TREND-CYCLE ESTIMATION USING FUZZY TRANSFORM OF HIGHER DEGREE

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Abstract. In this paper, we provide theoretical justification for the application of higher degree fuzzy transform in time series analysis. Under the assumption that a time series can be additively decomposed into a trend-cycle, a seasonal component and a random noise, we demonstrate that the higher degree fuzzy transform technique can be used for the estimation of the trend-cycle, which is one of the basic tasks in time series analysis. We prove that high frequencies appearing in the seasonal component can be arbitrarily suppressed and that random noise, as a stationary process, can be successfully decreased using the fuzzy transform of higher degree with a reasonable adjustment of parameters of a generalized uniform fuzzy partition.

1. Introduction

The fuzzy transform (F–transform) as an approximation technique was introduced by Perfilieva in [15] (see, also, [16]) and then generalized by means of polynomials to a higher degree in [18]. The core of the F–transform technique consists in a fuzzy partition of the real line or a real interval with help of fuzzy sets that satisfies a la Ruspini condition or the partition of a unity condition. The fuzzy sets of a fuzzy partition are called basic functions and are used to transform a function into a vector of F–transform components (polynomials). This step in the F–transform is called the *direct F–transform*. The approximation of the original function is provided by the *inverse F–transform* where the linear combination of the F–transform components with weights that are related to the function values of basic functions from the considered generalized uniform fuzzy partition is applied. An indisputable advantage of the F–transform technique is the good compression of information about a function accompanied by the reduction of undesirable effects, e.g., high frequencies or types of noise (see, e.g., [7, 12, 19]).

The first applications of the F–transform to time series analysis can be found in [13, 14], where the F–transform was used for trend-cycle (non-parametric) estimation. Note that the time series considered in these papers are additively decomposable into a trend-cycle, a seasonal component and a random noise. It is well known

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that trend (trend-cycle) estimation is one of the major tasks in time series analysis, and in the literature, there are many approaches for trend (trend-cycle) estimation such as model-based approaches (an ARIMA model or a state space model), non-parametric linear filtering (the Henderson, LOESS, and Hodrick-Prescott filters), or singular spectrum analysis. For a review of some modern methods, we refer to [1]. A successful estimation of the trend-cycle requires significant suppression or better elimination of high frequencies and random fluctuations. In the case of the F–transform technique, the problem of trend-cycle estimation has been theoretically investigated in [11, 12], where the authors demonstrated how high frequencies in a seasonal component can be suppressed and a random noise can be reduced using the F^0– and F^1–transform with respect to the triangle and raised cosine generalized uniform fuzzy partitions. It is well known that the higher degree F–transform demonstrates better approximation of functions, which is rather inconsistent with the suppression of high infrequences and the elimination of random noise in time series. Therefore, a natural question is whether the higher degree F–transform, mainly for the degree $m > 1$, can be a useful technique for trend-cycle estimation. In [4], we show several preliminary results on the suppression of high frequencies in time series using the higher degree F–transform technique and this contribution should complete our research on trend-cycle estimation with the help of the higher degree F–transform with respect to generalized uniform fuzzy partitions. We believe that the results offer a unified view on trend-cycle extraction possibilities using the higher degree F–transform technique.

The paper is structured as follows. In Section 2, we briefly review the basic concepts related to the higher degree F–transform technique and prove essential facts that are used in the next section. The main part of this contribution is Section 3, which is devoted to a theoretical justification of the application of the higher degree F–transform in trend-cycle estimation. More precisely, we show that, under an assumption on the smoothness of the trend-cycle function, high frequencies and a random noise can be efficiently removed with the help of higher degree F–transform by an appropriate setting of parameters of generalized uniform fuzzy partitions. Section 4 provides an illustration of the results on artificial time series and real time series including a comparison with non-parametric methods for trend-cycle estimation.[1]

2. Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, integers, reals and complex numbers, respectively. For any complex number $c \in \mathbb{C}$, we use $|c|$ to denote the absolute value of c. i.e., $|c| = (c \cdot \bar{c})^{\frac{1}{2}}$, where $\bar{c}$ is the complex conjugate of c. Moreover, for any $m \times n$ matrix A with complex elements, we define $\|A\| = \max\{|A_{ij}| \mid i = 1, \dots, m, j = 1, \dots, n\}$.

[1]We choose non-parametric methods for our comparison, for the sake of fairness, because estimation based on the higher degree F–transform belongs to this class of methods of trend-cycle estimation.

2.1. **Generalized Uniform Fuzzy Partition.** A fuzzy partition of an interval or the real line is a core of the (higher degree) F–transform. In this paper, we restrict ourselves to fuzzy partitions that are uniformly spread along the real line and are determined by generating functions.

Definition 2.1. A real–valued function $K : \mathbb{R} \to [0,1]$ is said to be a *generating function* if K is a continuous and even function that is non-increasing in $[0,1]$ and vanishing outside of $[-1,1]$.

Basic examples of generating functions frequently appearing in applications of the F–transform technique are the triangle and raised cosine functions.

Example 2.2. The functions $K^{tr}, K^{rc} : \mathbb{R} \to [0,1]$ defined by

$$K^{tr}(t) = \max(1 - |t|, 0),$$

$$K^{rc}(t) = \begin{cases} \frac{1}{2}(1 + \cos(\pi t)), & -1 \le t \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

for any $t \in \mathbb{R}$, are called the *triangle* and *raised cosine* generating functions, respectively.

The triangle generating function is a special case of the B-spline generating functions that are used in a little modified form in [9, 10].

Example 2.3. Let us define a rectangular pulse β^0 as follows:

$$\beta^0(t) = \begin{cases} 1, & -\frac{1}{2} < x < \frac{1}{2}, \\ \frac{1}{2}, & |x| = \frac{1}{2}, \\ 0, & \text{otherwise}. \end{cases}$$

A central B-spline of degree n denoted by β^n is constructed from the $(n+1)$-fold convolution of the rectangular pulse β^0:

$$\beta^n(t) = \underbrace{\beta^0 \star \beta^0 \star \cdots \star \beta^0(t)}_{(n+1) \text{ times}}.$$

A B-spline generating function of degree n is denoted by $K^{bs,n}(t)$ and defined by rescaling the support of $\beta^n(t)$, i.e.,

$$K^{bs,n}(t) = \beta^n\left(\frac{(n+1)\cdot t}{2}\right).$$

Obviously, it holds that $K^{bs,1}(t) = K^{tr}(t)$.

Let K be a generating function, and let h and r be positive constants. The parameters h and r are called the *bandwidth* and the *shift*, respectively. Let $k \in \mathbb{N}$. Define by $t_k = k \cdot r$ the k-th node of the real line. The function $A_{h,r,k} : \mathbb{R} \to [0,1]$ given by

$$A_{h,r,k}(t) = K\left(\frac{t - t_k}{h}\right)$$

is said to be a *scaled generating function placed at the k-th node of the real line.* Now, we can proceed to the definition of a generalized uniform fuzzy partition of $\mathbb{R}$ as has been proposed in [6].

Definition 2.4. Let K be a generating function, and let h and r be positive real constants. *A generalized uniform fuzzy partition of the real line determined by the triplet* (K, h, r) is the collection $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$ of scaled generating functions placed at all nodes $t_k = k \cdot r$, $k \in \mathbb{Z}$, which satisfies *a la Ruspini condition*:

$$\sum_{k\in\mathbb{Z}} A_{h,r,k}(t) = 1$$

for any $t \in \mathbb{R}$. A function $A_{h,r,k}$ is called the *k-th basic function* of the fuzzy partition.

The following theorem provides a sufficient condition for a generalized uniform fuzzy partition to be determined from a triplet $(\alpha \cdot K, h, r)$, where we define $(\alpha \cdot K)(x) = \alpha \cdot K(x)$ for $x \in \mathbb{R}$.

Theorem 2.5. *Let K be a generating function, and let $\gamma = \int_{-1}^{1} K(t)dt$. If K satisfies the γ–symmetry condition, i.e.,*

$$\sum_{k\in\mathbb{Z}} K(t - k\gamma) = 1, \quad t \in [0, 1],$$

and $\frac{\gamma h}{r} \in \mathbb{N}$ for $h, r > 0$, then the triplet $\left(\frac{r}{\gamma h} \cdot K, h, r\right)$ determines a generalized uniform fuzzy partition of $\mathbb{R}$.

Proof. See [4]. □

As a straightforward consequence of the previous theorem we find a sufficient condition for the determination of the raised cosine and B-spline generalized uniform fuzzy partitions.

Corollary 2.6. *If $\frac{h}{r} \in \mathbb{N}$ for $h, r > 0$, then the triplet $\left(\frac{r}{h} \cdot K^{rc}, h, r\right)$ determines a raised cosine generalized uniform fuzzy partition.*

Corollary 2.7. *If $\frac{2h}{r(n+1)} \in \mathbb{N}$ for $h, r > 0$ and a positive natural number n, then the triplet $\left(\frac{r(n+1)}{2h} \cdot K^{bs,n}, h, r\right)$ determines a B-spline generalized uniform fuzzy partition of degree n.*

In what follows, when we consider the raised cosine or B-spline generalized uniform fuzzy partition, we assume that its parameters satisfy the conditions stated in Corollaries 2.6 and 2.7.

2.2. **Higher Degree F–transform of Complex-valued Functions.** Let $L^2_{loc}(\mathbb{R})$ be a set of all complex-valued functions that are square integrable on any closed subinterval of the real line, and let K be a generating function. Then, the space $L^2_{loc}(\mathbb{R})$ endowed with the inner product $\langle \cdot, \cdot \rangle_K$ defined by

$$\langle f, g \rangle_K = \int_{-\infty}^{\infty} f(t)\overline{g(t)}K(t)dt, \quad f, g \in L^2_{loc}(\mathbb{R}),$$

where $\overline{g(t)}$ is the complex conjugate to $g(t)$ forms a weighted Hilbert space, which is denoted by $L^2(K)$. In addition, we say that f and g are *orthogonal in the space* $L^2(K)$, denoted by $f \perp g$, if $\langle f, g\rangle_K = 0$. Moreover, let M be a linear subspace of $L^2(K)$. A function f is said to be orthogonal with the subspace M, denoted by $f \perp M$, if $f \perp h$ for any $h \in M$. Finally, the set $M^T = \left\{ f \in L^2(K) \mid f \perp M \right\}$ is called the *orthogonal complement* to M. In what follows, we provide a lemma that is very important for the representation of the fuzzy transform.

Lemma 2.8. *Let K be a generating function, and let a square matrix Z_m be defined as follows:*

$$Z_m = \begin{bmatrix} \langle 1,1\rangle_K & \langle 1,t\rangle_K & \dots & \langle 1,t^m\rangle_K \\ \langle t,1\rangle_K & \langle t,t\rangle_K & \dots & \langle t,t^m\rangle_K \\ \dots & \dots & \dots & \dots \\ \langle t^m,1\rangle_K & \langle t^m,t\rangle_K & \dots & \langle t^m,t^m\rangle_K \end{bmatrix}. \tag{1}$$

Then, Z_m is an invertible matrix.

Proof. For simplicity, put $Z = Z_m$. The matrix Z is invertible if its rows are linearly independent. Denote by Z_r the r-th row, where $r_0 = 1, \dots, m+1$. Let us assume that there exists the row Z_{r_0}, $r_0 \in \{1, \dots, m+1\}$, such that

$$Z_{r_0} = c_1 Z_1 + \dots + c_{r_0-1} Z_{r_0-1} + c_{r_0+1} Z_{r_0+1} + \dots + c_{m+1} Z_{m+1}$$

holds for certain constant $c_1, \dots, c_{r_0-1}, c_{r_0+1}, \dots, c_{m+1} \in \mathbb{R}$. The j-th constituent of the vector Z_{r_0} has the form

$$Z_{r_0 j} = \langle t^{r_0-1}, t^{j-1}\rangle_K = \sum_{\substack{i=1\\ i\neq r_0}}^{m+1} c_i Z_{ij} = \sum_{\substack{i=1\\ i\neq r_0}}^{m+1} c_i \langle t^{i-1}, t^{j-1}\rangle_K = \Big\langle \sum_{\substack{i=1\\ i\neq r_0}}^{m+1} c_i t^{i-1}, t^{j-1} \Big\rangle_K,$$

where we used the linearity of the inner product. Hence, we obtain that the equation

$$\Big\langle \sum_{\substack{i=1\\ i\neq r_0}}^{m+1} c_i t^{i-1} - t^{r_0-1}, t^{j-1} \Big\rangle_K = 0$$

holds for any $j = 1, \dots, m+1$; therefore,

$$\sum_{\substack{i=1\\ i\neq r_0}}^{m+1} c_i t^{i-1} - t^{r_0-1}$$

belongs to the orthogonal complement to $V = \mathrm{Span}\{1, t, \dots, t^m\}$. Moreover,

$$\sum_{\substack{i=1\\ i\neq r_0}}^{m+1} c_i t^{i-1} - t^{r_0-1} \in V$$

which implies that

$$\sum_{\substack{i=1\\ i\neq r_0}}^{m+1} c_i t^{i-1} - t^{r_0-1} = 0.$$

But, this is a contraction with the fact that the monomials of $\{1, t, \ldots, t^m\}$ are linearly independent. Hence, the row Z_{r_0} is not a linear combination of the remaining rows and Z_m is an invertible matrix. □

Let $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$ be a generalized uniform fuzzy partition of the real line. As with the construction of the weighted Hilbert space $L^2(K)$, we use $L^2(A_{h,r,k})$ to denote the weighted Hilbert space of all functions of $L^2_{loc}(\mathbb{R})$ with the inner product defined by

$$\langle f, g\rangle_{A_{h,r,k}} = \int_{-\infty}^{\infty} f(t)\overline{g(t)}A_{h,r,k}(t)dt.$$

The norm on $L^2(A_{h,r,k})$ is defined by

$$||f||_{A_{h,r,k}} = \sqrt{\langle f, f\rangle_{A_{h,r,k}}}.$$

In what follows, we use $f \perp_k g$ and $f \perp_k M$ to denote the orthogonality of f and g and the orthogonality of f and a linear subspace M of $L^2(A_{h,r,k})$ with respect to the inner product $\langle\cdot,\cdot\rangle_{A_{h,r,k}}$, respectively. Finally, let us denote $P_k(m)$ the linear subspace of $L^2(A_{h,r,k})$ of all polynomials with complex coefficients of degrees at most equal to m. In [18], the direct fuzzy transform of higher degree was defined by means of the orthogonal projections of functions of $L^2_{loc}(\mathbb{R})$ to the linear subspaces $P_k(m)$.

Definition 2.9. Let $f \in L^2_{loc}(\mathbb{R})$, and let $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$ be a generalized uniform fuzzy partition of the real line. The *direct fuzzy transform of degree m (direct F^m–transform)* of f with respect to $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$ is the collection of polynomials

$$\left\{Q^k \mid Q^k \in P_k(m), (f - Q^k)\perp_k P_k(m),\ k \in \mathbb{Z}\right\}.$$

The polynomial $F^m_k[f] = Q^k$ is called the k-th component of the direct F–transform of degree m of f.

In the original paper [18] on the higher degree F–transform, orthogonal bases of polynomials derived by the Gram-Schmidt orthogonalization process with respect to basic functions of fuzzy partitions were used to find the components of the direct F^m–transform. In this paper, we consider another approach to the derivation of the direct F^m–transform components, one that is based on the monomial basis and special matrices. The following theorem shows how the components of the direct F^m–transform can be derived with the help of simple matrices.

Theorem 2.10. *Let $f \in L^2_{loc}(\mathbb{R})$, and let $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$ be a generalized uniform fuzzy partition of the real line determined by a triplet (K, h, r). Then, the k-th component of the direct F^m–transform of f with respect to $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$ has the following form*

$$F^m_k[f](t) = \beta_{k,0} + \beta_{k,1}(t - t_k) + \ldots + \beta_{k,m}(t - t_k)^m \tag{2}$$

determined by

$$(\beta_{k,0}, \beta_{k,1}, \ldots, \beta_{k,m})^T = H^{-1} \cdot Z_m^{-1} \cdot Y_k$$

where $H = \mathrm{diag}(1, h, \ldots, h^m)$, Z_m is the square matrix defined in (1), and $Y_k = (Y_{k,1}, \ldots, Y_{k,m+1})^T$ is defined by

$$Y_{k,i} = \langle f(th + t_k), t^{i-1}\rangle_K = \int_{-1}^{1} f(th + t_k) \cdot t^{i-1}K(t)dt.$$

Proof. Let $\{1, t-t_k, \dots, (t-t_k)^m\}$ be the basis of the linear space $P_k(m)$ and let the k-th component of the direct F^m–transform of f have the form in (2). From the definition, we find that $(f - F_k^m[f](t)) \perp_k (t-t_k)^i$ for any $i = 0, 1, \dots, m$. In what follows, for the simplicity, we use $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_{A_{h,r,k}}$. From the linearity of the inner product, we obtain

$$\langle F_k^m[f](t), (t-t_k)^i \rangle = \langle f, (t-t_k)^i \rangle$$

for any $i = 0, 1, \dots, m$. Substituting the expression of $F_k^m[f](t)$ in (2) into the previous equation, we obtain

$$\begin{aligned} &\beta_{k,0}\langle 1, (t-t_k)^i \rangle + \beta_{k,1}\langle (t-t_k), (t-t_k)^i \rangle + \dots \\ &+\beta_{k,m}\langle (t-t_k)^m, (t-t_k)^i \rangle = \langle f, (t-t_k)^i \rangle \end{aligned} \tag{3}$$

for any $i = 0, 1, \dots, m$. In addition, we have

$$\begin{aligned} \langle (t-t_k)^j, (t-t_k)^i \rangle &= \int_{t_k-h}^{t_k+h} (t-t_k)^{i+j} K\left(\frac{t-t_k}{h}\right) dt \\ &= h^{i+j+1} \int_{-1}^{1} t^{i+j} K(t) dt = h^{i+j+1} \langle t^i, t^j \rangle_K, \end{aligned}$$

and

$$\begin{aligned} \langle f, (t-t_k)^i \rangle &= \int_{t_k-h}^{t_k+h} f(t) \cdot (t-t_k)^i K\left(\frac{t-t_k}{h}\right) dt \\ &= h^{i+1} \int_{-1}^{1} f(th+t_k) \cdot t^i K(t) dt = h^{i+1} Y_{k,i+1} \end{aligned}$$

for any $i, j = 0, 1, \dots, m$. Hence, the equation (3) can be rewritten as follows

$$\beta_{k,0}\langle 1, t^i \rangle_K + h\beta_{k,1}\langle t, t^i \rangle_K + \dots + h^m \beta_{k,m}\langle t^m, t^i \rangle_K = Y_{k,i+1}.$$

Since $\langle t^j, t^i \rangle_K = \langle t^i, t^j \rangle_K$ for any $i, j = 0, 1, \dots, m$, we find that

$$\beta_{k,0}\langle t^i, 1 \rangle_K + h\beta_{k,1}\langle t^i, t \rangle_K + \dots + h^m \beta_{k,m}\langle t^i, t^m \rangle_K = Y_{k,i+1}.$$

for any $i = 0, 1, \dots, m$. This system of linear equations can be expressed in the matrix form as follows:

$$Z_m \cdot H \cdot \boldsymbol{\beta}_k = Y_k,$$

where $H = \mathrm{diag}(1, h, \dots, h^m)$, $\boldsymbol{\beta}_k = (\beta_{k,0}, \beta_{k,1}, \dots, \beta_{k,m})^T$, Z_m is defined in (1) and $Y_k = (Y_{k,1}, \dots Y_{k,m+1})^T$. Since the matrices Z_m and H are invertible (see Lemma 2.8), we obtain $\boldsymbol{\beta}_k = H^{-1} \cdot Z_m^{-1} \cdot Y_k$ and the proof is finished. □

A basic question is how well the components of the F^m–transform locally approximate the original function, because this fact directly influences the quality of approximation of the original function by the inverse F^m–transform that is introduced below. There are several results with respect to this issue; see, e.g., Theorem 5 in [17], Theorem 2 in [10] and elsewhere. In [10], the authors present an interesting result showing that the quality of approximation by the F^m–transform can be improved with the help of B-spline generalized uniform fuzzy partitions. In this paper, we provide an upper estimation for the pointwise approximation of the

F^m–transform component that includes the measure of uniform continuity of the original function, namely, the modulus of continuity. The use of the modulus of continuity in the formulation of the upper estimation is motivated by our further investigation of trend-cycle estimation, where the size of changes in the course of the trend-cycle influences the quality of its estimation more than the choice of a small bandwidth, which generally improves the quality of F^m–transform approximation. Indeed, the choice of a small bandwidth supports the reconstruction of seasonal and random constituents in time series, which has a negative effect on trend-cycle estimation. The modulus of continuity is defined for a function f as follows:

$$\omega(f,\delta) = \sup\{|f(x) - f(y)| \mid x, y \in \mathbb{R}, |x-y| \leq \delta\}$$

where $\delta > 0$.

Theorem 2.11. *Let $f \in L^2_{loc}(\mathbb{R})$, and let $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$ be a generalized uniform fuzzy partition of the real line determined by a triplet (K, h, r). Then, for any $m \in \mathbb{N}$, $k \in \mathbb{Z}$, and $t \in [t_k - h, t_k + h]$, the following inequality holds*

$$|f(t) - F^m_k[f](t)| \leq (m+1)\|Z_m^{-1}\| \cdot \theta(m, K) \cdot \omega(f, 2h), \tag{4}$$

where Z_m is defined in (1) and

$$\theta(m,K) = \sum_{\ell=1}^{m+1} \int_{-1}^{1} |s|^{\ell-1} K(s)ds.$$

Proof. Let $t \in [t_k - h, t_k + h]$ be arbitrary. From Theorem 2.10 and the fact that $F^m_k[c](t) = c$ holds for any complex constant function c, we obtain the following upper estimation

$$|f(t) - F^m_k[f](t)| = |f(t) \cdot F^m_k[1](t) - F^m_k[f](t)| =$$
$$|(\alpha_{k,0} - \beta_{k,0}) + (\alpha_{k,1} - \beta_{k,1})(t - t_k) + \ldots + (\alpha_{k,m} - \beta_{k,m})(t - t_k)^m| \leq$$
$$\sum_{j=0}^{m} |\alpha_{k,j} - \beta_{k,j})| \cdot |t - t_k|^j$$

where $(\alpha_{k,0}, \alpha_{k,1}, \ldots, \alpha_{k,m})$ and $(\beta_{k,0}, \beta_{k,1}, \ldots, \beta_{k,m})$ are determined by

$$(\alpha_{k,0}, \alpha_{k,1}, \ldots, \alpha_{k,m})^T = H^{-1} Z_m^{-1} A_k,$$
$$(\beta_{k,0}, \beta_{k,1}, \ldots, \beta_{k,m})^T = H^{-1} Z_m^{-1} B_k$$

with $A_k = (A_{k,j})_{j=1,\ldots,m+1}$ and $B_k = (B_{k,j})_{j=1,\ldots,m+1}$ that are the column matrices given by

$$A_{k,j} = f(t) \cdot \langle 1, s^{j-1}\rangle_K = \int_{-1}^{1} f(t) \cdot s^{j-1} K(s)ds,$$
$$B_{k,j} = \langle f(sh + t_k), s^{j-1}\rangle_K = \int_{-1}^{1} f(sh + t_k) \cdot s^{j-1} K(s)dt.$$

From (2.2) and (2.2), we obtain

$$(\alpha_{k,0} - \beta_{k,0}, \alpha_{k,1} - \beta_{k,1}, \ldots, \alpha_{k,m} - \beta_{k,m})^T = H^{-1} Z_m^{-1}(A_k - B_k).$$

Hence, for any $j = 0, \dots, m$, we find that

$$|\alpha_{k,j} - \beta_{k,j}| \leq \frac{\|Z_m^{-1}\|}{h^j} \sum_{\ell=1}^{m+1} |A_{k\ell} - B_{k\ell}| \leq$$

$$\frac{\|Z_m^{-1}\|}{h^j} \sum_{\ell=1}^{m+1} \int_{-1}^{1} |f(t) - f(sh + t_k)| \cdot |s|^{\ell-1} K(s) ds \leq$$

$$\frac{\|Z_m^{-1}\| \cdot \omega(f, 2h)}{h^j} \sum_{\ell=1}^{m+1} \int_{-1}^{1} |s|^{\ell-1} K(s) ds.$$

Using this inequality and the upper estimation of $|f(t) - F_k^m[f](t)|$, we obtain

$$|f(t) - F_k^m[f](t)| \leq \sum_{j=0}^{m} \|Z_m^{-1}\| \cdot \omega(f, 2h) \frac{|t - t_k|^j}{h^j} \sum_{\ell=1}^{m+1} \int_{-1}^{1} |s|^{\ell-1} K(s) ds \leq$$

$$(m+1)\|Z_m^{-1}\| \cdot \omega(f, 2h) \sum_{\ell=1}^{m+1} \int_{-1}^{1} |s|^{\ell-1} K(s) ds =$$

$$(m+1)\|Z_m^{-1}\| \cdot \theta(m, K) \cdot \omega(f, 2h)$$

and the proof is finished. □

From the previous theorem, one can see that the term $(m+1)\|Z_m^{-1}\| \cdot \theta(m, K)$ in (4) is constant for the given degree m of the higher degree F–transform and the generating function K. Therefore, the absolute difference between function values $f(t)$ and $F_k^m[f](t)$ is affected mainly by the value of the modulus of continuity of f. Obviously, lower values of the modulus of continuity of the original function lead to better quality of their local approximation by the components of the F^m–transform.

The inverse step of the higher degree F–transform is a linear-like combination of basic functions with coefficients represented by the F–transform components (polynomials).

Definition 2.12. Let $f \in L^2_{loc}(\mathbb{R})$, and let $\{Q_k \mid k \in \mathbb{Z}\}$ be a direct F^m–transform of the function f with respect to a generalized uniform fuzzy partition $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$. The *inverse fuzzy transform of degree m (inverse F^m–transform) of* f with respect to $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$ is defined by

$$F^m[f](t) = \hat{f}^m(t) = \sum_{k\in\mathbb{Z}} Q_k(t) A_{h,r,k}(t), \quad t \in \mathbb{R}.$$

In the following, we omit reference to the degree of the direct F–transform in all cases where no confusion can appear. Moreover, when we discuss the inverse F^m–transform of a function f with respect to a generalized uniform fuzzy partition and the direct F^m–transform is not explicitly mentioned, we assume that the direct F^m–transform of f is computed with respect to the same fuzzy partition.

Recall that the inverse F^m–transform is a linear mapping preserving constant functions, i.e., $F^m[\alpha f + \beta g] = \alpha F^m[f] + \beta F^m[g]$ and $F^m[c] = c$, where c is a constant function. For details, we refer to [18]. As a simple consequence of Theorem 2.11, the following statement provides an upper estimation for the pointwise approximation of the inverse F^m–transform to the original function based on the modulus of continuity that is used in the investigation of trend-cycle estimation, as discussed above.

Theorem 2.13. *Let $f \in L^2_{loc}(\mathbb{R})$, and let $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$ be a generalized uniform fuzzy partition of the real line determined by a triplet (K, h, r). Let $F^m[f]$ be the inverse F^m–transform of f with respect to $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$. Then, for any $t \in \mathbb{R}$, the following statement holds*

$$|f(t) - F^m[f](t)| \leq (m+1)\|Z_m^{-1}\| \cdot \theta(m, K) \cdot \omega(f, 2h),$$

where Z_m and $\theta(m, K)$ have the same meaning as in Theorem 2.11.

Proof. From Theorem 2.11, we have

$$|f(t) - F_k^m[f](t)| \leq (m+1)\|Z_m^{-1}\| \cdot \theta(m, K) \cdot \omega(f, 2h)$$

for any $k \in \mathbb{Z}$ and $t \in [t_k - h, t_k + h]$. Hence, using a la Ruspini condition, we obtain

$$\begin{gathered}
\big|f(t) - F^m[f](t)\big| = \left|\sum_{k\in\mathbb{Z}} f(t)A_{h,r,k}(t) - \sum_{k\in\mathbb{Z}} F_k^m[f](t)A_{h,r,k}(t)\right| = \\
\left|\sum_{k\in\mathbb{Z}} \left(f(t) - F_k^m[f](t)\right) A_{h,r,k}(t)\right| \leq \sum_{k\in\mathbb{Z}} \big|f(t) - F_k^m[f](t))\big| A_{h,r,k}(t) \leq \\
\sum_{k\in\mathbb{Z}} (m+1)\|Z_m^{-1}\| \cdot \theta(m, K) \cdot \omega(f, 2h) \cdot A_{h,r,k}(t) = \\
(m+1)\|Z_m^{-1}\| \cdot \theta(m, K) \cdot \omega(f, 2h) \sum_{k\in\mathbb{Z}} A_{h,r,k}(t) = \\
(m+1)\|Z_m^{-1}\| \cdot \theta(m, K) \cdot \omega(f, 2h)
\end{gathered}$$

for any $t \in \mathbb{R}$. □

2.3. **Higher-degree Fuzzy Transform of Complex-valued Stochastic Processes.** To analyse a random noise, we also consider the higher degree F–transform of a stationary stochastic process that was introduced in [5]. In this case, we replace the integration of complex-valued functions by the integration of complex-valued stochastic processes in the sense of the Riemann integral and use the same formulas for the direct and inverse F^m–transform. Note that the linearity of the F^m–transform for complex-valued stochastic processes remains true, but the equality is replaced in this case by the equality that almost surely holds, which is a simple consequence of the mean square calculus.

For completeness, let us recall basic definitions. We use $\mathbf{E}[\xi]$ and $\mathbf{Var}[\xi]$ to denote the expected value and the variance of a complex-valued random variable ξ, respectively. Let $\xi_1, \dots, \xi_q, \dots$ be a sequence of complex-valued random variables and ξ be a complex-valued random variable of the same probability space $(\Omega, \mathcal{F}, P)$. We say that $\xi_1, \dots, \xi_q, \dots$ converges to ξ in mean square and write

$$\operatorname*{l.i.m.}_{q\to\infty} \xi_q = \xi$$

provided that $\mathbf{E}[|\xi_q|^2] < \infty$ for any $q = 1, 2 \dots,$ $\mathbf{E}[|\xi|^2] < \infty$ and

$$\lim_{q\to\infty} \mathbf{E}\left[|\xi_q - \xi|^2\right] = 0.$$

Moreover, let $a_1, \dots, a_q, \dots$ be a sequence of real values. We say that the sequence ξ_q/a_q is stochastically bounded and write $\xi_q = \mathcal{O}_p(a_q)$, if, for any $\varepsilon > 0$, there exist $M > 0$ and q_0 such that

$$P\left(\left|\frac{\xi_q}{a_q}\right| > M\right) < \varepsilon$$

for any $q > q_0$.[2] Let $f(t)$ be an arbitrary real function, and let $\xi(t)$ be a stochastic process, $t \in \mathbb{R}$. The integral

$$\int_c^d \xi(t) f(t) dt$$

is defined as

$$\operatorname*{l.i.m.}_{n\to\infty} \sum_{j=2}^{n} \xi(t'_j) f(t'_j)(t_j - t_{j-1}),$$

where $c = t_1 < t_2 < \cdots < t_n = d$ and $t_{j-1} \le t'_j \le t_j$ holds for any $j = 2, \dots, n$. For details, we refer to [8, 21].

We say that ξ is equal to ξ' almost surely (denoted by $\xi = \xi'$ a.s.) if

$$P\left(\{\omega \in \Omega \mid \xi(\omega) = \xi'(\omega)\}\right) = 1.$$

Similarly, we say that ξ is less than or equal to ξ' almost surely (denoted by $\xi \le \xi'$ a.s.) if

$$P\left(\{\omega \in \Omega \mid \xi(\omega) \le \xi'(\omega)\}\right) = 1.$$

Remark 2.14. By Markov's inequality, it is easy to show that $E[|\xi - \xi'|^2] = 0$ (equality in mean square) implies $\xi = \xi'$ a.s. Moreover, one can prove that if $\xi_i = \xi'_i$ a.s. for $i = 1, \dots, n$, then $\sum_{i=1}^n \xi_i = \sum_{i=1}^n \xi'_i$ a.s. Finally, if $\xi = \xi'$ a.s. (or $\xi \le \xi'$ a.s.), then $\mathbf{E}[\xi] = \mathbf{E}[\xi']$ (or $\mathbf{E}[\xi] \le \mathbf{E}[\xi']$).

3. Trend-cycle Estimation Using $\mathbf{F}^m$–transform

3.1. **Assumptions on Time Series.** Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X(t)$, $t \in \mathbb{R}$, be a time series, and assume that $X(t)$ can be additively decomposed and written in the following form

$$X(t) = TC(t) + S(t) + R(t),$$

where $TC(t)$, $S(t)$ and $R(t)$ denote the trend-cycle, seasonal components and noise fluctuations, respectively. Naturally, we assume that $TC(t)$ is a smooth function

[2] Using the probability of an opposite event, this definition is equivalent to the following: for any $0 < \varepsilon \le 1$, there exists $M > 0$ and q_0 such that $P\left(|\xi_q| \le M|a_q|\right) \ge 1 - \varepsilon$ for any $q > q_0$.

with little change in its course; i.e, its modulus of continuity $\omega(TC, 2h)$ is a small number even for a larger bandwidth h. In addition, we assume that the seasonal component $S(t)$ is the sum of a finite number of basic waves as follows:

$$S(t) = \sum_{j=1}^{s} P_j e^{i\lambda_j t},$$

where P_j is a complex number, λ_j is the j-th frequency, and i denotes the imaginary unit. Furthermore, we assume that random noise $R(t)$ is a stationary stochastic process with zero mean. It is well known (see, e.g., [21]) that each stationary process $R(t)$ with zero mean can be approximated with arbitrary precision by a finite linear combination of harmonic oscillations, i.e.,

$$R(t) \approx \sum_{j=1}^{n} \xi_j e^{i\varphi_j t},$$

where $\xi_1, \ldots, \xi_n$ are pairwise uncorrelated random variables independent on time t with zero mean and finite variances, φ_j and i denote the j-th frequency and the imaginary unit, respectively. In this paper, we use this property and restrict our consideration to a random noise $R(t)$, which is a stationary process in the form

$$R(t) = \sum_{j=1}^{n} \xi_j e^{i\varphi_j t}. \tag{5}$$

3.2. **Analysis of $\mathbf{F}^m$–transform Components of Basic Wave $e^{i\lambda t}$.** Since the F^m–transform is linear, the quality of trend-cycle estimation by the F^m–transform clearly depends on how well a basic wave $e^{i\lambda t}$ in expression (3.1) of seasonal component $S(t)$ and expression (5) of random noise $R(t)$ can be suppressed by the F^m–transform. In this subsection, we provide several results concerning the suppression of the basic wave in terms of the F^m–transform components that are used in the next subsection, which is devoted to the suppression of high frequencies in time series.

Let $p(\lambda, t) = e^{i\lambda t}$, where t, λ and i denote the time, frequency and imaginary unit, respectively. The complex-valued function $p(\lambda, t)$ thus represents a basic wave constituent in expressions (3.1) and (5). Note that $p(\lambda, t) \in L^2_{loc}(\mathbb{R})$ and $p(\lambda, t)$ is complex Riemann integrable. The k-th component of the direct F^m–transform of a function f with respect to a generalized uniform fuzzy partition $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$ determined by a triplet (K, h, r) is denoted by

$$F^m_{k,(K,h)}[f]$$

to emphasize the type of generating function K and the bandwidth h of the fuzzy partition.

In the sequel, we consider an increasing sequence of positive real numbers $\{h_q\}_q$ such that $h_q \to \infty$ as $q \to \infty$. Moreover, we write $f(h_q) = \mathcal{O}(g(h_q))$ if and only if there exists a positive constant C and a natural number q_0 such that $|f(h_q)| \leq C|g(h_q)|$ for any $q > q_0$.

We start our analysis of the basic wave with the statement showing that the size of each F^m–transform component of $p(\lambda, t)$ can be arbitrarily small depending on the setting of the bandwidth of fuzzy partitions. First, let us recall the Riemann-Lebesgue lemma well known in the Fourier analysis. For details, we refer to [20].

Lemma 3.1. *Let f be a function of $L^1[a,b]$. Then, the following integrals*

$$\int_a^b f(x)\cos\lambda x dx \text{ and } \int_a^b f(x)\sin\lambda x dx$$

tend to zero as $|\lambda| \to \infty$.

Theorem 3.2. *Let $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$, $q = 1,2,\ldots$, be a generalized uniform fuzzy partition determined by a triplet (K, h_q, r_q). Then, for each $m \in \mathbb{N}$ and $t \in [t_k - h_q, t_k + h_q]$,*

$$\lim_{q\to\infty} F^m_{k,(K,h_q)}[p(\lambda,t)](t) = 0 \tag{6}$$

holds true.

Proof. Let $q = 1,2,\ldots$, and let $t \in [t_k-h_q, t_k+h_q]$ be arbitrary. From Theorem 2.10, we know that

$$F^m_{k,(K,h_q)}[p(\lambda,t)](t) = \beta_{k,0} + \beta_{k,1}(t-t_k) + \cdots + \beta_{k,m}(t-t_k)^m,$$

where

$$\boldsymbol{\beta}_k = (\beta_{k,0},\cdots,\beta_{k,m})^T = H^{-1}\cdot Z_m^{-1}\cdot Y_k.$$

Recall that $Y_k = (Y_{k,i})$ is the $(m+1)\times 1$ matrix determined in our case by

$$Y_{k,i} = \int_{-1}^1 p(\lambda, h_q t + t_k)\cdot t^{i-1}K(t)dt.$$

Then, one can simply find that

$$\left|F^m_{k,(K,h_q)}[p(\lambda,t)](t)\right| \le \sum_{j=0}^m |t-t_k|^j|\beta_{k,j}|.$$

Moreover, it holds that

$$|\beta_{k,j}| \le \frac{\|Z_m^{-1}\|}{h_q^j}\sum_{\ell=0}^m |Y_{k,\ell+1}|$$

for any $j = 0,1,\ldots,m$. Hence, we find that

$$\left|F^m_{k,(K,h_q)}[p(\lambda,t)](t)\right| \le \|Z_m^{-1}\| \sum_{j,\ell=0}^m \left|\frac{t-t_k}{h_q}\right|^j |Y_{k,\ell+1}| \le$$

$$(m+1)\|Z_m^{-1}\|\sum_{\ell=0}^m |Y_{k,\ell+1}|.$$

In addition, for any $\ell = 0,1,\ldots,m$, we have

$$Y_{k,\ell+1} = \int_{-1}^1 p(\lambda,h_q t+t_k)t^\ell K(t)dt = \int_{-1}^1 e^{i\lambda(h_q t+t_k)}t^\ell K(t)dt$$

$$= e^{i\lambda t_k}\int_{-1}^1 t^\ell K(t)e^{i\lambda h_q t}dt$$

$$= e^{i\lambda t_k}\left(\int_{-1}^1 t^\ell K(t)\cos(\lambda h_q t)dt + i\int_{-1}^1 t^\ell K(t)\sin(\lambda h_q t)dt\right);$$

therefore,

$$|Y_{k,\ell+1}| = \sqrt{\left(\int_{-1}^{1} t^\ell K(t)\cos(\lambda h_q t)dt\right)^2 + \left(\int_{-1}^{1} t^\ell K(t)\sin(\lambda h_q t)dt\right)^2}.$$

From the assumption that $h_q \to \infty$ as $q \to \infty$, Lemma 3.1 and the properties of limits, we find that

$$\lim_{q\to\infty} |Y_{k,\ell+1}| = 0$$

for any $\ell = 0, 1, \ldots, m$. Hence,

$$0 \le \lim_{q\to\infty} \left|F^m_{k,(K,h_q)}[p(\lambda,t)](t)\right| \le (m+1)\|Z_m^{-1}\| \sum_{\ell=0}^{m} \lim_{q\to\infty} |Y_{k,\ell+1}| = 0,$$

which implies the desired statement. □

It should be noted that a similar statement can be proved for high frequencies, namely, if a generalized uniform fuzzy partition is fixed, i.e., the generating function K and its bandwidth h is not changed, then

$$F^m_{k,(K,h)}[p(\lambda,t)](t) \to 0 \quad \text{as} \quad \lambda \to \infty.$$

Nevertheless, in this paper, we are not interested in this case, because we assume that the basic waves with their frequencies are prescribed.

Although, the previous theorem has a major importance for the high frequency suppression task, it says nothing about the speed in which F–transform component $F^m_{k,(K,h_q)}[p(\lambda,t)](t)$ tends to zero in (6) with respect to the bandwidth h_q. In the following, we show a rate of convergence, which is dependent on the differentiability of generating functions, and demonstrate that this rate of convergence can be even higher in the case of the B-spline generalized uniform fuzzy partitions.

We use the following notation and assumption. For $I \subseteq \mathbb{R}$, $C^N(I)$ (or $C^{\star N}(I)$) denotes the class of all complex-valued functions that are N-times differentiable on I and for which its N-th derivative is continuous (or integrable) on I.[3] If $\{\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}} \mid q = 1, 2, \ldots\}$ is a family of generalized uniform fuzzy partitions determined by $\{(K, h_q, r_q) \mid q = 1, 2, \ldots\}$, we always assume that r_q/h_q, $q = 1, 2, \ldots$, is a constant. Therefore, the same number of overlapping basic functions is considered for any fuzzy partition $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$ of the given family.

Theorem 3.3. *Let $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$, $q = 1, 2, \ldots$, be a generalized uniform fuzzy partition determined by a triplet (K, h_q, r_q). Let $M, N \in \mathbb{N}$ be such that $M \le N$ and put $L = \min(N, M + 2)$. If $K \in C^M(\mathbb{R}) \cap C^{\star N}([-1, 1])$, then*

$$F^m_{k,(K,h_q)}[p(\lambda,t)](t) = \mathcal{O}\left(h_q^{-L}\right)$$

holds true for any $t \in [t_k - h_q, t_k + h_q]$.

Proof. Let $q = 1, 2, \ldots$. By the same arguments used in the proof of Theorem 3.2, we find that

$$\left|F^m_{k,(K,h_q)}[p(\lambda,t)](t)\right| \le (m+1)\|Z_m^{-1}\| \sum_{\ell=0}^{m} |Y_{k,\ell+1}|, \tag{7}$$

[3]Note that $C^N(I) \subset C^{\star N}(I)$ because the continuity implies the integrability, but not vice versa.

where

$$|Y_{k,\ell+1}| = \left|\int_{-1}^{1} t^\ell K(t) e^{i\lambda h_q t} dt\right|.$$

Since $K \in C^M(\mathbb{R}) \cap C^{\star N}([-1,1])$ and $t^\ell \in C^\infty(\mathbb{R})$, one can show easily that $t^\ell K \in C^M(\mathbb{R}) \cap C^{\star N}([-1,1])$. Using the integration by parts, we obtain

$$\int_{-1}^{1} t^\ell K(t) e^{i\lambda h_q t} dt = \frac{(-1)^N}{(i \cdot \lambda h_q)^N} \cdot \int_{-1}^{1} (t^\ell K(t))^{(N)} \cdot e^{i\lambda h_q t} dt + e^{i\lambda h_q t} \sum_{j=0}^{N-1} \frac{(-1)^j}{(i\cdot\lambda h_q)^{j+1}} \cdot (t^\ell K(t))^{(j)} \Big|_{t=-1}^{t=1}. \tag{8}$$

Moreover, from the continuity of $K^{(j)}$ on $\mathbb{R}$ for any $j = 0, \ldots, M$, we find that $(t^\ell K(t))^{(j)}\big|_{t=\pm 1} = 0$ for any $j = 0, \ldots, M$. Therefore, if $N > M + 2$, we obtain

$$\int_{-1}^{1} t^\ell K(t) e^{i\lambda h_q t} dt = \frac{(-1)^N}{(i \cdot \lambda h_q)^N} \cdot \int_{-1}^{1} (t^\ell K(t))^{(N)} \cdot e^{i\lambda h_q t} dt + e^{i\lambda h_q t} \sum_{j=M+1}^{N-1} \frac{(-1)^j}{(i\cdot\lambda h_q)^{j+1}} \cdot (t^\ell K(t))^{(j)} \Big|_{t=-1}^{t=1}.$$

It follows that

$$|Y_{k,\ell+1}| = \left|\int_{-1}^{1} t^\ell K(t) e^{i\lambda h_q t} dt\right| \le \frac{1}{|h_q|^{M+2}} \cdot \left(\frac{1}{|h_q|^{N-M-2} \cdot |\lambda|^N} \int_{-1}^{1} \left|(t^\ell K(t))^{(N)}\right| dt + \sum_{j=M+1}^{N-1} \frac{1}{|h_q|^{j-M-1} \cdot |\lambda|^{j+1}} \cdot \left| (t^\ell K(t))^{(j)} \Big|_{t=-1}^{t=1} \right| \right).$$

From this inequality and (7), we simply obtain

$$F^m_{k,(K,h_q)}[p(\lambda,t)](t) = \mathcal{O}\left(h_q^{-(M+2)}\right). \tag{9}$$

If $N \le M+2$, we have $(t^\ell K(t))^{(j)}\big|_{t=\pm 1} = 0$ for any $j = 0, \ldots, N-2$. Consequently, the equation (8) becomes

$$\int_{-1}^{1} t^\ell K(t) e^{i\lambda h_q t} dt = \frac{(-1)^N}{(i \cdot \lambda h_q)^N} \cdot \int_{-1}^{1} (t^\ell K(t))^{(N)} \cdot e^{i\lambda h_q t} dt + \frac{(-1)^{N-1} e^{i\lambda h_q t}}{(i\cdot\lambda h_q)^N} \cdot (t^\ell K(t))^{(N-1)} \Big|_{t=-1}^{t=1}.$$

Hence, we obtain

$$|Y_{k,\ell+1}| = \left|\int_{-1}^{1} t^\ell K(t) e^{i\lambda h_q t} dt\right| \le \frac{1}{|h_q|^N \cdot |\lambda|^N} \cdot \left(\int_{-1}^{1} \left|(t^\ell K(t))^{(N)}\right| dt + \left| (t^\ell K(t))^{(N-1)} \Big|_{t=-1}^{t=1} \right| \right)$$

and from (7), we simply find that

$$F^m_{k,(K,h_q)}[p(\lambda,t)](t) = \mathcal{O}\left(h_q^{-N}\right). \tag{10}$$

Putting $L = \min(N, M+2)$, the statement is a straightforward consequence of (9) and (10). □

Since $K^{rc} \in C^1(\mathbb{R}) \cap C^{\star 3}([-1,1])$, we obtain the following rate of convergence for the raised cosine generating function as a consequence of the above mentioned theorem.

Corollary 3.4. *Let $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$, $q = 1,2,\ldots$, be a raised cosine generalized uniform fuzzy partition determined by a triplet $(\tilde{K}^{rc}, h_q, r_q)$, where $\tilde{K}^{rc} = \frac{r_q}{h_q}\cdot K^{rc}$. Then, for any $t \in [t_k - h_q, t_k + h_q]$, it holds that*

$$F^m_{k,(\tilde{K}^{rc},h_q)}[p(\lambda,t)](t) = \mathcal{O}\left(h_q^{-3}\right).$$

For the B-spline generating function of degree n, one can find that $K^{bs,n} \in C^{n-1}(\mathbb{R}) \cap C^{\star n-1}([-1,1])$. As a consequence of Theorem 3.3, we obtain that the rate of convergence is $h_q^{-(n-1)}$. Nevertheless, the following statement demonstrates that the result of Theorem 3.3 guarantees a "rough" rate of convergence, which need not be actually the highest one.

Theorem 3.5. *Let $\left\{A_{h_q,r_q,k}\right\}_{k\in\mathbb{Z}}$, $q = 1,2,\ldots$, be a B-spline generalized uniform fuzzy partition of degree n determined by the triplet $\left(\tilde{K}^{bs,n}, h_q, r_q\right)$, where $\tilde{K}^{bs,n} = \frac{r_q(n+1)}{2h_q}\cdot K^{bs,n}$. Then, for any $t \in [t_k - h_q, t_k + h_q]$, it holds that*

$$F^m_{k,(\tilde{K}^{bs,n},h_q)}[p(\lambda,t)](t) = \mathcal{O}\left(h_q^{-(n+1)}\right).$$

Proof. Let $f_q = -\frac{\lambda h_q}{2\pi}$. Recall that the Fourier transform (unitary with ordinary frequency) of $\beta^n(t)$ is the function $\mathrm{sinc}^{n+1}(f)$, where f is the frequency and $\mathrm{sinc}(f) = \frac{sin(\pi f)}{\pi f}$. By the definition of $\tilde{K}^{bs,n}$ and the basic properties of the Fourier transform, for any $\ell = 0,1,\ldots,m$, we obtain

$$\begin{aligned}\int_{-1}^{1} t^\ell \tilde{K}^{bs,n}(t) e^{i\lambda h_q t}dt &= \frac{r_q(n+1)}{2h_q}\int_{-1}^{1} t^\ell \beta^n\left(\frac{(n+1)t}{2}\right) e^{-i2\pi t f_q}dt \\ &= \frac{r_q(n+1)}{2h_q}\cdot \mathcal{F}\left[t^\ell \beta^n\left(\frac{(n+1)t}{2}\right)\right](f_q) = \\ &\frac{r_q}{h_q}\cdot\left(\frac{i}{2\pi}\right)^\ell \cdot \frac{d^\ell}{df^\ell}\mathrm{sinc}^{n+1}\left(\frac{2f}{n+1}\right)\Bigg|_{f=f_q},\end{aligned}$$

where $\frac{r_q}{h_q} = c$ for any $q = 1,2,\ldots$ by the assumption on the family of generalized uniform fuzzy partitions. Moreover, one can prove by a direct computation that

$$\frac{d^\ell}{df^\ell}\mathrm{sinc}^{n+1}\left(\frac{2f}{n+1}\right)\Bigg|_{f=f_q} = \mathcal{O}(f_q^{-(n+1)}),$$

for any $\ell = 0, 1, \ldots, m$. By $f_q = -\frac{\lambda h_q}{2\pi}$, we obtain

$$\left| \int_{-1}^{1} t^{\ell} \tilde{K}^{bs,n}(t) e^{i\lambda h_q t} dt \right| = \mathcal{O}(h_q^{-(n+1)}),$$

Using similar arguments considered in the proof of Theorem 3.3, we obtain the desired statement. □

As we have mentioned in Example 2.3, the triangular generalized uniform fuzzy partition is a B-spline generalized uniform fuzzy partition of degree $n = 1$. The following statement is then a consequence of the previous theorem.

Corollary 3.6. *Let $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$, $q = 1, 2, \ldots$, be a generalized uniform fuzzy partition determined by a triplet $(\tilde{K}^{tr}, h_q, r_q)$, where $\tilde{K}^{tr} = \frac{r_q}{h_q} \cdot K^{tr}$. Then, for any $t \in [t_k - h_q, t_k + h_q]$, it holds that*

$$F^m_{k,(\tilde{K}^{tr},h_q)}[p(\lambda, t)](t) = \mathcal{O}\left(h_q^{-2}\right).$$

3.3. **Suppression of High Frequencies in Time Series.** The aim of this subsection is to prove that the F^m–transform technique enables us to suppress high frequencies in time series. The main idea is to show that we can construct a generalized uniform fuzzy partition whereby seasonal component is significantly suppressed and noise is also highly reduced. From the previous subsection, we know that the F^m–transform can significantly suppress the basic wave $p(\lambda, t) = e^{i\lambda t}$. In the following two parts, we apply this fact to the seasonal and random components to achieve the desired statements.

3.3.1. *Seasonal Component.* Recall that the seasonal component $S(t)$ is given by

$$S(t) = \sum_{j=1}^{s} P_j e^{i\lambda_j t},$$

where P_j is a complex number, λ_j is the j-th frequency, and i denotes the imaginary unit. In the following theorem, we show how the seasonal component can be suppressed by the application of the F^m–transform technique.

Theorem 3.7. *Let $S(t)$ be a seasonal component of a time series of the form (3.1), and let $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$, $q = 1, 2, \ldots$, be a generalized uniform fuzzy partition determined by a triplet (K, h_q, r_q). If $\hat{S}^m_{(K,h_q)}(t)$ denotes the inverse F^m–transform of $S(t)$ with respect to $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$, then, for any $t \in \mathbb{R}$, it holds that*

$$\lim_{q\to\infty} \hat{S}^m_{(K,h_q)}(t) = 0. \tag{11}$$

Proof. By the linearity of the F^m–transform, for any $q = 1, 2, \ldots$ and $k \in \mathbb{Z}$, we find that

$$F^m_{k,(K,h_q)}[S](t) = \sum_{j=1}^{s} P_j \cdot F^m_{k,(K,h_q)}[p(\lambda_j, t)](t).$$

Hence, we obtain

$$\begin{aligned}\hat{S}^m_{(K,h_q)}(t) &= \sum_{k\in\mathbb{Z}} F^m_{k,(K,h_q)}[S](t) A_{h_q,r_q,k}(t)\\ &= \sum_{j=1}^{s} P_j \sum_{k\in\mathbb{Z}} F^m_{k,(K,h_q)}[p(\lambda_j,t)](t) A_{h_q,r_q,k}(t);\end{aligned}$$

therefore,

$$\left|\hat{S}^m_{(K,h_q)}(t)\right| \le \sum_{j=1}^{s} |P_j| \sum_{k\in\mathbb{Z}} \left|F^m_{k,(K,h_q)}[p(\lambda_j,t)](t)\right| A_{h_q,r_q,k}(t).$$

Since only a finite number of summands is applied from the infinite sum in the derivation of the inverse F^m–transform, statement (11) is a straightforward consequence of the previous inequality and Theorem 3.2. □

In Theorem 3.3, we have proved the influence of the differentiability of generating functions to a rate of convergence of the F^m–transform applied to the basic wave $p(\lambda,t)$. The same rate of convergence also holds true for the inverse F^m–transform of a seasonal component as the following theorem states.

Theorem 3.8. *Let $S(t)$ be a seasonal component of a time series represented by (3.1), and let $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$, $q = 1,2,\ldots$, be a generalized uniform fuzzy partition determined by a triplet (K,h_q,r_q) such that $K \in C^M(\mathbb{R}) \cap C^{\star N}([-1,1])$. If $\hat{S}^m_{(K,h_q)}(t)$ is the inverse F^m–transform of $S(t)$ with respect to $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$, then, for any $t \in \mathbb{R}$, it holds that*

$$\hat{S}^m_{(K,h_q)}(t) = \mathcal{O}(h_q^{-L}),$$

where $L = \min(N, M+2)$.

Proof. In the proof of Theorem 3.7, we have shown that

$$\left|\hat{S}^m_{(K,h_q)}(t)\right| \le \sum_{j=1}^{s} |P_j| \sum_{k\in\mathbb{Z}} \left|F^m_{k,(K,h_q)}[p(\lambda_j,t)](t)\right| A_{h_q,r_q,k}(t)$$

for any $q = 1,2,\ldots$. Since the infinite sum has only a finite number of non-zero summands, we can rewrite the previous inequality as follows

$$\left|\hat{S}^m_{(K,h_q)}(t)\right| \le \sum_{j=1}^{s} |P_j| \sum_{\ell=1}^{n} \left|F^m_{k_\ell,(K,h_q)}[p(\lambda_j,t)](t)\right| A_{h_q,r_q,k_\ell}(t).$$

According to Theorem 3.3, for any $j = 1,\ldots,s$ and $\ell = 1,\ldots,n$, there exist $C_{j\ell} > 0$ and $N_{j\ell} > 0$ such that

$$\left|F^m_{k_\ell,(K,h_q)}[p(\lambda_j,t)](t)\right| \le C_{j\ell} h_q^{-L} \tag{12}$$

holds for any $q > N_{j\ell}$. Put

$$\begin{aligned}C &= \max\{C_{j\ell} \mid j = 1,\ldots,s, \ell = 1,\ldots,n\},\\ N_0 &= \max\{N_{j\ell} \mid j = 1,\ldots,s, \ell = 1,\ldots,n\},\end{aligned}$$

we find from (12) that

$$\left| F^m_{k_\ell,(K,h_q)}[p(\lambda_j, t)](t) \right| \le C h_q^{-L}$$

for any $q > N_0$. Since there exists G such that $|P_j| < G$ for any $j = 1, \dots, s$, we obtain

$$\left| \hat{S}^m_{(K,h_q)}(t) \right| \le C h_q^{-L} \sum_{j=1}^{s} |P_j| \le C' h_q^{-L},$$

where $C' = s \cdot C \cdot G$, for any $q > N_0$, which proves the statement. □

As a consequence of Theorem 3.8 and by a simple application of Theorem 3.5, we obtain the statement for the F^m–transform with respect to the raised cosine and the B-spline generalized uniform fuzzy partitions.

Corollary 3.9. *Let $S(t)$ be a seasonal component of a time series of the form* (3.1)*, and let $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$, $q = 1, 2, \dots$, be a generalized uniform fuzzy partition determined by a triplet (K, h_q, r_q). Then, the following statements holds:*

(i) if $K = \tilde{K}^{rc}$, then $\hat{S}^m_{(K,h_q)}(t) = \mathcal{O}(h_q^{-3})$;
(ii) if $K = \tilde{K}^{bs,n}$, then $\hat{S}^m_{(K,h_q)}(t) = \mathcal{O}(h_q^{-(n+1)})$;
(iii) if $K = \tilde{K}^{tr}$, then $\hat{S}^m_{(K,h_q)}(t) = \mathcal{O}(h_q^{-2})$.

From Corollary 3.9, one can see that the use of the B-spline generalized uniform fuzzy partitions of the degree n, $n \ge 3$ should be favorable in the suppression of high frequencies against the application of the raised cosine generalized uniform fuzzy partition if we are able to set a really large bandwidth for the fuzzy partition. A further discussion on this issue is in section Illustrative examples.

3.3.2. *Random Component.* Now, we are interested in the noise reduction. Recall that the stochastic process $R(t)$ is given by

$$R(t) = \sum_{j=1}^{n} \xi_j e^{i\varphi_j t}$$

where $\xi_1, \dots, \xi_n$ are pairwise uncorrelated complex-valued random variables with zero mean independent on time t, i is the imaginary unit and $\varphi_1, \dots, \varphi_n$ are real constants.

Before we provide main theorems, we prove a useful lemma, which states one of the basic properties of the F^m–transform of stochastic processes.

Lemma 3.10. *Let $f \in L^2_{loc}(\mathbb{R})$, and let ξ be a random variable with finite variance. Let $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$ be a generalized uniform fuzzy partition determined by a triplet (K, r, h). Then, for any $k \in \mathbb{Z}$,*

$$F^m_{k,(K,h)}[\xi \cdot f] = \xi \cdot F^m_{k,(K,h)}[f] \quad a.s.$$

holds true.

Proof. First, let us show that

$$\int_{-1}^{1} \xi \cdot g(t)dt = \xi \int_{-1}^{1} g(t)dt \quad a.s. \tag{13}$$

for any Riemann integrable complex-valued function g on the interval $[-1, 1]$ and any random variable ξ with finite variance. By the definition of the integral of stochastic process, it is sufficient to show that

$$\Phi = \lim_{n\to\infty} \mathbf{E}\left[\left|\sum_{j=2}^{n} \xi g(t'_j)(t_j - t_{j-1}) - \xi \int_{-1}^{1} g(t)dt\right|^2\right] = 0,$$

where $-1 = t_1 < t_2 < \cdots < t_n = 1$ and $t_{j-1} \le t'_j \le t_j$ holds for any $j = 2, \ldots, n$. By a simple computation, we find that

$$\Phi = \mathbf{E}[\xi^2] \cdot \lim_{n\to\infty} \left|\sum_{j=2}^{n} g(t'_j)(t_j - t_{j-1}) - \int_{-1}^{1} g(t)dt\right|^2 = \mathbf{E}[\xi^2] \cdot 0 = 0.$$

Hence and from Remark 2.14, we obtain (13).

Let $F^m_{k,(K,h)}[\xi \cdot f] = \beta_{k,0} + \beta_{k,1}(t - t_k) + \ldots + \beta_{k,m}(t - t_k)^m$. From Theorem 2.10, we know that $(\beta_{k,0}, \cdots, \beta_{k,m})^T = H^{-1} \cdot Z_m^{-1} \cdot Y_k$, where, according to (13) and the fact that $f(th + t_k)$ is Riemann integrable on $[-1, 1]$ for any $k \in \mathbb{Z}$, we have

$$Y_{k,i} = \int_{-1}^{1} \xi \cdot f(th + t_k) \cdot t^{i-1} K(t)dt = \xi \int_{-1}^{1} f(th + t_k) \cdot t^{i-1} K(t)dt \quad a.s.$$

Put $Y'_{k,i} = \int_{-1}^{1} f(th + t_k) \cdot t^{i-1} K(t)dt$. Using the matrix notation, we can write Y_k as the scalar multiplication of ξ and Y'_k, i.e.,

$$Y_k = \xi \cdot Y'_k \quad a.s.$$

Hence and from the rules for matrix operations, we obtain

$$\begin{gathered}(\beta_{k,0}, \cdots, \beta_{k,m})^T = H^{-1} \cdot Z_m^{-1} \cdot Y_k = H^{-1} \cdot Z_m^{-1} \cdot (\xi \cdot Y'_k) = \\ \xi \cdot (H^{-1} \cdot Z_m^{-1} \cdot Y'_k) = \xi \cdot (\beta'_{k,0}, \cdots, \beta'_{k,m})^T \quad a.s.,\end{gathered}$$

where $F^m_{k,(K,h)}[f](t) = \beta'_{k,0} + \beta'_{k,1}(t - t_k) + \ldots + \beta'_{k,m}(t - t_k)^m$, which proves the desired statement. □

The following theorem confirms a natural expectation that the zero mean value of a stationary stochastic process is preserved and higher bandwidth values decrease the variance of a stationary stochastic process after the application of F^m–transform.

Theorem 3.11. *Let $R(t)$ be a stationary stochastic process defined by (5), and let $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$, $q = 1, 2, \ldots$, be a generalized uniform fuzzy partition determined by a triplet (K, h_q, r_q). Let $\hat{R}^m_{(K,h_q)}(t)$ be the inverse F^m–transform of $R(t)$ with respect to $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$. Then, for any $t \in \mathbb{R}$, it holds that*

(i) $\mathbf{E}\left[\hat{R}^m_{(K,h_q)}(t)\right] = 0$, $q = 1, 2, \ldots,$

(ii) $\lim_{q\to\infty} \mathbf{Var}\left[\hat{R}^m_{(K,h_q)}(t)\right] = 0.$

Proof. (i) Since $p(\varphi_j, t)$, $j = 1, \ldots, n$, is a non-random function, which is complex Riemann integrable on each closed subinterval of the real line, by the linearity of the F^m–transform and Lemma 3.10, we obtain

$$F^m_{k,(K,h_q)}[R](t) = \sum_{j=1}^{n} \xi_j F^m_{k,(K,h_q)}[p(\varphi_j, t)](t) \quad a.s.$$

for any $k \in \mathbb{Z}$, $q = 1, 2, \ldots$. Hence and from Remark 2.14, we find that

$$\mathbf{E}\left[F^m_{k,(K,h_q)}[R](t)\right] = \sum_{j=1}^{n} \mathbf{E}\left[\xi_j\right] F^m_{k,(K,h_q)}[p(\varphi_j, t)](t) = 0, \tag{14}$$

for any $t \in \mathbb{R}$, $k \in \mathbb{Z}$ and $q = 1, 2, \ldots$. Recall that

$$\hat{R}^m_{(K,h_q)}(t) = \sum_{k \in \mathbb{Z}} F^m_{k,(K,h_q)}[R](t) A_{h_q,r_q,k}(t).$$

Since the infinite sum in the derivation of the inverse F^m–transform has only a finite number of non-zero summands, from (14), we simply obtain

$$\mathbf{E}\left[\hat{R}^m_{(K,h_q)}(t)\right] = \sum_{k \in \mathbb{Z}} \mathbf{E}\left[F^m_{k,(K,h_q)}[R](t)\right] A_{h_q,r_q,k}(t) = 0,$$

(ii) Let $q = 1, 2, \ldots$. For any $k_1, k_2 \in \mathbb{Z}$, it follows from Remark 2.14 that

$$\mathbf{E}\left[F^m_{k_1,(K,h_q)}[R](t) \cdot \overline{F^m_{k_2,(K,h_q)}[R](t)}\right] =$$
$$\mathbf{E}\left[\sum_{j=1}^{n} \xi_j F^m_{k_1,(K,h_q)}[p(\varphi_j, t)](t) \cdot \sum_{\ell=1}^{n} \overline{\xi_\ell F^m_{k_2,(K,h_q)}[p(\varphi_\ell, t)](t)}\right].$$

By the assumption that $\xi_1, \ldots, \xi_n$ are pairwise uncorrelated complex-valued random variables with zero mean, we have $\mathbf{E}[\xi_\ell \overline{\xi_j}] = 0$ for any $\ell \neq j$. Therefore, the previous equation becomes

$$\mathbf{E}\left[F^m_{k_1,(K,h_q)}[R](t) \cdot \overline{F^m_{k_2,(K,h_q)}[R](t)}\right] =$$
$$\sum_{j=1}^{n} \mathbf{E}\left[\xi_j \overline{\xi_j}\right] F^m_{k_1,(K,h_q)}[p(\varphi_j, t)](t) \cdot \overline{F^m_{k_2,(K,h_q)}[p(\varphi_j, t)](t)} =$$
$$\sum_{j=1}^{n} \mathbf{Var}\left[\xi_j\right] F^m_{k_1,(K,h_q)}[p(\varphi_j, t)](t) \cdot \overline{F^m_{k_2,(K,h_q)}[p(\varphi_j, t)](t)}.$$

Using Theorem 3.2, we obtain that

$$\lim_{q \to \infty} F^m_{k_1,(K,h_q)}[p(\varphi_j, t)](t) = 0 \quad \text{and} \quad \lim_{q \to \infty} \overline{F^m_{k_2,(K,h_q)}[p(\varphi_j, t)](t)} = 0$$

for any $j = 1, \ldots, n$. Consequently,

$$\lim_{q \to \infty} \mathbf{E}\left[F^m_{k_1,(K,h_q)}[R](t) \cdot \overline{F^m_{k_2,(K,h_q)}[R](t)}\right] = 0, \tag{15}$$

for any $k_1, k_2 \in \mathbb{Z}$. Then, it holds that

$$\mathbf{Var}\left[\hat{R}^m_{(K,h_q)}(t)\right] = \mathbf{E}\left[\hat{R}^m_{(K,h_q)}(t) \cdot \overline{\hat{R}^m_{(K,h_q)}(t)}\right] =$$

$$\mathbf{E}\left[\sum_{k\in\mathbb{Z}} F^m_{k,(K,h_q)}[R](t) A_{h_q,r_q,k}(t) \cdot \sum_{\ell\in\mathbb{Z}} \overline{F^m_{\ell,(K,h_q)}[R](t)} A_{h_q,r_q,\ell}(t)\right] =$$

$$\sum_{k,\ell\in\mathbb{Z}} \mathbf{E}\left[F^m_{k,(K,h_q)}[R](t) \cdot \overline{F^m_{\ell,(K,h_q)}[R](t)}\right] A_{h_q,r_q,k}(t) A_{h_q,r_q,\ell}(t).$$

Again only a finite number of summands in the infinite sums is applied to derive the variance of $\hat{R}^m_{(K,h_q)}(t)$; therefore, the statement (ii) is a simple consequence of (15). □

A counterpart of Theorem 3.7 for the random component $R(t)$ is the following corollary showing that $R(t)$ can be removed by the application of the F^m–transform, of course, with an appropriate choice of the bandwidth.

Corollary 3.12. *Let the assumptions of Theorem 3.11 be satisfied. Let $\hat{R}^m_{(K,h_q)}(t)$, $q = 1, 2, \ldots$, be the inverse F^m–transforms of $R(t)$ with respect to $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$. Then,*

$$\operatorname*{l.i.m.}_{q\to\infty} \hat{R}^m_{(K,h_q)}(t) = 0. \tag{16}$$

Proof. From (ii) of Theorem 3.11 , it holds that

$$\lim_{q\to\infty} \mathbf{E}\left[\left|\hat{R}^m_{(K,h_q)}(t) - 0\right|^2\right] = \lim_{q\to\infty} \mathbf{Var}\left[\hat{R}^m_{(K,h_q)}(t)\right] = 0,$$

which proves the statement (16). □

A counterpart of Theorem 3.8 for the random component $R(t)$ is the following statement demonstrating the influence of the differentiability of generating functions on the rate of convergence of the F^m–transform of $R(t)$.

Theorem 3.13. *Let $R(t)$ be a stationary stochastic process defined by (5), and let $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$, $q = 1, 2, \ldots$, be a generalized uniform fuzzy partition determined by a triplet (K, r_q, h_q) such that $K \in C^M(\mathbb{R}) \cap C^{\star N}([-1, 1])$ and put $L = \min(N, M+2)$. Let $\hat{R}^m_{(K,h_q)}(t)$ be the inverse F^m–transform of $R(t)$ with respect to $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$. Then, for any $t \in \mathbb{R}$, it holds that*

$$\hat{R}^m_{(K,h_q)}(t) = \mathcal{O}_p\left(h_q^{-L}\right).$$

Proof. Let $q = 1, 2, \ldots$ We have to prove that, for any $\varepsilon > 0$, there exist $M > 0$ and q_0 such that

$$P\left(\left|\frac{\hat{R}^m_{(K,h_q)}(t)}{h_q^{-L}}\right| > M\right) < \varepsilon$$

for $q > q_0$. From Markov's inequality, we know that

$$P\left(\left|\frac{\hat{R}^m_{(K,h_q)}(t)}{h_q^{-L}}\right| > M\right) \le \frac{\mathbf{E}\left[\left|\hat{R}^m_{(K,h_q)}(t)\right|\right]}{M h_q^{-L}}.$$

Similarly to the proof of Theorem 3.11, from Remark 2.14, one can find that

$$\mathbf{E}\left[\left|\hat{R}^m_{(K,h_q)}(t)\right|\right] \leq \sum_{k\in\mathbb{Z}}\sum_{j=1}^{n} \mathbf{E}\left[|\xi_j|\right] |F^m_{k,(K,h_q)}[p(\varphi_j,t)](t)| A_{h_q,r_q,k}(t).$$

Similarly to the proof of Theorem 3.8 with help of Theorem 3.3, there exists $C > 0$ and q_0 such that

$$\mathbf{E}\left[\left|\hat{R}^m_{(K,h_q)}(t)\right|\right] \leq C h_q^{-L} \sum_{j=1}^{n} \mathbf{E}\left[|\xi_j|\right]$$

for any $q > q_0$. By Hölder's inequality, we have $\mathbf{E}[|\xi_j|] \leq \mathbf{Var}[\xi_j]^{1/2}$. From the finiteness of variance of ξ_j for any $j = 1,\ldots,n$, we obtain that $\sum_{j=1}^{n} \mathbf{E}\left[|\xi_j|\right] = D < \infty$. Hence, putting $C' = 2\cdot C\cdot D$, we find that $\mathbf{E}[|\hat{R}^m_{(K,h_q)}(t)|] < C' h_q^{-L}$ holds for any $q > q_0$.

Let $\varepsilon > 0$. Put $M = \frac{C'}{\varepsilon}$ and consider q_0. Then

$$P\left(\left|\frac{\hat{R}^m_{(K,h_q)}(t)}{h_q^{-L}}\right| > M\right) \leq \frac{\mathbf{E}\left[\left|\hat{R}^m_{(K,h_q)}(t)\right|\right]}{M h_q^{-L}} < \frac{C' h_q^{-L}}{M h_q^{-L}} = \varepsilon$$

holds for any $q > q_0$. Hence, we obtain the desired statement. □

In a similar way, one could reformulate Corollary 3.9 for the random component $\hat{R}^m_{(K,h_q)}$.

3.4. **Estimation of the Trend-cycle Using F^m–transform.** In this subsection, we provide two theorems showing that the F^m–transform technique can successfully estimate the trend-cycle of a time series under assumptions on the smoothness of its course.

Theorem 3.14. *Let $X(t)$ be a times series that can be additively decomposed into a trend-cycle TC, a seasonal component S and a random noise R, i.e.,*

$$X(t) = TC(t) + S(t) + R(t), \quad t \in \mathbb{R},$$

where the assumptions on TC, S and R are given in Subsection 3.1. Let $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$ be a generalized uniform fuzzy partition determined by a triplet (K,r,h), and let $\hat{X}^m(t)$, $\hat{S}^m(t)$ and $\hat{R}^m(t)$ denote the respective inverse F^m–transforms of TC, S and R with respect to $\{A_{h,r,k}\}_{k\in\mathbb{Z}}$. Then, for any $t\in\mathbb{R}$, it holds that

$$\left|\hat{X}^m(t) - TC(t)\right| \leq (m+1)\|Z_m^{-1}\| \cdot \theta(m,K)\cdot \omega(TC,2h) + |\hat{S}^m(t)| + |\hat{R}^m(t)| \quad a.s.,$$

where Z_m and $\theta(m,K)$ have the same meaning as in Theorem 2.11.

Proof. Let $t\in\mathbb{R}$. By the linearity of the inverse F^m–transform, we obtain

$$\hat{X}^m(t) = \widehat{TC}^m(t) + \hat{S}^m(t) + \hat{R}^m(t) \quad a.s.$$

It follows that

$$\left|\hat{X}^m(t) - \widehat{TC}^m(t)\right| \leq \left|\hat{S}^m(t)\right| + \left|\hat{R}^m(t)\right| \quad a.s.$$

Moreover, by Theorem 2.11, we have

$$\left|\widehat{TC}^m(t) - TC(t)\right| \leq (m+1)\|C^{-1}\| \cdot \theta(m,K) \cdot \omega(TC, 2h).$$

From the previous two inequalities, we obtain

$$\left|\hat{X}^m(t) - TC(t)\right| \leq \left|\hat{X}^m(t) - \widehat{TC}^m(t)\right| + \left|\widehat{TC}^m(t) - TC(t)\right| \leq$$
$$\left|\hat{S}^m(t)\right| + \left|\hat{R}^m(t)\right| + (m+1)\|C^{-1}\| \cdot \theta(m,K) \cdot \omega(TC, 2h) \quad a.s.$$

and the proof is finished. □

From Theorem 3.7 and Corollary 3.12, one can see that $|\hat{S}^m(t)|$ as well as $|\hat{R}^m(t)|$ converge to zero for an increasing bandwidth h, where the convergence for $\hat{R}^m(t)$ is in mean square.[4] Therefore, if we assume that the trend-cycle TC is a smooth function with the modulus of continuity $\omega(TC, 2h)$ that remains very small even for a large bandwidth h, the inverse F^m–transform of $X(t)$, i.e., $\hat{X}^m(t)$, can very well estimate the trend-cycle $TC(t)$.

The following theorem is a consequence of the previous theorem and Theorems 3.8 and 3.13 and provides the information about the stochastic boundedness of the difference between the trend-cycle and the inverse F^m– transform of $X(t)$.

Theorem 3.15. *Let $X(t)$ be a times series that is specified in Theorem 3.14. Let $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$, $q = 1, 2, \ldots,$ be a generalized uniform fuzzy partition determined by a triplet (K, h_q, r_q) such that $K \in C^M(\mathbb{R}) \cap C^{\star N}([-1, 1])$ and put $L = \min(N, M + 2)$. Let $\hat{X}^m_{(K,h_q)}(t)$ denote the inverse F^m–transform with respect to $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$. Then, for any $t \in \mathbb{R}$, it holds that*

$$TC(t) = \hat{X}^m_{(K,h_q)}(t) + \mathcal{O}_p\left(\omega(TC, 2h_q) + h_q^{-L}\right).$$

Proof. Let $t \in \mathbb{R}$. To show that

$$TC(t) - \hat{X}^m_{(K,h_q)}(t) = \mathcal{O}_p\left(\omega(TC, 2h_q) + h_q^{-L}\right),$$

we need to prove that for any $0 < \varepsilon \leq 1$, there exist $M > 0$ and q_0 such that

$$P\left(\left|TC(t) - \hat{X}^m_{(K,h_q)}(t)\right| \leq M\left|\omega(TC, 2h_q) + h_q^{-L}\right|\right) \geq 1 - \varepsilon$$

holds for any $q > q_0$.[5] Let $\hat{S}^m_{(K,h_q)}(t)$ and $\hat{R}^m_{(K,h_q)}(t)$ be inverse F^m–transform of S and R with respect to the fuzzy partition $\{A_{h_q,r_q,k}\}_{k\in\mathbb{Z}}$, respectively. From Theorems 3.8, there exist $M_1 > 0$ and q_1 such that

$$\left|\hat{S}^m_{(K,h_q)}(t)\right| \leq M_1 \cdot h_q^{-L}$$

for any $q > q_1$. Moreover, from Theorem 3.13, for any $\varepsilon > 0$, there exist $M_2 > 0$ and q_2 such that

$$P\left(\left|\hat{R}^m_{(K,h_q)}(t)\right| \leq M_2 \cdot h_q^{-L}\right) \geq 1 - \varepsilon.$$

[4]By Markov's inequality, we obtain that $|\hat{R}^m(t)|$ goes to zero as $h \to \infty$ also in probability.

[5]Against the original definition of stochastic boundedness, we consider here the definition based on the opposite event.

Let $M = \max\{(m+1)\|Z_m^{-1}\| \cdot \theta(m,K), 2M_1, 2M_2\}$ and $q_0 = \max\{q_1, q_2\}$. Put

$$A_q = (m+1)\|Z_m^{-1}\| \cdot \theta(m,K) \cdot \omega(TC, 2h_q) + \left|\hat{S}^m_{(K,h_q)}(t)\right| + \left|\hat{R}^m_{(K,h_q)}(t)\right|,$$

$$B_q = \omega(TC, 2h_q) + h_q^{-L}.$$

Note that $A_q = |A_q|$ for any $q = 1, 2, \dots$ Then

$$P\left(A_q \le M \cdot |B_q|\right) \ge P\left(\left|\hat{R}^m_{(K,h_q)}(t)\right| \le M_2 \cdot h_q^{-L}\right) \ge 1 - \varepsilon$$

for any $q > q_0$. From Theorem 3.14, we have

$$\left|TC(t) - \hat{X}^m_{(K,h_q)}(t)\right| \le A_q \quad a.s.$$

Hence and from $P(A_q \le M \cdot |\omega(TC, 2h_q) + h_q^{-L}|) \ge 1 - \varepsilon$ for any $q > q_0$, we find that

$$P\left(\left|TC(t) - \hat{X}^m_{(K,h_q)}(t)\right| \le M \cdot |B_q|\right) \ge 1 - \varepsilon$$

for any $q > q_0$, and the proof is finished. □

In a similar way, one could reformulate the previous statement for the special cases of generalized uniform fuzzy partitions – only the rate of convergence h^{-L} has to be changed appropriately.

From the previous theorems, one can see that the quality of the estimation of a trend-cycle by the F^m–transform depends on the smoothness of the trend-cycle and the size of frequencies presented in the seasonal and random components. Therefore, a natural question arises how to set the optimal bandwidth value to achieve an efficient estimation of a trend-cycle using the F^m–transform technique. Based on experimental results, a reasonable setting of the bandwidth value seems to be

$$h \ge \max\{T_j \mid j = 1, \dots, s\},$$

where T_j, $j = 1, \dots, s$, is one of the most significant periods in the time series detected with the help of periodogram.

4. **Illustrative Examples**

In this section, we demonstrate how the F^m–transform technique suppresses the high frequencies in time series and, consequently, provides an estimation of their trend-cycles. Moreover, we compare the results of the F^m–transform with results provided by classical methods used for the estimation of the trend-cycle. For this task, we choose two non-parametric methods: Seasonal Trend decomposition using Loess (STL) (see [2]) and Singular Spectrum Analysis (SSA) (see [3]). We also used the Butterworth low-pass filter, but the results were significantly worse in comparison with the three methods mentioned; therefore, we do not include this standard method in our analysis. We consider two types of data: artificial time series whose structure is exactly known and real time series taken from a free web database whose trend-cycles are unknown.

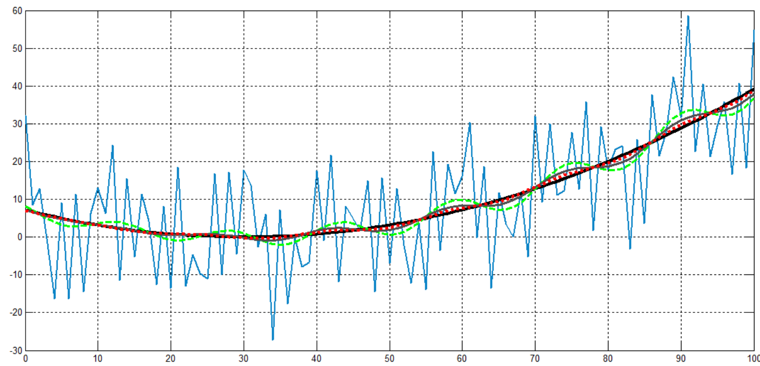

FIGURE 1. Estimations of the Known Trend-cycle (Black Line) Using the F^2-transform with Respect to the Triangle (Dotted Red Line), Raised Cosine (Grey Line) and B-spline of Degree 3 (Dashed Green Line) Generalized Uniform Fuzzy Partition

In the following, for the sake of simplicity, fuzzy partitions that are used for computation are specified in such a way that their shift is the half of the corresponding bandwidth, i.e., $r = h/2$; hence, we omit the shift value in the next text.

Example 4.1. Let the time series $X(t)$ be generated by the following formula

$$X(t) = 0.008(t-30)^2 + 5\sin(0.63t+1.5) + 5\sin(1.26t+0.35) \\ +15\sin(2.7t+1.12) + 7\cos(0.41t+0.79) + R(t)$$

on the set of integers $\{0, 1, \ldots, 100\}$. The trend-cycle is modelled by the non-periodic function, namely, $TC(t) = 0.008(t-30)^2$, and the noise $R(t)$ is white noise with $R(t) \sim \mathcal{N}(0,1)$. The remaining functions (waves) model the seasonal component $S(t)$. One can see that the longest period that appears in the seasonal component is $T = 15.3$ and corresponds to the lowest frequency $\omega = 0.41$. By (3.4), we choose $h = 16 > 15.3$. Further, we assume that the identical bandwidth is used for the triangle, raised cosine and B-spline generalized uniform fuzzy partitions of degree 3. Figure 1 depicts the estimations of the trend-cycle using the F^2–transform with respect to all the previously mentioned fuzzy partitions. From these estimations of the trend-cycle, one can see that the B-spline generalized uniform fuzzy partition provides the worst result (dashed green line). A significant improvement of the trend-cycle estimation for the B-spline generalized uniform fuzzy partition can be attained by a small enlargement of the bandwidth value to $h = 22$. In this case, the estimation is nearly identical to the original trend-cycle, as shown in Figure 2 together with two estimations provided by the SSA and STL methods. A comparison of all the estimations by means of RSME is depicted in Table 1.

From the previous example, one can see that the higher degree fuzzy transform technique is superior to the classical methods in the estimation of the trend-cycle of a time series. Moreover, a greater value of bandwidth had to be used in the case of the B-spline generalized uniform fuzzy partition of the degree 3 to improve

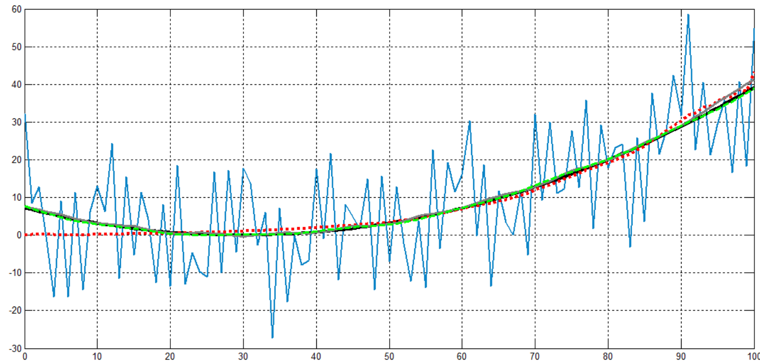

FIGURE 2. Estimations of the Known Trend-cycle (Black Line) Using the F^2-transform with Respect to the B-spline Generalized Uniform Fuzzy Partition of Degree 3 (Dashed Green Line), SSA (Dotted Red Line) and STL (Grey Line)

Method	$F^2_{tr,h=16}$	$F^2_{rc,h=16}$	$F^2_{bs,h=16}$	$F^2_{bs,h=22}$	SSA	STL
RMSE	0.4100	0.9822	2.0001	0.2830	1.9766	0.5605

TABLE 1. RMSE Errors of Trend-cycle Estimations

the trend-cycle estimation. However, this correction seems to be natural, since the F^m–transform with respect to B-spline generalized uniform fuzzy partitions of the degree n, $(n > 1)$ has very good approximation ability as demonstrated in [10]. Hence, a successful application of result (ii) of Corollary 3.9 is clearly attained by enlarging the bandwidth value for B-spline generalized uniform fuzzy partitions of the degree n, $(n > 1)$ against the adjusted bandwidth value for the triangle and raised cosine fuzzy partitions. A comparison of the quality of trend-cycle estimation by the F^m–transform with respect to the considered fuzzy partitions on different bandwidths is shown in Figure 3.

In the following example, we assume that the trend-cycle is a periodic function.

Example 4.2. Let time series $X(t)$ be determined by formula:

$$\begin{aligned} X(t) &= 20\sin(0.063t) + 5\sin(0.63t + 1.5) + 5\sin(1.26t + 0.35) \\ &+15\sin(2.7t + 1.12) + 7\cos(0.41t + 0.79) + R(t), \end{aligned}$$

where the only difference from the time series in Example 4.1 is the modelling of the trend-cycle, which is now the periodic function $TC(t) = 20\sin(0.063t)$. A demonstration of the results obtained under the same parameters as in Example 4.1 is depicted in Figures 4 and 5, where we omit the estimation for the B-spline generalized uniform fuzzy partition with respect to $h = 16$ because of the result against $h = 22$. A comparison with the help of RMSE is provided in Table 2.

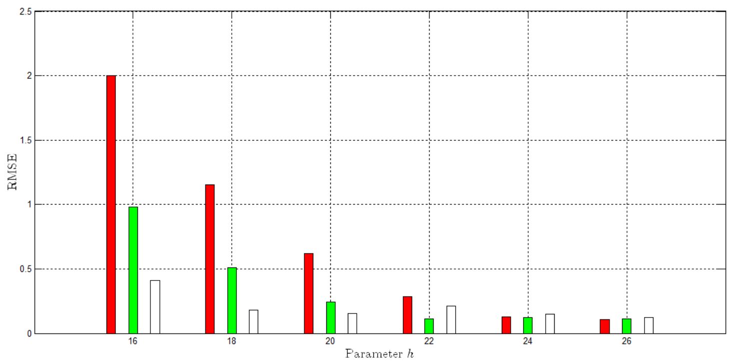

FIGURE 3. Errors of Trend-cycle Estimation Using the F^2–transform with Respect to the Triangle (White Columns), Raised Cosine (Green Columns) and B-spline of Degree 3 (Red Columns) Generalized Uniform Fuzzy Partitions with Differences Setting of the Bandwidth h

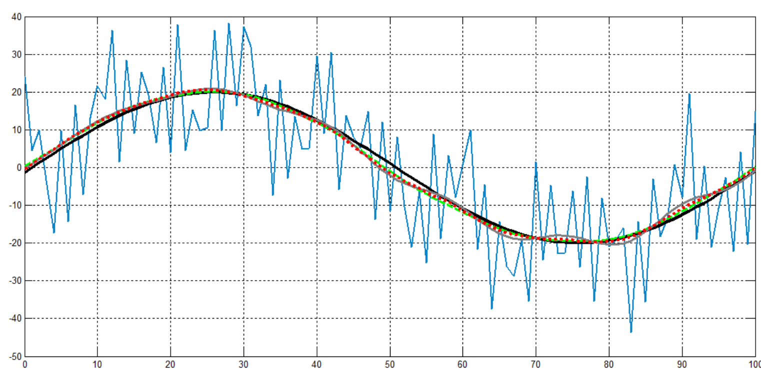

FIGURE 4. Estimations of the Known Trend-cycle (Black Line) Using the F^2-transform with Respect to the Triangle (Dotted Red Line), Raised Cosine (Grey Line) and B-spline of Degree 3 (Dashed Green Line) Generalized Uniform Fuzzy Partition

Method	$F^2_{tr,h=16}$	$F^2_{rc,h=16}$	$F^2_{bs,h=22}$	SSA	STL
RMSE	1.0416	1.3586	0.9771	1.1005	1.3934

TABLE 2. RMSE Errors of Trend-cycle Estimations

An analysis of the results in the previous example shows that the F^m–transform is really a favorable technique for trend-cycle estimation that can defeat the standard methods including the SSA and STL method. It should be noted that one can completely reduce the estimation error by an appropriate enlargement of the bandwidth.

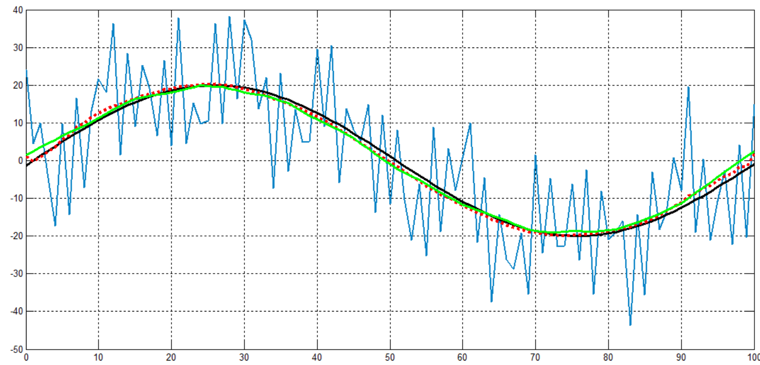

FIGURE 5. Estimations of the Known Trend-cycle (Black Line) Using the SSA (Dotted Red Line) and STL (Green Line) Methods

The final example shows the application of the F^m–transform technique on the estimation of the trend-cycle from two real time series.

Example 4.3. We choose two time series:[6]

(a) monthly Number of Slaughtered Pigs in Victoria (1981-Jul to 1994-Jun),
(b) monthly Lake Erie Levels (1956-Jan to 1968-Dec).

Both seem to have a clear trend-cycle. Nevertheless, the exact trend-cycle is unknown, which suggests that an objective assessment of the quality of the trend-cycle estimations is necessary. Therefore, we leave to the reader the task of assessing the quality of estimation by the F^m–transform compared to the other methods.

For (a), the trend-cycle is estimated using the F^2– and F^3–transforms with respect to the triangle generalized uniform fuzzy partition with the bandwidth $h = 14$. The results, including those for the SSA and STL methods, are depicted in Figure 6.

For (b), in addition to the SSA and STL methods, the trend-cycle is estimated by the F^2-transform with respect to all three types of generalized uniform fuzzy partitions: triangle and raised cosine with the same bandwidth $h = 14$, and the B-spline of degree 3 with the bandwidth $h = 18$. All the estimations are depicted in Figure 7.

The results in all three examples clearly support the application of higher degree fuzzy transform techniques for the estimation of the trend-cycle of a time series. The comparisons in Tables 1 and 2 shows that the F^m–transform technique is a serious alternative to the methods commonly used in time series analysis. Note that even better results could be attained by the F^m–transform technique after a deeper analysis of the optimal setting of the bandwidth. This task, however, has been left to future research.

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[6]Website $http : //www.comp - engine.org/$

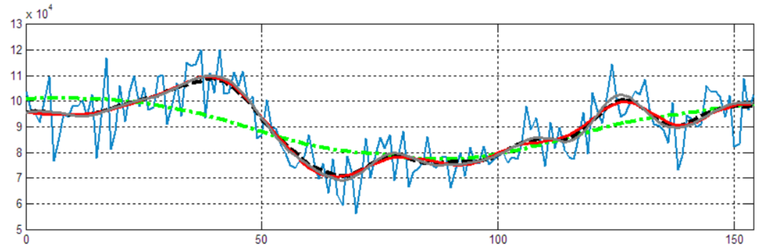

FIGURE 6. Estimations of the Unknown Trend-cycle Using the F^2–transform (Red Line), F^3–transform (Grey Line), SSA Method (Dash-dot Green Line), and STL Method (Dashed Black Line)

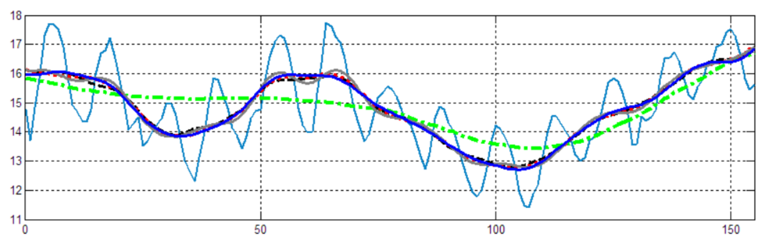

FIGURE 7. Estimations of the Unknown Trend-cycle Using the F^2-transform with Respect to Triangle (Dotted Red Line), Raised Cosine (Grey Line) and B-spline (Blue Line) Generalized Uniform Fuzzy Partition, Respectively. The Estimation Using SSA Method (Dash-Dot Green Line), and STL Method (Dashed Black Line)

5. **Conclusions**

In this paper, we proved that the high frequencies in time series present in the seasonal and random components can be successfully suppressed by the setting of parameters of the higher degree F–transform. We showed that the bandwidth and the choice of the generating function with respect to its differentiability are the most important parameters in this task. As a consequence of high frequency suppression by the F^m–transform we provided a theoretical justification of trend-cycle estimation using this technique. We illustrated trend-cycle estimation by the F^m–transform based on two artificial and two real time series including a comparison with the standard non-parametric methods, namely, Seasonal Trend decomposition using Loess (STL) and Singular Spectrum Analysis (SSA). The practical comparison with the standard methods strengthened our opinion that the F^m–transform technique is a suitable tool for time series analysis.

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