D \geq B - y_1} (\mathcal{T}(D'))' = \mathcal{T}(B)'.$$

Proof. (1) Let *LHS* and *RHS* denote the left side of the equality and the right side of the equality, respectively. It is easily to see that $LHS \leq RHS$. Suppose that $a \in M$ with $a \not\leq LHS$. Then there exists some D such that $y_\mu \not\leq D \geq B - y_1$ and $a \not\leq \mathcal{T}(D)'$. Let $\tilde{D} = D \vee \underline{B}(y)$. Then $y_\mu \not\leq \tilde{D} \geq B$. Since \mathcal{T} is a stratified

(L, M) -fuzzy topology, we have

$$\begin{aligned}
\mathcal{T}(\tilde{D}') &= (\mathcal{T}(D \vee \underline{B}(y))')' \\
&= (\mathcal{T}(D' \wedge \underline{B}(y))')' \\
&\leq \mathcal{T}(D')' \vee \mathcal{T}(\underline{B}(y))' \\
&= \mathcal{T}(D')'.
\end{aligned}$$

Hence $a \not\leq \mathcal{T}(\tilde{D}')$. Therefore $a \not\leq RHS$. By the arbitrariness of a , we obtain that $RHS \leq LHS$.

(2) Let LHS denote the left side of equality. Then $LHS \leq \mathcal{T}(B)'$ is trivial. It now suffices to prove that $(LHS)' \leq \mathcal{T}(B')$. By (1), we have

$$\begin{aligned}
(LHS)' &= \bigwedge_{y_\mu \not\leq B} \bigvee_{y_\mu \not\leq D \geq B} \mathcal{T}(D') \\
&= \bigvee_{f \in \prod_{y_\mu \not\leq B} B_{y_\mu}} \bigwedge_{y_\mu \not\leq B} \mathcal{T}(f(y_\mu)') \text{ (by CD law)} \\
&\leq \bigvee_{f \in \prod_{y_\mu \not\leq B} B_{y_\mu}} \mathcal{T} \left(\bigvee_{y_\mu \not\leq B} f(y_\mu)' \right) \text{ (by LMT3)} \\
&= \bigvee_{f \in \prod_{y_\mu \not\leq B} B_{y_\mu}} \mathcal{T} \left(\left(\bigwedge_{y_\mu \not\leq B} f(y_\mu) \right)' \right) \\
&= \mathcal{T}(B'),
\end{aligned}$$

where $B_{y_\mu} = \{D \mid y_\mu \not\leq D \geq B\}$ and $\bigwedge_{y_\mu \not\leq B} f(y_\mu) = B$ for all $y_\mu \not\leq B$ and $f \in \prod_{y_\mu \not\leq B} B_{y_\mu}$. \square

Theorem 3.4. Let $\mathcal{T} : L^X \rightarrow M$ be an (L, M) -fuzzy topology. Then

- (1) $\forall x_\lambda \in J(L^X)$, $d^{\mathcal{T}}(\underline{\perp}_L)(x_\lambda) = \perp_M$;
- (2) $\forall x_\lambda \in J(L^X)$, $d^{\mathcal{T}}(x_1)(x_\lambda) = \perp_M$;
- (3) $\forall A, B \in L^X$, $d^{\mathcal{T}}(A \vee B) = d^{\mathcal{T}}(A) \vee d^{\mathcal{T}}(B)$;

If \mathcal{T} is stratified, then

- (1)* $\forall x_\lambda \in J(L^X)$, $\forall \lambda \not\leq s$, $d^{\mathcal{T}}(\underline{s})(x_\lambda) = \perp_M$ and (1)* \Rightarrow (1).
- (4) $\forall A \in L^X$, $\forall x_\lambda \in J(L^X)$, $d^{\mathcal{T}}(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq A-x_1} \bigvee_{y_\mu \not\leq B} d^{\mathcal{T}}(B)(y_\mu)$.

Proof. Firstly, (1) and (2) are true trivially.

(3) By the definition of $d^{\mathcal{T}}$, it is easily seen that $d^{\mathcal{T}}(\cdot)$ is an order-preserving mapping. Then $d^{\mathcal{T}}(A) \vee d^{\mathcal{T}}(B) \leq d^{\mathcal{T}}(A \vee B)$. Hence it suffices to prove that $d^{\mathcal{T}}(A)(x_\lambda) \vee d^{\mathcal{T}}(B)(x_\lambda) \geq d^{\mathcal{T}}(A \vee B)(x_\lambda)$ for all $x_\lambda \in J(L^X)$. By (LMT2) and

$(A \vee B) - x_1 = (A - x_1) \vee (B - x_1)$, we have

$$\begin{aligned}
& d^{\mathcal{T}}(A)(x_\lambda) \vee d^{\mathcal{T}}(B)(x_\lambda) \\
&= \left(\bigwedge_{x_\lambda \not\leq D_1 \geq A - x_1} (\mathcal{T}(D'_1))' \right) \vee \left(\bigwedge_{x_\lambda \not\leq D_2 \geq B - x_1} (\mathcal{T}(D'_2))' \right) \\
&= \bigwedge_{x_\lambda \not\leq D_1 \geq A - x_1, x_\lambda \not\leq D_2 \geq B - x_1} \left((\mathcal{T}(D'_1))' \vee (\mathcal{T}(D'_2))' \right) \\
&= \bigwedge_{x_\lambda \not\leq D_1 \vee D_2 \geq (A \vee B) - x_1} (\mathcal{T}(D'_1) \wedge \mathcal{T}(D'_2))' \\
&\geq \bigwedge_{x_\lambda \not\leq D_1 \vee D_2 \geq (A \vee B) - x_1} (\mathcal{T}(D_1 \vee D_2))' \\
&\geq d^{\mathcal{T}}(A \vee B)(x_\lambda).
\end{aligned}$$

(1)* Since $\mathcal{T}(s) = \top_M$ for all $s \in L$, (1)* holds. (1)* \Rightarrow (1), clearly.

(4) By Lemma 3.3, we have

$$\begin{aligned}
d^{\mathcal{T}}(A)(x_\lambda) &= \bigwedge_{x_\lambda \not\leq B \geq A - x_1} (\mathcal{T}(B'))' \\
&= \bigwedge_{x_\lambda \not\leq B \geq A - x_1} \bigvee_{y_\mu \not\leq B} \bigwedge_{y_\mu \not\leq D \geq B - y_1} (\mathcal{T}(D'))' \\
&= \bigwedge_{x_\lambda \not\leq B \geq A - x_1} \bigvee_{y_\mu \not\leq B} d^{\mathcal{T}}(B)(y_\mu).
\end{aligned}$$

By Definition 3.1, we get a mapping $d^{\mathcal{T}} : L^X \longrightarrow M^{J(L^X)}$ defined by $A \longrightarrow d^{\mathcal{T}}(A)$. The mapping $d^{\mathcal{T}}$ has the above properties. If an arbitrary mapping $d : L^X \longrightarrow M^{J(L^X)}$ has these properties, then d is called an (L, M) -fuzzy derived operator. That is, \square

Definition 3.5. An (L, M) -fuzzy derived operator on X is a mapping $d : L^X \longrightarrow M^{J(L^X)}$ satisfying the following conditions:

(LMD1) $\forall x_\lambda \in J(L^X)$, $d(\perp_L)(x_\lambda) = \perp_M$;

(LMD2) $\forall x_\lambda \in J(L^X)$, $d(x_1)(x_\lambda) = \perp_M$;

(LMD3) $\forall A, B \in L^X$, $d(A \vee B) = d(A) \vee d(B)$;

(LMD4) $d(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq A - x_1} \bigvee_{y_\mu \not\leq B} d(B)(y_\mu)$.

The pair (X, d) is called an (L, M) -fuzzy derived space.

An (L, M) -fuzzy derived operator d is called stratified if it satisfies

(LMD1)* $\forall x_\lambda \in J(L^X)$, $\forall \lambda \not\leq s$, $d(s)(x_\lambda) = \perp_M$ and (LMD1)* \Rightarrow (LMD1).

The pair (X, d) is called a stratified (L, M) -fuzzy derived space if d satisfies (LMD1)* and (LMD2)-(LMD4).

A mapping $f : X \rightarrow Y$ between two (L, M) -fuzzy derived spaces (X, d_X) and (Y, d_Y) is called continuous if $\forall x_\lambda \in J(L^X), \forall A \in L^X$,

$$\begin{aligned} d_X(A)(x_\lambda) &\leq f_L^\rightarrow(A)(f_L^\rightarrow(x_\lambda)) \vee d_Y(f_L^\rightarrow(A))(f_L^\rightarrow(x_\lambda)) \\ &= \begin{cases} \top_M, & f(x)_\lambda \leq f_L^\rightarrow(A); \\ d_Y(f_L^\rightarrow(A))(f(x)_\lambda), & f(x)_\lambda \not\leq f_L^\rightarrow(A), \end{cases} \end{aligned}$$

where $f_L^\rightarrow(x_\lambda) = f(x)_\lambda$.

Theorem 3.6. *Stratified (L, M) -fuzzy derived spaces and their continuous mappings form a category, denoted by $LM\text{-SDS}$.*

Proof. It is easy to be checked and the details are omitted here. \square

The next theorem is obvious.

Theorem 3.7. *If a mapping $d : L^X \rightarrow M^{J(L^X)}$ is an (L, M) -fuzzy derived operator, then*

$$d(A)(x_\lambda) = d(A - x_1)(x_\lambda)$$

for any $A \in L^X$ and for all $x_\lambda \in J(L^X)$.

The following theorems give equivalent characterizations of (LMD4).

Theorem 3.8. *Let $d : L^X \rightarrow M^{J(L^X)}$ be a mapping satisfying (LMD1)*, (LMD2) and (LMD3). Then (LMD4) is equivalent to the following conditions:*

(LMD4) † $\forall x_\lambda \in J(L^X), d(A)(x_\lambda) = \bigwedge_{\mu \prec \lambda} d(A)(x_\mu)$;

(LMD4)* $\forall a \in J(M), \bigvee d(\bigvee d(A)_{[a]})_{[a]} \leq A \vee (\bigvee d(A)_{[a]})$.

Proof. Sufficiency. We need to check (LMD4).

First, by (LMD4) † , we have the following conclusion:

$$x_\lambda \leq \bigvee d(A)_{[a]} \Leftrightarrow x_\lambda \in d(A)_{[a]} \Leftrightarrow d(A)(x_\lambda) \geq a.$$

In fact, it suffices to prove that $x_\lambda \leq \bigvee d(A)_{[a]}$ implies $d(A)(x_\lambda) \geq a$. If $x_\lambda \leq \bigvee d(A)_{[a]}$, then $x_\gamma \prec x_\lambda \leq \bigvee d(A)_{[a]}$ for every $\gamma \prec \lambda$. This shows that there exists some $x_\nu \in d(A)_{[a]}$ such that $x_\gamma \prec x_\nu$. By (LMD4) † , we know that $d(A)(x_\gamma) \geq d(A)(x_\nu) \geq a$. Hence $\bigwedge_{\gamma \prec \lambda} d(A)(x_\gamma) = d(A)(x_\lambda) \geq a$.

Now we shall check (LMD4): $d(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq A - x_1} \bigvee_{y_\mu \not\leq B} d(B)(y_\mu)$. Let RHS denote the right side of the equality. It follows from (LMD3) that d is an order-preserving mapping. On one hand, for any $x_\lambda \not\leq B \geq A - x_1$, by Theorem 3.7, we know $d(A)(x_\lambda) = d(A - x_1)(x_\lambda) \leq d(B)(x_\lambda) \leq \bigvee_{y_\mu \not\leq B} d(B)(y_\mu)$. Hence $d(A)(x_\lambda) \leq RHS$. On the other hand, suppose that $a \in J(M)$ with $a \not\leq d(A)(x_\lambda)$. Then there exists some $b \in \beta^*(a)$ such that $b \not\leq d(A)(x_\lambda)$. By Theorem 3.7, we have $d(A - x_1)(x_\lambda) = d(A)(x_\lambda) \not\leq b$. This shows that $x_\lambda \notin d(A - x_1)_{[b]}$, i.e., $x_\lambda \not\leq \bigvee d(A - x_1)_{[b]}$. Further, $x_\lambda \not\leq (A - x_1) \vee (\bigvee d(A - x_1)_{[b]})$. Let $B = (A - x_1) \vee (\bigvee d(A - x_1)_{[b]})$. Then $x_\lambda \not\leq B \geq A - x_1$. By (LMD4)*, we get $y_\mu \not\leq \bigvee d(\bigvee d(A - x_1)_{[b]})_{[b]}$ for any $y_\mu \not\leq B$. Then $d(B)(y_\mu) = d(A - x_1)(y_\mu) \vee d(\bigvee d(A - x_1)_{[b]})(y_\mu) \not\leq b$ for any $y_\mu \not\leq B$, since $b \in J(M)$. Hence $\bigvee_{y_\mu \not\leq B} d(B)(y_\mu) \not\leq a$. Therefore $a \not\leq RHS$. By the arbitrariness of a , we obtain $RHS \leq d(A)(x_\lambda)$.

Necessity. We need to check (LMD4)[†] and (LMD4)*.

(LMD4)[†] From the definition of $d(A)(x_\lambda)$, it is not difficult to know that $d(A)(x_\lambda) \leq \bigwedge_{\mu \prec \lambda} d(A)(x_\mu)$. It suffices to show that $\bigwedge_{\mu \prec \lambda} d(A)(x_\mu) \leq d(A)(x_\lambda)$. Suppose that $a \in M$ with $a \not\leq d(A)(x_\lambda)$. Then there exists some B_a such that $x_\lambda \not\leq B_a \geq A - x_1$ and $\bigvee_{y_\mu \not\leq B_a} d(B_a)(y_\mu) \not\leq a$. Since $\lambda = \bigvee \beta(\lambda)$, we have that there exists some $\mu \in \beta(\lambda)$ such that $x_\mu \not\leq B_a$. This implies $d(A)(x_\mu) \not\leq a$. Hence $\bigwedge_{\mu \prec \lambda} d(A)(x_\mu) \not\leq a$. From the arbitrariness of a , we obtain $\bigwedge_{\mu \prec \lambda} d(A)(x_\mu) \leq d(A)(x_\lambda)$.

(LMD4)* Take any $x_\lambda \not\leq A \vee (\bigvee d(A)_{[a]})$, we have $x_\lambda \not\leq A$ and $x_\lambda \not\leq \bigvee d(A)_{[a]}$. Since (LMD4)[†] has been proved, we get $x_\lambda \not\leq \bigvee d(A)_{[a]}$ if and only if $d(A)(x_\lambda) \not\leq a$, which implies that there exists some B_a such that $x_\lambda \not\leq B_a \geq A - x_1$ and $\bigvee_{y_\mu \not\leq B_a} d(B_a)(y_\mu) \not\leq a$. This shows $d(B_a)(y_\mu) \not\leq a$ for any $y_\mu \not\leq B_a$. Let $D = B_a \vee A(x)$. Then $x_\lambda \not\leq D \geq A$. Next we shall check that $D \geq \bigvee d(A)_{[a]}$.

In fact, take any $z_t \not\leq D$, we have $z_t \not\leq B_a$ and $z_t \not\leq A(x)$. By (LMD1)*, we know $d(D)(z_t) = d(B_a \vee A(x))(z_t) = d(B_a)(z_t) \vee d(A(x))(z_t) = d(B_a)(z_t) \geq d(A)(z_t)$. It follows from $d(B_a)(z_t) \not\leq a$ that $d(A)(z_t) \not\leq a$. Then $z_t \not\leq \bigvee d(A)_{[a]}$. Hence $D \geq \bigvee d(A)_{[a]}$.

Note that $x_\lambda \not\leq D \geq \bigvee d(A)_{[a]} \geq \bigvee d(A)_{[a]} - x_1$ and $d(D)(z_t) = d(B_a)(z_t)$ for any $z_t \not\leq D$. Then $\bigvee_{z_t \not\leq D} d(D)(z_t) = \bigvee_{z_t \not\leq D} d(B_a)(z_t) \leq \bigvee_{y_\mu \not\leq B_a} d(B_a)(y_\mu) \not\leq a$. This implies $\bigvee_{z_t \not\leq D} d(D)(z_t) \not\leq a$, which means $d(\bigvee d(A)_{[a]})(x_\lambda) \not\leq a$, i.e., $x_\lambda \not\leq \bigvee d(\bigvee d(A)_{[a]})_{[a]}$. Therefore $\bigvee d(\bigvee d(A)_{[a]})_{[a]} \leq A \vee (\bigvee d(A)_{[a]})$. \square

Theorem 3.9. *Let $d : L^X \rightarrow M^{J(L^X)}$ be a mapping satisfying (LMD1)*, (LMD2) and (LMD3). If $\alpha(a \vee b) = \alpha(a) \cap \alpha(b)$ for each $a, b \in M$, then (LMD4) is also equivalent to the following conditions:*

(LMD4)[†] $\forall x_\lambda \in J(L^X)$, $d(A)(x_\lambda) = \bigwedge_{\mu \prec \lambda} d(A)(x_\mu)$;

(LMD4)** $\forall a \in \alpha(\perp_M)$, $\bigvee d(\bigvee d(A)^{[a]})_{[a]} \leq A \vee (\bigvee d(A)^{[a]})$.

Proof. Sufficiency. We need to check (LMD4).

First, by (LMD4)[†], we have

$$x_\lambda \leq \bigvee d(A)^{[a]} \Leftrightarrow x_\lambda \in d(A)^{[a]} \Leftrightarrow a \notin \alpha(d(A)(x_\lambda)).$$

In fact, it suffices to prove that $x_\lambda \leq \bigvee d(A)^{[a]}$ implies $a \notin \alpha(d(A)(x_\lambda))$. If $x_\lambda \leq \bigvee d(A)^{[a]}$, then $x_\gamma \prec x_\lambda \leq \bigvee d(A)^{[a]}$ for every $\gamma \prec \lambda$. This implies that there exists some $x_\nu \in d(A)^{[a]}$ such that $x_\gamma \prec x_\nu$. By (LMD4)[†], we have $d(A)(x_\gamma) \geq d(A)(x_\nu)$. Then $\alpha(d(A)(x_\gamma)) \subseteq \alpha(d(A)(x_\nu))$, since α is an order-reversing mapping. So $a \notin \alpha(d(A)(x_\gamma))$ for every $\gamma \prec \lambda$. Hence $a \notin \bigcup_{\gamma \prec \lambda} \alpha(d(A)(x_\gamma)) = \alpha(\bigwedge_{\gamma \prec \lambda} d(A)(x_\gamma)) = \alpha(d(A)(x_\lambda))$.

Now we shall check (LMD4): $d(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq A - x_1} \bigvee_{y_\mu \not\leq B} d(B)(y_\mu)$. Let RHS denote the right side of the equality. It suffices to prove that $RHS \leq d(A)(x_\lambda)$. Suppose that $a \in \alpha(\perp_M)$ with $a \in \alpha(d(A)(x_\lambda))$. Then there exists some $b \prec a$ such that $b \in \alpha(d(A)(x_\lambda))$. By Theorem 3.7, we have $b \in \alpha(d(A - x_1)(x_\lambda))$. This shows $x_\lambda \notin d(A - x_1)^{[b]}$, i.e., $x_\lambda \not\leq \bigvee d(A - x_1)^{[b]}$. Let $B = (A - x_1) \vee (\bigvee d(A - x_1)^{[b]})$. Then $x_\lambda \not\leq B \geq A - x_1$. By (LMD4)**, we

have $y_\mu \not\leq \bigvee d(\bigvee d(A - x_1)^{[b]})^{[b]}$ for any $y_\mu \not\leq B$. Then $b \in \alpha(d(A - x_1)(y_\mu)) \cap \alpha(d(\bigvee d(A - x_1)^{[b]})(y_\mu)) = \alpha(d(B)(y_\mu))$. Hence $a \in \alpha(\bigvee_{y_\mu \not\leq B} d(B)(y_\mu))$. This shows $a \in \bigcup_{x_\lambda \not\leq B \geq A - x_1} \alpha(\bigvee_{y_\mu \not\leq B} d(B)(y_\mu)) = \alpha(RHS)$. By the arbitrariness of a , we obtain $RHS \leq d(A)(x_\lambda)$.

Necessity. We need to check (LMD4)[†] and (LMD4)**.

(LMD4)[†] The proof is similar to that of Theorem 3.8.

(LMD4)** Take any $x_\lambda \not\leq A \vee (\bigvee d(A)^{[a]})$, we have $x_\lambda \not\leq A$ and $x_\lambda \not\leq \bigvee d(A)^{[a]}$, i.e., $a \in \alpha(d(A)(x_\lambda)) = \bigcup_{x_\lambda \not\leq B \geq A - x_1} \alpha(\bigvee_{y_\mu \not\leq B} d(B)(y_\mu))$, which implies that there exists some B_a such that $x_\lambda \not\leq B_a \geq A - x_1$ and $a \in \alpha(\bigvee_{y_\mu \not\leq B_a} d(B_a)(y_\mu))$. Since α is an order-reversing map, we have $a \in \alpha(d(B_a)(y_\mu))$ for any $y_\mu \not\leq B_a$. Let $D = B_a \vee A(x)$. Then $x_\lambda \not\leq D \geq A$. Next we shall check that $D \geq \bigvee d(A)^{[a]}$.

In fact, take any $z_t \not\leq D$, we have $z_t \not\leq B_a$ and $z_t \not\leq A(x)$. By (LMD1)*, we know $d(D)(z_t) = d(B_a \vee A(x))(z_t) = d(B_a)(z_t) \vee d(A(x))(z_t) = d(B_a)(z_t) \geq d(A)(z_t)$. This implies $a \in \alpha(d(B_a)(z_t)) \subseteq \alpha(d(A)(z_t))$. Thus $z_t \not\leq d(A)^{[a]}$, i.e., $z_t \not\leq \bigvee d(A)^{[a]}$. Hence $D \geq \bigvee d(A)^{[a]}$.

Note that $x_\lambda \not\leq D \geq \bigvee d(A)^{[a]} \geq \bigvee d(A)^{[a]} - x_1$ and $d(D)(z_t) = d(B_a)(z_t)$ for any $z_t \not\leq D$. Then $\bigvee_{z_t \not\leq D} d(D)(z_t) \leq \bigvee_{y_\mu \not\leq B_a} d(B_a)(y_\mu)$, which implies $a \in \alpha(\bigvee_{z_t \not\leq D} d(D)(z_t))$. Hence $a \in \alpha(d(\bigvee d(A)^{[a]})(x_\lambda))$, i.e. $x_\lambda \not\leq \bigvee d(\bigvee d(A)^{[a]})^{[a]}$. Therefore $\bigvee d(\bigvee d(A)^{[a]})^{[a]} \leq A \vee (\bigvee d(A)^{[a]})$. \square

Given an (L, M) -fuzzy derived operators $d : L^X \rightarrow M^{J(L^X)}$, we define two mappings $d_{[a]} : L^X \rightarrow L^X$ ($a \in J(M)$) and $d^{[a]} : L^X \rightarrow L^X$ ($a \in \alpha(\perp)$) as follows:

$$d_{[a]}(A) = \bigvee d(A)_{[a]} \text{ and } d^{[a]}(A) = \bigvee d(A)^{[a]}$$

for each $A \in L^X$, respectively.

In order to discuss the relationships among d , $d_{[a]}$ and $d^{[a]}$, we need the following lemma.

Lemma 3.10. *If $d : L^X \rightarrow M^{J(L^X)}$ is an (L, M) -fuzzy derived operator, then for each $A \in L^X$,*

- (1) $d_{[a]}(A) \leq d_{[b]}(A)$ for any $a, b \in J(M)$ with $b \in \beta^*(a)$ and $d_{[a]}(A) = \bigwedge_{b \in \beta^*(a)} d_{[b]}(A)$.
- (2) $d^{[b]}(A) \leq d^{[a]}(A)$ for any $a, b \in \alpha(\perp_M)$ with $b \in \alpha(a)$ and $d^{[a]}(A) = \bigvee_{b \in \alpha(a)} d^{[b]}(A)$.

Proof. (1) Take any $x_\lambda \in J(L^X)$ such that $x_\lambda \leq d_{[a]}(A) = \bigvee d(A)_{[a]}$. Then $d(A)(x_\lambda) \geq a$. For any $b \in \beta^*(a)$, we have $d(A)(x_\lambda) \geq b$. Then $x_\lambda \leq \bigvee d(A)_{[b]} = d_{[b]}(A)$. Hence $d_{[a]}(A) \leq d_{[b]}(A)$. On one hand, $d_{[a]}(A) \leq \bigwedge_{b \in \beta^*(a)} d_{[b]}(A)$ is trivial. On the other hand, take any $x_\lambda \in J(L^X)$ such that $x_\lambda \not\leq d_{[a]}(A)$, i.e., $d(A)(x_\lambda) \not\geq a$. Since $a = \bigvee_{b \in \beta^*(a)} b$, there exists some $b \in \beta^*(a)$ such that $d(A)(x_\lambda) \not\geq b$, i.e., $x_\lambda \not\leq d_{[b]}(A)$. Hence $x_\lambda \not\leq \bigwedge_{b \in \beta^*(a)} d_{[b]}(A)$. Therefore $\bigwedge_{b \in \beta^*(a)} d_{[b]}(A) \leq d_{[a]}(A)$.

(2) Its proof is similar to that of (1) and omitted here. \square

The following theorem shows the relations among d , $d_{[a]}$ and $d^{[a]}$.

Theorem 3.11. *Let $d : L^X \rightarrow M^{J(L^X)}$ be a mapping. Then the following statements are equivalent.*

- (1) d is a stratified (L, M) -fuzzy derived operator on X .
- (2) For each $a \in J(M)$, $d_{[a]}$ is a stratified L -derived operator on X .
- (3) If $\alpha(a \vee b) = \alpha(a) \cap \alpha(b)$ for any $a, b \in M$, then for each $a \in \alpha(\perp)$, $d^{[a]}$ is a stratified L -derived operator on X .

Proof. (1) \Rightarrow (2) For each $a \in J(M)$, we need to check $d_{[a]}$ satisfies the conditions of (LD1)* and (LD2)-(LD4).

(LD1)* For all $s \in L$, take any $x_\lambda \in J(L^X)$ such that $x_\lambda \not\leq s$. By (LMD1)*, we have $d(\underline{s})(x_\lambda) = \perp_M \not\leq a$. Then $x_\lambda \not\leq \bigvee d(\underline{s})_{[a]} = d_{[a]}(\underline{s})$. Hence $d_{[a]}(\underline{s}) \leq s$.

(LD2) By (LMD2), we have $d(x_1)(x_\lambda) = \perp_M \not\leq a$. Then $x_\lambda \not\leq \bigvee d(x_1)_{[a]}$. Hence $x_\lambda \not\leq d_{[a]}(x_1)$ for any $x_\lambda \in J(L^X)$.

(LD3) On one hand, it is easy to prove that $d_{[a]}(A) \leq d_{[a]}(B)$ whenever $A \leq B$, since $d(\cdot)$ is an order-preserving mapping. Then we have $d_{[a]}(A \vee B) \geq d_{[a]}(A) \vee d_{[a]}(B)$. On the other hand, take any $x_\lambda \in J(L^X)$ such that $x_\lambda \leq d_{[a]}(A \vee B)$. By (LMD3), we know $d(A \vee B)(x_\lambda) = d(A)(x_\lambda) \vee d(B)(x_\lambda) \geq a$. Note that $a \in J(M)$. Thus $d(A)(x_\lambda) \geq a$ or $d(B)(x_\lambda) \geq a$, which means $x_\lambda \leq \bigvee d(A)_{[a]}$ or $x_\lambda \leq \bigvee d(B)_{[a]}$. Then $x_\lambda \leq d_{[a]}(A) \vee d_{[a]}(B)$. Hence $d_{[a]}(A \vee B) \leq d_{[a]}(A) \vee d_{[a]}(B)$. Therefore $d_{[a]}(A \vee B) = d_{[a]}(A) \vee d_{[a]}(B)$.

(LD4) By (LMD4)*, we have $\bigvee d(\bigvee d(A)_{[a]})_{[a]} \leq A \vee (\bigvee d(A)_{[a]})$ for each $a \in J(M)$. This shows $d_{[a]}(d_{[a]}(A)) \leq A \vee d_{[a]}(A)$.

(2) \Rightarrow (1) Define $d : L^X \rightarrow M^{J(L^X)}$ by

$$d(A)(x_\lambda) = \bigvee \{a \in J(M) \mid x_\lambda \leq d_{[a]}(A)\}.$$

We need to check d satisfies the conditions of (LMD1)*, (LMD2), (LMD3), (LMD4)[†] and (LMD4)*.

(LMD1)* For any $x_\lambda \in J(L^X)$ and $\lambda \not\leq s$, by (LD1)*, we have $d(\underline{s})(x_\lambda) = \bigvee_{x_\lambda \leq d_{[a]}(\underline{s})} a = \bigvee \emptyset = \perp_M$.

(LMD2) For any $x_\lambda \in J(L^X)$, by (LD2), we have $d(x_1)(x_\lambda) = \bigvee_{x_\lambda \leq d_{[a]}(x_1)} a = \bigvee \emptyset = \perp_M$.

(LMD3) It suffices to show that $d(A \vee B)(x_\lambda) = d(A)(x_\lambda) \vee d(B)(x_\lambda)$. On one hand, if $A \leq B$, then $d(A)(x_\lambda) = \bigvee_{x_\lambda \leq d_{[a]}(A)} a \leq \bigvee_{x_\lambda \leq d_{[a]}(B)} a = d(B)(x_\lambda)$, since $d_{[a]}(\cdot)$ is an order-preserving mapping. Hence $d(A \vee B)(x_\lambda) \geq d(A)(x_\lambda) \vee d(B)(x_\lambda)$. On the other hand, suppose that $d(A \vee B)(x_\lambda) = \bigvee \{a \in J(M) \mid x_\lambda \leq d_{[a]}(A \vee B)\}$. Take any $a \in J(M)$ with $x_\lambda \leq d_{[a]}(A \vee B)$. By (LD3), we have $x_\lambda \leq d_{[a]}(A) \vee d_{[a]}(B)$. This implies $x_\lambda \leq d_{[a]}(A)$ or $x_\lambda \leq d_{[a]}(B)$. So $a \leq d(A)(x_\lambda)$ or $a \leq d(B)(x_\lambda)$, which means $a \leq d(A)(x_\lambda) \vee d(B)(x_\lambda)$. Hence $d(A \vee B)(x_\lambda) \leq d(A)(x_\lambda) \vee d(B)(x_\lambda)$. Therefore $d(A \vee B)(x_\lambda) = d(A)(x_\lambda) \vee d(B)(x_\lambda)$.

(LMD4)[†] It suffices to prove that $d(A)(x_\lambda) = \bigwedge_{\mu < \lambda} d(A)(x_\mu)$. On one hand, $d(A)(x_\lambda) \leq \bigwedge_{\mu < \lambda} d(A)(x_\mu)$ is trivial. On the other hand, firstly, we prove the following conclusion:

$$\forall b \in J(M), b \leq d(A)(x_\lambda) \Leftrightarrow x_\lambda \leq d_{[b]}(A).$$

In fact, if $b \leq d(A)(x_\lambda) = \bigvee_{x_\lambda \leq d_{[a]}(A)} a$, then for any $t \in \beta^*(b)$, there exists some $a \in J(M)$ such that $x_\lambda \leq d_{[a]}(A)$ and $t \prec a$. By Lemma 3.10, we have $d_{[a]}(A) \leq d_{[t]}(A)$ and $x_\lambda \leq \bigwedge_{t \prec b} d_{[t]}(A) = d_{[b]}(A)$. On the other hand, it is obvious.

Then we shall check $\bigwedge_{\mu \prec \lambda} d(A)(x_\mu) \leq d(A)(x_\lambda)$. Suppose that $a \in J(M)$ with $a \not\leq d(A)(x_\lambda)$. Then $x_\lambda \not\leq d_{[a]}(A)$. By $\lambda = \bigvee_{\mu \prec \lambda} \mu$, we know that there exists some $\mu \prec \lambda$ such that $x_\mu \not\leq d_{[a]}(A)$. Then $a \not\leq d(A)(x_\mu)$. Hence $a \not\leq \bigwedge_{\mu \prec \lambda} d(A)(x_\mu)$. By the arbitrariness of a , we obtain $\bigwedge_{\mu \prec \lambda} d(A)(x_\mu) \leq d(A)(x_\lambda)$.

Therefore $d(A)(x_\lambda) = \bigwedge_{\mu \prec \lambda} d(A)(x_\mu)$.

(LMD4)* Since $d_{[a]} = \bigvee d(A)_{[a]}$ and (LD4), $\bigvee d(\bigvee d(A)_{[a]})_{[a]} \leq A \vee (\bigvee d(A)_{[a]})$ holds clearly.

(1) \Rightarrow (3) Similarly.

(3) \Rightarrow (1) Define $d : L^X \rightarrow M^{J(L^X)}$ by

$$d(A)(x_\lambda) = \bigwedge \{a \in a \in \alpha(\perp_M) \mid x_\lambda \not\leq d^{[a]}(A)\}.$$

The other proofs are similar to (2) \Rightarrow (1). \square

Now we consider that a family of stratified L -derived operators forms a stratified (L, M) -fuzzy derived operator.

Theorem 3.12. *Let $\{d(a) : L^X \rightarrow L^X \mid a \in J(M)\}$ be a family of stratified L -derived operators on X satisfying $d(a)(A) = \bigwedge_{b \in \beta^*(a)} d(b)(A)$ for each $A \in L^X$.*

Then there exists a stratified (L, M) -fuzzy derived operator $d : L^X \rightarrow M^{J(L^X)}$ such that $d_{[a]} = d(a)$.

Proof. In a similar way of Theorem 3.11, we can prove that d defined by $d(A)(x_\lambda) = \bigvee \{a \in J(M) \mid x_\lambda \leq d(a)(A)\}$ is a stratified (L, M) -fuzzy derived operator and obtain the following conclusion:

$$a \leq d(A)(x_\lambda) \Leftrightarrow x_\lambda \leq d(a)(A).$$

By (LMD4), we have $x_\lambda \leq \bigvee d(A)_{[a]}$ if and only if $d(A)(x_\lambda) \geq a$. It follows from $d_{[a]}(A) = \bigvee d(A)_{[a]}$ that

$$x_\lambda \leq d(a)(A) \Leftrightarrow x_\lambda \leq d_{[a]}(A).$$

Hence $d(a)(A) = d_{[a]}(A)$ for any $A \in L^X$. Therefore $d_{[a]} = d(a)$. \square

Analogous to Theorem 3.12, we can obtain the following theorem.

Theorem 3.13. *Let $\{d(a) : L^X \rightarrow L^X \mid a \in \alpha(\perp)\}$ be a family of stratified L -derived operators on X satisfying $d(a)(A) = \bigvee_{b \in \alpha(a)} d(b)(A)$ for each $A \in L^X$. If $\alpha(a \vee b) = \alpha(a) \cap \alpha(b)$ for any $a, b \in M$, then there exists a stratified (L, M) -fuzzy derived operator $d : L^X \rightarrow M^{J(L^X)}$ such that $d^{[a]} = d(a)$.*

4. Categories Isomorphic to LM -SDS

In this section, we will show that the category LM -STop, the category LM -SCS and the category LM -SDS are all isomorphic to the category LM -SDS.

Firstly, we shall show that the category $LM\text{-STop}$ is isomorphic to the category $LM\text{-SDS}$. By Definition 3.1 and Theorem 3.4, we know that a stratified (L, M) -fuzzy topology can induce a stratified (L, M) -fuzzy derived operator. The following theorem shows the converse.

Theorem 4.1. *Let (X, d) be a stratified (L, M) -fuzzy derived space. Define $\mathcal{T}^d : L^X \rightarrow M$ by $\forall A \in L^X$,*

$$\mathcal{T}^d(A) = \bigwedge_{x_\lambda \not\leq A'} (d(A')(x_\lambda))'.$$

Then \mathcal{T}^d is a stratified (L, M) -fuzzy topology on X .

Proof. We need to check the axioms (LMT1)*, (LMT2) and (LMT3).

(LMT1)* By (LMD1)*, we know $\mathcal{T}^d(\underline{s}) = \bigwedge_{x_\lambda \not\leq \underline{s}'} (d(\underline{s}')(x_\lambda))' = \top_M$.

(LMT2) For all $A, B \in L^X$, we have

$$\begin{aligned} \mathcal{T}^d(A) \wedge \mathcal{T}^d(B) &= \left(\bigwedge_{x_\lambda \not\leq A'} (d(A')(x_\lambda))' \right) \wedge \left(\bigwedge_{y_\mu \not\leq B'} (d(B')(y_\mu))' \right) \\ &= \bigwedge_{x_\lambda \not\leq A', y_\mu \not\leq B'} \left((d(A')(x_\lambda))' \wedge (d(B')(y_\mu))' \right) \\ &\leq \bigwedge_{x_\lambda \not\leq A' \vee B'} (d(A')(x) \vee d(B')(x))' \\ &= \bigwedge_{x_\lambda \not\leq A' \vee B'} (d(A' \vee B')(x))' \quad (\text{by LMD3}) \\ &= \bigwedge_{x_\lambda \not\leq (A \wedge B)'} (d((A \wedge B)')(x_\lambda))' = \mathcal{T}^d(A \wedge B). \end{aligned}$$

(LMT3) By (LMD3), it is easy to see that d is an order-preserving mapping. For all $\{A_j \mid j \in J\} \subseteq L^X$, we have

$$\begin{aligned} \mathcal{T}^d \left(\bigvee_{j \in J} A_j \right) &= \bigwedge_{x_\lambda \not\leq (\bigvee_{j \in J} A_j)'} d \left(\left(\bigvee_{j \in J} A_j \right)' (x_\lambda) \right)' \\ &= \bigwedge_{j \in J} \bigwedge_{x_\lambda \not\leq A_j'} d \left(\bigwedge_{j \in J} A_j'(x_\lambda) \right)' \\ &\geq \bigwedge_{j \in J} \bigwedge_{x_\lambda \not\leq A_j'} d(A_j'(x_\lambda))' = \bigwedge_{j \in J} \mathcal{T}^d(A_j). \end{aligned}$$

In what follows, the relationships between the continuous mappings of (L, M) -fuzzy derived spaces and the continuous mappings of (L, M) -fuzzy topological spaces are discussed. \square

Theorem 4.2. *If $f : X \rightarrow Y$ is continuous between (L, M) -fuzzy derived spaces (X, d_X) and (Y, d_Y) , then f is continuous with respect to (L, M) -fuzzy topologies \mathcal{T}^{d_X} and \mathcal{T}^{d_Y} .*

Proof. In order to prove that $\mathcal{T}^{d_Y}(B) \leq \mathcal{T}^{d_X}(f_L^{\leftarrow}(B))$ for all $B \in L^Y$, we need to check $\bigwedge_{y_\mu \not\leq B'} (d_Y(B')(y_\mu))' \leq \bigwedge_{x_\lambda \not\leq f_L^{\leftarrow}(B')} (d_X(f_L^{\leftarrow}(B'))(x_\lambda))'$. It suffices to check $\bigvee_{x_\lambda \not\leq f_L^{\leftarrow}(B')} d_X(f_L^{\leftarrow}(B'))(x_\lambda) \leq \bigvee_{y_\mu \not\leq B'} d_Y(B')(y_\mu)$. Let $a \in M$ with $a \prec \bigvee_{x_\lambda \not\leq f_L^{\leftarrow}(B')} d_X(f_L^{\leftarrow}(B'))(x_\lambda)$. Then there exists some $x_\lambda \in J(L^X)$ such that $x_\lambda \not\leq f_L^{\leftarrow}(B')$ and $a \leq d_X(f_L^{\leftarrow}(B'))(x_\lambda)$. Note that $x_\lambda \not\leq f_L^{\leftarrow}(B')$ if and only if $f(x)_\lambda \not\leq B'$ and $f : (X, d_X) \rightarrow (Y, d_Y)$ is continuous. Then $a \leq d_X(f_L^{\leftarrow}(B'))(x_\lambda) \leq f_L^{\rightarrow}(f_L^{\leftarrow}(B'))(f(x)_\lambda) \vee d_Y(f_L^{\rightarrow}(f_L^{\leftarrow}(B'))(f(x)_\lambda)) \leq B'(f(x)_\lambda) \vee d_Y(B')(f(x)_\lambda) = d_Y(B')(f(x)_\lambda)$. This implies $a \leq \bigvee_{y_\mu \not\leq B'} d_Y(B')(y_\mu)$. By the arbitrariness of a , we obtain $\bigvee_{x_\lambda \not\leq f_L^{\leftarrow}(B')} d_X(f_L^{\leftarrow}(B'))(x_\lambda) \leq \bigvee_{y_\mu \not\leq B'} d_Y(B')(y_\mu)$. Therefore $f : (X, \mathcal{T}^{d_X}) \rightarrow (Y, \mathcal{T}^{d_Y})$ is continuous. \square

Theorem 4.3. *If $f : X \rightarrow Y$ is continuous between (L, M) -fuzzy topological spaces (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) , then f is continuous with respect to (L, M) -fuzzy derived operators $d^{\mathcal{T}_X}$ and $d^{\mathcal{T}_Y}$.*

Proof. In order to prove $d^{\mathcal{T}_X}(A)(x_\lambda) \leq f_L^{\rightarrow}(A)(f(x)_\lambda) \vee d^{\mathcal{T}_Y}(f_L^{\rightarrow}(A))(f(x)_\lambda)$, we only need to check that $d^{\mathcal{T}_X}(A)(x_\lambda) \leq d^{\mathcal{T}_Y}(f_L^{\rightarrow}(A))(f(x)_\lambda)$ whenever $f(x)_\lambda \not\leq f_L^{\rightarrow}(A)$. It suffices to prove that $\bigvee_{f(x)_\lambda \not\leq D \geq f_L^{\rightarrow}(A) - f(x)_1} \mathcal{T}_Y(D') \leq \bigvee_{x_\lambda \not\leq B \geq A - x_1} \mathcal{T}_X(B')$ whenever $f(x)_\lambda \not\leq f_L^{\rightarrow}(A)$. By Lemma 3.3, we have $\bigvee_{f(x)_\lambda \not\leq D \geq f_L^{\rightarrow}(A) - f(x)_1} \mathcal{T}_Y(D') = \bigvee_{f(x)_\lambda \not\leq D \geq f_L^{\rightarrow}(A)} \mathcal{T}_Y(D')$ whenever $f(x)_\lambda \not\leq f_L^{\rightarrow}(A)$. Let $a \in M$ such that $a \prec \bigvee_{f(x)_\lambda \not\leq D \geq f_L^{\rightarrow}(A)} \mathcal{T}_Y(D')$. Then there exists some $D \in L^Y$ such that $f(x)_\lambda \not\leq D \geq f_L^{\rightarrow}(A)$ and $a \prec \mathcal{T}_Y(D')$. This implies $x_\lambda \not\leq f_L^{\leftarrow}(D) \geq f_L^{\leftarrow}(f_L^{\rightarrow}(A)) \geq A \geq A - x_1$. By the continuity of $f : (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$, we have $a \prec \mathcal{T}_Y(D') \leq \mathcal{T}_X(f_L^{\leftarrow}(D')) = \mathcal{T}_X(f^{\leftarrow}(D)')$. Hence $a \leq \bigvee_{x_\lambda \not\leq B \geq A - x_1} \mathcal{T}_X(B')$. By the arbitrariness of a , we obtain $\bigvee_{f(x)_\lambda \not\leq D \geq f_L^{\rightarrow}(A) - f(x)_1} \mathcal{T}_Y(D') \leq \bigvee_{x_\lambda \not\leq B \geq A - x_1} \mathcal{T}_X(B')$ whenever $f(x)_\lambda \not\leq f_L^{\rightarrow}(A)$. Therefore $f : (X, d^{\mathcal{T}_X}) \rightarrow (Y, d^{\mathcal{T}_Y})$ is continuous. \square

The following theorem shows that there exists a one-to-one correspondence between stratified (L, M) -fuzzy derived operators and stratified (L, M) -fuzzy topologies.

Theorem 4.4. *If \mathcal{T} is a stratified (L, M) -fuzzy topology on X and $d : L^X \rightarrow M^{J(L^X)}$ is a stratified (L, M) -fuzzy derived operator on X , then $\mathcal{T}^{d^{\mathcal{T}}} = \mathcal{T}$ and $d^{\mathcal{T}^d} = d$.*

Proof. Firstly, by Lemma 3.3, $\mathcal{T}^{d^{\mathcal{T}}} = \mathcal{T}$ follows from

$$\begin{aligned} \mathcal{T}^{d^{\mathcal{T}}}(A) &= \bigwedge_{x_\lambda \not\leq A'} (d^{\mathcal{T}}(A')(x_\lambda))' \\ &= \bigwedge_{x_\lambda \not\leq A'} \bigvee_{x_\lambda \not\leq B \geq A' - x_1} \mathcal{T}(B') \\ &= \mathcal{T}(A'') \\ &= \mathcal{T}(A). \end{aligned}$$

Secondly, by (LMD4), $d^{\mathcal{T}^d} = d$ follows from

$$\begin{aligned} d^{\mathcal{T}^d}(A)(x_\lambda) &= \bigwedge_{x_\lambda \not\leq B \geq A - x_1} (\mathcal{T}^d(B'))' \\ &= \bigwedge_{x_\lambda \not\leq B \geq A - x_1} \bigvee_{y_\mu \not\leq B} d(B)(y_\mu) \\ &= d(A)(x_\lambda). \end{aligned}$$

By Definition 3.1, Theorem 3.4 and Theorem 4.1-4.4, we can obtain the following \square theorem.

Theorem 4.5. *The category $LM\text{-STop}$ is isomorphic to the category $LM\text{-SDS}$. There is a functor $\mathbb{F}: LM\text{-STop} \rightarrow LM\text{-SDS}$ such that*

$$\begin{cases} \mathbb{F}((X, \mathcal{T})) = (X, d^{\mathcal{T}}) & (X, \mathcal{T}) \in ob(LM\text{-STop}); \\ \mathbb{F}(f) = f & f \in mor(LM\text{-STop}). \end{cases}$$

and a functor $\mathbb{G}: LM\text{-SDS} \rightarrow LM\text{-STop}$ such that

$$\begin{cases} \mathbb{G}((X, d)) = (X, \mathcal{T}^d) & (X, d) \in ob(LM\text{-SDS}); \\ \mathbb{G}(g) = g & g \in mor(LM\text{-SDS}). \end{cases}$$

Secondly, we shall show that the category $LM\text{-SCS}$ is isomorphic to the category $LM\text{-SDS}$. The following theorem was presented for an (L, L) -fuzzy topological space. It can easily be transformed to an (L, M) -fuzzy topological space as follows.

Theorem 4.6. [14, 15] *Let (X, \mathcal{T}) be an (L, M) -fuzzy topological space and let (X, Cl) be an (L, M) -fuzzy closure operator on X .*

(1) *Define a mapping $Cl^{\mathcal{T}}: L^X \rightarrow M^{J(L^X)}$ by $\forall A \in L^X, \forall x_\lambda \in J(L^X)$,*

$$Cl^{\mathcal{T}}(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq A} (\mathcal{T}(B'))'.$$

Then $Cl^{\mathcal{T}}$ is an (L, M) -fuzzy closure operator on X .

(2) *Define a mapping $\mathcal{T}^{Cl}: L^X \rightarrow M$ by $\forall A \in L^X$,*

$$\mathcal{T}^{Cl}(A) = \bigwedge_{x_\lambda \not\leq A'} (Cl(A')(x_\lambda))'.$$

Then \mathcal{T}^{Cl} is an (L, M) -fuzzy topology on X .

Through the (L, M) -fuzzy topology as a bond, we get two mappings from Definition 3.1, Theorem 4.1 and Theorem 4.6. One is $d^{\mathcal{T}^{Cl}}: L^X \rightarrow M^{J(L^X)}$ defined by

$$d^{\mathcal{T}^{Cl}}(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq A - x_1} (\mathcal{T}^{Cl}(B'))' = \bigwedge_{x_\lambda \not\leq B \geq A - x_1} \bigvee_{y_\mu \not\leq B} Cl(B)(y_\mu).$$

The other is $Cl^{\mathcal{T}^d}: L^X \rightarrow M^{J(L^X)}$ defined by

$$Cl^{\mathcal{T}^d}(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq A} (\mathcal{T}^d(B'))' = \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq B} d(B)(y_\mu).$$

In order to simplify the formula of $Cl^{\mathcal{T}^d}(A)(x_\lambda)$, we need the following lemma.

Lemma 4.7. *Let $d : L^X \longrightarrow M^{J(L^X)}$ be a stratified (L, M) -fuzzy derived operator. Then $\forall x_\lambda \not\leq A$,*

$$\bigwedge_{x_\lambda \not\leq B \geq A - x_1} \bigvee_{y_\mu \not\leq B} d(B)(y_\mu) = \bigwedge_{x_\lambda \not\leq B \geq A} \bigvee_{y_\mu \not\leq B} d(B)(y_\mu).$$

Proof. Let *LHS* and *RHS* denote the left side of the equality and the right side of the equality, respectively. On one hand, $LHS \leq RHS$ is true trivially. On the other hand, suppose that $a \in M$ with $a \not\leq LHS$. Then there exists some B such that $x_\lambda \not\leq B \geq A - x_1$ and $a \not\leq \bigvee_{y_\mu \not\leq B} d(B)(y_\mu)$. Let $\tilde{B} = B \vee \underline{A(x)}$. For all $x_\lambda \not\leq A$, we have $x_\lambda \not\leq \tilde{B} \geq (A - x_1) \vee \underline{A(x)} \geq A$. Note that $y_\mu \not\leq \tilde{B}$ implies $y_\mu \not\leq B$ and $y_\mu \not\leq \underline{A(x)}$. Thus $d(\tilde{B})(y_\mu) = d(B)(y_\mu) \vee d(\underline{A(x)})(y_\mu) = d(B)(y_\mu)$. It follows that $\bigvee_{y_\mu \not\leq \tilde{B}} d(\tilde{B})(y_\mu) = \bigvee_{y_\mu \not\leq \tilde{B}} d(B)(y_\mu) \leq \bigvee_{y_\mu \not\leq B} d(B)(y_\mu)$. Hence $a \not\leq \bigvee_{y_\mu \not\leq \tilde{B}} d(\tilde{B})(y_\mu)$, which means $a \not\leq RHS$. By the arbitrariness of a , we obtain $RHS \leq LHS$. Therefore $RHS = LHS$. \square

By (LMC4), we have $d^{\mathcal{T}^{Cl}}(A)(x_\lambda) = Cl(A - x_1)(x_\lambda)$. By (LMD4) and Lemma 4.7, we know $Cl^{\mathcal{T}^d}(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq A - x_1} \bigvee_{y_\mu \not\leq B} d(B)(y_\mu) = d(A)(x_\lambda)$ whenever $x_\lambda \not\leq A$.

Therefore, the aforesaid mappings are actually $d^{Cl} : L^X \longrightarrow M^{J(L^X)}$ defined by $\forall A \in L^X, \forall x_\lambda \in J(L^X)$,

$$d^{Cl}(A)(x_\lambda) = Cl(A - x_1)(x_\lambda);$$

and $Cl^d : L^X \longrightarrow M^{J(L^X)}$ defined by $\forall A \in L^X, \forall x_\lambda \in J(L^X)$,

$$Cl^d(A)(x_\lambda) = \begin{cases} \top_M, & x_\lambda \leq A; \\ d(A)(x_\lambda), & x_\lambda \not\leq A. \end{cases}$$

Remark 4.8. Let $M = 2$ (namely, \mathcal{T} is a stratified L -topology). If $d^{Cl}(A)(x_\lambda) = \top_M$, then $x_\lambda \leq d^{Cl}(A)$ if and only if $x_\lambda \leq Cl(A - x_1)$. If $Cl^d(A)(x_\lambda) = \top_M$, then $Cl^d(A) = A \vee d(A)$. These are exactly the conclusions of [12] in L -topological spaces.

In [14, 15], F.G. Shi and B. Pang proved that the category of (L, L) -fuzzy topological spaces and their continuous mappings is isomorphic to the category of (L, L) -fuzzy closure spaces and their continuous mappings. It can easily be extended to that the category **LM-STop** is isomorphic to the category **LM-SCS**. By Theorem 4.5, we get the following theorem.

Theorem 4.9. *The category **LM-SCS** is isomorphic to the category **LM-SDS**. There is a functor $\mathbb{F} : LM\text{-SCS} \longrightarrow LM\text{-SDS}$ such that*

$$\begin{cases} \mathbb{F}((X, Cl)) = (X, d^{Cl}) & (X, Cl) \in ob(LM\text{-SCS}); \\ \mathbb{F}(f) = f & f \in mor(LM\text{-SCS}). \end{cases}$$

and a functor $\mathbb{G}: LM\text{-SDS} \longrightarrow LM\text{-SCS}$ such that

$$\begin{cases} \mathbb{G}((X, d)) = (X, Cl^d) & (X, d) \in ob(LM\text{-SDS}); \\ \mathbb{G}(g) = g & g \in mor(LM\text{-SDS}). \end{cases}$$

Finally, we shall show that the category $LM\text{-SQS}$ is isomorphic to the category $LM\text{-SDS}$. The following theorem was presented for an (L, L) -fuzzy topological space. It can easily be transformed to an (L, M) -fuzzy topological space as follows.

Theorem 4.10. [5] *Let (X, \mathcal{T}) be an (L, M) -fuzzy topological space and let (X, \mathcal{Q}) be an (L, M) -fuzzy quasi-neighborhood space.*

(1) *Define a set $\mathcal{Q}^{\mathcal{T}} = \{\mathcal{Q}_{x_\lambda}^{\mathcal{T}} \mid x_\lambda \in J(L^X)\}$ of mappings $\mathcal{Q}_{x_\lambda}^{\mathcal{T}} : L^X \longrightarrow M$ by $\forall A \in L^X$,*

$$\mathcal{Q}_{x_\lambda}^{\mathcal{T}}(A) = \begin{cases} \bigvee_{x_\lambda \not\leq B' \geq A'} \mathcal{T}(B), & x_\lambda \not\leq A'; \\ \perp_M, & x_\lambda \leq A'. \end{cases}$$

Then the set $\mathcal{Q}^{\mathcal{T}}$ is an (L, M) -fuzzy quasi-neighborhood system on X .

(2) *Define a mapping $\mathcal{T}^{\mathcal{Q}} : L^X \longrightarrow M$ by $\forall A \in L^X$,*

$$\mathcal{T}^{\mathcal{Q}}(A) = \bigwedge_{x_\lambda \not\leq A'} \mathcal{Q}_{x_\lambda}(A).$$

Then $\mathcal{T}^{\mathcal{Q}}$ is an (L, M) -fuzzy topology on X .

Through the (L, M) -fuzzy topology as a bond, we get a mapping and a set from Definition 3.1, Theorem 4.1 and Theorem 4.10. The mapping is $d^{\mathcal{T}^{\mathcal{Q}}} : L^X \longrightarrow M^{J(L^X)}$ defined by

$$d^{\mathcal{T}^{\mathcal{Q}}}(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq B \geq A - x_1} (\mathcal{T}^{\mathcal{Q}}(B'))' = \bigwedge_{x_\lambda \not\leq B \geq A - x_1} \bigvee_{y_\mu \not\leq B''} \mathcal{Q}_{y_\mu}(B').$$

The set is $\mathcal{Q}^{\mathcal{T}^d} = \{\mathcal{Q}_{x_\lambda}^{\mathcal{T}^d} \mid x_\lambda \in J(L^X)\}$ of mappings $\mathcal{Q}_{x_\lambda}^{\mathcal{T}^d} : L^X \longrightarrow M$ defined by

$$\mathcal{Q}_{x_\lambda}^{\mathcal{T}^d}(A) = \begin{cases} \bigvee_{x_\lambda \not\leq B' \geq A'} \mathcal{T}^d(B) = \bigvee_{x_\lambda \not\leq B' \geq A'} \bigwedge_{y_\mu \not\leq B'} (d(B')(y_\mu))', & x_\lambda \not\leq A'; \\ \perp_M, & x_\lambda \leq A'. \end{cases}$$

By (LMQ4), we know $d^{\mathcal{T}^{\mathcal{Q}}}(A)(x_\lambda) = (\mathcal{Q}_{x_\lambda}(A - x_1))'$. By (LMD4) and Lemma 4.7, we have $\mathcal{Q}_{x_\lambda}^{\mathcal{T}^d}(A) = \left(\bigwedge_{x_\lambda \not\leq B' \geq A' - x_1} \bigvee_{y_\mu \not\leq B'} d(B')(y_\mu) \right)' = (d(A')(x_\lambda))'$ whenever $x_\lambda \not\leq A'$.

Therefore, the aforesaid mapping is actually $d^{\mathcal{Q}} : L^X \longrightarrow M^{J(L^X)}$ defined by $\forall A \in L^X, \forall x_\lambda \in J(L^X)$,

$$d^{\mathcal{Q}}(A)(x_\lambda) = (\mathcal{Q}_{x_\lambda}(A - x_1))';$$

and the aforesaid set is actually $\mathcal{Q}^d = \{\mathcal{Q}_{x_\lambda}^d \mid x_\lambda \in J(L^X)\}$ of mappings $\mathcal{Q}_{x_\lambda}^d : L^X \longrightarrow M$ defined by $\forall A \in L^X$,

$$\mathcal{Q}_{x_\lambda}^d(A) = \begin{cases} (d(A')(x_\lambda))', & x_\lambda \not\leq A'; \\ \perp_M, & x_\lambda \leq A'. \end{cases}$$

In [5], J.M. Fang proved that the category of (L, L) -fuzzy topological spaces and their continuous mappings is isomorphic to the category of (L, L) -fuzzy quasi-neighborhood spaces and their continuous mappings. It can easily be extended to that the category $LM\text{-STop}$ is isomorphic to the category $LM\text{-SQS}$. By Theorem 4.5, we get the following theorem.

Theorem 4.11. *The category $LM\text{-SQS}$ is isomorphic to the category $LM\text{-SDS}$. There is a functor $\mathbb{F}: LM\text{-SQS} \rightarrow LM\text{-SDS}$ such that*

$$\begin{cases} \mathbb{F}((X, \mathcal{Q})) = (X, d^{\mathcal{Q}}) & (X, \mathcal{Q}) \in ob(LM\text{-SQS}); \\ \mathbb{F}(f) = f & f \in mor(LM\text{-SQS}). \end{cases}$$

and a functor $\mathbb{G}: LM\text{-SDS} \rightarrow LM\text{-SQS}$ such that

$$\begin{cases} \mathbb{G}((X, d)) = (X, \mathcal{Q}^d) & (X, d) \in ob(LM\text{-SDS}); \\ \mathbb{G}(g) = g & g \in mor(LM\text{-SDS}). \end{cases}$$

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