

ADAPTIVE BACKSTEPPING CONTROL OF UNCERTAIN FRACTIONAL ORDER SYSTEMS BY FUZZY APPROXIMATION APPROACH

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ABSTRACT. In this paper, a novel problem of observer-based adaptive fuzzy fractional control for fractional order dynamic systems with commensurate orders is investigated; the control scheme is constructed by using the backstepping and adaptive technique. Dynamic surface control method is used to avoid the problem of explosion of complexity which is caused by backstepping design process. Fuzzy logic systems are used to approximate the unknown nonlinear functions. A fractional order Lyapunov function is defined at each stage and the negativity of an overall Lyapunov function is ensured by proper selection of the control law. It is proven that the proposed controller guarantees the boundedness property for all the signals and also the tracking error can converge to a small neighborhood of the origin. Simulation examples are given to demonstrate the effectiveness and robustness of the proposed controllers.

1. Introduction

Fractional Order calculus has a history of about 300 years. Before the 20th century, it was not used in physics and engineering for a long period but in the past 20 years, fractional order calculus has attracted increasing attention of physicists and engineers from an application point of view. A distinguished feature of fractional-order systems is their memory and hereditary properties which can be utilized to characterize some engineering plants and processes more precisely, and fractional order calculus has attracted increasing attention from control view because of more potential advantages and design freedom than integer order one [2, 8, 16, 18–20, 22, 28, 38].

Generally, from the viewpoint of practice it is impossible to obtain the exact model of a controlled plant. Uncertainty reduces the performance of the control system and may cause instability. So it is appropriate to take into account the effects of the system uncertainties. As a result, analysis and synthesis of systems with model uncertainties has been one of the critical issues in the control field. There are several methods to deal with the uncertainties that can be divided into two major categories, namely model-based and model-free methods. State/output feedback control [24], adaptive control [10], Sliding Mode Control [42], H_∞ control [26] and the fractional order adaptive control are model-based methods and CRONE

Received: February 2017; Revised: August 2017; Accepted: December 2017

Key words and phrases: Adaptive fuzzy control, Backstepping design technique, Fractional systems, Dynamic Surface Control.

control [29], fuzzy control [25] and $PI^\lambda D^\mu$ control [23] are model free methods to solve uncertain problems.

Backstepping is a recursive nonlinear control method which has been widely applied in practical applications [15]. Through this technique, the Lyapunov function is constructed and the virtual control input is designed [21, 48, 49]. When this recursive process completes, a feedback design for the real control input results [31]. Backstepping method is very effective if system parameters are uncertain. However, backstepping is more restricted to the classical integer-order nonlinear systems and there are few results on fractional order backstepping technique [2, 7]. Backstepping can guarantee stabilities, good tracking, and transient performance for strict feedback systems, so it is an appropriate method for adaptive nonlinear control.

Adaptive backstepping is a good solution to control uncertain fractional order systems [27, 43]. Fractional order adaptive backstepping control approach is firstly proposed in [2, 5, 7]. However, a restriction of adaptive backstepping works is that they cannot be used to nonlinear systems where the structure of system functions is not completely known. To overcome this problem, a fuzzy logic system, as a universal function approximation, is used to approximate the completely unknown nonlinear functions and the need for exact knowledge of system nonlinearities is removed. Using the FLS, various adaptive backstepping control strategies have been developed to control uncertain nonlinear systems [12, 34–37].

In the above mentioned papers, an adaptive approach is used to estimate the fuzzy logic systems and then the backstepping technique is applied to construct the desired controllers. There are a few articles about fuzzy adaptive backstepping on fractional systems. In [39, 40] a fractional order adaptive fuzzy backstepping control approach was used to design robust controllers for SISO nonlinear strict feedback systems with unknown nonlinearities. In nonlinear systems, if the states are not all available for feedback, an observer based output-feedback control law is derived, an adaptive fuzzy output feedback control method was investigated in [3, 33, 36].

The backstepping design procedure has the problem of explosion of complexity caused by the repeated differentiations of virtual controllers [32, 46] and, the complexity of a controller drastically grows as the order of the system increases. In order to avoid this problem, dynamic surface control (DSC) technique has been proposed for a class of strict-feedback nonlinear system with unknown system function, by introducing a first-order low-pass filter at each step of the conventional backstepping design procedure [32, 41, 46].

Although the tools and approaches of fractional order systems and fuzzy adaptive backstepping control are not new, implementation of fuzzy fractional adaptive backstepping control for fractional order system dynamics is new. In the controller design process, the dynamic feedback strategy begins with an input-driven filter, and by combining fractional fuzzy adaptive systems with backstepping and DSC technique, a fractional fuzzy adaptive output feedback controller is constructed recursively. This method is used for a class of fractional order systems in strict-feedback form. Furthermore guarantees semi-global uniform ultimate boundedness of all the signals in the closed-loop system. Those mentioned above motivate the

study of this paper.

The contributions of this paper are summarized as follows.

- Fractional Lyapunov stability theorems are used for stability analysis of uncertain fractional-order nonlinear systems.

- Fuzzy logic system, as a universal function approximation, is used to approximate the completely unknown nonlinear functions and fractional-order adaptation laws are designed to update the fuzzy parameters.

- With [6,17,30,44] different, in our work, an observer is designed to estimate the unknown states, and also a dynamic surface control is compounded with traditional backstepping to avoid from the explosion of complexity.

- With [6] different, in this work, the system is nonlinearly parameterized and non-linear functions is unknown, but the system in [6] is linearly parameterized and theta is an unknown constant.

The rest of this paper is organized as follows. In Section 2, some definitions in fractional calculus and fuzzy logic are presented. In Section 3, the Output Feedback and Adaptive Fuzzy controller is systematically designed. In Section 4, some numerical simulations are provided to illustrate the validity of the proposed algorithm. Some concluding remarks are made in Section 5.

2. Preliminaries

2.1. Fractional Calculus. Fractional order calculus is the generalization concept of the conventional integer order calculus. There are three main commonly used definitions for D^β ($\beta > 0$): GrnwaldLetnikov definition, RiemannLiouville definition and Caputo's definition. Among these definitions the Caputo's definition is used in most of the engineering applications, since in this definition, integer order derivatives $f(t)$ (i.e. $f(t)$; $f^{(1)}$; $f^{(2)}$ and so on) are used as the initial conditions, and therefore easily interpreted from physical data and observations. The Caputo fractional derivative of $f(t)$ a continuous function is defined as

$$\begin{cases} D^\beta f(t) = \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\beta-m+1}} & (m-1 \leq \beta < m) \\ \frac{d^m}{dt^m} f(t) & \beta = m \end{cases} \quad (1)$$

where $m \in N$, and $\Gamma(\beta) = \int_0^\infty x^{\beta-1} e^{-x}$ is the Gamma function.

Definition 2.1. [22] The two parameter function of the Mittag-Leffler function type is defined by the series expansion

$$E_{\beta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \gamma)}, \quad \beta, \gamma > 0 \quad (2)$$

The Laplace transform of (2) is

$$\mathcal{L}\left\{t^{\gamma-1} E_{\beta,\gamma}(-at^\beta)\right\} = \frac{s^{\beta-\gamma}}{s^\beta + 1} \quad (3)$$

Now, we give three useful lemmas that will be used for proving the stability theorem1.

Lemma 2.2. [22] Let γ is a complex number. If $0 < \beta < 2$ and $\frac{\pi\beta}{2} < t < \min\{\pi, \pi\beta\}$, then

$$E_{\beta,\gamma}(z) = -\sum_{j=1}^n \frac{1}{\Gamma(\gamma - \beta j)z^j} + o\left(\frac{1}{|z|^{n+1}}\right), \quad (4)$$

when $|z| \rightarrow \infty$ and $t \leq |\arg(z)| \leq \pi$.

Lemma 2.3. [22] Let $0 < \beta < 2$ and $\gamma \in \mathfrak{R}$. If $\frac{\pi\beta}{2} < t < \min\{\pi, \pi\beta\}$, and $C > 0$ is a real constant, then

$$|E_{\beta,\gamma}(z)| \leq \frac{C}{1+|z|}, \quad t \leq |\arg(z)| \leq \pi \text{ and } |z| \geq 0 \quad (5)$$

Lemma 2.4. [13] Suppose that $x(t) = 0$ is the equilibrium point of the following system

$$D_t^\beta x(t) f(t, x(t)) \quad (6)$$

If there exist a Lyapunov function $V(t, x(t))$ and a class- K function $g_i, i = 1, 2, 3$ such that

$$g_1(\|x(t)\|) \leq V(t, x(t)) \leq g_2(\|x(t)\|) \quad (7)$$

$$D_t^\beta V(t, x(t)) \leq -g_3(\|x(t)\|) \quad (8)$$

Then system (6) is asymptotically stable.

Lemma 2.5. [1] Let $x(t) \in R^n$ be a smooth function. Then, for any $t > 0$

$$\frac{1}{2} D_t^\beta x^T(t) \leq x^T(t) D_t^\beta x(t)$$

2.2. System Description. In the paper, we consider the following n -dimensional fractional-order system with model uncertainties, external disturbances and control inputs in strict feedback form

$$\begin{cases} D^\beta x_1 = x_2 + f_1(x_1) + d_1(t), \\ D^\beta x_2 = x_3 + f_2(x_1, x_2) + d_2(t), \\ \vdots \\ D^\beta x_{n-1} = x_n + f_{n-1}(x_1, \dots, x_{n-1}) + d_{n-1}(t), \\ D^\beta x_n = u + f_n(x) + d_n(t), \\ y = x_1 \end{cases} \quad (9)$$

$x = [x_1, x_2, \dots, x_n]^T \in \mathfrak{R}^n$ denotes the system state and u is the control input, $f_i(\cdot) (i = 1, 2, \dots, n)$ are unknown smooth nonlinear functions. $d_i(\cdot) (i = 1, 2, \dots, n)$ are the external disturbance uncertainties of the system, which satisfy $|d_i(t)| \leq \bar{d}_i$, with \bar{d}_i as a constant and the fractional commensurate order β belong to $(0, 1)$. The control objective is to design a controller such that the system output y can track the reference signal y_d . The reference signal y_d and its time derivatives up to the n th order $y_d^{(n)}$ is assumed to be available, bounded and continuous.

2.3. Fuzzy Logic Systems. The basic configuration of a fuzzy logic system consists of a fuzzifier, a fuzzy inference engine working on fuzzy IF-THEN rules and a defuzzifier. The i th fuzzy rule is written as

$$R_i : \text{IF } x_1 \text{ is } F_1^i \text{ and } \dots, x_n \text{ is } F_n^i \text{ THEN } y \text{ is } B^i, 1 \leq i \leq n$$

where $x = [x_1, x_2, \dots, x_n]^T$, $x \in \mathfrak{R}^n$, and y are the FLS input signal and output signal, respectively. Fuzzy sets F_j^i and B^i , related with the fuzzy functions $\mu_{F_j^i}(x_j)$ and $\mu_{B^i}(y)$, respectively. N is the rules number. By singleton function, center average defuzzification and product inference, the FLS can be expressed as

$$y(x) = \frac{\sum_{i=1}^N \bar{W}_i \prod_{j=1}^n \mu_{F_j^i}(x_j)}{\sum_{i=1}^N \prod_{j=1}^n \mu_{F_j^i}(x_j)}$$

where $\bar{W}_i = \max_{y \in R} \mu_{B^i}(y)$.

Define the fuzzy basis functions as

$$\rho_i(x) = \frac{\prod_{j=1}^n \mu_{F_j^i}(x_j)}{\sum_{i=1}^N \prod_{j=1}^n \mu_{F_j^i}(x_j)}$$

It is assumed that fuzzy basis functions are selected so that there is always at Least one active rule. Denoting $W = [\bar{W}_1, \bar{W}_2, \dots, \bar{W}_N]^T$ is the adjustable parameter vector and $\xi(x) = [\rho_1(x), \rho_2(x), \dots, \rho_N(x)]$, then FLS can be rewritten as

$$y(x) = W^T \xi(x) \quad (10)$$

Lemma 2.6. [11] *Let $f(x)$ be a continuous function defined on a compact set Ω . Then for any constant $\varepsilon > 0$, there exists a fuzzy logic system (10) such as*

$$\text{SUP}_{x \in \Omega} |f(x) - W^T \xi(x)| \leq \varepsilon$$

The fuzzy inference engine uses the fuzzy IF-THEN rules to perform a mapping from an input vector $x = [x_1, x_2, \dots, x_n]^T$ to an output y .

Remark 2.7. In approximation-based adaptive controller designing, adaptive techniques will be used to estimate the ideal weight vectors W_i , for $i = 1, 2, \dots, n$ which are unknown parameters. Usually, two methods can be used to do this. The first one estimates each element of W_i [33]; the second one estimates its norm [45]. We will use the second method in this manuscript.

With an increase of fuzzy rules, the number of parameters to be estimated will increase significantly. As a result, the online learning time becomes prohibitively large. To solve this problem, we consider the norm of the ideal weighting vector in fuzzy logic systems as the estimation parameter instead of the elements of weighting vector. Thus, the number of adaptation laws is reduced considerably.

3. Output Feedback and Adaptive Fuzzy Control Design

In fuzzy controller, when full system states are not available, output feedback or observer based fuzzy controller, which will be discussed in subsection 3.1, can be

used. In this section, it is assumed that the states of system (9) are not available for feedback, so a state observer for estimating the states should be determinate, and then fuzzy fractional adaptive output feedback control scheme via backstepping is considered.

3.1. Observer Design. For system (9), each of states x_i is not available for feedback control design; thus, a full order observer is needed to estimate x_1, \dots, x_n and generate some signals for controller design. Let us consider the linear observer [9] for system (9):

$$\begin{cases} D^\beta \hat{x}_i = \hat{x}_{i+1} - k_i \hat{x}_1, & 1 \leq i \leq n-1 \\ D^\beta \hat{x}_n = u - k_n \hat{x}_1 \end{cases} \quad (11)$$

where \hat{x}_i is the estimation of x_i . The design parameter k_i is chosen such that the matrix

$$\mathbf{A} = \begin{bmatrix} -k_1 & & I_{n-1} \\ \vdots & & \\ -k_n & \dots & 0 \end{bmatrix}$$

is strict Hurwitz matrix, Thus, given a $Q > 0$, there exists a $P > 0$ satisfying

$$A_c^T P + P A_c = -2Q \quad (12)$$

Define the state error as $e_i = x_i - \hat{x}_i$, $1 \leq i \leq n-1$. Then we can have

$$\begin{cases} D^\beta e_i = e_{i+1} - k_i e_1 + f_i(\bar{x}_i) + d_i(t) + k_i y, \\ D^\beta e_n = -k_n e_1 + f_n(\bar{x}_n) + d_n(t) + k_n y \end{cases}$$

and the error dynamic can be expressed as

$$D^\beta e = A_c e + F(\bar{x}) + D(t) \quad (13)$$

with

$$\begin{aligned} e &= (e_1, e_2, \dots, e_n)^T \\ D(t) &= (d_1(t), \dots, d_n(t)) \\ F(\bar{x}) &= (f_1(\bar{x}_1) + k_1 y, \dots, f_n(\bar{x}_n) + k_n y)^T \\ &= (F_1(\bar{x}), \dots, F_n(\bar{x})) \end{aligned}$$

3.2. Fuzzy Adaptive Control Design and Stability Analysis. In this section, we will incorporate the DSC technique proposed in [41] into the backstepping-based fuzzy fractional adaptive output feedback controller and stability analysis procedure for the system described by (9). By incorporating the DSC technique, the proposed control scheme is simplified and the repeated differentiation of v_i is avoided. Similar to the traditional backstepping design method, the recursive design procedure contains steps and is based on the change of coordinates, given as

$$\begin{cases} z_1 = x_1 - y_d, \\ z_i = \hat{x}_i - v_{if}, & i = 2, 3, \dots, n \end{cases}$$

where v_{i-1} and v_{if} are the input and output of the first order filter respectively. In each step, a virtual control function \hat{v}_i is proposed, and the real tracking control law u is designed at the last step [47]. The virtual control function and the real control law will be designed as:

$$v_i(Z_i) = -\frac{1}{2a_i^2}z_i\hat{\theta}, \quad i = 1, \dots, n-1 \quad (14)$$

$$u = -\frac{1}{2a_n^2}z_n\hat{\theta} \quad (15)$$

with

$$Z_1 = (x_1, y_d, D^\beta y_d)^T, Z_i = (\hat{x}_i, v_{if}, D^\beta v_{if})^T, \quad i = 2, \dots, n-1$$

$$\hat{x}_i = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_i)$$

θ is an unknown constant, where defines as

$$\theta = \max\{\|W_i\|^2 : i = 0, 1, 2, \dots, n\}.$$

and $\hat{\theta}$ is the estimate of θ and adaptive law defined as

$$D^\beta \hat{\theta} = \sum_{i=1}^n \frac{\gamma}{2a_i^2} z_i^2 - k_0 \hat{\theta} \quad (16)$$

where positive constants $a_i (i = 1, \dots, n)$, γ are k_0 design parameters.

Theorem 3.1. *Consider the nonlinear system (9) and construct linear observer (11) with the control law (15), the intermediate virtual-control function v_i for $1 \leq i \leq n-1$, described as (14), and the adaptive laws (16). Then the closed-loop system is semi-globally bounded, and the tracking error will eventually be arbitrary small if appropriate design parameters are chosen.*

Proof. Step 1:

Define the tracking error for the system as

$$z_1 = x_1 - y_d$$

Expressing x_2 in terms of its estimate as $x_2 = \hat{x}_2 + e_2$, we obtain

$$\begin{aligned} D^\beta z_1 &= D^\beta x_1 - D^\beta y_d \\ &= x_2 + f_1(x_1) + d_1 - D^\beta y_d \\ &= \hat{x}_2 + e_2 + f_1(x_1) + d_1 - D^\beta y_d \end{aligned} \quad (17)$$

In order to avoid repeatedly differentiating v_1 , a new state variable v_{2f} is introduced. Let v_1 pass through a first order filter with time constant τ_2 to obtain v_{2f} as

$$\tau_2 D^\beta v_{2f} + v_{2f} = v_1, \quad v_{2f}(0) = v_1(0).$$

Let $s_2 = v_{2f} - v_1$ be the output error of this filter; then, one has $D^\beta v_{2f} = -\frac{s_2}{\tau_2}$, and

$$D^\beta s_2 = D^\beta v_{2f} - D^\beta v_1 = -\frac{s_2}{\tau_2} + C_2(Z_1)$$

and we define $\hat{x}_2 = z_2 + v_{2f}$. By substituting v_{2f} in to the above relation, we have

$$\hat{x}_2 = z_2 + s_2 + v_1 \quad (18)$$

Now consider the following Lyapunov functions:

$$V_1 = e^T P e + \frac{1}{2} z_1^2 + \frac{1}{2\gamma} \tilde{\theta}^2 \quad (19)$$

where $\tilde{\theta} = \theta - \hat{\theta}$ and $\gamma > 0$ is a design constant. From (12), (17) and (18), the time fractional derivative of V_1 is given by

$$\begin{aligned} D^\beta V_1 &= \frac{1}{2} D^\beta e^T P e + \frac{1}{2} e^T P D^\beta e + z_1 D^\beta z_1 + \frac{1}{\gamma} \tilde{\theta} D^\beta \tilde{\theta} \\ &= \frac{1}{2} [P A^T + A P] e + z_1 (\hat{x}_2 + e_2 + f_1 + d_1 - D^\beta y_d) + e^T P (F + D) + \frac{1}{\gamma} \tilde{\theta} D^\beta \tilde{\theta} \\ &= -e^T Q e + z_1 (z_2 + s_2 + v_1 + e_2 + f_1 + d_1 - D^\beta y_d) + e^T P (F + D) - \frac{1}{\gamma} \tilde{\theta} D^\beta \tilde{\theta} \end{aligned} \quad (20)$$

As $F(\bar{x}) = (f_1 + k_1 y, \dots, f_i + k_i y, \dots, f_n + k_n y)$ is an unknown function, by Lemma 2.6, for any given ε_0 , there exists a fuzzy logic system $W_0^T \xi_0(\bar{x})$ such that

$$F(\bar{x}) = W_0^T \xi_0(\bar{x}) + \delta_0(\bar{x}), \quad |\delta_0(\bar{x})| \leq \varepsilon_0$$

As $\xi_0^T \xi_0 \leq 1$, and according to the definition of θ , we know $\|W_0\|^2 \leq \theta$. Therefore, the following inequality holds:

$$\begin{aligned} e^T P F &\leq e^T P (W_0^T \xi_0(\bar{x}) + \delta_0(\bar{x})) \\ &\leq \frac{1}{2} \|e\|^2 + \frac{1}{2} \|P\|^2 \theta + \frac{1}{2} \|P\|^2 \varepsilon_0^2 \end{aligned} \quad (21)$$

In addition, by the definition of D , we have

$$e^T P D \leq \frac{1}{2} \|e\|^2 + \frac{1}{2} \|P\|^2 \|\bar{D}\|^2 \quad (22)$$

where $\bar{D} = (\bar{d}_1, \bar{d}_2, \dots, \bar{d}_n)$.

By using the inequality $2ab \leq a^2 + b^2$ we have

$$e_2 z_1 \leq \frac{1}{2} e_2^2 + \frac{1}{2} z_1^2 \leq \frac{1}{2} \|e\|^2 + \frac{1}{2} z_1^2 \quad (23)$$

Using Young's inequality [4], we have

$$z_1 d_1 \leq \frac{1}{2} \zeta^{-2} z_1^2 + \frac{1}{2} \zeta^2 \bar{d}_1^2 \quad (24)$$

where ζ is a positive constant. Substituting the inequalities (21)-(24) into (20), the fractional time derivative of V_1 is rewritten as

$$\begin{aligned} D^\beta V_1 &\leq -[\lambda_{\min}(Q) - 3/2] \|e\|^2 + z_1 (z_2 + s_2 + v_1 + \bar{f}_1) + \frac{1}{2} \zeta^2 \bar{d}_1^2 \\ &\quad - \frac{1}{2} z_1^2 - \frac{1}{\gamma} \tilde{\theta} D^\beta \tilde{\theta} + \frac{1}{2} \|P\|^2 (\theta + \varepsilon_0^2 + \|\bar{D}\|^2) \end{aligned} \quad (25)$$

where

$$\bar{f}_1(Z_1) = f_1 + \frac{z_1}{2}(1 + \zeta^{-2}) - D^\beta y_d + \frac{1}{2}z_1 + k_1 z_1$$

However, $\bar{f}_1(Z_1)$ is an unknown nonlinear function as it contains $f_1(x_1)$, which cannot be implemented in practice. Therefore, according to Lemma 2.6, for any given constant $\varepsilon_1 > 0$, there exists a fuzzy logic system $W_1^T \xi_1(Z_1)$ such that

$$\bar{f}_1(Z_1) = W_1^T \xi_1(Z_1) + \delta_1(Z_1), \quad |\delta_1(Z_1)| \leq \varepsilon_1$$

From the definition of θ and v_1 , we have

$$\begin{aligned} z_1 \bar{f}_1 &= z_1 \frac{W_1^T}{\|W_1\|} \xi_1 \|W_1\| + z_1 \delta_1 \\ &\leq \frac{1}{2a_1^2} z_1^2 \theta + \frac{1}{2} a_1^2 + \frac{1}{2} z_1^2 + \frac{1}{2} \varepsilon_1^2 \end{aligned} \quad (26)$$

$$z_1 v_1 = -\frac{1}{2a_1^2} z_1^2 \hat{\theta} \quad (27)$$

We use the inequality $\xi_1^T(X_1)\xi_1(X_1) \leq 1$. Then substituting (26) and (27) into (25) yields

$$\begin{aligned} D^\beta V_1 &\leq -[\lambda_{\min}(Q) - 3/2] \|e\|^2 + z_1(z_2 + s_2) - k_1 z_1^2 \\ &\quad + \frac{1}{\gamma} \tilde{\theta} \left(\frac{\gamma}{2a_1^2} z_1^2 - D^\beta \hat{\theta} \right) + \Delta_1 \end{aligned} \quad (28)$$

where

$$\Delta_1 = \frac{1}{2} \|P\|^2 (\theta + \varepsilon_0^2 + \|\bar{D}^2\|) + \frac{1}{2} \zeta^2 \bar{d}_1^2 + \frac{1}{2} a_1^2 + \frac{1}{2} \varepsilon_1^2.$$

Step 2:

We previously defined $z_2 = \hat{x}_2 - v_{2f}$ and then derived it as

$$\begin{aligned} D^\beta z_2 &= D^\beta \hat{x}_2 - D^\beta v_{2f} \\ &= \hat{x}_3 - l_2 \hat{x}_1 - D^\beta v_{2f} \end{aligned} \quad (29)$$

We have

$$D^\beta s_2 = D^\beta v_{2f} - D^\beta v_1 = -\frac{s_2}{\tau_2} + C_2(Z_1)$$

Next, let v_2 pass through a first order filter with time constant τ_3 to obtain v_{3f}

$$\tau_3 D^\beta v_{3f} + v_{3f} = v_2, \quad v_{3f}(0) = v_2(0).$$

Then, defining $s_3 = v_{3f} - v_2$ as the output error of this filter, we have $D^\beta v_{3f} = -\frac{s_3}{\tau_3}$, and

$$D^\beta s_3 = D^\beta v_{3f} - D^\beta v_2 = -\frac{s_3}{\tau_3} + C_3(Z_2)$$

and we define $\hat{x}_3 = z_3 + v_{3f}$.

By substituting v_{3f} in the above relation, we have

$$\hat{x}_3 = z_3 + s_3 + v_2 \quad (30)$$

Now consider the following Lyapunov function:

$$V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{2}s_2^2$$

The fractional time derivative of V_2 is

$$\begin{aligned} D^\beta V_2 &= D^\beta V_1 + z_2 D^\beta z_2 + s_2 D^\beta s_2 \\ &\leq -[\lambda_{\min}(Q) - 3/2] \|e\|^2 + z_1 s_2 + \frac{1}{\gamma} \tilde{\theta} \left(\frac{\gamma}{2a_1^2} z_1^2 - D^\beta \hat{\theta} \right) \\ &\quad + \Delta_1 + z_2(z_3 + s_3 + v_2 + \bar{f}_2) - \frac{1}{2}z_2^2 - \frac{s_2^2}{\tau_2} + s_2 C_2(Z_1) \end{aligned} \quad (31)$$

where

$$\bar{f}_2(Z_2) = -l_2 \hat{x}_1 - D^\beta v_{2f} + z_1 + \frac{1}{2}z_2 + k_2 z_2$$

Since $\bar{f}_2(Z_2)$ is also an unknown nonlinear function, the fuzzy logic system $W_2^T \xi_2(Z_2)$ is now employed to approximate it. For any given constant $\varepsilon_2 > 0$, there exists a fuzzy logic system $W_2^T \xi_2(Z_2)$ such that

$$\bar{f}_2(Z_2) = W_2^T \xi_2(Z_2) + \delta_2(Z_2), \quad |\delta_2(Z_2)| \leq \varepsilon_2$$

Using a similar process as in (26) and (27), we can get

$$z_2 \bar{f}_2 \leq \frac{1}{2a_2^2} z_2^2 \theta + \frac{1}{2}a_2^2 + \frac{1}{2}z_2^2 + \frac{1}{2}\varepsilon_2^2 \quad (32)$$

$$z_2 v_2 = -\frac{1}{2a_2^2} z_2^2 \hat{\theta} \quad (33)$$

Then, by substituting (32) and (33) into (31), we have

$$\begin{aligned} D^\beta V_2 &\leq -[\lambda_{\min}(Q) - 3/2] \|e\|^2 + z_1 s_2 - k_2 z_2^2 + z_2(z_3 + s_3) \\ &\quad + \frac{1}{\gamma} \tilde{\theta} \left(\sum_{i=1}^2 \frac{\gamma}{2a_i^2} z_i^2 - D^\beta \hat{\theta} \right) - \frac{s_2^2}{\tau_2} + s_2 C_2(Z_1) + \Delta_2 \end{aligned} \quad (34)$$

where

$$\Delta_2 = \frac{1}{2} \|P\|^2 (\theta + \varepsilon_0^2 + \|\bar{D}^2\|) + \frac{1}{2} \zeta^2 \bar{d}_1^2 + \frac{1}{2} \sum_{i=1}^2 a_i^2 + \frac{1}{2} \sum_{i=1}^2 \varepsilon_i^2.$$

Step m: ($3 \leq m \leq n-1$): We previously defined $z_m = \hat{x}_m - v_{mf}$ and then derived it as

$$\begin{aligned} D^\beta z_m &= D^\beta \hat{x}_m - D^\beta v_{mf} \\ &= \hat{x}_{m+1} - l_m \hat{x}_1 - D^\beta v_{mf} \end{aligned} \quad (35)$$

We have

$$D^\beta s_m = D^\beta v_{mf} - D^\beta v_{m-1} = -\frac{s_m}{\tau_m} + C_m(Z_{m-1}) \quad (36)$$

Next, introduce a new variable v_{m+1f} , and let v_m pass through a first order filter with time constant τ_{m+1} to obtain v_{m+1f}

$$\tau_{m+1}D^\beta v_{m+1f} + v_{m+1f} = v_m, \quad v_{m+1f}(0) = v_m(0).$$

Then, defining $s_{m+1} = v_{m+1f} - v_m$ as the output error of this filter, we have $D^\beta v_{m+1f} = -\frac{s_{m+1}}{\tau_{m+1}}$, and

$$D^\beta s_{m+1} = D^\beta v_{m+1f} - D^\beta v_m = -\frac{s_{m+1}}{\tau_{m+1}} + C_{m+1}(Z_m)$$

and we define $\hat{x}_{m+1} = z_{m+1} + v_{m+1f}$.

By substituting v_{m+1f} in the above relation, we have

$$\hat{x}_{m+1} = z_{m+1} + s_{m+1} + v_m \quad (37)$$

Choose the Lyapunov function candidate

$$V_m = V_{m-1} + \frac{1}{2}z_m^2 + \frac{1}{2}s_m^2$$

The fractional time derivative of V_m is

$$\begin{aligned} D^\beta V_m &= D^\beta V_{m-1} + z_m(\hat{x}_{m+1} - l_m \hat{x}_1 - D^\beta v_{mf}) + s_m D^\beta s_m \\ &\leq -[\lambda_{\min}(Q) - 3/2]\|e\|^2 + \sum_{i=1}^{m-1} z_i s_{i+1} + \frac{1}{\gamma} \tilde{\theta} \left(\sum_{i=1}^{m-1} \frac{\gamma}{2a_i^2} z_i^2 - D^\beta \hat{\theta} \right) \\ &\quad + z_m(z_{m+1} + s_{m+1} + v_m + \bar{f}_m) - \frac{1}{2}z_m^2 + \Delta_{m-1} \\ &\quad - \sum_{i=1}^{m-1} \left(\frac{s_{i+1}^2}{\tau_{i+1}} + s_{i+1} C_{i+1}(Z_i) \right) \end{aligned} \quad (38)$$

where

$$\bar{f}_m(Z_m) = -l_m \hat{x}_1 - D^\beta v_{mf} + z_{m-1} + \frac{1}{2}z_m + k_m z_m$$

Similarly, $\bar{f}_m(Z_m)$ can be approximated by the fuzzy logic system $W_m^T \xi_m(Z_m)$ as

$$\bar{f}_m(Z_m) = W_m^T \xi_m(Z_m) + \delta_m(Z_m), \quad |\delta_m(Z_m)| \leq \varepsilon_m$$

Using a similar process as in (26) and (27), we can get

$$z_m \bar{f}_m \leq \frac{1}{2a_m^2} z_m^2 \theta + \frac{1}{2} a_m^2 + \frac{1}{2} z_m^2 + \frac{1}{2} \varepsilon_m^2 \quad (39)$$

$$z_m v_m = -\frac{1}{2a_m^2} z_m^2 \hat{\theta} \quad (40)$$

Then, by substituting (39) and (40) into (38), we have

$$\begin{aligned} D^\beta V_m &\leq -[\lambda_{\min}(Q) - 3/2]\|e\|^2 + \sum_{i=1}^{m-1} z_i s_{i+1} - \sum_{i=1}^m k_i z_i^2 + z_m(z_{m+1} + s_{m+1}) \\ &\quad + \frac{1}{\gamma} \tilde{\theta} \left(\sum_{i=1}^m \frac{\gamma}{2a_i^2} z_i^2 - D^\beta \hat{\theta} \right) - \sum_{i=1}^{m-1} \left(\frac{s_{i+1}^2}{\tau_{i+1}} + s_{i+1} C_{i+1}(Z_i) \right) + \Delta_m \end{aligned} \quad (41)$$

where

$$\Delta_m = \frac{1}{2} \|P\|^2 (\theta + \varepsilon_0^2 + \|\bar{D}^2\|) + \frac{1}{2} \zeta^2 \bar{d}_1^2 + \frac{1}{2} \sum_{i=1}^m a_i^2 + \frac{1}{2} \sum_{i=1}^m \varepsilon_i^2.$$

Step n: We previously defined $z_n = \hat{x}_n - v_{nf}$ and then derived it as

$$\begin{aligned} D^\beta z_n &= D^\beta \hat{x}_n - D^\beta v_{nf} \\ &= u - l_n \hat{x}_1 - D^\beta v_{nf} \end{aligned} \quad (42)$$

We have

$$D^\beta s_n = D^\beta v_{nf} - D^\beta v_{n-1} = -\frac{s_n}{\tau_n} + C_n(Z_{n-1}) \quad (43)$$

Choose the Lyapunov function candidate

$$V_n = V_{n-1} + \frac{1}{2} z_n^2 + \frac{1}{2} s_n^2$$

The fractional time derivative of V_n is

$$\begin{aligned} D^\beta V_n &= D^\beta V_{n-1} + z_n (u - l_n \hat{x}_1 - D^\beta v_{nf}) + s_n D^\beta s_n \\ &\leq -[\lambda_{\min}(Q) - 3/2] \|e\|^2 + \sum_{i=1}^{n-1} z_i s_{i+1} + \frac{1}{\gamma} \tilde{\theta} \left(\sum_{i=1}^{n-1} \frac{\gamma}{2a_i^2} z_i^2 - D^\beta \hat{\theta} \right) \\ &\quad + z_{n-1} z_n + \Delta_{n-1} + z_n (u + \bar{f}_n) - \frac{1}{2} z_n^2 - \sum_{i=1}^{n-1} \left(\frac{s_{i+1}^2}{\tau_{i+1}} + s_{i+1} C_{i+1}(Z_i) \right) \end{aligned} \quad (44)$$

where

$$\bar{f}_n(Z_n) = -l_n \hat{x}_1 - D^\beta v_{nf} + z_{n-1} + \frac{1}{2} z_n + k_n z_n$$

Similarly, $\bar{f}_n(Z_n)$ can be approximated by the fuzzy logic system $W_n^T \xi_n(Z_n)$ as

$$\bar{f}_n(Z_n) = W_n^T \xi_n(Z_n) + \delta_n(Z_n), \quad |\delta_n(Z_n)| \leq \varepsilon_n$$

and we can obtain

$$z_n \bar{f}_n \leq \frac{1}{2a_n^2} z_n^2 \theta + \frac{1}{2} a_n^2 + \frac{1}{2} z_n^2 + \frac{1}{2} \varepsilon_n^2 \quad (45)$$

$$z_n u = -\frac{1}{2a_n^2} z_n^2 \hat{\theta} \quad (46)$$

Then, by substituting (45) and (46) into (44), we have

$$\begin{aligned} D^\beta V_n &\leq -[\lambda_{\min}(Q) - 3/2] \|e\|^2 + \sum_{i=1}^{n-1} z_i s_{i+1} - \sum_{i=1}^n k_i z_i^2 \\ &\quad + \frac{1}{\gamma} \tilde{\theta} \left(\sum_{i=1}^m \frac{\gamma}{2a_i^2} z_i^2 - D^\beta \hat{\theta} \right) - \sum_{i=1}^{n-1} \left(\frac{s_{i+1}^2}{\tau_{i+1}} + s_{i+1} C_{i+1}(Z_i) \right) + \Delta_n \end{aligned} \quad (47)$$

where

$$\Delta_n = \frac{1}{2} \|P\|^2 (\theta + \varepsilon_0^2 + \|\bar{D}^2\|) + \frac{1}{2} \zeta^2 \bar{d}_1^2 + \frac{1}{2} \sum_{i=1}^n a_i^2 + \frac{1}{2} \sum_{i=1}^n \varepsilon_i^2.$$

By the definition $D^\beta \hat{\theta}$, we can get

$$\begin{aligned} D^\beta V_n \leq & -[\lambda_{\min}(Q) - 3/2] \|e\|^2 + \sum_{i=1}^{n-1} z_i s_{i+1} - \sum_{i=1}^n k_i z_i^2 \\ & + \frac{k_0}{\gamma} \tilde{\theta} \hat{\theta} - \sum_{i=1}^n \left(\frac{s_{i+1}^2}{\tau_{i+1}} + s_{i+1} C_{i+1}(Z_i) \right) + \Delta_n \end{aligned} \quad (48)$$

By using Young's inequality, we have

$$\begin{aligned} z_i s_{i+1} & \leq \frac{1}{2} z_i^2 + \frac{1}{2} s_{i+1}^2 \\ |s_{i+1} C_{i+1}| & \leq \frac{s_{i+1}^2 C_{i+1}}{2\pi} + 2\pi \\ \tilde{\theta} \hat{\theta} = \tilde{\theta}(\theta - \tilde{\theta}) & \leq -\frac{1}{2} \tilde{\theta}^2 + \frac{1}{2} \theta^2 \end{aligned} \quad (49)$$

where $\pi > 0$ is a design constant. Substituting (49) into (48), one has

$$\begin{aligned} D^\beta V_n \leq & -[\lambda_{\min}(Q) - 3/2] \|e\|^2 - \sum_{i=1}^n \left(k_i - \frac{1}{2} \right) z_i^2 \\ & + \frac{k_0}{2\gamma} \tilde{\theta}^2 - \sum_{i=1}^{n-1} \left(\frac{1}{\tau_{i+1}} - \frac{1}{2} - \frac{C_{i+1}^2}{2\pi} \right) + \bar{\Delta}_n \\ \bar{\Delta}_n = \Delta_n & + \frac{k_0}{2\gamma} \theta^2 + 2\pi \end{aligned} \quad (50)$$

Let $\lambda_{\min}(Q) - 3/2 > 0$, and denote

$$c = \min \left\{ \frac{\lambda_{\min}(Q) - 3/2}{\lambda_{\max}(P)}, 2 \left(k_i - \frac{1}{2} \right), 2 \left(\frac{1}{\tau_{i+1}} - \frac{1}{2} - \frac{C_{i+1}^2}{2\pi} \right), k_0 \right\}, \quad i = 1, \dots, n.$$

To ensure $c > 0$, the control gains k_i are τ_{i+1} chosen to satisfy the following conditions: $k_i - \frac{1}{2} > 0$, and $\frac{1}{\tau_{i+1}} - \frac{1}{2} - \frac{C_{i+1}^2}{2\pi} > 0$ Therefore

$$D^\beta V_n \leq -cV_n + \bar{\Delta}_n. \quad (51)$$

There exists a nonnegative function $N(t)$ satisfying .

$$D^\beta V_n + N(t) = -cV_n + \bar{\Delta}_n. \quad (52)$$

Taking the Laplace transform of (52) gives

$$s^\beta V(s) - V(0)s^{\beta-1} + N(s) = -cV(s) + \bar{\Delta}_n \quad (53)$$

where $V_n(0)$ is a nonnegative constant and $V_n(s) = L\{V_n(t)\}$. It then follows that

$$V_n(s) = \frac{V_n(0)s^{\beta-1} - N(s)}{s^\beta + c} + \frac{\bar{\Delta}_n}{s(s^\beta + c)} \quad (54)$$

If $x(0) = 0$, namely $V_n(0) = 0$; if $x(0) \neq 0$, then $V_n(0) > 0$. Because $V_n(t, x)$ is locally Lipschitz with respect to x , it follows from the fractional uniqueness and existence theorem [1], and the inverse Laplace transform that unique solution of (54) is as

$$V_n(t) = V_n(0)E_\beta(-ct^\beta) + \bar{\Delta}_n t^\beta E_{\beta, \beta+1}(-ct^\beta) - N(t) * t^{\beta-1} E_{\beta, \beta}(-ct^\beta) \quad (55)$$

where $*$ is convolution operator. Since both $t^{\beta-1}$ and $E_{\beta, \beta}(-ct^\beta)$ [49] are non-negative functions, it follows that

$$V_n(t) \leq V_n(0)E_\beta(-ct^\beta) + \bar{\Delta}_n t^\beta E_{\beta, \beta+1}(-ct^\beta) \quad (56)$$

Noting that $\arg(-ct^\beta) = -\pi$, $|-ct^\beta| \geq 0$ for all $t \geq 0$ and $\beta \in (0, 2)$ and Lemma 2.3, one knows that there exists a positive constant C such that

$$|E_\beta(-ct^\beta)| \leq \frac{C}{1+ct^\beta} \quad (57)$$

It follows from (53) that

$$\lim_{t \rightarrow \infty} |V_n(0)|E_\beta(-ct^\beta) = 0 \quad (58)$$

Hence, for every $\bar{\varepsilon} > 0$, there exists a constant $t_1 > 0$ such that $t > t_1$, which implies

$$|V_n(0)|E_\beta(-ct^\beta) \leq \bar{\varepsilon} \quad (59)$$

In the other hand, using Lemma 2.2, one has

$$E_{\beta, \beta+1}(-ct^\beta) = \frac{1}{\Gamma(1)ct^\beta} + o\left(\frac{1}{|ct^\beta|^{1+1}}\right) \quad (60)$$

where the integer n in Lemma 2.2 is chosen as $n = 1$. From (60), for every $\bar{\varepsilon} > 0$, there exists a positive constant t_2 such that

$$\bar{\Delta}_n t^\beta E_{\beta, \beta+1}(-ct^\beta) \leq \frac{\bar{\Delta}_n}{c} + \bar{\varepsilon} \quad (61)$$

For all $t > t_2$. As the design parameters can be adjusted such that $(\frac{\bar{\Delta}_n}{c}) \leq \bar{\varepsilon}$, it follows from (56), (59), and (61) that

$$|V_n(t)| \leq 3\bar{\varepsilon} \quad (62)$$

This means that $V(t)$ is bounded by $3\bar{\varepsilon}$. Thus, all signals in the closed-loop system are bounded. Moreover, by increasing the design parameter c , i.e., adjusting k_i and τ_{i+1} , the errors in the controlled closed-loop system can be made arbitrarily small. This concludes the proof. \square

Remark 3.2. Different from the adaptive fractional order backstepping design [6], that the uncertain system parameters are only assumed to be unknown constants, the adaptive fuzzy fractional-order backstepping scheme introduced in our work assumes that the nonlinear system model can be fully unknown.

4. Simulation Results

Example 4.1. In order to assess the robustness of the proposed control method, we compare the performance of the controller in the presence of measurement noise and without it. In the first test, the following nonlinear fractional nominal system is evaluated by our proposed method and the simulation was performed according to the controller parameters in Table 1.

$$\begin{cases} D^\beta x_1 = x_2 + 0.1x_1 e^{-0.5x_1^2} \cos x_1 \\ D^\beta x_2 = u + x_2 \sin\left(\frac{0.2}{1+x_1^2}\right) \\ y = x_1 + n(t) \end{cases} \quad (63)$$

where the reference signal is given as $y_d = \sin(t)$ and $n(t)$ is the measurement noise. According to Theorem 3.1, the virtual control function v_1 and the control law u are chosen, respectively, as

$$v_1 = -\frac{1}{2a_1^2} z_1 \hat{\theta}, \quad u = -\frac{1}{2a_1^2} z_2 \hat{\theta}$$

where $z_1 = y - y_d$, $z_2 = \hat{x}_2 - v_{2f}$. The adaptive law is given as

$$D^\beta \hat{\theta} = \sum_{i=1}^2 \frac{\gamma}{2a_i^2} z_i^2 - k_0 \hat{\theta}$$

In the second test, simulation experiments have been performed on the system in the presence of measurement noise and step perturbation. We perform our proposed method according to the controller parameters in Table 1. Figure. 1 shows the measurement noise, noise histogram and step perturbation. The white noise power equal to 0.001.

In the above mentioned tests, the simulation is run under the initial conditions $[x_1(0), x_2(0), \hat{x}_1(0), \hat{x}_2(0)]^T = [1, 0.8, 0, 0]^T$ and $\hat{\theta} = 0$

Parameter	a_1	a_2	l_1, l_2	γ	k_0, τ_2	β
Value	0.4	40	10	3	0.001	0.8

TABLE 1. Controller Parameters

The control objective of these simulations is to construct an adaptive fuzzy fractional output feedback control scheme to guarantee that all the closed-loop signals remain bounded, and the tracking error converge to a neighborhood of the origin.

These two tests are compared with each other. The simulation results are shown in Figures. 2-4. Figure. 2 shows the tracking performance of the control signals in the presence of measurement noise and without it. Figures. 3 and 4 show that the control input signal and the tracking error respectively. Simulation results show our proposed method is robust because it has an acceptable performance in the presence of measurement noise and perturbation.

Clearly, the simulation results verify our theoretical findings and show the effectiveness of our control scheme.

Example 4.2. In this example, in order to assess the effectiveness of the proposed control design, we compare our method with the fractional adaptive backstepping controller in [6].

First, the following fractional order controlled gyroscope model with external disturbance [6] is evaluated by our proposed method and the simulation was performed according to the system parameters in Table 2 and controller parameters in Table 3.

$$\begin{cases} D^\beta x_1 = x_2 \\ D^\beta x_2 = -p(t)x_1 - c_1x_2 - c_2x_2^3 + q(t)x_1^3 + u + d(t) \\ y = x_1 \end{cases} \quad (64)$$

where $p(t) = \frac{\alpha^2}{4} - f \sin(\omega t)$, $q(t) = \frac{\alpha^2}{12} - \frac{v}{6} - \frac{f \sin(\omega t)}{6}$. The additive disturbance is $d(t) = 0.5 \cos(\pi t) + 0.1 \sin(3t)$, which come from the modeling errors and other types of unknown nonlinearities in the system. Also is the control input.

Parameter	α	v	ω	f	c_1	c_2	β
Value	10	1	25	35.5	0.005	0.05	0.8

TABLE 2. System Parameters

The control objective is to design an adaptive fuzzy controller such that all signals in the closed loop system remain bounded and the system output is regulated.

According to Theorem 1, construct the virtual control law, the true control law and the adaptive laws as follows:

$$v_1 = -\frac{1}{2a_1^2} z_1 \hat{\theta}, \quad u = -\frac{1}{2a_1^2} z_2 \hat{\theta}$$

where $z_1 = y - y_d$, $z_2 = \hat{x}_2 - v_{2f}$ The adaptive law is given as

$$D^\beta \hat{\theta} = \sum_{i=1}^2 \frac{\gamma}{2a_i^2} z_i^2 - k_0 \hat{\theta}$$

Second, we perform the method in [6], according to the system parameters in Table 2. In the both control methods, the simulation is run under the initial conditions $[x_1(0), x_2(0), \hat{x}_1(0), \hat{x}_2(0)]^T = [1, 1, 0, 0]^T$ and $\hat{\theta} = 0$

Parameter	a_1	a_2	l_1, l_2	γ	k_0	τ_2	β
Value	0.25	20	20	4.2	0.05	0.09	0.8

TABLE 3. System Parameters

The control objective of this simulation is to construct an adaptive fuzzy fractional output feedback control scheme to guarantee that all the closed-loop signals remain bounded, and the regulating error converge to a neighborhood of the origin. The simulation results are shown in Figures. 5-10. Figures. 5-7 show the comparison of the proposed method and the method in [6] for $x_1(t)$, $x_2(t)$ and $u(t)$ respectively. As can be seen in the Figures 5 and 6, the convergence speed signals in the proposed method is faster than the method in [6]. Also Figure. 7 shows that control signal in proposed method has a less oscillation and amplitude than control signal in [6]. To examine the efficiency of the controller against fractional order variation, the error signal with the fractional order variation are compared in Figure. 8. Clearly, the nonlinear controller derived above is robust against system order reduction resulting from employing fractional derivatives. Also the control input signals with fractional order variation are compared in Figure. 9. $\hat{\theta}$ signal with fractional order variation is shown in Figure. 10. From simulation results it is clear that as we decrease order of fractional order operator beta, the overshoot in the control input signal increases. The overshoot represents extra control effort which compensates error due initial conditions.

From the simulation results, it can clearly be seen that the proposed controller guarantees the boundedness of all signals in the closed loop system, and also achieves the good tracking performance.

5. Conclusion

In this paper, the problem of observer-based adaptive fuzzy control scheme was investigated for fractional order dynamic systems in strict-feedback structure. The control scheme has been constructed by using the backstepping and adaptive technique. DSC method was used to avoid the problem of explosion of complexity which was caused by backstepping design process. Fuzzy logic systems were used to approximate the unknown nonlinear functions. The proposed control method guarantees that the closed-loop system is stable in the sense of semiglobally uniformly ultimately bounded and tracking error converge to an adjustable neighborhoods of the origin. Two simulation examples were provided to illustrate the effectiveness of the control scheme. In the first example by adding the noise and disturbance, the robustness of the controller was demonstrated. Simulation results showed that the proposed controller has a good performance against noise and disturbance. In the second example, fractional order gyroscope system with additive disturbance was considered. Performance of the proposed method was compared with the method in [6] under the same condition. Simulation results showed better performance of the proposed controller compared with the method in [6].

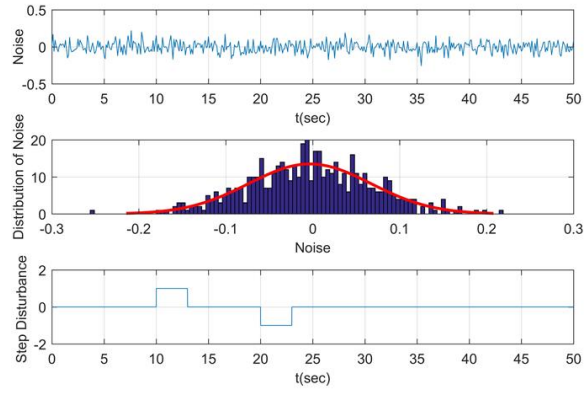


FIGURE 1. The Measurement Noise, Noise Histogram and Step Perturbation

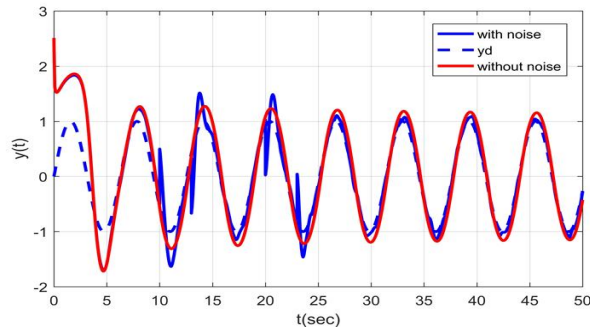


FIGURE 2. The System Output y with Noise, the Reference Signal y_d and the System Output y without Noise

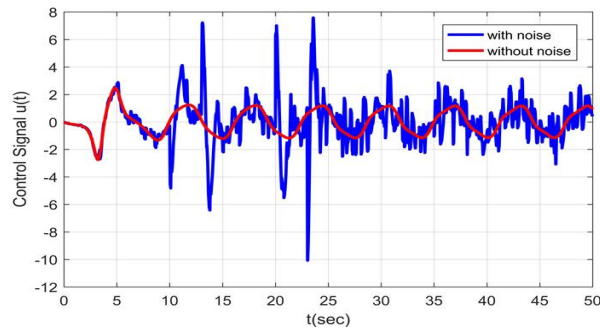


FIGURE 3. The Control Input Signal $u(t)$

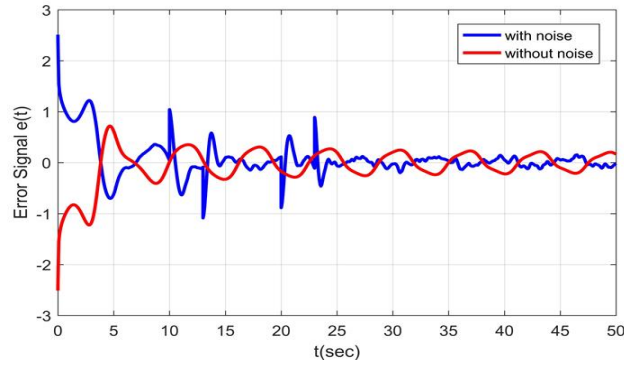
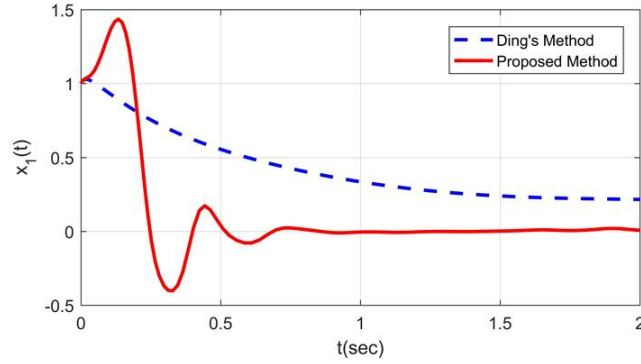
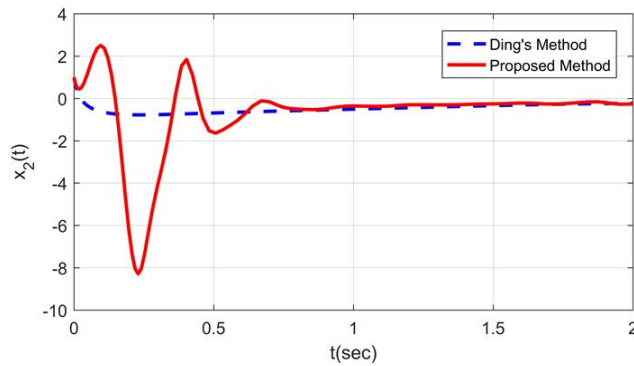


FIGURE 4. The Tracking Error

FIGURE 5. Comparison of the Proposed Method and the Method in [6] for $x_1(t)$ FIGURE 6. Comparison of the Proposed Method and the Method in [6] for $x_2(t)$

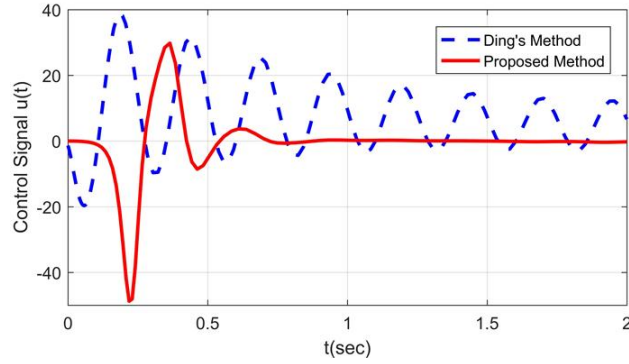


FIGURE 7. Comparison of the Proposed Method and the Method in [6] for $u(t)$

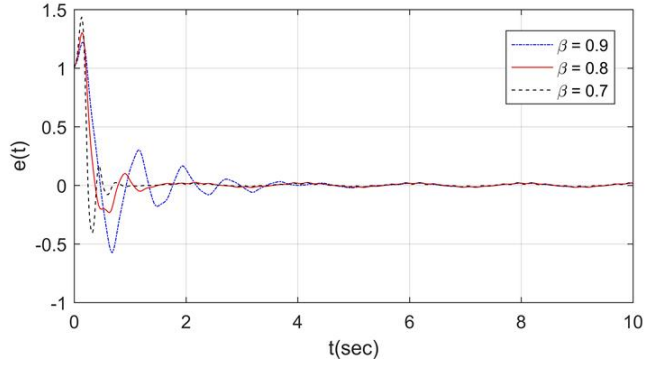


FIGURE 8. Error Signal with Fractional Order Variation

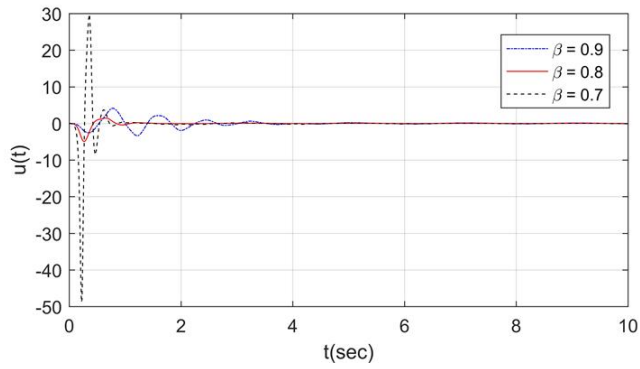
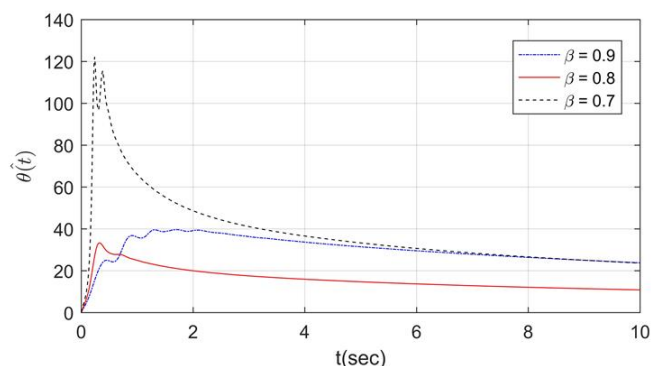


FIGURE 9. Control Input Signal with Fractional Order Variation

FIGURE 10. $\hat{\theta}$ Signal with Fractional Order Variation

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