STRATIFIED (L, M)-SEMIUNIFORM CONVERGENCE TOWER SPACES

J. LI

Abstract. The notion of stratified (L, M)-semiuniform convergence tower spaces is introduced, which extends the notions of probabilistic semiuniform convergence spaces and lattice-valued semiuniform convergence spaces. The resulting category is shown to be a strong topological universe. Besides, the relations between our category and that of stratified (L, M)-filter tower spaces are studied.

1. Introduction

In the theory of topological spaces, "uniform concepts" such as Cauchy filter, completeness , uniform continuous etc, cannot be described. Uniform spaces were introduced by Weil [26] in 1937 for describing "uniform concepts". But the category of uniform spaces and uniformly continuous mappings is not Cartesian closed. Cook and Fisher [3] generalized uniform spaces to uniform limit spaces, which was slightly modified by Wyler [27] in 1974. The resulting category is Cartesian closed. By omitting some axioms of uniform limit spaces, Preuss [23, 24] established the category of semiuniform convergence spaces and uniformly continuous mappings. The category of semiuniform convergence spaces is a strong topological universe including uniform spaces and topological spaces, and it is possible to study both topological and uniform aspects within this framework.

For the lattice-valued case, lattice-valued uniform convergence spaces are introduced in [19]. The category of lattice-valued uniform convergence spaces is a Cartesian closed supercategory of the category of lattice-valued uniform spaces [10]. In [4], the lattice context of these spaces was generalized from complete Heyting algebras to the case of enriched lattices. By making use of the lattice-valued inclusion order of stratified L-filters, Fang [6] proposed the concept of stratified L-ordered quasi-uniform limit structure. Fang [5] extended semiuniform convergence spaces to the lattice-valued case by relaxing the axioms of lattice-valued uniform convergence spaces, which was called stratified L-semiuniform convergence spaces. The category of stratified L-semiuniform convergence spaces is a strong topological universe when L is a completely distributive lattice.

In 1971, Frank [9] introduced the notion of a probabilistic topological space by using "θ-closure". For categorical consederation, some subsequent models were

Received: June 2005; Revised: July 2006; Accepted: November 2006
Key words and phrases: Stratified L-filters, Stratified (L, M)-semiuniform convergence tower spaces, Lattice-valued semiuniform convergence spaces, Strong topological universe.

combinations of probabilistic ideas with notions of convergence spaces, filter spaces, and uniform convergence spaces [8, 9, 11, 20, 22, 25]. These probabilistic spaces were subsequently extended. Yang and Li [28] introduced stratified (L, M)-filter tower spaces, which extended the notions of probabilistic filter spaces and stratified (L, M)-filter spaces. Flores, Mohapatra and Richardson [7] proposed an alternative set of axioms for the study of lattice-valued convergence spaces [17, 18], which extended the notion of probabilistic convergence spaces.

This paper starts from this kind of extension idea and proposes the notion of stratified (L, M)-semiuniform convergence tower spaces as a kind of extension of probabilistic semiuniform convergence spaces and stratified L-semiuniform convergence spaces. The category of stratified (L, M)-semiuniform convergence tower spaces and uniformly continuous mappings is shown to be a strong topological universe. Moreover, the relations between stratified (L, M)-semiuniform convergence spaces and stratified (L, M)-filter tower spaces are studied.

This paper is organized as follows. In Section 2, we give the necessary lattice-theoretic backgrounds, the notations and results about stratified L-filters as well as some concepts related to categorical theory. In Section 3, we show that the category of stratified (L, M)-semiuniform convergence tower spaces is a well-fibred topological category, and establish the relations between stratified L-semiuniform convergence spaces and stratified (L, L)-semiuniform convergence tower spaces. In Section 4, we show that the category of stratified (L, M)-semiuniform convergence tower spaces is Cartesian closed. Section 5 presents the extensionality of the category of stratified (L, M)-semiuniform convergence tower spaces. In Section 6, we show that the category of stratified (L, M)-semiuniform convergence tower spaces is closed under the formation of products of quotient mappings. Finally, we conclude that the category of stratified (L, M)-semiuniform convergence tower spaces is a strong topological universe. In Section 7, we investigate the relations between stratified (L, M)-semiuniform convergence tower spaces and stratified (L, M)-filter tower spaces.

2. **Preliminaries**

Throughout this paper, L(resp.,M) denotes a complete Heyting algebra, i.e., a complete lattice equipped with an implication $\rightarrow: L \times L \rightarrow L$ such that $a \wedge b \leqslant c \Leftrightarrow a \leqslant b \rightarrow c$ for all $a, b, c \in L$. The smallest element and the largest element in L (resp., M)are denoted by 0 and 1 respectively. For a nonempty set X, L^X denotes the set of all L-subsets on X. L^X is also a complete Heyting algebra, when it inherits the structure of the lattice L in a natural way. The smallest element and the largest element in L^X are denoted by 0_X and 1_X respectively. For each $a \in L$, we define the L-subset a_X by $a_X(x) = a$ for all $x \in X$.

Definition 2.1. [12, 17] A mapping $\mathcal{F} : L^X \longrightarrow L$ is called a stratified L-filter on X if it satisfies:

(LF1) $\mathcal{F}(0_X) = 0$, $\mathcal{F}(1_X) = 1$.
(LF2) $A \leqslant B \Longrightarrow \mathcal{F}(A) \leqslant \mathcal{F}(B)$.
(LF3) $\mathcal{F}(A) \wedge \mathcal{F}(B) \leqslant \mathcal{F}(A \wedge B)$.

(LF4) $a \wedge \mathcal{F}(A) \leqslant \mathcal{F}(a_X \wedge A)$.

The family of all stratified L-filters on X is denoted by $\mathcal{F}_L^s(X)$. On the set $\mathcal{F}_L^s(X)$, we define an order by $\mathcal{F} \leqslant \mathcal{G}$ if $\mathcal{F}(A) \leqslant \mathcal{G}(A)$ for all $A \in L^X$. Every nonempty family $\{\mathcal{F}_i\}_{i\in I}$ of stratified L-filters has an infimum $\bigwedge_{i\in I} \mathcal{F}_i$, which can be calculated as $\forall A \in L^X$, $(\bigwedge\limits_{i\in I} \mathcal{F}_i)(A) = \bigwedge\limits_{i\in I} \mathcal{F}_i(A)$. Let $f : X \longrightarrow Y$ be a mapping. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$, the mapping $f^{\Rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$, $A \mapsto \mathcal{F}(f^{\leftarrow}(A))$ is a stratified L-filter on Y and is called the image of $\mathcal{F}$ under f in [12]. For each $\mathcal{G} \in \mathcal{F}_L^s(Y)$, Jäger [17] proved that the mapping $f^{\Leftarrow}(\mathcal{G}) : L^X \longrightarrow L$ defined by $f^{\Leftarrow}(\mathcal{G}) = \bigvee\limits_{f^{\leftarrow}(B)\leqslant A} \mathcal{G}(B)$ is a stratified L-filter on X if and only if $f^{\leftarrow}(B) = 0_X$ implies $\mathcal{G}(B) = 0$. For a family $\{\mathcal{F}_i\}_{i\in I}$, $\forall i \in I$, $\mathcal{F}_i \in \mathcal{F}_L^s(X_i)$, Jäger [17] proposed that the product $\prod_i \mathcal{F}_i$ is defined by $\prod_i \mathcal{F}_i = \bigvee_i (Pr_{X_i})^{\Leftarrow}(\mathcal{F}_i)$. In particular, for $\mathcal{F} \in \mathcal{F}_L^s(X_1)$ and $\mathcal{G} \in \mathcal{F}_L^s(X_2)$, we have $(\mathcal{F} \times \mathcal{G})(A) = \bigvee\limits_{A_1\times A_2\leqslant A} \mathcal{F}(A_1) \wedge \mathcal{G}(A_2)$.

Example 2.2. [12, 17] For each point $x \in X$, the mapping $[x] : L^X \longrightarrow L$, $A \mapsto A(x)$ is a stratified L-filter on X, called the point L-filter of x.

For a stratified L-filter $\mathcal{F}$ on $X \times X$, a stratified L-filter $\mathcal{F}^{-1}$ [19] was defined by $\mathcal{F}^{-1}(A) = \mathcal{F}(A^{-1})$ for each $A \in L^{X\times X}$, where $A^{-1}(x, y) = A(y, x)$ for all $(x, y) \in X \times X$.

Lemma 2.3. [19] *If $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X \times X)$, then $\mathcal{F} \leqslant \mathcal{G} \Longrightarrow \mathcal{F}^{-1} \leqslant \mathcal{G}^{-1}$.*

Lemma 2.4. [19] *If $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X \times X)$, then $(\mathcal{F} \times \mathcal{G})^{-1} = \mathcal{G} \times \mathcal{F}$.*

For mappings $f : X \longrightarrow Z$ and $g : Y \longrightarrow W$, the product mapping $f \times g : X \times Y \longrightarrow Z \times W$ is defined by $(f \times g)(x, y) = (f(x), g(y))$, $\forall (x, y) \in X \times Y$. Furthermore, $(f \times g)^{\Rightarrow}(\mathcal{F} \times \mathcal{G}) = f^{\Rightarrow}(\mathcal{F}) \times g^{\Rightarrow}(\mathcal{G})$.

Lemma 2.5. [5] *Let $f : X \longrightarrow Y$ be a mapping. Then*

(1) *$((f \times f)^{\Rightarrow}(\mathcal{F}))^{-1} = (f \times f)^{\Rightarrow}(\mathcal{F}^{-1})$ for all $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$.*

(2) *$(f \times f)^{\Leftarrow}(\mathcal{H})$ exists if and only if $(f \times f)^{\Leftarrow}(\mathcal{H}^{-1})$ exists, further $((f \times f)^{\Leftarrow}(\mathcal{H}))^{-1} = (f \times f)^{\Leftarrow}(\mathcal{H}^{-1})$ for all $\mathcal{H} \in \mathcal{F}_L^s(Y \times Y)$.*

Definition 2.6. [1] A category **C** is called a topological category over **Set** provided that for any set X, any class J, any family $((X_j, \xi_j))_{j\in J}$ of **C**-objects and any family $(f_j : X \longrightarrow X_j)_{j\in J}$ of mappings, there exists a unique **C**-structure ξ on X which is initial with respect to the source $(f_j : X \longrightarrow (X_j, \xi_j))_{j\in J}$. This means that for a **C**-object (Y, η), a mapping $g : (Y, \eta) \longrightarrow (X, \xi)$ is a **C**-morphism if and only if for all $j \in J$, $f_j \circ g : (Y, \eta) \longrightarrow (X_j, \xi_j)$ is a **C**-morphism.

Definition 2.7. [1] A subcategory **A** of **B** is said to be reflective in **B** if for each **B**-object B, there exists an **A**-object C and a **B**-morphism $f : B \longrightarrow C$ such that for any **B**-morphism $g : B \longrightarrow A$ from B to an **A**-object A, there exists a unique **A**-morphism $h : C \longrightarrow A$ with $h \circ f = g$.

3. **Stratified (L, M)-semiuniform Convergence Tower Spaces**

In this section, the definition of stratified (L, M)-semiuniform convergence tower spaces is introduced. It is shown that the category of stratified L-semiuniform convergence spaces is isomorphic to a reflective subcategory of our category when $L = M$.

Definition 3.1. Let X be a non-void set and $\bar{\mathcal{T}} = \{\mathcal{T}_\lambda \mid \lambda \in M\}$ a nonempty family of subsets of $\mathcal{F}_L^s(X \times X)$. The pair $(X, \bar{\mathcal{T}})$ is called a stratified (L, M)-semiuniform convergence tower space if it satisfies

(UCT1) For all $x \in X$, $\lambda \in M$, $[x] \times [x] \in \mathcal{T}_\lambda$.

(UCT2) $\mathcal{G} \in \mathcal{T}_\lambda$ whenever $\mathcal{F} \in \mathcal{T}_\lambda$ and $\mathcal{F} \leqslant \mathcal{G}$.

(UCT3) $\mathcal{F}^{-1} \in \mathcal{T}_\lambda$ whenever $\mathcal{F} \in \mathcal{T}_\lambda$.

(P1) $\mathcal{T}_\lambda \leqslant \mathcal{T}_\mu$ whenever $\mu \leqslant \lambda$.

(P2) $\mathcal{T}_0 = \mathcal{F}_L^s(X \times X)$.

The pair $(X, \bar{\mathcal{T}})$ is said to be left continuous if it satisfies $\bigcap_{\nu \in A} \mathcal{T}_\nu = \mathcal{T}_{\vee A}$ for any nonempty set $A \subseteq M$.

A mapping $f : (X, \bar{\mathcal{T}}^X) \longrightarrow (Y, \bar{\mathcal{T}}^Y)$ between two stratified (L, M)-semiuniform convergence tower spaces is called uniformly continuous if $(f \times f)^{\Rightarrow}(\mathcal{F}) \in \mathcal{T}_\lambda^Y$ for all $\mathcal{F} \in \mathcal{T}_\lambda^X$ and for all $\lambda \in M$. The category of stratified (L, M)-semiuniform convergence tower spaces(resp., left continuous stratified (L, M)-semiuniform convergence tower spaces) and uniformly continuous mappings is denoted by $\mathbf{S}(L, M)$-**SUConvTr** (resp., **LC-**$\mathbf{S}(L, M)$-**SUConvTr**).

Remark 3.2. A stratified (L, M)-semiuniform convergence tower space is not necessarily left continuous.

Let $L = M = X = [0, 1]$. Define $\bar{\mathcal{T}} = \{\mathcal{T}_\lambda \mid \lambda \in L\}$ as follows: $\mathcal{T}_1 = \{\mathcal{F} \in \mathcal{F}_L^s(X \times X) \mid$ there exists $x \in X$ such that $[x] \times [x] \leqslant \mathcal{F}\}$, $\lambda \in [0, 0.5)$, $\mathcal{T}_\lambda = \mathcal{F}_L^s(X \times X)$, and $\lambda \in [0.5, 1)$, $\mathcal{T}_\lambda = \mathcal{T}_1$. It is easily checked that $(X, \bar{\mathcal{T}})$ is a stratified (L, M)-semiuniform convergence tower space, but it is not left continuous.

Example 3.3. For $L = \{0, 1\}$ and $M = [0, 1]$ we can identify the notion of $(2, M)$-semiuniform convergence tower spaces with the notion of probabilistic semiuniform convergence spaces [22].

Example 3.4. Let $L = \{0, 1\}$ and $M = [0, \infty]$ with the opposite order. Then a left continuous stratified (L, M)-semiuniform convergence tower space is a semi-approach uniform convergence space in the definition of Nauwelaerts [21].

Example 3.5. For $L = M$ we can identify the notion of left continuous (L, L)-semiuniform convergence tower spaces with the notion of lattice-valued semiuniform convergence spaces [5] (see Theorem 3.14) .

Example 3.6. [13] Let (X, Λ) be a lattice-valued uniform convergence space. Define $\bar{\Lambda} = \{\Lambda_\lambda \mid \lambda \in L\}$, where $\Lambda_\lambda \subseteq \mathcal{F}_L^s(X \times X)$ is called the λ-level structure defined by $\mathcal{F} \in \Lambda_\lambda \Longleftrightarrow \Lambda(\mathcal{F}) \geqslant \lambda$. Then $\bar{\Lambda}$ satisfies (UCT1) – (UCT3), (P1) and (P2). Therefore, the pair $(X, \bar{\Lambda})$ is a stratified (L, L)-semiuniform convergence tower space.

Example 3.7. Let $L = \{0, 1\}$ and $M = \triangle^+$ be the frame of distance distribution functions with the pointwise minimum and supremum. Then a probabilistic uniform convergence space in the definition of Ahsanullah and Jäger[2] is a stratified (L, M)-semiuniform convergence tower space.

Theorem 3.8. *The category* $\mathbf{S}(L, M)$-**SUConvTr** *is a well-fibred category over* **Set**.

Proof. For the fibre-smallness of $\mathbf{S}(L, M)$-**SUConvTr**, it is obvious that the class of all (L, M)-semiuniform convergence tower structures on a set X is a set. For the terminal separator property of $\mathbf{S}(L, M)$-**SUConvTr**, let $X = \{x\}$. Then for each $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$, $\mathcal{F} \geqslant [x] \times [x]$. By (UCT1) and (UCT2), we have $\mathcal{F} \in \mathcal{T}_\lambda \Longleftrightarrow \mathcal{F} \geqslant [x] \times [x]$. Hence, there is a unique stratified (L, M)-semiuniform convergence tower structure on $\{x\}$. □

Theorem 3.9. *The category* $\mathbf{S}(L, M)$-**SUConvTr** *is a topological category over* **Set**.

Proof. Let $f_i : X \longrightarrow (X_i, \bar{\mathcal{T}}_i)$ be a mapping, where $(X_i, \bar{\mathcal{T}}_i)$ is a stratified (L, M)-semiuniform convergence tower space for all $i \in I$. For each $\lambda \in M$, define a set $\mathcal{T}_\lambda \subseteq \mathcal{F}_L^s(X \times X)$ by

$$\mathcal{T}_\lambda = \{\mathcal{F} \in \mathcal{F}_L^s(X \times X) \mid (f_i \times f_i)^{\Rightarrow}(\mathcal{F}) \in (\mathcal{T}_i)_\lambda, \forall i \in I\}.$$

It can be easily proved that $\bar{\mathcal{T}} = \{\mathcal{T}_\lambda \mid \lambda \in M\}$ is the initial structure of the source $(f_i : X \longrightarrow (X_i, \bar{\mathcal{T}}_i))_{i \in I}$. □

By Theorem 3.9 we know there exists a unique final structure with respect to a sink $\{f_i : (X_i, \bar{\mathcal{T}}_i) \longrightarrow X\}_{i \in I}$ in the category $\mathbf{S}(L, M)$-**SUConvTr**. Now we explore the concrete form of the final structure.

Proposition 3.10. *Let* $\{f_i : (X_i, \bar{\mathcal{T}}_i) \longrightarrow X\}_{i \in I}$ *be a family of mappings, where* $(X_i, \bar{\mathcal{T}}_i)$ *is a stratified* (L, M)*-semiuniform convergence tower space. Define* $\bar{\mathcal{T}} = \{\mathcal{T}_\lambda \mid \lambda \in M\}$ *by*
$\mathcal{T}_\lambda = \{\mathcal{G} \in \mathcal{F}_L^s(X \times X) \mid \exists i \in I, \mathcal{F} \in (\mathcal{T}_i)_\lambda$ *such that* $(f_i \times f_i)^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G}\} \bigcup \{\mathcal{H} \in \mathcal{F}_L^s(X \times X) \mid \exists x \in X,$ *such that* $\mathcal{H} \geqslant [x] \times [x]\}$. *Then* $\bar{\mathcal{T}}$ *is the unique final structure with respect to the sink* $\{f_i : (X_i, \bar{\mathcal{T}}_i) \longrightarrow X\}_{i \in I}$. *Further, if the sink* $\{f_i : (X_i, \bar{\mathcal{T}}_i) \longrightarrow X\}_{i \in I}$ *is surjective, that is,* $X = \bigcup_{i \in I} f_i[X_i]$, *then* $\mathcal{T}_\lambda = \{\mathcal{G} \in \mathcal{F}_L^s(X \times X) \mid \exists i \in I, \mathcal{F} \in (\mathcal{T}_i)_\lambda$ *such that* $(f_i \times f_i)^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G}\}$.

Proof. The proof is routine and omitted. □

Let $(X, \bar{\mathcal{T}})$ be a stratified (L, M)-semiuniform convergence tower space. For a surjective mapping $f : (X, \bar{\mathcal{T}}) \longrightarrow Y$, we call $(Y, \bar{\xi})$ a quotient space of $(X, \bar{\mathcal{T}})$, f a quotient mapping, where $\bar{\xi} = \{\xi_\lambda \mid \lambda \in M\}$ is the final structure with respect to f. According to Proposition 3.10, $\xi_\lambda = \{\mathcal{G} \in \mathcal{F}_L^s(Y \times Y) \mid \exists \mathcal{F} \in \mathcal{T}_\lambda, (f \times f)^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G}\}$.

Dually, we can define a subspace and an initial mapping.

Since the category $\mathbf{S}(L, M)$-**SUConvTr** is a topological category over **Set**, there is the product of stratified (L, M)-semiuniform convergence tower spaces in the category $\mathbf{S}(L, M)$-**SUConvTr**. We now give the definition of the product of stratified (L, M)-semiuniform convergence tower spaces.

Definition 3.11. Let $\{(X_i, \bar{\mathcal{T}_i})\}_{i\in I}$ be a family of stratified (L, M)-semiuniform convergence tower spaces and $\{p_j : \prod_{i\in I} X_i \longrightarrow (X_j, \bar{\mathcal{T}_j})\}_{j\in I}$ be the source formed by the family of the projection mappings $\{p_j : \prod_{i\in I} X_i \longrightarrow X_j\}_{j\in I}$. The stratified (L, M)-semiuniform convergence tower structure on $X = \prod_{i\in I} X_i$, denoted by $\prod_{i\in I} \bar{\mathcal{T}_i}$, that is initial with respect to $\{p_j : X = \prod_{i\in I} X_i \longrightarrow (X_j, \bar{\mathcal{T}_j})\}_{j\in I}$, is called the product stratified (L, M)-semiuniform convergence tower structure and the pair $(X, \prod_{i\in I} \bar{\mathcal{T}_i})$ is called the product space briefly. Thus we have

$$\left(\prod_{i\in I} \bar{\mathcal{T}_i}\right)_\lambda = \{\mathcal{F} \in \mathcal{F}_L^s(X \times X) \mid (p_i \times p_i)^{\Rightarrow}(\mathcal{F}) \in (\mathcal{T}_i)_\lambda, \forall i \in I\}.$$

For the product space of stratified (L, M)-semiuniform convergence tower spaces $(X, \bar{\mathcal{T}^X})$ and $(Y, \bar{\mathcal{T}^Y})$, we write $(X \times Y, \bar{\mathcal{T}^X} \times \bar{\mathcal{T}^Y})$ for $(X, \bar{\mathcal{T}^X}) \times (Y, \bar{\mathcal{T}^Y})$.

Theorem 3.12. **LC-S**(L, M)**-SUConvTr** *is a reflective subcategory of* **S**(L, M)**-SUConvTr**.

Proof. Let $(X, \bar{\mathcal{T}}) \in$|**S**$(L, M)$**-SUConvTr**|. Define $L\bar{\mathcal{T}} = \{(L\mathcal{T})_\lambda \mid \lambda \in M\}$ as follows:
$(L\mathcal{T})_\lambda = \{\mathcal{F} \in \mathcal{F}_L^s(X \times X) \mid$ there exists $A \subseteq M$, $\bigvee A = \lambda$ and $\mathcal{F} \in \mathcal{T}_\mu$ for each $\mu \in A\}$. Then it is easily checked that $(X, L\bar{\mathcal{T}}) \in$|**LC-S**$(L, M)$**-SUConvTr**|. Further, we show that $id_X : (X, \bar{\mathcal{T}}) \to (X, L\bar{\mathcal{T}})$ is the **LC-S**(L, M)**-SUConvTr**-reflection.

(1) Since $\mathcal{T}_\lambda \subseteq (L\mathcal{T})_\lambda$ for each $\lambda \in M$, $id_X : (X, \bar{\mathcal{T}}) \longrightarrow (X, L\bar{\mathcal{T}})$ is uniformly continuous.

(2) Assume that $f : (X, \bar{\mathcal{T}}) \longrightarrow (Y, \bar{\xi})$ is uniformly continuous, where $(Y, \bar{\xi}) \in$|**LC-S**(L, M)**-SUConvTr**|. Let $\mathcal{F} \in (L\mathcal{T})_\lambda$. Then there exists $A \subseteq M$ such that $\bigvee A = \lambda$ and $\mathcal{F} \in \mathcal{T}_\mu$ for each $\mu \in A$. Since $f : (X, \bar{\mathcal{T}}) \longrightarrow (Y, \bar{\xi})$ is uniformly continuous, $(f \times f)^{\Rightarrow}(\mathcal{F}) \in \xi_\mu$ for each $\mu \in A$. As $(Y, \bar{\xi})$ is left continuous, we have $(f \times f)^{\Rightarrow}(\mathcal{F}) \in \xi_\lambda$. This shows $f : (X, L\bar{\mathcal{T}}) \longrightarrow (Y, \bar{\xi})$ is uniformly continuous. □

Lattice-valued semiuniform convergence structures were introduced by Fang [5] as follows:

Definition 3.13. [5] Let X be a non-void set. A mapping $T^X : \mathcal{F}_L^s(X \times X) \longrightarrow L$ is called a stratified L-semiuniform convergence structure on X if it fulfills

(UC1) $T^X([x] \times [x]) = 1$ for all $x \in X$.

(UC2) $\mathcal{F} \leqslant \mathcal{G} \Longrightarrow T^X(\mathcal{F}) \leqslant T^X(\mathcal{G})$ for all $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X \times X)$.

(UC3) $T^X(\mathcal{F}) \leqslant T^X(\mathcal{F}^{-1})$ for all $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$.

The pair (X, T^X) is called a stratified L-semiuniform convergence space.

A mapping $f : (X, T^X) \longrightarrow (Y, T^Y)$ between two stratified L-semiuniform convergence spaces is called uniformly continuous if $T^X(\mathcal{F}) \leqslant T^Y((f \times f)^{\Rightarrow}(\mathcal{F}))$, for all $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$.

The category of stratified L-semiuniform convergence spaces and uniformly continuous mappings is denoted by **S**L**-SUConv**.

In the following, it is proved that Fang's category is isomorphic to a reflective subcategory of **S**(L, L)**-SUConvTr**.

Define a mapping φ : **S**L-**SUConv** $\longrightarrow$ **LC- S**(L, L)-**SUConvTr** by $\varphi(f) = f$ and $\varphi(X, T) = (X, \bar{\mathcal{T}}_T)$, where

$$\bar{\mathcal{T}}_T = \{(\mathcal{T}_T)_\lambda \mid \lambda \in L\}$$

and

$$\mathcal{F} \in (\mathcal{T}_T)_\lambda \Longleftrightarrow T(\mathcal{F}) \geqslant \lambda.$$

It is easily checked that $(X, \bar{\mathcal{T}}_T) \in$|**LC-S**$(L, L)$-**SUConvTr**|.

Conversely, define a mapping ψ : **LC-S**(L, L)-**SUConvTr** $\longrightarrow$ **S**L-**SUConv** by $\psi(f) = f$ and $\psi(X, \bar{\mathcal{T}}) = (X, T_{\bar{\mathcal{T}}})$, where for each $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$,

$$T_{\bar{\mathcal{T}}}(\mathcal{F}) = \bigvee\{\lambda \in L \mid \mathcal{F} \in \mathcal{T}_\lambda\}.$$

It is easily checked that $(X, T_{\bar{\mathcal{T}}}) \in$| **S**$L$ -**SUConv** |.

Theorem 3.14. LC-S(L, L)-**SUConvTr** *is isomorphic to* **S**L-**SUConv** .

Proof. Firstly, we prove that φ and ψ are functors. Assume that $f : (X, T^X) \longrightarrow (Y, T^Y)$ is uniformly continuous. If $\mathcal{F} \in (\mathcal{T}_{T^X})_\lambda$, then $T^X(\mathcal{F}) \geqslant \lambda$. Since $f : (X, T^X) \longrightarrow (Y, T^Y)$ is uniformly continuous, we have $T^Y((f \times f)^\Rightarrow(\mathcal{F})) \geqslant T^X(\mathcal{F}) \geqslant \lambda$. Thus $(f \times f)^\Rightarrow(\mathcal{F}) \in (\mathcal{T}_{T^Y})_\lambda$. Therefore, $f : (X, \bar{\mathcal{T}}_{T^X}) \longrightarrow (Y, \bar{\mathcal{T}}_{T^Y})$ is uniformly continuous and φ is a functor. Conversely, assume that $f : (X, \bar{\mathcal{T}}) \longrightarrow (Y, \bar{\xi})$ is uniformly continuous. Since $f : (X, \bar{\mathcal{T}}) \longrightarrow (Y, \bar{\xi})$ is uniformly continuous, for each $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$, we have $T_{\bar{\mathcal{T}}}(\mathcal{F}) = \bigvee\{\lambda \in L \mid \mathcal{F} \in \mathcal{T}_\lambda\} \leqslant \bigvee\{\lambda \in L \mid (f \times f)^\Rightarrow(\mathcal{F}) \in \xi_\lambda\} = T_{\bar{\xi}}((f \times f)^\Rightarrow(\mathcal{F}))$. Therefore, $f : (X, T_{\bar{\mathcal{T}}}) \longrightarrow (Y, T_{\bar{\xi}})$ is uniformly continuous and ψ is a functor.

It remains to show that $\varphi \circ \psi = id_{\mathbf{LC-S}(L,L)-\mathbf{SUConvTr}}$ and $\psi \circ \varphi = id_{\mathbf{S}L-\mathbf{SUConv}}$.

Let $(X, \bar{\mathcal{T}}) \in$|**LC-S**$(L, L)$-**SUConvTr**|. Then $\bar{\mathcal{T}}_{T_{\bar{\mathcal{T}}}} = \bar{\mathcal{T}}$. This follows from the fact: for each $\lambda \in L$, $\mathcal{F} \in (\mathcal{T}_{T_{\bar{\mathcal{T}}}})_\lambda \Longleftrightarrow T_{\bar{\mathcal{T}}}(\mathcal{F}) \geqslant \lambda \Longleftrightarrow \bigvee\{\mu \in L \mid \mathcal{F} \in \mathcal{T}_\mu\} \geqslant \lambda \Longleftrightarrow \mathcal{F} \in \mathcal{T}_\lambda$. This shows $\varphi \circ \psi = id_{\mathbf{LC-S}(L,L)-\mathbf{SUConvTr}}$.

Conversely, let $(X, T) \in$|**S**L-**SUConv**|. Then $T_{\bar{\mathcal{T}}_T} = T$. This follows from the fact: for each $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$, $T_{\bar{\mathcal{T}}_T}(\mathcal{F}) = \bigvee\{\lambda \in L \mid \mathcal{F} \in (\mathcal{T}_T)_\lambda\} = \bigvee\{\lambda \in L \mid T(\mathcal{F}) \geqslant \lambda\} = T(\mathcal{F})$. This shows $\psi \circ \varphi = id_{\mathbf{S}L-\mathbf{SUConv}}$.

□

4. Cartesian-closedness of S(L, M)-SUConvTr

Recall a category **C** is called Cartesian-closed [1] provided that the following conditions are satisfied:

(1) For each pair (X, Y) of **C**-objects there exists a product $X \times Y$ in **C**,

(2) For each pair of **C**-objects X and Y, there exists a **C**-object Y^X (called power object) and a **C**-morphism $ev_{X,Y} : Y^X \times X \longrightarrow Y$ (called evaluation morphism) such that for each C-object Z and each C-morphism $f : Z \times X \longrightarrow Y$, there exists a unique C-morphism $g : Z \longrightarrow Y^X$ such that $ev_{X,Y} \circ (g \times id_X) = f$.

Since **S**(L, M)-**SUConvTr** is topological, the condition (1) is fulfilled. We now explore the concrete form of power objects in **S**(L, M)-**SUConvTr**.

Given two stratified (L, M)-semiuniform convergence tower spaces $(X, \bar{\mathcal{T}})$ and $(Y, \bar{\xi})$. Put $[X, Y] = \{f \mid f : X \longrightarrow Y$ is uniformly continuous$\}$. Let $ev_{X,Y}$: $[X, Y] \times X \longrightarrow Y$ be the evaluation mapping such that $(f, x) \mapsto f(x)$ and j :

$([X,Y] \times [X,Y]) \times (X \times X) \longrightarrow ([X,Y] \times X) \times ([X,Y] \times X)$ be the canonical bijection such that $((f,g),(x,x')) \mapsto ((f,x),(g,x'))$.

For each $\lambda \in M$, define $\varepsilon_\lambda \subseteq \mathcal{F}_L^s([X,Y] \times [X,Y])$ as follows: $\varepsilon_\lambda = \{\mathcal{F} \in \mathcal{F}_L^s([X,Y] \times [X,Y]) \mid \mu \leqslant \lambda, ((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}(\mathcal{F} \times \mathcal{G}) \in \xi_\mu (\forall \mathcal{G} \in \mathcal{T}_\mu)\}$.

Lemma 4.1. [5] *Let $f : X \longrightarrow Y$, $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$. Then $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}(([f] \times [f]) \times \mathcal{F}) \geqslant (f \times f)^{\Rightarrow}(\mathcal{F})$.*

Lemma 4.2. [19] *Let $\mathcal{H} \in \mathcal{F}_L^s((X \times Y) \times (X \times Y))$, $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$, $\mathcal{G} \in \mathcal{F}_L^s(Y \times Y)$, $P_X : X \times Y \longrightarrow X$, $P_Y : X \times Y \longrightarrow Y$ be the projection mappings , and $m : (X \times X) \times (Y \times Y) \longrightarrow (X \times Y) \times (X \times Y)$ be the bijection with $((x_1,x_2),(y_1,y_2)) \mapsto ((x_1,y_1),(x_2,y_2))$. Then*

(1) $m^{\Rightarrow}((P_X \times P_X)^{\Rightarrow}(\mathcal{H}) \times (P_Y \times P_Y)^{\Rightarrow}(\mathcal{H})) \leqslant \mathcal{H}$.

(2) $(P_X \times P_X)^{\Rightarrow}(m^{\Rightarrow}(\mathcal{F} \times \mathcal{G})) \geqslant \mathcal{F}$ and $(P_Y \times P_Y)^{\Rightarrow}(m^{\Rightarrow}(\mathcal{F} \times \mathcal{G})) \geqslant \mathcal{G}$.

(3) $m^{\Rightarrow}(\mathcal{F}^{-1} \times \mathcal{G}) = (m^{\Rightarrow}(\mathcal{F} \times \mathcal{G}^{-1}))^{-1}$.

Corollary 4.3. *Let $\mathcal{F} \in \mathcal{T}_\lambda$ and $\mathcal{G} \in \xi_\lambda$. Then $m^{\Rightarrow}(\mathcal{F} \times \mathcal{G}) \in (\bar{\mathcal{T}} \times \bar{\xi})_\lambda$.*

Proposition 4.4. *$([X,Y], \bar{\varepsilon})$ is a stratified (L,M)-semiuniform convergence tower space.*

Proof. (UCT1) For all $f \in [X,Y]$ and $\lambda \in M$, let $\mu \leqslant \lambda$ and $\mathcal{G} \in \mathcal{T}_\mu$. Then $(f \times f)^{\Rightarrow}(\mathcal{G}) \in \xi_\mu$. By Lemma 4.1, we know $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}(([f] \times [f]) \times \mathcal{G}) \geqslant (f \times f)^{\Rightarrow}(\mathcal{G})$. Thus $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}(([f] \times [f]) \times \mathcal{G}) \in \xi_\mu$. This shows $[f] \times [f] \in \varepsilon_\lambda$.

(UCT2) is trivial.

(UCT3) Assume that $\mathcal{F} \in \varepsilon_\lambda$. Then for all $\mu \leqslant \lambda$, $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}(\mathcal{F} \times \mathcal{G}) \in \xi_\mu$ for each $\mathcal{G} \in \mathcal{T}_\mu$. Thus $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}(\mathcal{F}^{-1} \times \mathcal{G}) = (((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}(\mathcal{F} \times \mathcal{G}^{-1}))^{-1} \in \xi_\mu$. This shows $\mathcal{F}^{-1} \in \varepsilon_\lambda$.

(P_1) and (P_2) are trivial. □

Proposition 4.5. *Let $(X, \bar{\mathcal{T}}), (Y, \bar{\xi})$ be stratified (L,M)-semiuniform convergence tower spaces. Then the evaluation mapping $ev_{X,Y} : [X,Y] \times X \longrightarrow Y$ is uniformly continuous.*

Proof. Let $\mathcal{H} \in (\bar{\varepsilon} \times \bar{\mathcal{T}})_\lambda$. Then $(P_{[X,Y]} \times P_{[X,Y]})^{\Rightarrow}(\mathcal{H}) \in \varepsilon_\lambda$ and $(P_X \times P_X)^{\Rightarrow}(\mathcal{H}) \in \mathcal{T}_\lambda$. By construction of ε_λ, $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}((P_{[X,Y]} \times P_{[X,Y]})^{\Rightarrow}(\mathcal{H}) \times (P_X \times P_X)^{\Rightarrow}(\mathcal{H})) \in \xi_\lambda$. Since $j^{\Rightarrow}((P_{[X,Y]} \times P_{[X,Y]})^{\Rightarrow}(\mathcal{H}) \times (P_X \times P_X)^{\Rightarrow}(\mathcal{H})) \leqslant \mathcal{H}$, $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}((P_{[X,Y]} \times P_{[X,Y]})^{\Rightarrow}(\mathcal{H}) \times (P_X \times P_X)^{\Rightarrow}(\mathcal{H})) \leqslant (ev_{X,Y} \times ev_{X,Y})^{\Rightarrow}(\mathcal{H})$. This implies $(ev_{X,Y} \times ev_{X,Y})^{\Rightarrow}(\mathcal{H}) \in \xi_\lambda$. Therefore the evaluation mapping $ev_{X,Y} : [X,Y] \times X \longrightarrow Y$ is uniformly continuous. □

Let $f : Z \times X \longrightarrow Y$ be a mapping and $m_1 : (Z \times Z) \times (X \times X) \longrightarrow (Z \times X) \times (Z \times X)$ the canonical bijection. Define $f_* : Z \longrightarrow Y^X$ by $f_*(z)(x) = f(z,x)$.

Lemma 4.6. [19] *For each $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$ and $z \in Z$, it holds that $(f_*(z) \times f_*(z))^{\Rightarrow}(\mathcal{F}) \geqslant ((f \times f) \circ m_1)^{\Rightarrow}(([z] \times [z]) \times \mathcal{F})$.*

Lemma 4.7. [19] *Let $\mathcal{F} \in \mathcal{F}_L^s(Z \times Z)$ and $\mathcal{G} \in \mathcal{F}_L^s(X \times X)$. Then for each mapping $f : Z \times X \longrightarrow Y$, $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}((f_* \times f_*)^{\Rightarrow}(\mathcal{F}) \times \mathcal{G}) = ((f \times f) \circ m_1)^{\Rightarrow}(\mathcal{F} \times \mathcal{G})$.*

By Lemmas 4.6, 4.7 and Corollary 4.3, we have the following Proposition.

Proposition 4.8. *Let* $(X,\bar{\mathcal{T}}),(Y,\bar{\xi}),(Z,\bar{\eta})$ *be stratified* (L,M)*-semiuniform convergence tower spaces. If* $f: Z\times X \longrightarrow Y$ *is uniformly continuous, then so is* $f_*: Z \longrightarrow ([X,Y],\bar{\varepsilon})$.

By Propositions 4.4, 4.5 and 4.8, we obtain the main result in this section.

Theorem 4.9. $\mathbf{S}(L,M)$-**SUConvTr** *is Cartesian-closed.*

5. Extensionality of $\mathbf{S}(L,M)$-SUConvTr

In this section, we will explore the extensionality of the category $\mathbf{S}(L,M)$-**SUConvTr**.

Recall in a topological category C, a partial morphism from Y to X is a C-morphism $f: Z\to X$ whose domain is a subobject of Y. A topological category C is called extensional provided that for every C-object X has a one-point extension X^*, in the sense that every C-object X can be embedded via the addition of a single point ∞ into a C-object X^* such that for every partial morphism $f: Z\to X$, the mapping $f^*: Y\longrightarrow X^*$ defined by

$$f^*(x)=\begin{cases} f(x), & x\in Z;\\ \infty, & x\notin Z.\end{cases}$$

is a C-morphism.

In this section, i denotes the inclusion mapping on X.

First we need some technical results of Fang [5].

Lemma 5.1. [5] *Let* X *be a non-void set. Put* $X^*=X\bigcup\{\infty\}$, $\infty\notin X$*. Define a mapping:* $\inf_\infty: L^{X^*\times X^*}\longrightarrow L$ *as follows:*

$$\inf_{\infty}(A)=\inf\{A(x,y)\mid (x,y)\in(\{\infty\}\times X^*)\bigcup(X^*\times\{\infty\})\}, A\in L^{X^*\times X^*}.$$

Then $\inf_\infty$ *is a stratified* L*-filter on* $X^*\times X^*$.

Lemma 5.2. [5] *Let* $(X^*)^2\backslash X^2$ *denote the set* $(\{\infty\}\times X^*)\bigcup(X^*\times\{\infty\})$*. Then for each* $\mathcal{H}\in\mathcal{F}^s_L(X^*\times X^*)$, $(i\times i)^{\Leftarrow}(\mathcal{H})$ *does not exist if and only if* $\mathcal{H}(1_{(X^*)^2\backslash X^2})\neq 0$*. In general, let* $Z\subseteq Y$ *and* $k: Z\longrightarrow Y$ *be the inclusion mapping. Then* $(k\times k)^{\Leftarrow}(\mathcal{H})$ *does not exist if and only if* $\mathcal{H}(1_{Y^2\backslash Z^2})\neq 0$ *for each* $\mathcal{H}\in\mathcal{F}^s_L(Y\times Y)$.

Lemma 5.3. [5] *Let* $\mathcal{F}\in\mathcal{F}^s_L(X\times X)$*, then* $(i\times i)^{\Leftarrow}((i\times i)^{\Rightarrow}(\mathcal{F})\wedge inf_\infty)=\mathcal{F}$.

Theorem 5.4. *Let* $(X,\bar{\mathcal{T}})$ *be a stratified* (L,M)*-semiuniform convergence tower space, define* $\bar{\xi}=\{\xi_\lambda\mid\lambda\in M\}$ *by* $\xi_\lambda=\{\mathcal{G}\in\mathcal{F}^s_L(X^*\times X^*)\mid (i\times i)^{\Leftarrow}(\mathcal{G})\in\mathcal{T}_\lambda,\mathcal{G}(1_{(X^*)^2\backslash X^2})=0\}\bigcup\{\mathcal{G}\in\mathcal{F}^s_L(X^*\times X^*)\mid\mathcal{G}(1_{(X^*)^2\backslash X^2})\neq 0)\}$*. Then* $(X^*,\bar{\xi})$ *is a stratified* (L,M)*-semiuniform convergence tower space.*

Proof. (UCT1) For each $x\in X$, since $[x]\times[x](1_{(X^*)^2\backslash X^2})=0$ and $(i\times i)^{\Leftarrow}([x]\times[x])=[x]\times[x]\in\mathcal{T}_\lambda$, $[x]\times[x]\in\xi_\lambda$. In addition, $([\infty]\times[\infty])(1_{(X^*)^2\backslash X^2})\neq 0$, so $[\infty]\times[\infty]\in\xi_\lambda$.

(UCT2) Suppose that $\mathcal{F} \in \xi_\lambda$ and $\mathcal{G} \geqslant \mathcal{F}$. If $\mathcal{G}(1_{(X^*)^2\setminus X^2}) \neq 0$, then $\mathcal{G} \in \xi_\lambda$. Otherwise, we have $\mathcal{F}(1_{(X^*)^2\setminus X^2}) = 0$, and thus $(i \times i)^{\Leftarrow}(\mathcal{F}) \in \mathcal{T}_\lambda$. Since $(i \times i)^{\Leftarrow}(\mathcal{G}) \geqslant (i \times i)^{\Leftarrow}(\mathcal{F})$, $(i \times i)^{\Leftarrow}(\mathcal{G}) \in \mathcal{T}_\lambda$, which means $\mathcal{G} \in \xi_\lambda$.

(UCT3) is proved from the facts that $\mathcal{F}(1_{(X^*)^2\setminus X^2}) \neq 0 \Longleftrightarrow \mathcal{F}^{-1}(1_{(X^*)^2\setminus X^2}) \neq 0$ and $(i \times i)^{\Leftarrow}(\mathcal{F}^{-1}) = ((i \times i)^{\Leftarrow}(\mathcal{F}))^{-1}$.

(P_1) and (P_2) are trivial. □

Theorem 5.5. **S**(L,M)-**SUConvTr** *is extensional.*

Proof. Let $(X, \bar{\mathcal{T}})$ be a stratified (L,M)-semiuniform convergence tower space and $(X^*, \bar{\xi})$ be defined as above. By Theorem 5.4, $(X^*, \bar{\xi})$ is a stratified (L,M)-semiuniform convergence tower space. It suffice to show that $(X^*, \bar{\xi})$ is the one-point extension of $(X, \bar{\mathcal{T}})$.

For this it suffices to prove:

(1) $(X, \bar{\mathcal{T}})$ is a subspace of $(X^*, \bar{\xi})$.

(2) $(X^*, \bar{\xi})$ is the one-point extension of $(X, \bar{\mathcal{T}})$.

(1) For each $\lambda \in M$, denote $\varepsilon_\lambda = \{\mathcal{F} \mid (i \times i)^{\Rightarrow}(\mathcal{F}) \in \xi_\lambda\}$. We need to prove $\varepsilon_\lambda = \mathcal{T}_\lambda$. Let $\mathcal{F} \in \mathcal{T}_\lambda$, since $(i \times i)^{\Leftarrow}((i \times i)^{\Rightarrow}(\mathcal{F})) = \mathcal{F}$, $(i \times i)^{\Rightarrow}(\mathcal{F}) \in \xi_\lambda$, which means $\mathcal{F} \in \varepsilon_\lambda$. Conversely, assume that $\mathcal{F} \in \varepsilon_\lambda$, then $(i \times i)^{\Rightarrow}(\mathcal{F}) \in \xi_\lambda$. Since $(i \times i)^{\Rightarrow}(\mathcal{F})(1_{(X^*)^2\setminus X^2}) = 0$, $(i \times i)^{\Leftarrow}((i \times i)^{\Rightarrow}(\mathcal{F})) = \mathcal{F} \in \mathcal{T}_\lambda$.

(2) Suppose that f is a partial morphism from $(Y, \bar{\mathcal{T}}^Y)$ to $(X, \bar{\mathcal{T}})$, i.e., there exists a stratified (L,M)-semiuniform convergence tower space $(Z, \bar{\mathcal{T}}^Z)$, which is a subspace of $(Y, \bar{\mathcal{T}}^Y)$ such that $f : (Z, \bar{\mathcal{T}}^Z) \longrightarrow (X, \bar{\mathcal{T}})$ is uniformly continuous. We now show f^* is uniformly continuous. Let $\mathcal{F} \in (\mathcal{T}^Y)_\lambda$.

Case 1 $(i \times i)^{\Leftarrow}(\mathcal{F})$ does not exist. Then $\mathcal{F}(1_{Y^2\setminus Z^2}) \neq 0$. However,
$(f^* \times f^*)^{\Rightarrow}(\mathcal{F})(1_{(X^*)^2\setminus X^2}) = \mathcal{F}(1_{Y^2\setminus Z^2}) \neq 0$. This means $(f^* \times f^*)^{\Rightarrow}(\mathcal{F}) \in \xi_\lambda$.

Case 2 $(i \times i)^{\Leftarrow}(\mathcal{F})$ exists. Since $(k \times k)^{\Rightarrow}((k \times k)^{\Leftarrow}(\mathcal{F})) \geqslant \mathcal{F} \in (\mathcal{T}^Y)_\lambda$ and $(Z, \bar{\mathcal{T}}^Z)$ is a subspace of $(Y, \bar{\mathcal{T}}^Y)$, $(k \times k)^{\Leftarrow}(\mathcal{F}) \in (\mathcal{T}^Z)_\lambda$. As f is uniformly continuous, $(f \times f)^{\Rightarrow}((k \times k)^{\Leftarrow}(\mathcal{F})) \in \mathcal{T}_\lambda$. Moreover, $(i \times i)^{\Leftarrow}((i \times i)^{\Rightarrow}((f \times f)^{\Rightarrow}((k \times k)^{\Leftarrow}(\mathcal{F})) \wedge \inf_\infty)) = (f \times f)^{\Rightarrow}((k \times k)^{\Leftarrow}(\mathcal{F})) \in \mathcal{T}_\lambda$. It follows that $(i \times i)^{\Rightarrow}((f \times f)^{\Rightarrow}((k \times k)^{\Leftarrow}(\mathcal{F})) \wedge \inf_\infty) \in \xi_\lambda$. Since $(i \times i)^{\Rightarrow}((f \times f)^{\Rightarrow}((k \times k)^{\Leftarrow}(\mathcal{F})) \wedge \inf_\infty) \leqslant (f^* \times f^*)^{\Rightarrow}(\mathcal{F})$ (See the proof of [[5], Theorem 7.5]), $(f^* \times f^*)^{\Rightarrow}(\mathcal{F}) \in \xi_\lambda$. This shows that f^* is uniformly continuous. □

6. Products of Quotient Mappings in S(L, M)-SUConvTr

In this section, we show that products of quotient mappings are quotient mappings, and conclude that **S**(L,M)-**SUConvTr** is a strong topological universe.

Lemma 6.1. [5] *Let* $\{X_i\}_{i\in I}$ *be a family of sets,* $P_{X_i} : \prod_i X_i \longrightarrow X_i$ *the projection mapping and* $j : \prod_i X_i \times X_i \longrightarrow \prod_i X_i \times \prod_i X_i$ *the bijection. Then*

(1) $\mathcal{G}_i \leqslant ((P_{X_i} \times P_{X_i}) \circ j)^{\Rightarrow}(\prod_i \mathcal{G}_i)$, $\mathcal{G}_i \in \mathcal{F}_L^s(X_i \times X_i)$, $\forall i \in I$.

(2) $j^{\Rightarrow}(\prod_i (P_{X_i} \times P_{X_i})^{\Rightarrow}(\mathcal{H})) \leqslant \mathcal{H}$ *for each* $\mathcal{H} \in \mathcal{F}_L^s(\prod_i X_i \times \prod_i X_i)$.

Lemma 6.2. [5] *Let* $\{f_i : X_i \longrightarrow Y_i\}_{i\in I}$ *be a family of surjective mappings and for all* $i \in I$, $\mathcal{G}_i \in \mathcal{F}_L^s(X_i \times X_i)$. *Then* $((\prod_i f_i \times \prod_i f_i) \circ j)^{\Rightarrow}(\prod_i \mathcal{G}_i) \leqslant k^{\Rightarrow}(\prod_i (f_i \times f_i)^{\Rightarrow}(\mathcal{G}_i))$, *where* $k : \prod_i (Y_i \times Y_i) \longrightarrow \prod_i Y_i \times \prod_i Y_i$ *is the bijection.*

Theorem 6.3. *Let $\{f_i : (X_i, \bar{\mathcal{T}}_i) \longrightarrow (Y_i, \bar{\xi}_i)\}_{i\in I}$ be a family of quotient mappings, $(X, \bar{\mathcal{T}})$ the product of $\{(X_i, \bar{\mathcal{T}}_i)\}_{i\in I}$ and $(Y, \bar{\xi})$ the product of $\{(Y_i, \bar{\xi}_i)\}_{i\in I}$. Then $\prod_i f_i : (X, \bar{\mathcal{T}}) \longrightarrow (Y, \bar{\xi})$ is a quotient mapping.*

Proof. Obviously, $\prod_i f_i$ is surjective. Put $\varepsilon_\lambda = \{\mathcal{H} \in \mathcal{F}_L^s(Y\times Y) \mid \exists \mathcal{G} \in \mathcal{T}_\lambda, (\prod_i f_i \times \prod_i f_i)^{\Rightarrow}(\mathcal{G}) \leqslant \mathcal{H}$. It suffices to prove that $\varepsilon_\lambda = \xi_\lambda$ for all $\lambda \in M$.

Assume that $\mathcal{H} \in \varepsilon_\lambda$, then there exists $\mathcal{G} \in \mathcal{T}_\lambda$ such that $(\prod_i f_i\times\prod_i f_i)^{\Rightarrow}(\mathcal{G}) \leqslant \mathcal{H}$. Thus $(P_{Y_i}\times P_{Y_i})^{\Rightarrow}((\prod_i f_i\times\prod_i f_i)^{\Rightarrow}(\mathcal{G})) \leqslant (P_{Y_i}\times P_{Y_i})^{\Rightarrow}(\mathcal{H})$. Note that $(P_{Y_i}\times P_{Y_i})\circ(\prod_i f_i \times \prod_i f_i) = (f_i \times f_i)\circ(P_{X_i} \times P_{X_i})$, we have $(f_i \times f_i)^{\Rightarrow}((P_{X_i} \times P_{X_i})^{\Rightarrow}(\mathcal{G})) \leqslant (P_{Y_i} \times P_{Y_i})^{\Rightarrow}(\mathcal{H})$. Since f_i is a quotient mapping and $(P_{X_i} \times P_{X_i})^{\Rightarrow}(\mathcal{G}) \in (\mathcal{T}_i)_\lambda$, $(P_{Y_i}\times P_{Y_i})^{\Rightarrow}(\mathcal{H}) \in (\xi_i)_\lambda$. As $(Y, \bar{\xi})$ is the product of $\{(Y_i, \bar{\xi}_i)\}_{i\in I}$, we obtain $\mathcal{H} \in \xi_\lambda$. Conversely, let $\mathcal{H} \in \xi_\lambda$, then $(P_{Y_i} \times P_{Y_i})^{\Rightarrow}(\mathcal{H}) \in (\xi_i)_\lambda$. Since f_i is a quotient mapping, there exists $\mathcal{G}_i \in (\mathcal{T}_i)_\lambda$ such that $(f_i \times f_i)^{\Rightarrow}(\mathcal{G}_i) \leqslant (P_{Y_i} \times P_{Y_i})^{\Rightarrow}(\mathcal{H})$. By $\mathcal{G}_i \leqslant ((P_{X_i}\times P_{X_i})\circ j)^{\Rightarrow}(\prod_i \mathcal{G}_i)$, we have $((P_{X_i}\times P_{X_i})\circ j)^{\Rightarrow}(\prod_i \mathcal{G}_i) \in (\mathcal{T}_i)_\lambda$. As $(X, \bar{\mathcal{T}})$ is the product of $\{(X_i, \bar{\mathcal{T}}_i)\}_{i\in I}$, we have $j^{\Rightarrow}(\prod_i \mathcal{G}_i) \in \mathcal{T}_\lambda$. By Lemma 6.1 (2) and Lemma 6.2, the inequality $((\prod_i f_i\times\prod_i f_i)\circ j)^{\Rightarrow}(\prod_i \mathcal{G}_i)) \leqslant k^{\Rightarrow}(\prod_i(f_i\times f_i)^{\Rightarrow}(\mathcal{G}_i)) \leqslant k^{\Rightarrow}(\prod_i(P_{Y_i} \times P_{Y_i})^{\Rightarrow}(\mathcal{H})) \leqslant \mathcal{H}$ holds. Then we have $\mathcal{H} \in \varepsilon_\lambda$. □

Recall that the following several convenient properties for a topological category C are proposed by Preuss in the book [24]:

(CP1) C is Cartesian closed.

(CP2) C is extensional.

(CP3) In C product of quotient mappings is a quotient mapping. Moreover, C is called

(1) strongly Cartesian closed provided that it fulfills (CP1) and (CP3).

(2) is a topological universe provided that it fulfills (CP1) and (CP2).

(3) is a strong topological universe provided that it fulfills (CP1), (CP2) and (CP3).

By Theorem 6.3, 5.5 and 4.9, we have the following Theorem.

Theorem 6.4. **S**(L, M)-**SUConvTr** *is a strong topological universe.*

7. The Relations Between S(L, M)-SUConvTr and S(L, M)-FilTr

In this section, we study the relations between stratified (L, M)-filter tower spaces and stratified (L, M)-semiuniform convergence tower spaces.

The concept of stratified (L, M)-filter tower spaces was introduced in [28] as follows:

Definition 7.1. Let X be a nonempty set. If $\bar{\gamma} = \{\gamma_\lambda \mid \lambda \in M\}$, where, $\gamma_\lambda \subseteq \mathcal{F}_L^s(X)$, satisfies the following:

(LMFT1) For each $x \in X, \lambda \in M$, $[x] \in \gamma_\lambda$;

(LMFT2) $\mathcal{G} \in \gamma_\lambda$ whenever $\mathcal{F} \in \gamma_\lambda$ and $\mathcal{F} \leqslant \mathcal{G}$;

(LFT1) $\gamma_\lambda \leqslant \gamma_\mu$ whenever $\mu \leqslant \lambda$;

(LFT2) $\gamma_0 = \mathcal{F}_L^s(X)$,

then the pair $(X, \bar{\gamma})$ is called a stratified (L, M)-filter tower space.

Definition 7.2. A mapping $f : (X, \bar{\gamma}) \longrightarrow (Y, \bar{\eta})$ between two stratified (L, M)-filter tower spaces is called uniformly continuous if $f^{\Rightarrow}(\mathcal{F}) \in \eta_\lambda$ for all $\mathcal{F} \in \gamma_\lambda$ and for all $\lambda \in M$.

The category of stratified (L, M)-filter tower spaces and uniformly continuous mappings is denoted by $\mathbf{S}(L, M)$-$\mathbf{FilTr}$.

Lemma 7.3. *Let $(X, \bar{\mathcal{T}})$ be a stratified (L, M)-semiuniform convergence tower space. Then $(X, \bar{\gamma}_{\bar{\mathcal{T}}})$ is a stratified (L, M)-filter tower space, where $\bar{\gamma}_{\bar{\mathcal{T}}} = \{(\gamma_{\bar{\mathcal{T}}})_\lambda \mid \lambda \in M\}$,*

$$(\gamma_{\bar{\mathcal{T}}})_\lambda = \{\mathcal{F} \mid \mathcal{F} \in \mathcal{F}_L^s(X), \mathcal{F} \times \mathcal{F} \in \mathcal{T}_\lambda\}.$$

Proof. (LMFT1) For each $x \in X$, $\lambda \in M$, since $[x] \times [x] \in \mathcal{T}_\lambda$, $[x] \in (\gamma_{\bar{\mathcal{T}}})_\lambda$.

(LMFT2) If $\mathcal{F} \in (\gamma_{\bar{\mathcal{T}}})_\lambda$ and $\mathcal{F} \leqslant \mathcal{G}$, then $\mathcal{F} \times \mathcal{F} \in \mathcal{T}_\lambda$ and $\mathcal{F} \times \mathcal{F} \leqslant \mathcal{G} \times \mathcal{G}$. By (UCT2), we have $\mathcal{G} \times \mathcal{G} \in \mathcal{T}_\lambda$. Thus $\mathcal{G} \in (\gamma_{\bar{\mathcal{T}}})_\lambda$.

(LMT1) and (LMT2) can be easily proved by (P1) and (P2) respectively. □

Lemma 7.4. *Let $(X, \bar{\gamma})$ be a stratified (L, M)-filter tower space. Then $(X, \bar{\mathcal{T}}\gamma)$ is a stratified (L, M)-semiuniform convergence tower space, where*

$$(\mathcal{T}_{\bar{\gamma}})_\lambda = \{\mathcal{F} \mid \mathcal{F} \in \mathcal{F}_L^s(X \times X), \exists \mathcal{H} \in \gamma_\lambda : \mathcal{F} \geqslant \mathcal{H} \times \mathcal{H}\}.$$

Proof. (UCT1) For each $x \in X$ and $\lambda \in M$, Since $[x] \in \gamma_\lambda$, $[x] \times [x] \in (\mathcal{T}_{\bar{\gamma}})_\lambda$.

(UCT2) If $\mathcal{F} \in (\mathcal{T}_{\bar{\gamma}})_\lambda$ and $\mathcal{F} \leqslant \mathcal{G}$, then there exists $\mathcal{H} \in \gamma_\lambda$ such that $\mathcal{H} \times \mathcal{H} \leqslant \mathcal{F} \leqslant \mathcal{G}$. So $\mathcal{G} \in (\mathcal{T}_{\bar{\gamma}})_\lambda$.

(UCT3) $\mathcal{F} \in (\mathcal{T}_{\bar{\gamma}})_\lambda \Longrightarrow \exists \mathcal{H} \in \gamma_\lambda : \mathcal{H} \times \mathcal{H} \leqslant \mathcal{F} \Longrightarrow (\mathcal{H} \times \mathcal{H})^{-1} \leqslant \mathcal{F}^{-1} \Longrightarrow \mathcal{H} \times \mathcal{H} \leqslant \mathcal{F}^{-1} \Longrightarrow \mathcal{F}^{-1} \in (\mathcal{T}_{\bar{\gamma}})_\lambda$.

(P1) and (P2) follow from (LMT1) and (LMT2) respectively. □

Define a mapping $\theta : \mathbf{S}(L, M)$-$\mathbf{SUConvTr} \longrightarrow \mathbf{S}(L, M)$-$\mathbf{FilTr}$ by $(X, \bar{\mathcal{T}}) \longmapsto (X, \bar{\gamma}_{\bar{\mathcal{T}}})$ and $f \longmapsto f$, where $\bar{\gamma}_{\bar{\mathcal{T}}} = \{(\gamma_{\bar{\mathcal{T}}})_\lambda \mid \lambda \in M\}$,

$$(\gamma_{\bar{\mathcal{T}}})_\lambda = \{\mathcal{F} \mid \mathcal{F} \in \mathcal{F}_L^s(X), \mathcal{F} \times \mathcal{F} \in \mathcal{T}_\lambda\}.$$

Conversely, define a mapping $\delta : \mathbf{S}(L, M)$-$\mathbf{FilTr} \longrightarrow \mathbf{S}(L, M)$-$\mathbf{SUConvTr}$ by $(X, \bar{\gamma}) \longmapsto (X, \bar{\mathcal{T}}\gamma)$ and $f \longmapsto f$, where $\bar{\mathcal{T}}\gamma = \{(\mathcal{T}_{\bar{\gamma}})_\lambda \mid \lambda \in M\}$,

$$(\mathcal{T}_{\bar{\gamma}})_\lambda = \{\mathcal{F} \mid \mathcal{F} \in \mathcal{F}_L^s(X \times X), \exists \mathcal{H} \in \gamma_\lambda : \mathcal{F} \geqslant \mathcal{H} \times \mathcal{H}\}.$$

Theorem 7.5. (1) *θ and δ are functors.*

(2) *$\delta \circ \theta \leqslant id_{\mathbf{S}(L,M)-\mathbf{SUConvTr}}$ and $\theta \circ \delta = id_{\mathbf{S}(L,M)-\mathbf{FilTr}}$.*

Proof. (1) We first prove that θ is a functor. Assume that $f : (X, \bar{\mathcal{T}}^X) \longrightarrow (Y, \bar{\mathcal{T}}^Y)$ is uniformly continuous. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $\lambda \in M$, if $\mathcal{F} \in (\gamma_{\bar{\mathcal{T}}^X})_\lambda$, then $\mathcal{F} \times \mathcal{F} \in \mathcal{T}_\lambda^X$. Since $f : (X, \bar{\mathcal{T}}^X) \longrightarrow (Y, \bar{\mathcal{T}}^Y)$ is uniformly continuous, we have $(f \times f)^{\Rightarrow}(\mathcal{F} \times \mathcal{F}) = f^{\Rightarrow}(\mathcal{F}) \times f^{\Rightarrow}(\mathcal{F}) \in \mathcal{T}_\lambda^Y$. This implies $f^{\Rightarrow}(\mathcal{F}) \in (\gamma_{\bar{\mathcal{T}}^Y})_\lambda$. Therefore $f : (X, \bar{\gamma}_{\bar{\mathcal{T}}^X}) \longrightarrow (Y, \bar{\gamma}_{\bar{\mathcal{T}}^Y})$ is uniformly continuous and θ is a functor.

We next prove that δ is a functor. If $f : (X, \bar{\gamma}^X) \longrightarrow (Y, \bar{\gamma}^Y)$ is uniformly continuous, then $f : (X, \bar{\mathcal{T}}_{\bar{\gamma}^X}) \longrightarrow (Y, \bar{\mathcal{T}}_{\bar{\gamma}^Y})$ is uniformly continuous can be proved from the following: for each $\lambda \in M, \mathcal{F} \in (\mathcal{T}_{\bar{\gamma}})_\lambda \Longrightarrow \exists \mathcal{H} \in \gamma_\lambda : \mathcal{H} \times \mathcal{H} \leqslant \mathcal{F} \Longrightarrow$

$(f \times f)^{\Rightarrow}(\mathcal{H} \times \mathcal{H}) \leqslant (f \times f)^{\Rightarrow}(\mathcal{F}) \Longrightarrow f^{\Rightarrow}(\mathcal{H}) \times f^{\Rightarrow}(\mathcal{H}) \leqslant (f \times f)^{\Rightarrow}(\mathcal{F}) \Longrightarrow (f \times f)^{\Rightarrow}(\mathcal{F}) \in (\mathcal{T}_{\bar{\gamma}})_\lambda$.

(2) Let $(X, \bar{\mathcal{T}}) \in |$ **S**(L, M)-**SUConvTr** $|$ and $(X, \bar{\gamma}) \in |$ **S**(L, M)-**FilTr** $|$. We now prove that $\bar{\mathcal{T}}_{\bar{\gamma}_{\bar{\mathcal{T}}}} \leqslant \bar{\mathcal{T}}$ and $\bar{\gamma}_{\bar{\mathcal{T}}_{\bar{\gamma}}} = \bar{\gamma}$. It is easily checked that $\bar{\mathcal{T}}_{\bar{\gamma}_{\bar{\mathcal{T}}}} \leqslant \bar{\mathcal{T}}$. For $\bar{\gamma}_{\bar{\mathcal{T}}_{\bar{\gamma}}} = \bar{\gamma}$, obviously, $\bar{\gamma}_{\bar{\mathcal{T}}_{\bar{\gamma}}} \geqslant \bar{\gamma}$. Conversely, $\mathcal{F} \in (\gamma_{\bar{\mathcal{T}}_{\bar{\gamma}}})_\lambda \Longrightarrow \mathcal{F} \times \mathcal{F} \in (\mathcal{T}_{\bar{\gamma}})_\lambda \Longrightarrow \exists \mathcal{H} \in \gamma_\lambda$: $\mathcal{F} \times \mathcal{F} \geqslant \mathcal{H} \times \mathcal{H} \Longrightarrow \forall A \in L^X, \mathcal{F}(A) = \mathcal{F} \times \mathcal{F}(A \times 1_X) \geqslant \mathcal{H} \times \mathcal{H}(A \times 1_X) = \mathcal{H}(A) \Longrightarrow \mathcal{F} \geqslant \mathcal{H} \Longrightarrow \mathcal{F} \in \gamma_\lambda$. Hence $\bar{\gamma}_{\bar{\mathcal{T}}_{\bar{\gamma}}} \leqslant \bar{\gamma}$. This implies $\delta \circ \theta \leqslant id_{\mathbf{S}(L,M)-\mathbf{SUConvTr}}$ and $\theta \circ \delta = id_{\mathbf{S}(L,M)-\mathbf{FilTr}}$ respectively. □

Corollary 7.6. **S**(L, M)-**FilTr** *can be embedded in* **S**(L, M)-**SUConvTr** *as a bicoreflective subcategory.*

8. Conclusions

We defined the notion of stratified (L, M)-semiuniform convergence tower spaces. The resulting category is a strong topological universe, hence, it is a suitable framework for studying probabilistic semiuniform convergence spaces and lattice-valued semiuniform convergence spaces. Moreover, the relations between stratified (L, M)-semiuniform convergence tower spaces and stratified (L, M)-filter tower spaces [28] are studied. It is shown that **S**(L, M)-**FilTr** can be embedded in **S**(L, M)-**SUConvTr** as a bicoreflective subcategory.

Recently, Jäger [14, 15, 16] extended the notion of a stratified L-filter and introduced the notion of a s-stratified LM-filter. This motivates us to extend our notion in this paper to the s-stratified LM-filter case in our future work.

Acknowledgements. The author would like to express her sincere thanks to the anonymous reviewers for their careful reading and valuable suggestions. This work is supported by NSFC (11501177) and Foundation of the Education Department Henan province (15A110034).

References

[1] J. Adămek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Category*, Wiley, New York, 1990.

[2] T. M. G. Ahsanullah and G. Jäger, *Probabilistic uniform convergence spaces redefined*, Acta Math. Hungar., **146(2)** (2015), 376–390.

[3] C. H. Cook and H. R. Fischer, *Uniform convergence structures*, Math. Ann., **173(4)** (1967), 290–306.

[4] A. Craig and G. Jäger, *A common framework for lattice-valued uniform spaces and probabilistic uniform limit spaces*, Fuzzy Sets and Syst., **160(9)** (2009), 1177–1203.

[5] J. M. Fang, *Lattice-valued semiuniform convergence spaces*, Fuzzy Sets and Syst., **195(7)** (2012), 33–57.

[6] J. M. Fang, *Stratified L-ordered quasiuniform limit spaces*, Fuzzy Sets and Syst., **227** (2013), 51–73.

[7] P. V. Flores, R. N. Mohapatrs and G. Richardson, *Lattice-valued spaces: Fuzzy convergence*, Fuzzy Sets and Syst., **157(20)** (2006), 2706–2714.

[8] L. C. Florescu, *Probabilistic convergence structures*, Aequationes Math., **38(2-3)** (1989), 123–145.

[9] M. J. Frank, *Probabilistic topological spaces*, J. Math. Anal. Appl., **34(1)** (1971), 67–81.
[10] J. Gutiérrez García, A Unified Approach to the Concept of Fuzzy L-uniform Space, Thesis, Universidad del Pais Vasco, Bilbao, Spain, 2000.
[11] H. Herrlich and D. Zhang, *Categorical properties of probabilistic convergence spaces* , Appl. Categ. Str., **6** (1998), 495–513.
[12] U. Höhle and A. P. Šostak, *Axiomatic foundations of fixed-basis fuzzy topology*, in: U.Höhle, S.E.Rodabaugh(Eds.), Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, in: Handbook Series, vol.3, Kluwer Academic Publishers, Dordrecht, 1999, pp.123–173.
[13] G. Jäger, *Level spaces for lattice-valued uniform convergence spaces*, Quaest. Math., **31(3)** (2008), 255–277.
[14] G. Jäger, *Stratified LMN-convergence tower spaces*, Fuzzy Sets and Syst., **282(C)** (2016), 62–73.
[15] G. Jäger, *On stratified lattice-valued convergence spaces*, Iranian Journal of Fuzzy Systems, **14(6)** (2017), 149–164.
[16] G. Jäger, *A note on stratified LM-filters*, Iranian Journal of Fuzzy Systems, **10(4)** (2013), 135–142.
[17] G. Jäger, *A category of L-fuzzy convergence spaces*, Quaest. Math., **24(4)** (2001), 501–517.
[18] G. Jäger, *Subcategory of lattice-valued convergence spaces*, Fuzzy Sets and Syst., **156(1)** (2005), 1–24.
[19] G. Jäger and M. H. Burton, *Stratified L-uniform convergence spaces*, Quaest. Math., **28(1)** (2005), 11–36.
[20] J. Minkler, G. Minkler and G. Richardson, *Subcategories of filter tower spaces*, Appl. Categ. Str., **9(4)** (2001), 369–379.
[21] M. Nauwelaerts, *(Quasi-)semi-approach uniform convergence spaces*, Scientiae Math. Japonicae, **4** (2001), 669–697.
[22] H. Nusser, *A generalization of probabilistic uniform spaces*, Appl. Categ. Str., **10(1)** (2002), 81–98.
[23] G. Preuss, *Semiuniform convergence spaces*, Math. Japon., **41(4)** (1995), 465–491.
[24] G. Preuss, *Foundations of Topology–An Approach to Convenient Topology*, Kluwer Academic Publisher, Dordrecht, Boston, London, 2002.
[25] G. D. Richardson and D. C. Kent, *Probabilistic convergence spaces*, J. Austral. Math. Soc., **61(3)** (1996), 400–420.
[26] A. Weil, *Sur les éspaces à structures uniforms et sur la topologie générale*, Actualites Scientifiques et Industrielles, no.551, Hermann, Paris, (1937), 3–40.
[27] O. Wyler, *Filter space monads, regularity, completions*, in: TOPO 1972- General Topology and its Applications, Lecture Notes in Mathematics, vol. 378, Springer, Berlin, Heidelberg, New York, (1974), 591–637.
[28] X. F. Yang and S. G. Li, *Completion of stratified (L, M)-filter tower spaces*, Fuzzy Sets and Syst., **210** (2013), 22–38.

Juan Li, Department of Mathematics, Henan Normal University, Xinxiang 453007, P. R. China and School of Mathematics, Beijing Institute of Technology, Beijing 100081, P. R. China
E-mail address: `lijuan@htu.edu.cn`