TRANSPORT EQUATION WITH FUZZY DATA

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Abstract. In this paper, we use the generalized differentiability concept to study the fuzzy transport equation. We consider transport equation in the homogeneous and non-homogeneous cases with fuzzy initial condition. We also present the solution when speed parameter is a fuzzy number. Our method is based on the construction of the solutions by employing Zadeh's extension principle.

1. **Introduction**

Transport phenomena and fluid dynamics, such as heat and mass transfer, play a vitally important role in human life. Convection and diffusion are responsible for temperature fluctuations and transport of pollutants in air, water or soil. The ability to understand, predict and control transport phenomena is essential for many industrial applications, such as aerodynamic shape design, oil recovery from an underground reservoir, or multiphase flows in furnaces, heat exchangers and chemical reactors.

The traditional approach to investigation of a physical process is based on observations, experiments and measurements. The amount of information that can be obtained in this way is usually very limited and subject to vagueness. Alternatively, an analytical or computational study can be performed on the basis of a suitable mathematical model. As a rule, such a model consists of several differential and algebraic equations which make it possible to predict how the quantities of interest evolve and interact with one another. A drawback to this approach is the fact that complex physical phenomena give rise to complex mathematical equations that cannot be solved analytically.
A much newer theory, fuzzy sets theory, is a natural way to model dynamical systems subject to uncertainties. Fuzzy sets theory has been explored in various fields due to its great applicability and functionality. As soon as the idea of a function with fuzzy values was born, it raised the idea of some kind of fuzzy derivatives and fuzzy differential equations (FDEs) as well. Since then, many researches have been done by several authors in theory of ordinary and partial fuzzy differential equations (see e.g. [1, 2, 4, 6, 7, 8, 9, 10]).

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In this paper, we study the linear first order fuzzy transport equation. Let $u(x,t)$ represent the density or the concentration of a physical quantity Q and $q(u)$ be its flux function. The equation

$$u_t + q(u)_x = 0,$$

express the law of mass conservation in physics. The type of flux function depending on the involving model should be established. Let consider a simple convection-diffusion model of a pollutant on the surface or narrow channel. A water stream of constant speed v transports the pollutant along the direction of the x axis. In this case the flux function is $q(u) = vu$ and we obtain the simple transport equation

$$u_t + vu_x = 0,$$

where v is the stream speed. To determine the evolution of the concentration u, we need its initial profile, $u(x,0)$. A nonhomogenous transport equation

$$u_t + vu_x = f(x,t),$$

describes the effect of an external distributed source along the channel. The function $f(x,t)$ represents the intensity of the source, measured in concentration per unit time. In some more realistic models, the speed or initial profile are linguistic variables for instance, fast, slow, far, close, and etc. In this case the mass distribution $u(x,t)$ becomes a linguistic variable and the transport equation is changed to a fuzzy transport equation. In this study, we intend to investigate the effect of uncertainties on the mass distribution. We focus on fuzzy transport equation in homogenous and nonhomogenous cases. Our method is based on construction of solutions.

The remainder of this paper is organized as follows. In Section 2, we provide some preliminary information. In Section 3, we study homogeneous transport equations. In Section 4, the solution of non-homogeneous equation is given. Transport equation by non-precise speed is studied in Section 5.

2. **Preliminaries**

In this section we gather together some definitions and results from the literature, which we will use throughout the paper.

The space of fuzzy numbers (see e.g. [5]), denoted by $\mathbb{R}_{\mathcal{F}}$, is the set of functions $u : \mathbb{R} \to [0,1]$ such that satisfy the following properties:

(i) u is normal, i.e. there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex, i.e. $u(tx + (1-t)y) \geq min\{u(x), u(y)\}, \ \forall x, y \in \mathbb{R}, \ t \in [0,1]$,

(iii) u is upper semicontinuous,

(iv) $[u]^0 = cl\{x \in \mathbb{R} \mid u(x) > 0\}$ is compact, here clA denotes the closure of A.

For $0 < \alpha \leq 1$, α-cuts of $u \in \mathbb{R}_{\mathcal{F}}$ is defined by $[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \geq \alpha\}$. Then from (i)-(iv) it follows that for any $\alpha \in [0,1]$, $[u]^\alpha$ is a bounded closed interval and we denote it by $[u]^\alpha = [u_-^\alpha, u_+^\alpha]$. For $u, v \in \mathbb{R}_{\mathcal{F}}, \ \lambda \in \mathbb{R}$, we define the addition $u + v$ and scalar multiplication λu as $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$ and $[\lambda u]^\alpha = \lambda [u]^\alpha$, where $[u]^\alpha + [v]^\alpha$ and $\lambda[u]^\alpha$, mean the usual addition of two subsets of $\mathbb{R}$ and the usual product between a scalar and a subset of $\mathbb{R}$ respectively (see e.g. [5]).

The fuzzy number u is called positive if $supp(u) \subset (0, \infty)$ where $supp(u) = \{x \in$

$\mathbb{R}|u(x) > 0\}$. We denote by $\mathbb{R}^+_{\mathcal{F}}$, the space of all positive fuzzy numbers. Similarly we denote $\mathbb{R}^-_{\mathcal{F}}$, the space of all negative fuzzy numbers where $u \in \mathbb{R}^-_{\mathcal{F}}$ if and only if $supp(u) \subset (-\infty, 0)$.
Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists a unique fuzzy number $w \in \mathbb{R}_{\mathcal{F}}$ such that $v + w = u$, then w is called the H-difference of u, v and is denoted by $u \ominus v$ (see e.g. [5]).

If $u, v \in \mathbb{R}_{\mathcal{F}}$, the distance between u and v is defined by

$$D(u, v) = \sup_{\alpha\in[0,1]} max\{|u^\alpha_- - v^\alpha_-|, |u^\alpha_+ - v^\alpha_+|\}.$$

The following properties of distance D are well-known (see e.g. [5])

$$D(u_1 + u_2, u_1 + u_3) = D(u_2, u_3), \quad \forall u_1, u_2, u_3 \in \mathbb{R}_{\mathcal{F}},$$
$$D(\lambda u_1, \lambda u_2) = |\lambda| D(u_1, u_2), \quad \forall \lambda \in \mathbb{R},\ \forall u_1, u_2 \in \mathbb{R}_{\mathcal{F}},$$
$$D(u_1 + u_2, u_3 + u_4) \leq D(u_1, u_3) + D(u_2, u_4), \quad \forall u_1, u_2, u_3, u_4 \in \mathbb{R}_{\mathcal{F}},$$

also if there exist H-differences $u_1 \ominus u_2$ and $u_3 \ominus u_4$, we conclude

$$D(u_1 \ominus u_2, u_3 \ominus u_4) = D(u_1 + u_4, u_3 + u_2), \quad \forall u_1, u_2, u_3, u_4 \in \mathbb{R}_{\mathcal{F}},$$

and $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space.

Lemma 2.1. *(See e.g.* [5].*)*
(i) If we denote $\tilde{0} = \chi_{\{0\}}$, *then* $\tilde{0} \in \mathbb{R}_{\mathcal{F}}$ *is neutral element with respect to +, i.e.* $u + \tilde{0} = \tilde{0} + u = u$, *for all* $u \in \mathbb{R}_{\mathcal{F}}$.
(ii) With respect to $\tilde{0}$, *none of* $u \in \mathbb{R}_{\mathcal{F}} \setminus \mathbb{R}$, *has opposite in* $\mathbb{R}_{\mathcal{F}}$*(with respect to +).*
(iii) For any $a, b \in \mathbb{R}$ *with* $a, b \geq 0$ *or* $a, b \leq 0$ *and any* $u \in \mathbb{R}_{\mathcal{F}}$, *we have* $(a + b)u = au + bu$. *For general* $a, b \in \mathbb{R}$, *the above property does not hold.*
(iv) For any $\lambda \in \mathbb{R}$ *and any* $u, v \in \mathbb{R}_{\mathcal{F}}$, *we have* $\lambda(u + v) = \lambda u + \lambda v$.
(v) For any $\lambda, \mu \in \mathbb{R}$ *and any* $u \in \mathbb{R}_{\mathcal{F}}$, *we have* $\lambda(\mu u) = (\lambda\mu)u$.

Definition 2.2. (See e.g. [5].) Let $f : (a, b) \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in (a, b)$. We say f is strongly generalized differentiable at x_0, if there exists an element $f'(x_0) \in \mathbb{R}_{\mathcal{F}}$, such that
(i) for all $h > 0$ sufficiently small, there exist $f(x_0 + h) \ominus f(x_0), f(x_0) \ominus f(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h\searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h\searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

or
(ii) for all $h > 0$ sufficiently small, there exist $f(x_0) \ominus f(x_0 + h), f(x_0 - h) \ominus f(x_0)$ and the limits (in the metric D)

$$\lim_{h\searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h\searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0),$$

or
(iii) for all $h > 0$ sufficiently small, there exist $f(x_0 + h) \ominus f(x_0), f(x_0 - h) \ominus f(x_0)$ and the limits (in the metric D)

$$\lim_{h\searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h\searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0),$$

or
(iv) for all $h>0$ sufficiently small, there exist $f(x_0)\ominus f(x_0+h), f(x_0)\ominus f(x_0-h)$ and the limits (in the metric D)

$$\lim_{h\searrow 0}\frac{f(x_0)\ominus f(x_0+h)}{-h}=\lim_{h\searrow 0}\frac{f(x_0)\ominus f(x_0-h)}{h}=f'(x_0).$$

Remark 2.3. We say that a function is (i)-differentiable if it is differentiable as in the above Definition 2.2(i), and similarly we can define $(ii),(iii)$ and (iv)-differentiability at x_0.

Remark 2.4. In this paper, for the integral concept, we will use the fuzzy Riemann integral, see [5].

Lemma 2.5. *(See [5].) Let $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$ be a continuous fuzzy-valued function. Then $F(x)=\int_a^x f(t)dt$ is (i)-differentiable and we have $F'(x)=f(x)$.*

Theorem 2.6. *(See [4].) For $t_0\in\mathbb{R}$, the fuzzy differential equation $y'=g(t,y)$, $y(t_0)=y_0\in\mathbb{R}_{\mathcal{F}}$ where $g:\mathbb{R}\times\mathbb{R}_{\mathcal{F}}\to\mathbb{R}_{\mathcal{F}}$ is supposed to be continuous, is equivalent to one of the integral equations:*

$$y(t)=y_0+\int_{t_0}^{t}g(s,y(t))ds,\ \forall[t_0,t_1],$$

or

$$y(t)=y_0\ominus(-1)\int_{t_0}^{t}g(s,y(t))ds,\ \forall[t_0,t_1],$$

on some interval $(t_0,t_1)\subset\mathbb{R}$, depending on the strongly differentiability considered, (i) or (ii), respectively.

Lemma 2.7. *Let $f:\mathbb{R}\times(0,\infty)\to\mathbb{R}_{\mathcal{F}}$ be strongly generalized differentiable w.r.t x. Suppose there exists a continuous real-valued function $g(x,t)$ such that $D(\frac{\partial f}{\partial x}(x,t),\chi_{\{0\}})\le g(x,t)$ for $x\in\mathbb{R},t\ge 0$. Then $F(x,t)=\int_0^t f(x,s)ds$ is strongly generalized differentiable w.r.t x and we have $\frac{\partial F}{\partial x}(x,t)=\int_0^t\frac{\partial f}{\partial x}(x,s)ds$.*

Proof. Let $f:\mathbb{R}\times(0,\infty)\to\mathbb{R}_{\mathcal{F}}$ with $[f(x,t)]^\alpha=[f_-^\alpha(x,t),f_+^\alpha(x,t)]$ be (i)-differentiable. We first notice that due to the our assumptions we have the following derivatives uniformly w.r.t α.

$$\frac{\partial}{\partial x}\int_0^t f_-^\alpha(x,s)ds=\int_0^t\frac{\partial f_-^\alpha}{\partial x}(x,s)ds,$$
$$\frac{\partial}{\partial x}\int_0^t f_+^\alpha(x,s)ds=\int_0^t\frac{\partial f_+^\alpha}{\partial x}(x,s)ds.$$

This fact yields

$$\begin{aligned}\lim_{h\searrow 0}\frac{F(x+h,t)\ominus F(x,t)}{h}&=\lim_{h\searrow 0}\frac{\int_0^t f(x+h,s)ds\ominus\int_0^t f(x,s)ds}{h}\\&=\lim_{h\searrow 0}\frac{\int_0^t\left(f(x+h,s)\ominus f(x,s)\right)ds}{h}\\&=\int_0^t\lim_{h\searrow 0}\frac{f(x+h,s)\ominus f(x,s)ds}{h}=\int_0^t\frac{\partial f}{\partial x}(x,s)ds.\end{aligned}$$

It is easy to show for $F(x,t) \ominus F(x-h,t)$. In a similar way, we can prove the assertion for (ii)-differentiability. $\square$

Lemma 2.8. *Let $f : \mathbb{R} \times (0,\infty) \to \mathbb{R}_{\mathcal{F}}$ be a continuous fuzzy-valued function and $b \in (0,\infty)$. Then*

$$\lim_{h \searrow 0} \frac{1}{h} \int_t^{t+h} f(x+(t-s)b+hb, s)ds = f(x,t).$$

Proof. Let $(x,t) \in \mathbb{R} \times (0,\infty)$; for given $\epsilon > 0$ there exists $\delta_1 > 0$ such that for all $0 \le \alpha \le 1$ and $|x-y| < \delta_1, |t-s| < \delta_1$, we have $|f_-^\alpha(y,t) - f_-^\alpha(x,s)| < \epsilon, |f_+^\alpha(y,t) - f_+^\alpha(x,s)| < \epsilon$. Let $\delta = \min\{\frac{\delta_1}{2b}, \delta_1\}$. For $|h| \le \delta$, we have

$$\frac{1}{h} \int_t^{t+h} |f_-^\alpha(x+(t-s)b+hb, s) - f_-^\alpha(x,t)| ds \le \epsilon,$$

and the same result is valid for f_+^α for all $\alpha \in [0,1]$. It shows that

$$\lim_{h \searrow 0} D(\frac{1}{h} \int_t^{t+h} f(x+(t-s)b+hb, s)ds, f(x,t)) = 0.$$

This completes the proof. $\square$

It is well-known that a fuzzy-valued function $f : (a,b) \to \mathbb{R}_{\mathcal{F}}$ is (i)-differentiable if and only if the functions $f_-^\alpha(x)$ and $f_+^\alpha(x)$ are continuously differentiable with respect to x, uniformly with respect to $\alpha \in [0,1]$, provided that $[(f_-^\alpha)'(x), (f_+^\alpha)'(x)]$ defines a fuzzy number $f'(x) \in \mathbb{R}_{\mathcal{F}}$. Similarly, f is (ii)-differentiable if and only if the functions $f_-^\alpha(x)$ and $f_-^\alpha(x)$ are continuously differentiable with respect to x, uniformly with respect to $\alpha \in [0,1]$, provided that $[(f_+^\alpha)'(x), (f_-^\alpha)'(x)]$ defines a fuzzy number $f'(x) \in \mathbb{R}_{\mathcal{F}}$. In the following, we present a result only for existence of Hukuhara differences related to strongly generalized differentiability.

Theorem 2.9. *Let $f : (a,b) \to \mathbb{R}_{\mathcal{F}}$ be such that $[f(x)]^\alpha = [f_-^\alpha(x), f_+^\alpha(x)]$. Suppose that real-valued functions f_-^α, f_+^α are differentiable w.r.t x.*

(i) *If the intervals $[(f_-^\alpha)'(x), (f_+^\alpha)'(x)]$ for all $\alpha \in [0,1]$ and $x \in (a,b)$, determine valid $\alpha-$cuts of a fuzzy number, then the H-differences $f(x+h) \ominus f(x)$ and $f(x) \ominus f(x-h)$ exist for all $h > 0$ sufficiently small.*

(ii) *If the intervals $[(f_+^\alpha)'(x), (f_-^\alpha)'(x)]$ for all $\alpha \in [0,1]$ and $x \in (a,b)$, determine valid $\alpha-$cuts of a fuzzy number, then the H-differences $f(x) \ominus f(x+h)$ and $f(x-h) \ominus f(x)$ exist for all $h > 0$ sufficiently small.*

Proof. We prove (i) in details and the proof of (ii) is similar. Indeed, we show that the following intervals

$$[f_-^\alpha(x+h) - f_-^\alpha(x), f_+^\alpha(x+h) - f_+^\alpha(x)], \quad \forall \alpha \in [0,1],$$

are valid $\alpha-$cuts of a fuzzy number for any $x \in (a,b)$. For this end, let $diam(f(x)) = f_+^\alpha(x) - f_-^\alpha(x)$. Since $[(f_-^\alpha)'(x), (f_+^\alpha)'(x)]$ for all $x \in (a,b)$, determine a fuzzy number, $(diam(f))'(x) = (f_+^\alpha)'(x) - (f_-^\alpha)'(x) \ge 0$. This states that $diam(f)$ is an nondecreasing function w.r.t x, i.e., $diam(f(x+h)) \ge diam(f(x))$ and we have

$$f_-^\alpha(x+h) - f_-^\alpha(x) \le f_+^\alpha(x+h) - f_+^\alpha(x).$$

Now let $\alpha \leq \beta$. Define $G(x) = f_-^\alpha(x) - f_-^\beta(x)$. Since $[(f_-^\alpha)'(x), (f_+^\alpha)'(x)]$ forms a fuzzy number, $(f_-^\alpha)'(x)$ is nondecreasing w.r.t α. Therefore we have $G'(x) \leq 0$ which means that G is nonincreasing respect to x and then

$$f_-^\alpha(x+h) - f_-^\beta(x+h) \leq f_-^\alpha(x) - f_-^\beta(x).$$

Thus $f_-^\alpha(x+h) - f_-^\alpha(x)$ is nondecreasing respect to α. Similarly we can show that $f_+^\alpha(x+h) - f_+^\alpha(x)$ is nonincreasing w.r.t α. This completes the assertion. □

Remark 2.10. The assertion is not correct if we replace differentiability of f_-^α, f_+^α on (a,b) with local differentiability at a certain point. For example

$$f(x) = \begin{cases} (-1,0,1)(1 + x^2 \sin \frac{1}{x}), & x \neq 0, \\ (-1,0,1), & x = 0, \end{cases}$$

where at point $x = 0$ the $[(f_-^\alpha)'(0), (f_+^\alpha)'(0)] = [0,0]$ defines a fuzzy number, while none of the H-differences exists for $h > 0$ sufficiently small(see [3]).

3. Homogenous Transport Equation

In this section, we intend to study the transport of a mass without any source with a fuzzy initial value and a positive speed (it can be negative too):

$$u_t = bu_x, \quad u(x,0) = g(x), \tag{1}$$

where g is a fuzzy-valued function and b is a constant real number. We say $u(x,t)$ is a solution of (1) on $\mathbb{R} \times (0,\infty)$ if u is strongly differentiable with respect to x, t and it verifies (1). We notice that when we say u is strongly differentiable with respect to x on $\mathbb{R}$, we mean that it is only differential with respect to a certain type of the derivatives stated in Definition (2.2) in the whole of the domain. Since a (iii)- or (iv)-differentiable fuzzy-valued function on $\mathbb{R}$ is a real-valued function, we confine our attention only to (i)- or (ii)-differentiability in Definition (2.2). Our method is based on the constructive method in which we introduce the solution and examine that it verifies the problem.

Theorem 3.1. *Suppose g is a strongly generalized differentiable function on $\mathbb{R}$. Then $u(x,t) = g(x+bt)$ is a solution of (1) where b is a constant positive real number.*

Proof. First, let g be (i)-differentiable. Then $g(x+tb+bh) \ominus g(x+tb)$ and $g(x+tb) \ominus g(x+tb-bh)$ for h sufficiently small exist and we have

$$\begin{aligned} u_t &= \lim_{h \searrow 0} \frac{u(x,t+h) \ominus u(x,t)}{h} = \lim_{h \searrow 0} \frac{g(x+tb+bh) \ominus g(x+tb)}{h} \\ &= \lim_{h \searrow 0} b\frac{g(x+tb+bh) \ominus g(x+tb)}{bh} = bg'(x+th). \end{aligned}$$

Similarly

$$\begin{aligned} u_t &= \lim_{h \searrow 0} \frac{u(x,t) \ominus u(x,t-h)}{h} = \lim_{h \searrow 0} \frac{g(x+tb) \ominus g(x+tb-bh)}{h} \\ &= \lim_{h \searrow 0} b\frac{g(x+tb) \ominus g(x+tb-bh)}{bh} = bg'(x+th). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} u_x &= \lim_{h\searrow 0} \frac{u(x+h,t)\ominus u(x,t)}{h} = \lim_{h\searrow 0} \frac{g(x+tb+h)\ominus g(x+tb)}{h} \\ &= g'(x+tb). \end{aligned}$$

The same argument can be given to $\lim_{h\searrow 0} \frac{u(x,t)\ominus u(x-h,t)}{h}$. These conclude the (i)-differentiability of u w.r.t x and t. It is clear that u verifies the equation and initial condition. When g is (ii)-differentiable, similarly we can show that it is a solution. □

4. Nonhomogeneous

In this section, we focus on the following nonhomogeneous fuzzy transport equation for which the function and initial value can be fuzzy

$$\begin{aligned} &u_t = bu_x + f(x,t) \ in\ \mathbb{R}\times(0,\infty), \\ &u(x,0) = g(x), \end{aligned} \tag{2}$$

where g and f are fuzzy-valued functions.

Theorem 4.1. *Let $f:\mathbb{R}\times(0,\infty)\to\mathbb{R}_{\mathcal{F}}$ be $(i)-$differentiable w.r.t x, t and let $g:\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$ and b is a positive real number.*

(1) *If g is (i)-differentiable, then*

$$u_1(x,t) = g(x+tb) + \int_0^t f(x+(t-s)b,s)ds,$$

is a (i)-differentiable solution of (2).

(2) *If g is (ii)-differentiable, then*

$$u_2(x,t) = g(x+tb)\ominus(-1)\int_0^t f(x+(t-s)b,s)ds,$$

is a (ii)-differentiable solution of (2) provided the H-difference exists.

Proof. First we suppose g is (i)-differentiable. We show that $u_1(x,t)$ is (i)-differentiable and verifies the problem.

$$\frac{\partial u_1}{\partial x}(x,t) = \lim_{h\searrow 0}\frac{u_1(x+h,t)\ominus u_1(x,t)}{h}.$$

The above identity equals to

$$= \lim_{h\searrow 0}\frac{\Big(g(x+h+tb)+\int_0^t f(x+h+(t-s)b,s)ds\Big)\ominus\Big(g(x+tb)+\int_0^t f(x+(t-s)b,s)ds\Big)}{h}.$$

Therefore, we have

$$\begin{aligned} &\frac{\partial u_1}{\partial x}(x,t) = \lim_{\searrow 0}\frac{u_1(x+h,t)\ominus u_1(x,t)}{h} \\ &= \lim_{h\searrow 0}\frac{(g(x+h+tb)\ominus g(x+tb)) + \Big(\int_0^t f(x+(t-s)b+h,s)\ominus f(x+(t-s)b,s)ds\Big)}{h} \\ &= g'(x+tb) + \lim_{h\searrow 0}\frac{1}{h}\int_0^t f(x+(t-s)b+h,s)\ominus f(x+(t-s)b,s)ds. \end{aligned}$$

Employing Lemma 2.7 and uniform assumptions, we deduce

$$\frac{\partial u_1}{\partial x}(x,t) = g'(x+tb) + \int_0^t \frac{\partial f}{\partial x}(x+(t-s)b, s)ds.$$

For derivative w.r.t t, from Lemma 2.8, we have

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= \lim_{h\searrow 0} \frac{u_1(x,t+h)\ominus u_1(x,t)}{h} \\
&= bg'(x+tb) + \lim_{h\searrow 0}\frac{1}{h}\int_0^t f(x+(t-s)b+hb,s)\ominus f(x+(t-s)b,s)ds \\
&+ \lim_{h\searrow 0}\frac{1}{h}\int_t^{t+h} f(x+(t-s)b+hb,s)ds \\
&= bg'(x+tb) + b\int_0^t \frac{\partial f}{\partial x}(x+(t-s)b,s)ds + f(x,t).
\end{aligned}$$

Therefore, u_1 is a $(i)-$differentiable solution of (2).
Now let g is (ii)-differentiable. By Theorem 4 in [4] and Lemma 2.7, u_2 is (ii)-differentiable w.r.t x and we have

$$\frac{\partial u_2}{\partial x}(x,t) = g'(x+tb) + (-1)\int_0^t -\frac{\partial f}{\partial x}(x+(t-s)b,s)ds.$$

Now we compute (ii)-derivative of u_2 w.r.t t. First we have

$$\begin{aligned}
&u_2(x,t)\ominus u_2(x,t+h) = \\
&\quad= \left(g(x+tb)\ominus \int_0^t -f(x+(t-s)b,s)ds\right) \\
&\quad\ominus \left(g(x+tb+bh)\ominus \int_0^{t+h} -f(x+(t-s)b+hb,s)ds\right) \\
&\quad= (g(x+tb)\ominus g(x+tb+bh)) \\
&\quad+ \left(\int_0^{t+h} -f(x+(t-s)b+bh,s)ds \ominus \int_0^t -f(x+(t-s)b,s)ds\right) \\
&\quad= (g(x+tb)\ominus g(x+tb+bh)) \\
&\quad+ \int_0^t \left(-f(x+(t-s)b+bh,s)\ominus -f(x+(t-s)b,s)\right) ds \\
&\quad+ \int_t^{t+h} -f(x+(t-s)b+bh,s)ds.
\end{aligned}$$

Then multiplying with $\frac{1}{-h}$ and passing to limit with $h\to 0^+$, we obtain

$$\begin{aligned}
\frac{\partial u_2}{\partial t}(x,t) &= \lim_{h\searrow 0}\frac{u_2(x,t)\ominus u_2(x,t+h)}{-h} \\
&= bg'(x+tb) \\
&+ \lim_{h\searrow 0}\frac{1}{-h}\int_0^t -f(x+(t-s)b+hb,s)\ominus -f(x+(t-s)b,s)ds \\
&+ \lim_{h\searrow 0}\frac{1}{-h}\int_t^{t+h} -f(x+(t-s)b+hb,s)ds \\
&= bg'(x+tb) + b\int_0^t \frac{\partial f}{\partial x}(x+(t-s)b,s)ds + f(x,t).
\end{aligned}$$

The similar argument can be given for another limit. □

5. Transport by Non-Precise Speed

In this section, we study the following homogeneous fuzzy transport equation for which the speed and initial value can be fuzzy

$$u_t = bu_x \text{ in } \mathbb{R} \times (0, \infty),$$
$$u(x,0) = c\mathfrak{g}(x), \tag{3}$$

where $b, c \in \mathbb{R}_{\mathcal{F}}$ and $\mathfrak{g}$ is a real-valued function.
In the following theorem, using Zadeh's extension principle, we fuzzify the real function $\mathfrak{g} : \mathbb{R} \to \mathbb{R}$ to fuzzy function $\tilde{\mathfrak{g}} : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$. Since $\mathfrak{g}$ is a continuous function, we have $[\tilde{\mathfrak{g}}(X)]^\alpha = \mathfrak{g}([X]^\alpha)$. For instance, let $X = x + tb$ and $\mathfrak{g}$ be nondecreasing function. Then we have

$$[\tilde{\mathfrak{g}}(x + tb)]^\alpha = \mathfrak{g}([x + tb]^\alpha) = [\mathfrak{g}(x + tb_-^\alpha), \mathfrak{g}(x + tb_+^\alpha)].$$

Theorem 5.1. *Suppose $\mathfrak{g} \in C^2(\mathbb{R})$ is an integrable nonnegative monotone function and $b, c \in \mathbb{R}_{\mathcal{F}}$. Let $\tilde{\mathfrak{g}} : \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ be the Zadeh's extension of $\mathfrak{g}$. Consider*

$$u : \mathbb{R} \times (0, \infty) \to \mathbb{R}_{\mathcal{F}}$$

$$u(x,t) = c\tilde{\mathfrak{g}}(x + tb).$$

(i) *If $b, c \in \mathbb{R}_{\mathcal{F}}^+$ and $\mathfrak{g}, \mathfrak{g}'$ are nondecreasing functions, then u is (i)-differentiable w.r.t both t, x and it is a solution of (3).*
(ii) *If $b \in \mathbb{R}_{\mathcal{F}}^-$, $c \in \mathbb{R}_{\mathcal{F}}^+$ and $\mathfrak{g}$ is nonincreasing and $\mathfrak{g}'$ is nondecreasing function, then u is (i)-differentiable w.r.t t and (ii)-differentiable w.r.t x and satisfies (3).*
(iii) *If $b \in \mathbb{R}_{\mathcal{F}}^+$, $c \in \mathbb{R}_{\mathcal{F}}^-$ and $\mathfrak{g}, \mathfrak{g}'$ are nondecreasing functions, then u is (i)-differentiable w.r.t both t, x and satisfies (3).*
(iv) *If $b, c \in \mathbb{R}_{\mathcal{F}}^-$ and $\mathfrak{g}$ is nonincreasing and $\mathfrak{g}'$ is nondecreasing function, then u is (i)-differentiable w.r.t t and (ii)-differentiable w.r.t x and satisfies (3).*

Proof. Case (i): It is easy to check that $\alpha-$cuts $[u]^\alpha = [c_-^\alpha \mathfrak{g}(x + tb_-^\alpha), c_+^\alpha \mathfrak{g}(x + tb_+^\alpha)]$ satisfy conditions in case (i)-Theorem 2.9. Indeed, $[c_-^\alpha b_-^\alpha \mathfrak{g}'(x + tb_-^\alpha), c_+^\alpha b_+^\alpha \mathfrak{g}'(x + tb_+^\alpha)]$ are valid $\alpha-$cuts of a fuzzy number. Then by Theorem 2.9, H-differences $u(x, t + h) \ominus u(x,t)$ and $u(x,t) \ominus u(x, t - h)$ exist. In a similar way, since $[c_-^\alpha \mathfrak{g}'(x + tb_-^\alpha), c_+^\alpha \mathfrak{g}'(x + tb_+^\alpha)]$ forms a fuzzy number, H-differences $u(x + h, t) \ominus u(x,t)$ and $u(x,t) \ominus u(x - h, t)$ exist.
Now we show that the following limits are uniformly with respect to $\alpha \in [0, 1]$:

$$\lim_{h \searrow 0} \frac{c_-^\alpha \mathfrak{g}(x + (t + h)b_-^\alpha) - c_-^\alpha \mathfrak{g}(x + tb_-^\alpha)}{h} = c_-^\alpha b_-^\alpha \mathfrak{g}'(x + tb_-^\alpha), \tag{4}$$

$$\lim_{h \searrow 0} \frac{c_+^\alpha \mathfrak{g}(x + (t + h)b_+^\alpha) - c_+^\alpha \mathfrak{g}(x + tb_+^\alpha)}{h} = c_+^\alpha b_+^\alpha \mathfrak{g}'(x + tb_+^\alpha), \tag{5}$$

$$\lim_{h \searrow 0} \frac{c_-^\alpha \mathfrak{g}(x + tb_-^\alpha) - c_-^\alpha \mathfrak{g}(x + (t - h)b_-^\alpha)}{h} = c_-^\alpha b_-^\alpha \mathfrak{g}'(x + tb_-^\alpha), \tag{6}$$

$$\lim_{h \searrow 0} \frac{c_+^\alpha \mathfrak{g}(x + tb_+^\alpha) - c_+^\alpha \mathfrak{g}(x + (t - h)b_+^\alpha)}{h} = c_+^\alpha b_+^\alpha \mathfrak{g}'(x + tb_+^\alpha). \tag{7}$$

For (4), we have

$$\begin{aligned}
&\lim_{h\searrow 0}\sup_{\alpha\in[0,1]}\Big|\frac{c_-^\alpha\mathfrak{g}(x+(t+h)b_-^\alpha)-c_-^\alpha\mathfrak{g}(x+tb_-^\alpha)}{h}-c_-^\alpha b_-^\alpha\mathfrak{g}'(x+tb_-^\alpha)\Big|\\
=&\lim_{h\searrow 0}\sup_{\alpha\in[0,1]}|c_-^\alpha b_-^\alpha|\Big|\frac{\mathfrak{g}(x+(t+h)b_-^\alpha)-\mathfrak{g}(x+tb_-^\alpha)}{hb_-^\alpha}-\mathfrak{g}'(x+tb_-^\alpha)\Big|\\
=&\lim_{h\searrow 0}\sup_{\alpha\in[0,1]}|c_-^\alpha b_-^\alpha||\mathfrak{g}'(x+tb_-^\alpha+\zeta(h,\alpha))-\mathfrak{g}'(x+tb_-^\alpha)|\\
=&\lim_{h\searrow 0}\sup_{\alpha\in[0,1]}|c_-^\alpha b_-^\alpha||\mathfrak{g}''(x+tb_-^\alpha+\eta(h,\alpha))\zeta(h,\alpha)|\\
\le&\lim_{h\searrow 0}c_0b_0^2M(x,t)h=0,
\end{aligned}$$

where $\zeta(h,\alpha)$ is a point on the line segment between 0 , hb_-^α and $\eta(h,\alpha)$ is a point on the line segment between 0 and $\zeta(h,\alpha)$. And also, $|b_-^\alpha|\le\max\{|b_-^0|,|b_+^0|\}=b_0$ and $|c_-^\alpha|\le\max\{|c_-^0|,|c_+^0|\}=c_0$, for all $\alpha\in[0,1]$. Then

$$x-(t+h)b_0\le x+tb_-^\alpha+\eta(h,\alpha))\le x+(t+h)b_0,\ |\mathfrak{g}''(x+tb_-^\alpha+\eta(h,\alpha))|\le M(x,t).$$

This concludes that $[u_t(x,t)]^\alpha=[c_-^\alpha b_-^\alpha\mathfrak{g}'(x+tb_-^\alpha),c_+^\alpha b_+^\alpha\mathfrak{g}'(x+tb_+^\alpha)]$. The proof of other cases (5)-(7) are similar. Finally we deduce $u_t=bu_x$ with $u(x,0)=c\mathfrak{g}(x)$.
In the case (ii), according to Zadeh's extension principle, α-level sets u is defined as

$$[u]^\alpha=[c_-^\alpha\mathfrak{g}(x+tb_+^\alpha),c_+^\alpha\mathfrak{g}(x+tb_-^\alpha)].$$

Also since $\mathfrak{g}'$ is non-positive function, then $[c_-^\alpha b_+^\alpha\mathfrak{g}'(x+tb_+^\alpha),c_+^\alpha b_-^\alpha\mathfrak{g}'(x+tb_-^\alpha)]$ forms a fuzzy number. By Theorem 2.9, H-differences $u(x,t+h)\ominus u(x,t)$ and $u(x,t)\ominus u(x,t-h)$ exist. In a similar way, since $[c_+^\alpha\mathfrak{g}'(x+tb_-^\alpha),c_-^\alpha\mathfrak{g}'(x+tb_+^\alpha)]$ forms a fuzzy number, H-differences $u(x,t)\ominus u(x+h,t)$ and $u(x-h,t)\ominus u(x,t)$ exist.
In the case (iii), according to Zadeh's extension principle, α level sets u is defined as

$$[u]^\alpha=[c_-^\alpha\mathfrak{g}(x+tb_+^\alpha),c_+^\alpha\mathfrak{g}(x+tb_-^\alpha)].$$

Also since $\mathfrak{g}'$ is non-negative function, then $[c_-^\alpha b_+^\alpha\mathfrak{g}'(x+tb_+^\alpha),c_+^\alpha b_-^\alpha\mathfrak{g}'(x+tb_-^\alpha)]$ forms a fuzzy number. By Theorem 2.9, H-differences $u(x,t+h)\ominus u(x,t)$ and $u(x,t)\ominus u(x,t-h)$ exist. In a similar way, since $[c_-^\alpha\mathfrak{g}'(x+tb_+^\alpha),c_+^\alpha\mathfrak{g}'(x+tb_-^\alpha)]$ forms a fuzzy number, H-differences $u(x+h,t)\ominus u(x,t)$ and $u(x,t)\ominus u(x-h,t)$ exist.
In the case (iv), according to Zadeh's extension principle, α-level sets u is defined as

$$[u]^\alpha=[c_-^\alpha\mathfrak{g}(x+tb_-^\alpha),c_+^\alpha\mathfrak{g}(x+tb_+^\alpha)].$$

On the other hand, since $\mathfrak{g}'$ is non-positive function, then $[c_-^\alpha b_-^\alpha\mathfrak{g}'(x+tb_-^\alpha),c_+^\alpha b_+^\alpha\mathfrak{g}'(x+tb_+^\alpha)]$ forms a fuzzy number. By Theorem 2.9, H-differences $u(x,t+h)\ominus u(x,t)$ and $u(x,t)\ominus u(x,t-h)$ exist. In similar way, since $[c_+^\alpha\mathfrak{g}'(x+tb_+^\alpha),c_-^\alpha\mathfrak{g}'(x+tb_-^\alpha)]$ forms a fuzzy number, H-differences $u(x,t)\ominus u(x+h,t)$ and $u(x-h,t)\ominus u(x,t)$ exist. □

Remark 5.2. As we see the uncertainty of the solution depends on the space variable x and time variable t. It means that the membership function of the solution changes with time. For example in the case (ii), at a point x, uncertainty of the solution increases with time and at time t uncertainty decreases when x increases.

Example 5.3. We examine a simple convection model of a pollutant on the surface of a narrow channel. A water stream of constant speed about $b_0 = 2$ transports the pollutant along the negative direction of the x axis. We suppose the concentration of the pollutant at time $t = 0$ is maximum at point $x = 0$ about c_0 and we only know it decays when x goes up as $\exp(-x)$. Now we are interested in mass concentration at any time t at point x. Since the speed and initial mass are not precise, we have to find the mass concentration $u(x,t)$, as a solution of the initial value problem (3) with $b = (b_0 - a, b_0, b_0 + a)$, $c = (c_0 - d, c_0, c_0 + d)$ and $\mathfrak{g}(x) = \exp(-x)$. As a measure of the uncertainty we have used the length of the 0-level set. So, if we say increasing uncertainty, we understand increasing length of the 0-level set. Then a and d can be large or small depending on the uncertainty varieties in the speed and initial value. For instance, let

$$\mathfrak{g}(x) = \exp(-x), b = (-3,-2,-1), c = (1,2,3),$$

then $u(x,t) = c\tilde{\mathfrak{g}}(x+tb)$ is the solution with $\alpha-$cuts

$$[u(x,t)]_\alpha = [c\mathfrak{g}(x+tb)]_\alpha = [(1+\alpha)\exp(-x+t(1+\alpha)), (3-\alpha)\exp(-x+t(3-\alpha))].$$

Therefore, it is easy to check that $diam(u(x,t))$ increases with t and decreases with x for any $\alpha \in [0,1]$. For instance, for $\alpha = 0$ we obtain (see Figure 1)

$$u_+^0(x,t) - u_-^0(x,t) = 3\exp(-x+3t) - \exp(-x+t).$$

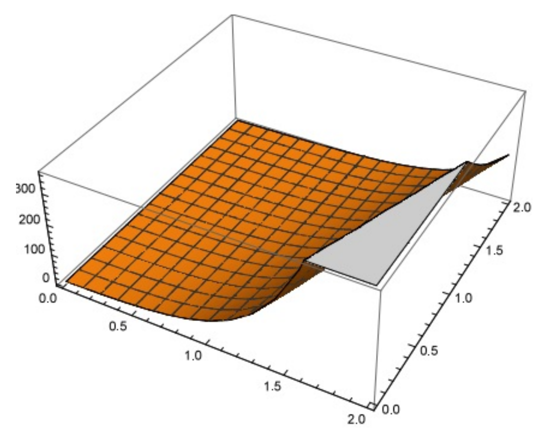

FIGURE 1. Diameter of the Solution for $\alpha = 0$

It means the resulting mass concentration is precise for limited time and at infinity the results are non-precise.

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