OPTIMAL STATISTICAL TESTS BASED ON FUZZY RANDOM VARIABLES

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Abstract. A novel approach is proposed for the problem of testing statistical hypotheses about the fuzzy mean of a fuzzy random variable. The concept of the (uniformly) most powerful test is extended to the (uniformly) most powerful fuzzy-valued test in which the test function is a fuzzy set representing the degrees of rejection and acceptance of the hypothesis of interest. For this purpose, the concepts of fuzzy test statistic and fuzzy critical value have been defined using the α cuts (levels) of the fuzzy observations and fuzzy parameter. In order to make a decision as a fuzzy test, a well-known method is employed to compare the observed fuzzy test statistic and the fuzzy critical value. In this work, we focus on the case in which the fuzzy data are observations of a normal fuzzy random variable. The proposed approach is general so that it can be applied to other kinds of fuzzy random variables as well. Numerical examples, including a lifetime testing problem, are provided to illustrate the proposed optimal tests.

1. Introduction

Testing statistical hypothesis is a main branch of statistical inference. In the classical parametric approaches, the methods required for the solution of a specific statistical testing problem depend strongly on certain basic assumptions that determine: to which class the distribution of a random quantity X is assumed to belong, whether all possible decisions are crisp, how exact the available data are, how exact the parameters involved in the underlying model are, and whether the hypotheses of interest are crisp. In practice, however, these assumptions are usually not warranted. In practical studies, it is frequently impossible to assume that the parameter, for which the distribution of X is determined, has a precise value; the value of the random quantity X is recorded as a precise value; the hypotheses of interest are presented as exact relations; and so on. We, therefore, need novel methods to test statistical hypotheses in the absence of some of these rigid assumptions.

Since its introduction by Zadeh [44], the fuzzy set theory has been developed and employed in some statistical contexts to deal with the above situations. In this regard, many statistical methods are being developed to analyze (descriptively and, mostly, inferentially) fuzzy-valued data (see for instance [3, 4, 5, 8, 20, 29, 33, 34] for a review). Also, an associated R package, $SAFD$ [36], has been developed to

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simplify and facilitate the performance of the standard operations on the class of fuzzy numbers; to compute some distances, sample means, variances and covariances, and sample correlations; and to perform bootstrap tests for the equality of means and a linear regression problem. Testing statistical hypotheses in fuzzy environments has been extensively studied. Below is a brief review of some of the studies relevant to the present work.

Arnold [1, 2] presented an approach for testing fuzzily formulated hypotheses based on crisp data, in which he proposed and considered generalized definitions for the probabilities of both type I and II errors. Viertl [37, 38] developed a procedure based on fuzzy observations for drawing inferences about an unknown parameter and used the extension principle to obtain the generalized estimators for a crisp parameter based on fuzzy data. He also drew some other statistical inferences for the crisp parameter, such as generalized confidence intervals, based on fuzzy data. Taheri and Behboodian [31] formulated the problem of testing fuzzy hypotheses when the observations are crisp. They presented new definitions for the probabilities of type I and II errors, and proved an extended version of the Neyman-Pearson Lemma. Torabi et al. [35] extended this approach to the case in which the data are fuzzy, too. Taheri and Behboodian [32] also studied the problem of testing hypotheses from a Bayesian point of view when the observations are ordinary and the hypotheses are fuzzy. Taheri and Arefi [30] presented an approach to the problem of testing fuzzy hypotheses, based on the so-called fuzzy critical regions. Yosefi et al. [42] presented an approach to testing fuzzy hypotheses based on a likelihood ratio test statistic they presented.

On the other hand, the topic of statistical inference, including testing hypotheses and confidence intervals, based on fuzzy random variables (FRVs), has been studied by some authors over the last two decades. For instance, Grzegorzewski [12, 13, 14] suggested some fuzzy tests for testing statistical hypotheses with vague data in parametric and non-parametric cases. Using a generalized metric for fuzzy numbers, Montenegro et al. [23, 24] proposed a method for testing hypotheses about the fuzzy mean of a FRV in one and two population settings. Gil et al. [10] and González-Rodríguez et al. [11] introduced a bootstrap approach to the one-sample and multi-sample test of means for imprecisely valued sample data. Chachi and Taheri [6] introduced a new approach to constructing fuzzy confidence intervals for the fuzzy mean of a FRV and used the approach to test hypotheses in a fuzzy environment [7]. Filzmoser and Viertl [9] and Parchami et al. [25, 26] presented p-value-based approaches to the problem of testing hypothesis, when the available data or the hypotheses of interest are fuzzy. Hryniewicz [15] and Taheri and Hesamian [33] defined some non-parametric tests in fuzzy environments. Hryniewicz [16] investigated the concept of p-value in a possibilistic context in which the concept of p-value is generalized to the case of imprecisely defined statistical hypotheses and vague statistical data. Lubiano et al. [21] focused on testing different hypotheses about means of data obtained by using the fuzzy rating scale [20]. They observed that the shape of the fuzzy assessment would scarcely affect statistical hypothesis testing conclusions about the mean values of the involved fuzzy data sets [22].

In this paper, a new approach is proposed to the problem of testing statistical hypotheses for the case when the parameter of interest is fuzzy and the available data are observations of FRVs. In our approach, the most powerful and uniformly most powerful tests are extended to the most powerful and uniformly most powerful fuzzy-valued tests, respectively. In the proposed procedure, the statistician should decide whether to accept or reject the hypothesis of interest based on a fuzzy test function.

The rest of the paper is organized as follows: In Section 2, we provide some preliminaries on fuzzy sets, fuzzy numbers, and FRVs. We define simple and one-sided hypotheses about a fuzzy parameter in Section 3. We introduce and investigate the (uniformly) most powerful fuzzy test for testing hypotheses about the mean of a normal FRV in Sections 4-6. Some numerical and practical examples are then provided in Section 7 to clarify the proposed procedures. Finally, a brief conclusion and some proposals for further study conclude the paper.

2. Preliminaries

2.1. **Fuzzy Sets and Fuzzy Numbers.** Let $\mathbb{X}$ be a universal set, which is assumed in this paper to be the set of real numbers; i.e, $\mathbb{X} = \mathbb{R}$. A fuzzy subset (briefly, a fuzzy set) $\widetilde{A}$ of $\mathbb{R}$ is defined by its membership function $\widetilde{A} : \mathbb{R} \to [0,1]$. For each $\alpha \in (0,1]$, the $\alpha-$level set (cut) of $\widetilde{A}$ is defined by $\widetilde{A}_\alpha = \{x \in \mathbb{R} \,|\, \widetilde{A}(x) \geq \alpha\}$, and $\widetilde{A}_0$ is the closure of the set $\{x \in \mathbb{R} \,|\, \widetilde{A}(x) > 0\}$. The fuzzy set $\widetilde{A}$ is called a fuzzy number if each $\widetilde{A}_\alpha$ is a nonempty closed interval for all $\alpha \in (0,1]$. The α-level of each fuzzy number $\widetilde{A}$ is usually denoted by $\widetilde{A}_\alpha = [a^l_\alpha, a^u_\alpha]$, where $a^l_\alpha = \inf\{x \in \mathbb{R} \mid A(x) \geq \alpha\}$ and $a^u_\alpha = \sup\{x \in \mathbb{R} \,|\, A(x) \geq \alpha\}$. The set of all fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R})$.

A special kind of fuzzy number is the triangular fuzzy number denoted by $\widetilde{A} = (a, a_l, a_r)_T$, where a, a_l, and a_r are the center, left, and right spreads of $\widetilde{A}$, respectively. The membership function of a triangular fuzzy number $\widetilde{A}$ is as follows:

$$\widetilde{A}(x) = \begin{cases} \frac{x-(a-a_l)}{a_l} & \text{if} \quad x \in [a-a_l, a], \\ \frac{(a+a_r)-x}{a_r} & \text{if} \quad x \in (a, a+a_r]. \end{cases}$$

It can be easily shown that the relation below captures the $\alpha-$level sets of a triangular fuzzy number $\widetilde{A}$:

$$\begin{aligned} \widetilde{A}_\alpha &= [a-(1-\alpha)a_l, a+(1-\alpha)a_r] \\ &= [a^l_\alpha, a^u_\alpha], \qquad \alpha \in [0,1], \end{aligned}$$

$\widetilde{A}$ is called a fuzzy point (crisp number) with the value m if its membership function is $\widetilde{A}(x) = I_{\{m\}}(x)$. Using the extension principle, arithmetic operations on fuzzy numbers are defined by $(\widetilde{A} \bigcirc \widetilde{B})(z) = \sup_{x,y\,:\,x\circ y=z} \min\{\widetilde{A}(x), \widetilde{B}(y)\}$, where $\bigcirc$ is any kind of the extended arithmetic operations $\oplus$, $\ominus$, $\otimes$, and $\oslash$, and $\circ$ is any kind of the arithmetic operations $+$, $-$, $\times$, and $/$. It is well-known that if $\widetilde{A}$ and $\widetilde{B}$ are two fuzzy numbers, then $\widetilde{A} \oplus \widetilde{B}$ is also a fuzzy number and $(\widetilde{A} \oplus \widetilde{B})_\alpha = [a^l_\alpha + b^l_\alpha, a^u_\alpha + b^u_\alpha]$, for all $\alpha \in (0,1]$ (for more details on fuzzy arithmetic, see e.g. [17]).

A well-known ordering on fuzzy numbers to be used in the sections below is defined as follows [40]:

(1) $\widetilde{A} = \widetilde{B}$, if $a^l_\alpha = b^l_\alpha$ and $a^u_\alpha = b^u_\alpha$ for any $\alpha \in (0,1]$.
(2) $\widetilde{A} \preceq (\prec)\widetilde{B}$, if $a^l_\alpha \leq (<) b^l_\alpha$ and $a^u_\alpha \leq (<) b^u_\alpha$ for any $\alpha \in (0,1]$.
(3) $\widetilde{A} \succeq (\succ)\widetilde{B}$, if $a^l_\alpha \geq (>) b^l_\alpha$ and $a^u_\alpha \geq (>) b^u_\alpha$ for any $\alpha \in (0,1]$.

In the sequel, we explore a criterion for comparing the observed fuzzy statistics and the observed fuzzy critical values. There are many ways to carry out this comparison. We use Yuan's approach since it is reasonable and has certain relevant properties [39, 43].

Definition 2.1. Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy numbers. Let also:

$$\Delta_{\widetilde{A}\widetilde{B}} = \int_{a^u_\alpha > b^l_\alpha} (a^u_\alpha - b^l_\alpha)\, d\alpha + \int_{a^l_\alpha > b^u_\alpha} (a^l_\alpha - b^u_\alpha)\, d\alpha.$$

Then, the truth degree of "$\widetilde{A}$ is greater than $\widetilde{B}$" is defined as $\mathcal{D}(\widetilde{A} > \widetilde{B}) = \frac{\Delta_{\widetilde{A}\widetilde{B}}}{\Delta_{\widetilde{A}\widetilde{B}} + \Delta_{\widetilde{B}\widetilde{A}}}$, and the truth degree of "$\widetilde{A}$ is smaller than $\widetilde{B}$" is defined as $\mathcal{D}(\widetilde{A} < \widetilde{B}) = 1 - \mathcal{D}(\widetilde{A} > \widetilde{B})$.

2.2. **Fuzzy Random Variables.** Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space, $\mathcal{X} : \Omega \to \mathcal{F}(\mathbb{R})$ be a fuzzy-valued function, X be a random variable having a distribution of $f_{\boldsymbol{\theta}}$ with the parameters $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p) \in \Theta^p$; i.e., $X \sim f_{\boldsymbol{\theta}}$ and $\Theta^p \subset \mathbb{R}^p$ be the parameter space, where $p \geq 1$. Throughout this paper, we assume that all random variables have the same probability space $(\Omega, \mathcal{A}, \mathcal{P})$.

The fuzzy-valued function $\mathcal{X} : \Omega \to \mathcal{F}(\mathbb{R})$ is called a FRV if $X^l_\alpha : \Omega \to (\mathbb{R})$ and $X^u_\alpha : \Omega \to (\mathbb{R})$ are two real valued random variables for all $\alpha \in (0,1]$ (where $\forall \omega \in \Omega;\ \mathcal{X}(\omega)_\alpha = [X^l_\alpha(\omega), X^u_\alpha(\omega)]$) [27].

FRVs $\mathcal{X}$ and $\mathcal{Y}$ are called identically distributed if X^l_α and Y^l_α are identically distributed, and X^u_α and Y^u_α are identically distributed, for all $\alpha \in (0,1]$. They are called independent if each random variable in the set $\{X^l_\alpha,\, X^u_\alpha :\ \alpha \in (0,1]\}$ is independent of each random variable in the set $\{Y^l_\alpha,\, Y^u_\alpha :\ \alpha \in (0,1]\}$ [41].

FRV $\mathcal{X}$ is said to have the same distribution as X with fuzzy parameters $\widetilde{\boldsymbol{\theta}} = (\widetilde{\theta}_1, \ldots, \widetilde{\theta}_p)$ if for all $\alpha \in (0,1]$, $X^l_\alpha \sim f_{\boldsymbol{\theta}^l_\alpha}$ and $X^u_\alpha \sim f_{\boldsymbol{\theta}^u_\alpha}$, where $\boldsymbol{\theta}^l_\alpha = (\theta^l_{1\alpha}, \ldots, \theta^l_{p\alpha})$, $\boldsymbol{\theta}^u_\alpha = (\theta^u_{1\alpha}, \ldots, \theta^u_{p\alpha})$, and $\widetilde{\theta}_{j\alpha} = [\theta^l_{j\alpha}, \theta^u_{j\alpha}]$, $j = 1, \ldots, p$ [41].

For example, $\mathcal{X}$ is normally distributed with the fuzzy parameters $\widetilde{\theta}$ and $\widetilde{\sigma}^2$ if and only if $X^l_\alpha \sim N(\theta^l_\alpha, \sigma^{2l}_\alpha)$ and $X^u_\alpha \sim N(\theta^u_\alpha, \sigma^{2u}_\alpha)$, for all $\alpha \in (0,1]$.

Definition 2.2. [41] We say that $\boldsymbol{\mathcal{X}} = (\mathcal{X}_1, \cdots, \mathcal{X}_n)$ is a normal fuzzy random sample of size n with the fuzzy parameters $\widetilde{\theta}$ and $\widetilde{\sigma}^2$, if $\mathcal{X}_i$'s are independent and identically distributed normal FRVs with fuzzy parameters $\widetilde{\theta}$ and $\widetilde{\sigma}^2$ for all $i = 1, \ldots, n$. In this case, we write $\mathcal{X}_1, \cdots, \mathcal{X}_n \overset{i.i.d.}{\sim} N(\widetilde{\theta}, \widetilde{\sigma}^2)$.

Corollary 2.3. [41] *Let $\mathcal{X}_1, \cdots, \mathcal{X}_n \overset{i.i.d.}{\sim} N(\widetilde{\theta}, \widetilde{\sigma}^2)$, then $X^l_{1\alpha}, \cdots, X^l_{n\alpha} \overset{i.i.d.}{\sim} N(\theta^l_\alpha, \sigma^{2l}_\alpha)$ and $X^u_{1\alpha}, \cdots, X^u_{n\alpha} \overset{i.i.d.}{\sim} N(\theta^u_\alpha, \sigma^{2u}_\alpha)$ for all $\alpha \in (0,1]$. In the case of $\widetilde{\sigma}^2 = I_{\{\sigma^2\}}$ (i.e., $\widetilde{\sigma}^2$ is a crisp parameter), we also see that for all $\alpha \in (0,1]$, $X^l_{1\alpha}, \ldots, X^l_{n\alpha} \overset{i.i.d.}{\sim}$*

$N(\theta^l_\alpha, \sigma^2)$ and $X^u_{1\alpha}, \ldots, X^u_{n\alpha} \overset{i.i.d.}{\sim} N(\theta^u_\alpha, \sigma^2)$. In this case, we write $\mathcal{X}_1, \ldots, \mathcal{X}_n \overset{i.i.d.}{\sim} N(\widetilde{\theta}, \sigma^2)$.

2.3. **Testing Statistical Hypotheses: the Classical Approach.** Here, we briefly review the essentials of the traditional method of testing statistical hypotheses, (For more details, see [19]). Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random sample with the observed value $\mathbf{x} = (x_1, \ldots, x_n)$ from a distribution P_θ with an unknown parameter θ. A hypothesis testing problem may be regarded as a decision-making problem where decisions have to be made about the truth of two propositions: the null hypothesis $H_0 : \theta \in \Theta_0$, and the alternative $H_1 : \theta \in \Theta_1$, based on a sample $\mathbf{X} = \mathbf{x}$ (where $\Theta = \Theta_0 \cup \Theta_1$ is the parameter space). The test function (decision rule) is then usually based on a test statistic $T(\mathbf{X})$ which is evaluated for the sample, resulting in the value $t(\mathbf{x})$.

In such problems, every Borel-measurable mapping $\varphi : \mathbb{R}^n \to \{0, 1\}$ is known as a (non-randomized) test function. The power function of $\varphi(\mathbf{X})$ is defined by $\pi_\varphi(\theta) = E[\varphi(\mathbf{X})] = P_\theta\{\text{Rejection of } H_0\}$. A test φ is said to be a test of the significance level $\delta \in [0, 1]$ if $\sup_{\theta \in \Theta_0} \pi_\varphi(\theta) = \delta$. The δ level test φ is called a uniformly most powerful (UMP) test at level δ if its power function dominates that of any δ level test, uniformly on $\theta \in \Theta_1$.

Usually, in a procedure of testing hypotheses, the space of values of the test statistic T is decomposed into a rejection (critical) region R and its complement R^c, the acceptance region. The hypothesis H_0 is completely rejected if and only if the value $t(\mathbf{x})$ falls within the rejection region R.

In the sequel, we introduce a test function that is a fuzzy set representing the degrees of rejection and acceptance of a hypothesis of interest.

3. Types of Hypotheses for a Fuzzy Parameter, and the Fuzzy Test Function

Definition 3.1. Let $\widetilde{\Theta} = \mathcal{F}(\Theta)$ be the class of all fuzzy numbers on the parameter space Θ and $\widetilde{\theta}_0$ be a known fuzzy number in Θ. Then:

(1) Any hypothesis of the form "$H : \widetilde{\theta} = \widetilde{\theta}_0$" is called a simple hypothesis.
(2) Any hypothesis of the form "$H : \widetilde{\theta} \succ \widetilde{\theta}_0$" is called a right one-sided hypothesis.
(3) Any hypothesis of the form "$H : \widetilde{\theta} \prec \widetilde{\theta}_0$" is called a left one-sided hypothesis.

Note that, in the above framework, the possible values of the parameter of interest are expressed as linguistic variables [17]. The unknown fuzzy parameter $\widetilde{\theta}$ may be treated as a fuzzy perception of the usual unknown fuzzy parameter θ [18, 14].

Example 3.2. Let $\widetilde{\Theta} = \mathcal{F}(\mathbb{R})$ be the parameter space for the mean of a normal fuzzy random variable $\mathcal{X}$. Then:

(1) The hypothesis "$H : \widetilde{\theta} = (1, 1, 2)_T$" is a simple hypothesis.
(2) The hypothesis "$H : \widetilde{\theta} \succ (1, 1, 2)_T$" is a right one-sided hypothesis. This hypothesis is equivalent to "$H : \widetilde{\theta} \in \widetilde{\Theta}_1 = \{\widetilde{\theta} \in \widetilde{\Theta} \mid \theta^l_\alpha > \alpha,\ \theta^u_\alpha > 3 -$

$2\alpha;\ \forall\alpha \in (0,1]\}$", where, $[\alpha, 3-2\alpha]$ is the $\alpha-$level set of the triangular fuzzy number $(1,1,2)_T$.

(3) The hypothesis "$H : \widetilde{\theta} \prec (1,1,2)_T$" is a left one-sided hypothesis. This hypothesis is equivalent to "$H : \widetilde{\theta} \in \widetilde{\Theta}_1 = \{\widetilde{\theta} \in \widetilde{\Theta} \mid \theta^l_\alpha < \alpha,\ \theta^u_\alpha < 3 - 2\alpha;\ \forall\alpha \in (0,1]\}$".

In the problem of testing a fuzzy hypothesis based on the fuzzy random sample $(\mathcal{X}_1, \cdots, \mathcal{X}_n)$, we propose the following definition of a fuzzy test function:

Definition 3.3. The fuzzy-valued function $\widetilde{\varphi} : \mathcal{F}^n(\mathbb{R}) \to \mathcal{F}(\{0,1\})$ is called a fuzzy test function whenever $\widetilde{\varphi}(\boldsymbol{\mathcal{X}})$, as a fuzzy set on $\{0,1\}$, shows the degrees of rejection and acceptance of the hypothesis H_0 for any $\boldsymbol{\mathcal{X}}$. Here, "0" and "1" stand for acceptance and rejection of the null hypothesis, respectively, and $\mathcal{F}(\{0,1\})$ is the class of all fuzzy sets on $\{0,1\}$.

4. **Testing Simple Hypotheses About the Mean of a Normal FRV**

In this section, based on the fuzzy random sample $\mathcal{X}_1, \ldots, \mathcal{X}_n \overset{i.i.d.}{\sim} N(\widetilde{\theta}, \sigma^2)$ where σ^2 is known, we develop a procedure for testing simple hypotheses

$$H_0 : \widetilde{\theta} = \widetilde{\theta}_0 \quad versus \quad H_1 : \widetilde{\theta} = \widetilde{\theta}_1, \tag{1}$$

in which, it is assumed that $\widetilde{\theta}_0 \prec \widetilde{\theta}_1$ or $\widetilde{\theta}_0 \succ \widetilde{\theta}_1$.

4.1. **Case I: $\widetilde{\theta}_0 \prec \widetilde{\theta}_1$.** Based on the fuzzy random sample $\mathcal{X}_1, \ldots, \mathcal{X}_n \overset{i.i.d.}{\sim} N(\widetilde{\theta}, \sigma^2)$, and according to Corollary 2.3, we have following crisp samples for any $\alpha \in (0,1]$:

$$X^l_{1\alpha}, \ldots, X^l_{n\alpha} \overset{i.i.d.}{\sim} N(\theta^l_\alpha, \sigma^2), \tag{2}$$

$$X^u_{1\alpha}, \ldots, X^u_{n\alpha} \overset{i.i.d.}{\sim} N(\theta^u_\alpha, \sigma^2). \tag{3}$$

So, based on the random samples in (2) and (3), for any $\alpha \in (0,1]$, we could decompose the hypothesis testing problem (1) into the following testing problems

$$H^l_0 : \theta^l_\alpha = \theta^l_{0\alpha} \quad versus \quad H^l_1 : \theta^l_\alpha = \theta^l_{1\alpha}, \tag{4}$$

$$H^u_0 : \theta^u_\alpha = \theta^u_{0\alpha} \quad versus \quad H^u_1 : \theta^u_\alpha = \theta^u_{1\alpha}, \tag{5}$$

As is already known, the classical test functions, on the significance level δ, for testing hypotheses (4) and (5) are given by

$$\varphi^l_\alpha(X^l_{1\alpha}, \ldots, X^l_{n\alpha}) = \begin{cases} 1 & \bar{X}^l_\alpha \geq \theta^l_{0\alpha} + z_{1-\delta}\frac{\sigma}{\sqrt{n}} \\ 0 & \bar{X}^l_\alpha < \theta^l_{0\alpha} + z_{1-\delta}\frac{\sigma}{\sqrt{n}} \end{cases}, \tag{6}$$

$$\varphi^u_\alpha(X^u_{1\alpha}, \ldots, X^u_{n\alpha}) = \begin{cases} 1 & \bar{X}^u_\alpha \geq \theta^u_{0\alpha} + z_{1-\delta}\frac{\sigma}{\sqrt{n}} \\ 0 & \bar{X}^u_\alpha < \theta^u_{0\alpha} + z_{1-\delta}\frac{\sigma}{\sqrt{n}} \end{cases}, \tag{7}$$

where, $\bar{X}^l_\alpha = \frac{1}{n}\sum_{i=1}^n X^l_{i\alpha}$, $\bar{X}^u_\alpha = \frac{1}{n}\sum_{i=1}^n X^u_{i\alpha}$, and z_δ is the $\delta-$quintile of the standard normal distribution; i.e., $\Phi(z_\delta) = \delta$.

To aggregate the results of the above crisp-testing problems appropriately, let us define a fuzzy critical value and a fuzzy test statistic. Assume that $c^l_\alpha = \theta^l_{0\alpha} + z_{1-\delta}\frac{\sigma}{\sqrt{n}}$ and $c^u_\alpha = \theta^u_{0\alpha} + z_{1-\delta}\frac{\sigma}{\sqrt{n}}$. Clearly, $c^l_\alpha \leq c^u_\alpha$ for any $\alpha \in (0, 1]$, so $[c^l_\alpha, c^u_\alpha]$ can be taken as $\alpha-$level set of a fuzzy set like $\widetilde{C}$. We call $\widetilde{C}$ as the fuzzy critical value for testing hypothesis (1). On the contrary, the related fuzzy test statistic can be defined as the fuzzy set $\bar{\mathcal{X}} = \frac{1}{n} \otimes (\mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_n)$, with $\alpha-$level sets $[\bar{X}^l_\alpha, \bar{X}^u_\alpha]$. Finally, to test the hypothesis (1), we need a method for comparing the observed fuzzy test statistic $\bar{\mathcal{X}}$ with the observed fuzzy critical value $\widetilde{C}$.

Definition 4.1. For the problem of testing hypothesis (1), based on the normal fuzzy sample $(\mathcal{X}_1, \ldots, \mathcal{X}_n) \in \mathcal{F}^n(\mathbb{R})$ (in case $\widetilde{\theta}_0 \prec \widetilde{\theta}_1$), the test function is defined as in the following fuzzy set

$$\widetilde{\varphi}(\mathcal{X}_1, \ldots, \mathcal{X}_n) = \{\frac{\mathcal{D}(\bar{\mathcal{X}} < \widetilde{C})}{0}, \frac{\mathcal{D}(\bar{\mathcal{X}} > \widetilde{C})}{1}\}.$$

This test accepts the null hypothesis with the degree of acceptance $\mathcal{D}(\bar{\mathcal{X}} < \widetilde{C})$ and rejects it with the degree of rejection $\mathcal{D}(\bar{\mathcal{X}} > \widetilde{C}) = 1 - \mathcal{D}(\bar{\mathcal{X}} < \widetilde{C})$.

Note that, using the fuzzy arithmetic, we have $\widetilde{C}_\alpha = \widetilde{\theta}_{0\alpha} \oplus I_{\{z_{1-\delta}\frac{\sigma}{\sqrt{n}}\}}\alpha$, for any $\alpha \in (0, 1]$. So, $\widetilde{C} = \widetilde{\theta}_0 \oplus I_{\{z_{1-\delta}\frac{\sigma}{\sqrt{n}}\}}$ and the above fuzzy test function becomes

$$\widetilde{\varphi}(\mathcal{X}_1, \ldots, \mathcal{X}_n) = \{\frac{\mathcal{D}(\bar{\mathcal{X}} < \widetilde{\theta}_0 \oplus I_{\{z_{1-\delta}\frac{\sigma}{\sqrt{n}}\}})}{0}, \frac{\mathcal{D}(\bar{\mathcal{X}} > \widetilde{\theta}_0 \oplus I_{\{z_{1-\delta}\frac{\sigma}{\sqrt{n}}\}})}{1}\}. \quad (8)$$

Remark 4.2. It is easily seen that our test leads to a fuzzy decision. This decision combines both uncertainties due to randomness and fuzziness [4]. The significance level comes from the probabilistic nature of sampling. On the other hand, degrees of acceptance and rejection are related to the fuzziness considered in the data and parameter. It should be mentioned that the method of comparison presented in Definition 2.1 relies on reasonable and robust properties proved in [39] and [43]. Moreover, the comparison approach used in our definition of the test is subjective, so that nothing of the main results of the present work will be lost by altering this definition to one which fits decision makers' requirements.

Definition 4.3. The power of the fuzzy test $\widetilde{\varphi}$ in $\widetilde{\theta}_1$ is defined to be the fuzzy set $\pi_{\widetilde{\varphi}}(\widetilde{\theta}_1)$ with $\alpha-$level sets $[\pi^l_\alpha(\widetilde{\theta}_1), \pi^u_\alpha(\widetilde{\theta}_1)]$, where

$$\pi^l_\alpha(\widetilde{\theta}_1) = \min\{\inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_\alpha}\left(\theta^l_{1\beta}\right), \sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_\alpha}\left(\theta^u_{1\beta}\right)\},$$

$$\pi^u_\alpha(\widetilde{\theta}_1) = \max\{\inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_\alpha}\left(\theta^l_{1\beta}\right), \sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_\alpha}\left(\theta^u_{1\beta}\right)\},$$

in which, $\pi_{\varphi^l_\alpha}(\cdot)$ and $\pi_{\varphi^u_\alpha}(\cdot)$ are the powers of tests φ^l_α and φ^u_α for testing hypotheses (4) and (5), respectively; i.e.,

$$\pi_{\varphi^l_\alpha}(\theta^l_{1\alpha}) = \Phi[\frac{\sqrt{n}}{\sigma}(\theta^l_{1\alpha} - \theta^l_{0\alpha}) - z_{1-\delta}],$$

$$\pi_{\varphi^u_\alpha}(\theta^u_{1\alpha}) = \Phi[\frac{\sqrt{n}}{\sigma}(\theta^u_{1\alpha} - \theta^u_{0\alpha}) - z_{1-\delta}].$$

Definition 4.4. In the problem of testing hypothesis (1), we say that

(1) the fuzzy test $\widetilde{\varphi}$ is of size δ, whenever $\pi_{\widetilde{\varphi}}(\widetilde{\theta}_0) = I_{\{\delta\}}$,

(2) the fuzzy test $\widetilde{\varphi}$ is at level δ, whenever $\pi_{\widetilde{\varphi}}(\widetilde{\theta}_0) = I_{\{\delta'\}}$ and $\delta' \leq \delta$.

Definition 4.5. Consider the problem of testing hypothesis (1). We say that the fuzzy test $\widetilde{\varphi}$ is a most powerful (MP) fuzzy test of size δ, if

(1) $\widetilde{\varphi}$ is of size δ, and

(2) for any fuzzy test $\widetilde{\varphi}_*$ of size δ, $\pi_{\widetilde{\varphi}_*}(\widetilde{\theta}_1) \preceq \pi_{\widetilde{\varphi}}(\widetilde{\theta}_1)$.

Theorem 4.6. *For the problem of testing hypothesis (1) (with $\widetilde{\theta}_0 \prec \widetilde{\theta}_1$), the fuzzy test $\widetilde{\varphi}$ in (8) is a MP fuzzy test of size δ.*

Proof. It is clear that, for testing hypotheses (4) and (5), the test functions in (6) and (7) are at the significance level δ. Since for any $\alpha \in (0,1]$ we have

$$[\pi_{\varphi^l_\alpha}(\theta^l_{0\alpha}), \pi_{\varphi^u_\alpha}(\theta^u_{0\alpha})] = [\delta, \delta] = \delta,$$

it holds that $\pi_{\widetilde{\varphi}}(\widetilde{\theta}_0) = I_{\{\delta\}}$.
Now, let $\widetilde{\varphi}_*$ be any fuzzy test of size δ for testing hypothesis (1). We have to show that $\pi_{\widetilde{\varphi}_*}(\widetilde{\theta}_1) \preceq \pi_{\widetilde{\varphi}}(\widetilde{\theta}_1)$, (with the $\alpha-$level sets in Definition 4.3). For any $\alpha \in (0,1]$, let $\varphi^l_{*\alpha}$ and $\varphi^u_{*\alpha}$ be test functions for testing hypotheses (4) and (5), using the two random samples (2) and (3), respectively. Note that $\widetilde{\theta}_0 \prec \widetilde{\theta}_1$, therefore $\theta^l_{0\alpha} < \theta^l_{1\alpha}$ and $\theta^u_{0\alpha} < \theta^u_{1\alpha}$ for any $\alpha \in (0,1]$. Since φ^l_α and φ^u_α are MP tests for testing hypotheses (4) and (5), respectively, then for the test functions $\varphi^l_{*\alpha}$ and $\varphi^u_{*\alpha}$, we have $\pi_{\varphi^l_{*\alpha}}(\theta^l_{1\alpha}) \leq \pi_{\varphi^l_\alpha}(\theta^l_{1\alpha})$ and $\pi_{\varphi^u_{*\alpha}}(\theta^u_{1\alpha}) \leq \pi_{\varphi^u_\alpha}(\theta^u_{1\alpha})$, for any $\alpha \in (0,1]$. As $\pi_{\varphi^l_\alpha}(\theta^l_{1\beta})$ and $\pi_{\varphi^u_\alpha}(\theta^u_{1\beta})$ are the increasing functions of β and decreasing functions of β, respectively, we have

$$\inf_{\alpha\leq\beta\leq1} \pi_{\varphi^l_{*\alpha}}(\theta^l_{1\beta}) \leq \pi_{\varphi^l_{*\alpha}}(\theta^l_{1\alpha}) \leq \pi_{\varphi^l_\alpha}(\theta^l_{1\alpha}) \leq \pi_{\varphi^l_\alpha}(\theta^l_{1\beta}) \qquad \forall\alpha \leq \beta \leq 1,$$

$$\sup_{\alpha\leq\beta\leq1} \pi_{\varphi^u_\alpha}(\theta^u_{1\beta}) \geq \pi_{\varphi^u_\alpha}(\theta^u_{1\alpha}) \geq \pi_{\varphi^u_{*\alpha}}(\theta^u_{1\alpha}) \geq \pi_{\varphi^u_{*\alpha}}(\theta^u_{1\beta}) \qquad \forall\alpha \leq \beta \leq 1.$$

Therefore,

$$\inf_{\alpha\leq\beta\leq1} \pi_{\varphi^l_{*\alpha}}(\theta^l_{1\beta}) \leq \inf_{\alpha\leq\beta\leq1} \pi_{\varphi^l_\alpha}(\theta^l_{1\beta}),$$

$$\sup_{\alpha\leq\beta\leq1} \pi_{\varphi^u_{*\alpha}}(\theta^u_{1\beta}) \leq \sup_{\alpha\leq\beta\leq1} \pi_{\varphi^u_\alpha}(\theta^u_{1\beta}).$$

Now

$$\min\{\inf_{\alpha\leq\beta\leq1} \pi_{\varphi^l_{*\alpha}}(\theta^l_{1\alpha}), \sup_{\alpha\leq\beta\leq1} \pi_{\varphi^u_{*\alpha}}(\theta^u_{1\alpha})\} \leq \inf_{\alpha\leq\beta\leq1} \pi_{\varphi^l_{*\alpha}}(\theta^l_{1\alpha}) \leq \inf_{\alpha\leq\beta\leq1} \pi_{\varphi^l_\alpha}(\theta^l_{1\alpha}),$$

$$\min\{\inf_{\alpha\leq\beta\leq1} \pi_{\varphi^l_{*\alpha}}(\theta^l_{1\alpha}), \sup_{\alpha\leq\beta\leq1} \pi_{\varphi^u_{*\alpha}}(\theta^u_{1\alpha})\} \leq \sup_{\alpha\leq\beta\leq1} \pi_{\varphi^u_{*\alpha}}(\theta^u_{1\alpha}) \leq \sup_{\alpha\leq\beta\leq1} \pi_{\varphi^u_\alpha}(\theta^u_{1\alpha}).$$

And thus

$$\min\{\inf_{\alpha\leq\beta\leq1} \pi_{\varphi^l_{*\alpha}}(\theta^l_{1\alpha}), \sup_{\alpha\leq\beta\leq1} \pi_{\varphi^u_{*\alpha}}(\theta^u_{1\alpha})\} \leq \min\{\inf_{\alpha\leq\beta\leq1} \pi_{\varphi^l_\alpha}(\theta^l_{1\alpha}), \sup_{\alpha\leq\beta\leq1} \pi_{\varphi^u_\alpha}(\theta^u_{1\alpha})\}.$$

Similarly, we have

$$\inf_{\alpha\leq\beta\leq1} \pi_{\varphi^l_{*\alpha}}(\theta^l_{1\alpha}) \leq \inf_{\alpha\leq\beta\leq1} \pi_{\varphi^l_\alpha}(\theta^l_{1\alpha}) \leq \max\{\inf_{\alpha\leq\beta\leq1} \pi_{\varphi^l_\alpha}(\theta^l_{1\alpha}), \sup_{\alpha\leq\beta\leq1} \pi_{\varphi^u_\alpha}(\theta^u_{1\alpha})\},$$

$$\sup_{\alpha\leq\beta\leq1} \pi_{\varphi^u_{*\alpha}}(\theta^u_{1\alpha}) \leq \sup_{\alpha\leq\beta\leq1} \pi_{\varphi^u_\alpha}(\theta^u_{1\alpha}) \leq \max\{\inf_{\alpha\leq\beta\leq1} \pi_{\varphi^l_\alpha}(\theta^l_{1\alpha}), \sup_{\alpha\leq\beta\leq1} \pi_{\varphi^u_\alpha}(\theta^u_{1\alpha})\},$$

and it follows that

$$\max\{\inf_{\alpha\leq\beta\leq 1}\pi_{\varphi^l_{*\alpha}}(\theta^l_{1\alpha}),\ \sup_{\alpha\leq\beta\leq 1}\pi_{\varphi^u_{*\alpha}}(\theta^u_{1\alpha})\}\leq \max\{\inf_{\alpha\leq\beta\leq 1}\pi_{\varphi^l_{\alpha}}(\theta^l_{1\alpha}),\ \sup_{\alpha\leq\beta\leq 1}\pi_{\varphi^u_{\alpha}}(\theta^u_{1\alpha})\}.$$

So, for any fuzzy test $\widetilde{\varphi}_*$ of size δ, $\pi_{\widetilde{\varphi}_*}(\widetilde{\theta}_1)\preceq\pi_{\widetilde{\varphi}}(\widetilde{\theta}_1)$, which completes the proof. □

4.2. **Case II: $\widetilde{\theta}_0\succ\widetilde{\theta}_1$.** Similar to the argument in Section 4.1, in case $\widetilde{\theta}_0\succ\widetilde{\theta}_1$, the fuzzy test function for testing hypothesis (1) can be obtained as follows

$$\widetilde{\varphi}(\mathcal{X}_1,\ldots,\mathcal{X}_n)=\{\frac{D(\bar{\mathcal{X}}>\widetilde{\theta}_0\ominus I_{\{z_{1-\delta}\frac{\sigma}{\sqrt{n}}\}})}{0},\frac{D(\bar{\mathcal{X}}<\widetilde{\theta}_0\ominus I_{\{z_{1-\delta}\frac{\sigma}{\sqrt{n}}\}})}{1}\}. \tag{9}$$

Theorem 4.7. *For the problem of testing hypothesis (1) (with $\widetilde{\theta}_0\succ\widetilde{\theta}_1$), the fuzzy test $\widetilde{\varphi}$ in (9) is a MP fuzzy test of size δ.*

Proof. The proof is similar to that of Theorem 4.6. □

Remark 4.8. It may be noted that if the hypotheses of interest reduce to crisp ones, then the optimal test in Theorem 4.6 reduces to the MP test in the classical hypothesis testing. Note that based on Corollary 2.3, using the indicator functions $I_{\{\theta\}}$ and $I_{\{X_i\}}$ as the membership functions for the fuzzy parameter $\widetilde{\theta}$ and fuzzy data $\mathcal{X}_i$, $i=1,\ldots,n$, respectively, reduces the fuzzy random sample $\mathcal{X}_1,\ldots,\mathcal{X}_n\overset{i.i.d.}{\sim}N(\widetilde{\theta},\sigma^2)$ to the crisp sample $X_1,\ldots,X_n\overset{i.i.d.}{\sim}N(\theta,\sigma^2)$. Also, note that the two samples in equations (2) and (3) are the same as above random sample because for any $\alpha\in(0,1]$, we have

$$\begin{aligned}\mathcal{X}_{i\alpha}&=[X^l_{i\alpha},X^u_{i\alpha}]=[X_i,X_i]=X_i\qquad i=1,\ldots,n,\\ \widetilde{\theta}_\alpha&=[\theta^l_\alpha,\theta^u_\alpha]=[\theta,\theta]=\theta.\end{aligned}$$

Now, consider the problem of testing the following fuzzy hypotheses

$$H_0:\widetilde{\theta}=\widetilde{\theta}_0\qquad versus\qquad H_1:\widetilde{\theta}\succ\widetilde{\theta}_0,$$

where $\widetilde{\theta}_0$ has the indicator function $I_{\{\theta_0\}}$ as its membership function. According to equations (6) and (7), the above hypothesis testing problem can be decomposed into the following testing problems

$$\begin{aligned}H^l_0:\theta^l_\alpha=\theta^l_{0\alpha}\quad &versus\quad H^l_1:\theta^l_\alpha>\theta^l_{0\alpha},\\ H^u_0:\theta^u_\alpha=\theta^u_{0\alpha}\quad &versus\quad H^u_1:\theta^u_\alpha>\theta^u_{0\alpha}.\end{aligned}$$

Since $\theta^l_\alpha=\theta^u_\alpha=\theta$ and $\theta^l_{0\alpha}=\theta^u_{0\alpha}=\theta_0$, it can be concluded that the above hypotheses are equivalent to the following ones:

$$H_0:\theta=\theta_0\quad versus\quad H_1:\theta>\theta_0,$$

It is clear that, based on the random sample $X_1,\ldots,X_n\overset{i.i.d.}{\sim}N(\theta,\sigma^2)$, the classical most powerful test function at the significance level δ for testing the above hypotheses will be as follows

$$\varphi(X_1,\ldots,X_n)=\begin{cases}1 & \bar{X}\geq\theta_0+z_{1-\delta}\frac{\sigma}{\sqrt{n}}\\ 0 & \bar{X}<\theta_0+z_{1-\delta}\frac{\sigma}{\sqrt{n}}\end{cases}.$$

Thus, we showed that the testing hypothesis problem with the usual (precise) hypotheses and crisp data is a special case of the testing fuzzy hypotheses problem considered in Theorem 4.6. Similar results may be concluded for the other theorems provided in this paper.

5. Testing Simple Hypothesis Versus Right One-sided Hypothesis

In this section, we exploit the fuzzy random sample $\mathcal{X}_1, \ldots, \mathcal{X}_n \overset{i.i.d.}{\sim} N(\widetilde{\theta}, \sigma^2)$ with a known variance σ^2 to investigate the problem of testing the following fuzzy hypotheses

$$H_0 : \widetilde{\theta} = \widetilde{\theta}_0 \qquad versus \qquad H_1 : \widetilde{\theta} \succ \widetilde{\theta}_0, \tag{10}$$

In this case, we propose a so-called uniformly most powerful fuzzy test.

First, note that the test functions (6) and (7) are also test functions at the significance level δ for testing the following hypotheses [19]

$$H_0^l : \theta_\alpha^l = \theta_{0\alpha}^l \qquad versus \qquad H_1^l : \theta_\alpha^l > \theta_{0\alpha}^l, \tag{11}$$

$$H_0^u : \theta_\alpha^u = \theta_{0\alpha}^u \qquad versus \qquad H_1^u : \theta_\alpha^u > \theta_{0\alpha}^u. \tag{12}$$

Based on the procedure employed for testing the above hypotheses for all $\alpha \in (0, 1]$, we can define the fuzzy test (8) as a fuzzy one for testing hypothesis (10). Now, let us define the concepts of power function and UMP fuzzy test for evaluating such a fuzzy test.

Definition 5.1. For the problem of testing hypothesis (10), the power function of any fuzzy test $\widetilde{\varphi}$ is defined as the fuzzy-valued function $\pi_{\widetilde{\varphi}} : \widetilde{\Theta}_1 \to \mathcal{F}([0, 1])$, where at each $\widetilde{\theta} \in \widetilde{\Theta}_1$, $\pi_{\widetilde{\varphi}}(\widetilde{\theta})$ is a fuzzy set with $\alpha-$level sets $[\pi_\alpha^l(\widetilde{\theta}), \pi_\alpha^u(\widetilde{\theta})]$, where

$$\pi_\alpha^l(\widetilde{\theta}) = \min\{\inf_{\alpha\leq\beta\leq1} \pi_{\varphi_\alpha^l}\left(\theta_\beta^l\right), \sup_{\alpha\leq\beta\leq1} \pi_{\varphi_\alpha^u}\left(\theta_\beta^u\right)\},$$

$$\pi_\alpha^u(\widetilde{\theta}) = \max\{\inf_{\alpha\leq\beta\leq1} \pi_{\varphi_\alpha^l}\left(\theta_\beta^l\right), \sup_{\alpha\leq\beta\leq1} \pi_{\varphi_\alpha^u}\left(\theta_\beta^u\right)\},$$

in which, $\pi_{\varphi_\alpha^l}(\cdot)$ and $\pi_{\varphi_\alpha^u}(\cdot)$ are the power functions of tests φ_α^l and φ_α^u for testing hypotheses (11) and (12), respectively.

Definition 5.2. In the problem of testing hypothesis (10), we say that the fuzzy test $\widetilde{\varphi}$ is the uniformly most powerful (UMP) fuzzy test of size δ, if

(1) $\widetilde{\varphi}$ is of size δ, and
(2) for any fuzzy test $\widetilde{\varphi}_*$ of size δ, $\pi_{\widetilde{\varphi}_*}(\widetilde{\theta}) \preceq \pi_{\widetilde{\varphi}}(\widetilde{\theta})$ for each $\widetilde{\theta} \in \widetilde{\Theta}_1$.

Theorem 5.3. *In the problem of testing hypothesis (10), a sufficient condition for the fuzzy test $\widetilde{\varphi}$ to be a UMP fuzzy test of size δ is that for any $\alpha \in (0, 1]$, the test functions φ_α^l and φ_α^u should be UMP tests for testing hypotheses (11) and (12), respectively.*

Proof. It is clear that if φ_α^l and φ_α^u are δ level tests, then $[\pi_{\varphi_\alpha^l}(\theta_{0\alpha}^l), \pi_{\varphi_\alpha^u}(\theta_{0\alpha}^u)] = [\delta, \delta] = \delta$, and so $\pi_{\widetilde{\varphi}}(\widetilde{\theta}_0) = I_{\{\delta\}}$.
Now, let $\widetilde{\varphi}_*$ be any fuzzy test of size δ. It is sufficient to show that $\pi_{\widetilde{\varphi}_*}(\widetilde{\theta}) \preceq \pi_{\widetilde{\varphi}}(\widetilde{\theta})$

(with $\alpha-$level sets defined in Definition 5.1), for any $\widetilde{\theta} \in \widetilde{\Theta}_1$. For $\alpha \in (0,1]$, let $\varphi^l_{*\alpha}$ and $\varphi^u_{*\alpha}$ be the test functions for testing hypotheses (11) and (12), based on the two random samples (2) and (3), respectively. For $\widetilde{\theta} \in \widetilde{\Theta}_1$, we have $\theta^l_\alpha > \theta^l_{0\alpha}$ and $\theta^u_\alpha > \theta^u_{0\alpha}$ for all $\alpha \in (0,1]$. Note that φ^l_α and φ^u_α are UMP tests for testing hypotheses (11) and (12), respectively. So, for the test functions $\varphi^l_{*\alpha}$ and $\varphi^u_{*\alpha}$, we have $\pi_{\varphi^l_{*\alpha}}(\theta^l_\alpha) \leq \pi_{\varphi^l_\alpha}(\theta^l_\alpha)$ and $\pi_{\varphi^u_{*\alpha}}(\theta^u_\alpha) \leq \pi_{\varphi^u_\alpha}(\theta^u_\alpha)$, for any $\alpha \in (0,1]$. Since $\pi_{\varphi^l_\alpha}(\theta^l_\beta)$ and $\pi_{\varphi^u_\alpha}(\theta^u_\beta)$ are increasing and decreasing functions of β, respectively, it follows that

$$\inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_{*\alpha}}(\theta^l_\beta) \leq \pi_{\varphi^l_{*\alpha}}(\theta^l_\alpha) \leq \pi_{\varphi^l_\alpha}(\theta^l_\alpha) \leq \pi_{\varphi^l_\alpha}(\theta^l_\beta) \qquad \forall \alpha \leq \beta \leq 1,$$
$$\sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_\alpha}(\theta^u_\beta) \geq \pi_{\varphi^u_\alpha}(\theta^u_\alpha) \geq \pi_{\varphi^u_{*\alpha}}(\theta^u_\alpha) \geq \pi_{\varphi^u_{*\alpha}}(\theta^u_\beta) \qquad \forall \alpha \leq \beta \leq 1.$$

Therefore, $\inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_{*\alpha}}(\theta^l_\beta) \leq \inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_\alpha}(\theta^l_\beta)$ and $\sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_{*\alpha}}(\theta^u_\beta) \leq \sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_\alpha}(\theta^u_\beta)$. Now

$$\min\{\inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_{*\alpha}}(\theta^l_\alpha), \sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_{*\alpha}}(\theta^u_\alpha)\} \leq \inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_{*\alpha}}(\theta^l_\alpha) \leq \inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_\alpha}(\theta^l_\alpha),$$
$$\min\{\inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_{*\alpha}}(\theta^l_\alpha), \sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_{*\alpha}}(\theta^u_\alpha)\} \leq \sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_{*\alpha}}(\theta^u_\alpha) \leq \sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_\alpha}(\theta^u_\alpha),$$

and thus

$$\min\{\inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_{*\alpha}}(\theta^l_\alpha), \sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_{*\alpha}}(\theta^u_\alpha)\} \leq \min\{\inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_\alpha}(\theta^l_\alpha), \sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_\alpha}(\theta^u_\alpha)\}.$$

Similarly, we have

$$\inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_{*\alpha}}(\theta^l_\alpha) \leq \inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_\alpha}(\theta^l_\alpha) \leq \max\{\inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_\alpha}(\theta^l_\alpha), \sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_\alpha}(\theta^u_\alpha)\},$$
$$\sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_{*\alpha}}(\theta^u_\alpha) \leq \sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_\alpha}(\theta^u_\alpha) \leq \max\{\inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_\alpha}(\theta^l_\alpha), \sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_\alpha}(\theta^u_\alpha)\},$$

and thus

$$\max\{\inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_{*\alpha}}(\theta^l_\alpha), \sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_{*\alpha}}(\theta^u_\alpha)\} \leq \max\{\inf_{\alpha\leq\beta\leq 1} \pi_{\varphi^l_\alpha}(\theta^l_\alpha), \sup_{\alpha\leq\beta\leq 1} \pi_{\varphi^u_\alpha}(\theta^u_\alpha)\}.$$

So, for any fuzzy test $\widetilde{\varphi}_*$ of size δ and any $\widetilde{\theta} \in \widetilde{\Theta}_1$, we showed that $\pi_{\widetilde{\varphi}_*}(\widetilde{\theta}) \preceq \pi_{\widetilde{\varphi}}(\widetilde{\theta})$, which completes the proof. □

Theorem 5.4. *The fuzzy test $\widetilde{\varphi}$ in (8) is a UMP fuzzy test of size δ for testing the right one-sided hypothesis (10).*

Proof. In constructing the fuzzy test $\widetilde{\varphi}$ in (8), we used the $\delta-$level UMP tests φ^l_α and φ^u_α for testing hypotheses (11) and (12). Based on Theorem 5.3, it is, therefore, clear that $\widetilde{\varphi}$ is an UMP fuzzy test for testing hypothesis (10) of size δ. □

6. Testing Simple Versus Left One-sided Hypotheses

Similar to the arguments in Section 5 and based on a normal fuzzy random sample $\mathcal{X}_1, \ldots, \mathcal{X}_n$, we can investigate the problem of testing simple versus left one-sided hypotheses as follows

$$H_0 : \widetilde{\theta} = \widetilde{\theta}_0 \qquad versus \qquad H_1 : \widetilde{\theta} \prec \widetilde{\theta}_0. \tag{13}$$

Theorem 6.1. *The fuzzy test $\widetilde{\varphi}$ in (9) is a UMP fuzzy test for testing hypothesis (13) of size δ.*

Proof. The proof is similar to that of Theorem 4.6. □

7. Numerical Examples

In this section, numerical examples are used to illustrate our proposed method.

Example 7.1. Let $\mathcal{X}_1, \ldots, \mathcal{X}_4$ be a fuzzy random sample from $N(\widetilde{\theta}, 9)$. Consider the problem of testing hypothesis $H_0 : \widetilde{\theta} = (-1, 1, 1)_T$ versus $H_1 : \widetilde{\theta} = (0, 1.5, 2)_T$ of size $\delta = 0.05$. Using the procedure introduced in Section 4, for any $\alpha \in (0, 1]$, we have

$$\widetilde{\theta}_{0\alpha} = [-2 + \alpha, -\alpha], \qquad \widetilde{\theta}_{1\alpha} = [-1.5 + 1.5\alpha, 2 - 2\alpha],$$

$$H_0^l : \theta_\alpha^l = -2 + \alpha \qquad versus \qquad H_1^l : \theta_\alpha^l = -1.5 + 1.5\alpha,$$

$$H_0^u : \theta_\alpha^u = -\alpha \qquad versus \qquad H_1^u : \theta_\alpha^u = 2 - 2\alpha.$$

For these simple hypotheses, the related tests of size $\delta = 0.05$ are given by

$$\varphi_\alpha^l(X_{1\alpha}^l, \ldots, X_{4\alpha}^l) = \begin{cases} 1 & \bar{X}_\alpha^l \geq \alpha + 0.46 \\ 0 & \bar{X}_\alpha^l < \alpha + 0.46 \end{cases},$$

$$\varphi_\alpha^u(X_{1\alpha}^u, \ldots, X_{4\alpha}^u) = \begin{cases} 1 & \bar{X}_\alpha^u \geq -\alpha + 2.46 \\ 0 & \bar{X}_\alpha^u < -\alpha + 2.46 \end{cases},$$

for which the power functions are $\pi_{\varphi_\alpha^l}(\theta_{1\alpha}^l) = \Phi[\frac{1}{3}(1 + \alpha) - 1.64]$ and $\pi_{\varphi_\alpha^u}(\theta_{1\alpha}^u) = \Phi[\frac{2}{3}(2 - \alpha) - 1.64]$, respectively.
Here, $\widetilde{C}_\alpha = [\alpha + 0.46, -\alpha + 2.46]$, so $\widetilde{C} = (1.46, 1, 1)_T$. According to relations (4) to (8), the MP fuzzy test of size $\delta = 0.05$ is obtained as

$$\widetilde{\varphi}(\mathcal{X}_1, \ldots, \mathcal{X}_4) = \left\{ \frac{\mathcal{D}(\bar{\mathcal{X}} < (1.46, 1, 1)_T)}{0}, \frac{\mathcal{D}(\bar{\mathcal{X}} > (1.46, 1, 1)_T)}{1} \right\}.$$

In this example, since $\pi_{\varphi_\alpha^l}(\theta_{1\alpha}^l) \leq \pi_{\varphi_\alpha^u}(\theta_{1\alpha}^u)$, for any $\alpha \in (0, 1]$, the $\alpha-$level sets of the fuzzy set $\pi_{\widetilde{\varphi}}(\widetilde{\theta})$ are

$$\left[\pi_\alpha^l\left(\widetilde{\theta}_1\right), \pi_\alpha^u\left(\widetilde{\theta}_1\right)\right] = \left[\Phi\left(\frac{1}{3}(1 + \alpha) - 1.64\right), \Phi\left(\frac{2}{3}(2 - \alpha) - 1.64\right)\right].$$

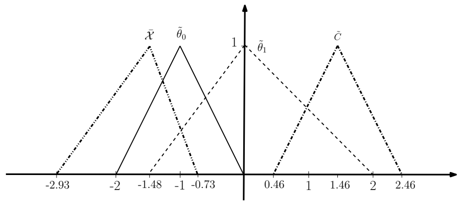

FIGURE 1. Membership Functions of Fuzzy Numbers in Example 7.1

FIGURE 2. The Fuzzy Power of the Fuzzy Test $\widetilde{\varphi}$ in Example 7.1
at the Fuzzy Point $\widetilde{\theta}_1 = (0, 1.5, 2)_T$

The membership functions of the fuzzy parameters $\widetilde{\theta}_0$ and $\widetilde{\theta}_1$, fuzzy statistic $\widetilde{\mathcal{X}}$, and fuzzy critical value $\widetilde{C}$ are shown in Figure 1. Also, the membership function of the power of fuzzy test $\widetilde{\varphi}$, $\pi_{\widetilde{\varphi}}(\widetilde{\theta}_1)$ is shown in Figure 2.

Let us suppose that the observed random sample consists of the following fuzzy numbers

$$\mathcal{X}_1 = (-0.29, 1.90, 0.46)_T, \qquad \mathcal{X}_2 = (-2.50, 1.21, 0.97)_T,$$
$$\mathcal{X}_3 = (0.88, 1.78, 1.52)_T, \qquad \mathcal{X}_4 = (-4.01, 0.91, 0.04)_T.$$

In this case, we have $\bar{\mathcal{X}} = (-1.48, 1.45, 0.75)_T$ and

$$\bar{\mathcal{X}}_\alpha = [-2.93 + 1.45\alpha, -0.73 - 0.75\alpha].$$

Here, we obtain $\Delta_{\widetilde{C}\bar{\mathcal{X}}} = 6.23$ and $\Delta_{\bar{\mathcal{X}}\widetilde{C}} = 0$, so $\mathcal{D}(\bar{\mathcal{X}} > (1.46, 1, 1)_T) = 0$ and $\mathcal{D}(\bar{\mathcal{X}} < (1.46, 1, 1)_T) = 1$. Therefore, based on the above observations, the MP fuzzy test for testing the hypothesis of interest is $\widetilde{\varphi}(\boldsymbol{\mathcal{X}}) = \left\{\frac{1}{0}, \frac{0}{1}\right\}$, which means that there is no reason for rejecting $H_0 : \widetilde{\theta} = (-1, 1, 1)_T$.

Example 7.2. In Example 7.1, suppose the following fuzzy random sample is reported

$$\mathcal{X}_1 = (4.20, 1.90, 0.12)_T \qquad \mathcal{X}_2 = (1.18, 1.83, 0.70)_T,$$
$$\mathcal{X}_3 = (0.71, 0.82, 1.62)_T \qquad \mathcal{X}_4 = (-1.50, 1.78, 0.22)_T.$$

Here, $\bar{\mathcal{X}} = (1.15, 1.58, 0.66)_T$, $\bar{\mathcal{X}}_\alpha = [-0.43 + 1.58\alpha, 1.81 - 0.66\alpha]$, $\Delta_{\widetilde{C}\bar{\mathcal{X}}} = 1.63$, and $\Delta_{\bar{\mathcal{X}}\widetilde{C}} = 0.55$ (Figure 3). Since $\mathcal{D}(\bar{\mathcal{X}} > (1.46, 1, 1)_T) = 0.25$, the MP fuzzy test of size $\delta = 0.05$ is $\widetilde{\varphi}(\boldsymbol{\mathcal{X}}) = \left\{\frac{0.75}{0}, \frac{0.25}{1}\right\}$.

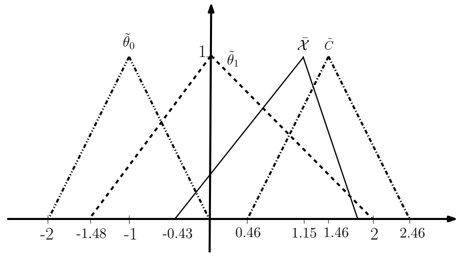

FIGURE 3. Membership Functions of Fuzzy Numbers in Example 7.2

$(33978, 712, 911)_T$	$(32617, 524, 638)_T$	$(33052, 467, 735)_T$	$(32611, 891, 886)_T$
$(33418, 612, 490)_T$	$(32455, 478, 579)_T$	$(33463, 368, 668)_T$	$(32466, 523, 746)_T$
$(31624, 881, 836)_T$	$(33070, 901, 898)_T$	$(33127, 712, 945)_T$	$(33543, 643, 792)_T$
$(33224, 537, 684)_T$	$(30881, 554, 564)_T$	$(32597, 412, 589)_T$	$(31565, 378, 672)_T$
$(34036, 613, 735)_T$	$(34053, 845, 823)_T$	$(32584, 945, 958)_T$	$(31838, 893, 901)_T$
$(32290, 779, 774)_T$	$(32800, 866, 645)_T$	$(33844, 784, 605)_T$	$(34157, 693, 817)_T$

TABLE 1. The Lifetime of Tiers in Example 7.3

In such a case, we are not fully convinced either to reject or to accept $H_0 : \widetilde{\theta} = (-1, 1, 1)_T$ but we merely indicate that the degrees of conviction one should reject or accept the null hypothesis are 0.25 and 0.75, respectively.

Example 7.3. The marketing department of a company intended to study the average lifetime of a tire as a new product of the company. A sample of 24 new tires were tested. Six cars, all of the same model and brand, were used to test the tires. Because of some unexpected situations, we cannot measure the lifetime of the tire precisely, and we just obtain approximate numbers for the tire lives. Suppose that the tire lives are reported in triangular fuzzy numbers as in Table 1 (the data set is due to [40]). In order to investigate that fuzzy random sample $\boldsymbol{\mathcal{X}} = (\mathcal{X}_1, \cdots, \mathcal{X}_n)$ is a normal fuzzy random sample of size n with a fuzzy parameter $\widetilde{\theta}$ and crisp parameter σ^2, the Shapiro-Wilk test of normality [28] is performed for $\boldsymbol{X}^l_\alpha = (X^l_{1\alpha}, \cdots, X^l_{n\alpha})$ and $\boldsymbol{X}^l_\alpha = (X^u_{1\alpha}, \cdots, X^u_{n\alpha})$ for all $\alpha \in (0, 1]$. In Table 2 the Shapiro-Wilk test of normality is performed for the observed values $\boldsymbol{x}^l_\alpha = (x^l_{1\alpha}, \cdots, x^l_{n\alpha})$ and $\boldsymbol{x}^l_\alpha = (x^u_{1\alpha}, \cdots, x^u_{n\alpha})$ for $\alpha = 0.0, 0.1, 0.2, \ldots, 0.9, 1.0$. For instance, for $\alpha = 1.0$, the Shapiro-Wilk test statistic and the $p - value$ of the test are obtained as 0.9605 and 0.4482, respectively. Also Fig. 4 shows the normal QQ plot of the center values. Based on the obtained $p - value$ the normality of the observations is accepted. Same result can be concluded from the Shapiro-Wilk test of normality for the other values of α given in Table 2. It means that $\boldsymbol{X}^l_\alpha$ and $\boldsymbol{X}^u_\alpha$ have normal distribution, for all values of $\alpha \in (0, 1]$. Therefore fuzzy random sample $\boldsymbol{\mathcal{X}} = (\mathcal{X}_1, \cdots, \mathcal{X}_n)$ is a normal fuzzy random sample of size n with a fuzzy parameter $\widetilde{\theta}$ and crisp parameter $\sigma^2 = 747000$. We are now going to test the hypothesis $H_0 : \widetilde{\theta} = (32000, 2000, 2000)_T$ against the one-sided alternative $H_1 : \widetilde{\theta} \succ (32000, 2000, 2000)_T$, of size $\delta = 0.05$.

	$\boldsymbol{x}^l_\alpha = (x^l_{1\alpha}, \cdots, x^l_{n\alpha})$		$\boldsymbol{x}^u_\alpha = (x^u_{1\alpha}, \cdots, x^u_{n\alpha})$	
α	Shapiro-Wilk statistic	P-value	Shapiro-Wilk statistic	P-value
0.0	0.960	0.455	0.968	0.628
0.1	0.961	0.468	0.967	0.604
0.2	0.962	0.487	0.966	0.583
0.3	0.962	0.489	0.966	0.572
0.4	0.961	0.472	0.965	0.567
0.5	0.960	0.450	0.965	0.560
0.6	0.969	0.436	0.964	0.543
0.7	0.959	0.426	0.964	0.525
0.8	0.959	0.429	0.963	0.504
0.9	0.959	0.427	0.961	0.474
1.0	0.960	0.448	0.960	0.448

TABLE 2. Testing Normality of the Data Given in Example 7.3

Applying the procedure proposed in Section 5, we have

$$\begin{aligned}
\frac{\sigma}{\sqrt{n}} z_{1-\delta} &= 290.22, \\
\widetilde{C} &= \widetilde{\theta}_0 \oplus I_{\{290.22\}} = (32290.22, 2000, 2000)_T, \\
\widetilde{C}_\alpha &= [30290.22 + 2000\alpha, 34290.22 - 2000\alpha], \\
\bar{\mathcal{X}} &= (32887, 667, 745)_T, \\
\bar{\mathcal{X}}_\alpha &= [32220 + 667\alpha, 33632 - 745\alpha], \\
\Delta_{\widetilde{C}\bar{\mathcal{X}}} &= 803.49, \\
\Delta_{\bar{\mathcal{X}}\widetilde{C}} &= 2036.05, \\
\mathcal{D}(\bar{\mathcal{X}} > \widetilde{C}) &= 0.72, \\
\mathcal{D}(\bar{\mathcal{X}} < \widetilde{C}) &= 0.28.
\end{aligned}$$

Therefore, the UMP fuzzy test of size $\delta = 0.05$ for testing the hypothesis under consideration is $\widetilde{\varphi}(\boldsymbol{\mathcal{X}}) = \left\{\frac{0.28}{0}, \frac{0.72}{1}\right\}$. Now, with respect to the observed fuzzy observations and of size $\delta = 0.05$, the hypothesis $H_0 : \widetilde{\theta} = (32000, 2000, 2000)_T$ is rejected with a degree of 0.72 and it is accepted with a degree of 0.28. Such a result may be interpreted as "we tend to reject H_0". The membership of the related fuzzy numbers used in this example and that of the fuzzy power of the fuzzy test at the fuzzy point $\widetilde{\theta} = (32500, 1500, 1500)_T$ are shown in Figures 5 and 6, respectively.

Remark 7.4. A problem that has been frequently investigated in connection with the use of fuzzy numbers is how to assess/choose/determine curves formalizing the imprecise valuations/perceptions, etc. Also, different mostly context-dependent responses may be found in the literature to the question "Do exact shapes of fuzzy sets matter?" [22]. On the one hand, the exact shape seems not to matter in any real sense as indicated by several studies related to hypothesis testing problems and according to certain criteria commonly used in such contexts. For instance, Lubiano et al. [22] demonstrated that the shape of the fuzzy assessment scarcely affects statistical conclusions.

The approach proposed in the study can be formulated for assessing any kind of membership function for fuzzy concepts. The approach uses triangular fuzzy numbers for assessing fuzzy numbers to model imprecise data associated with random experiments.

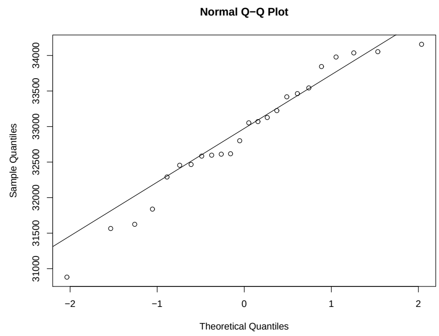

FIGURE 4. The Normal Quantile-quantile Plot of the Centers of the Observed Fuzzy Random Sample in Example 7.3

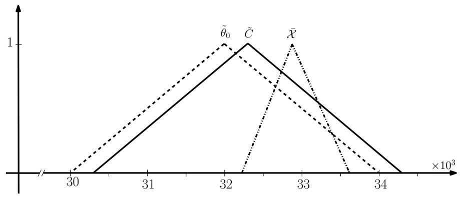

FIGURE 5. Membership Functions of Fuzzy Numbers in Example 7.3

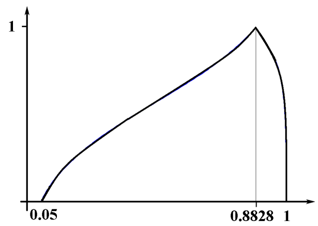

FIGURE 6. Membership function of the fuzzy power at the fuzzy point $\widetilde{\theta} = (32500, 1500, 1500)_T$ in Example 7.3

Triangular fuzzy numbers have proved to be a commonly welcome choice because of their following capabilities [22]:

(1) Ease of handling when used for most methodological and practical computations, whenever no differentiability of fuzzy numbers is required;
(2) Ease of interpreting results in a natural way;
(3) Ease of practical application; this becomes especially noticeable when those assessing fuzzy numbers against data have a poor knowledge, limited background and little or no expertise in using fuzzy sets.

8. **Conclusions**

A new approach was proposed for testing hypotheses about the mean of a (normal) fuzzy random variable. Invoking the usual methods of the theory of testing statistical hypotheses in a classical setting, we proposed a generalization of the classical tests for testing such hypotheses. In this approach, the concepts of the fuzzy critical value and fuzzy test statistic were defined to extend the concepts of the most powerful and the uniformly most powerful tests to the most powerful and uniformly most powerful fuzzy tests, respectively. Based on the proposed procedure, the hypothesis of interest can be accepted or rejected with degrees of conviction between 0 and 1; i.e., a fuzzy decision, unlike the classical crisp test, which leads to a binary (acceptance or rejection) decision. The proposed fuzzy tests are natural generalizations of the classical tests so that they reduce to the classical tests if all the data and parameters are crisp.

The proposed approach is applicable to practical situations, where the observed data are imprecise and/or the hypotheses of interest are formalized in linguistic terms. Although we focused on testing hypotheses about the mean of a normal fuzzy random variable, the proposed method is general and can be applied to testing hypotheses about any fuzzy parameters of arbitrary distributions.

The topic of testing fuzzy hypothesis using the fuzzy confidence intervals, and applying the proposed method to inferences about the parameters of fuzzy regression models form interesting topics for future studies.

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