AN INVESTIGATION ON THE CO-ANNIHILATORS IN TRIANGLE ALGEBRAS

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Abstract. In this paper, we introduce the notion of co-annihilator of a subset in a triangle algebra. It is shown that the co-annihilator of a subset is an interval valued residuated lattice (IVRL)-filter. Also, a special set of a triangle algebra is defined and the relationship between this set and co-annihilator of a subset in triangle algebra is considered. Finally, co-annihilators preserving congruence relation, or CP-congruence are defined and some results of them are given.

1. Introduction

In the original, and still most popular, approach to fuzzy set theory introduced by L. A. Zadeh, membership values are drawn from the unit interval, equipped with the usual ordering, and intersection and union are modeled by minimum and maximum, respectively. The concept of a commutative residuated lattice was firstly introduced by Ward and Dilworth [7] as generalization of ideal lattices of rings. Ono considered residuated lattices as an algebraic structure of substructural logics in [2]. Van Gass et al. introduced triangle algebras as a variety of residuated lattices equipped with approximation operators and with a third angular point u, different from 0,1 [6]. They defined some types of filters in triangle algebras and obtained some interesting results [5].

Here, we are going to study triangle algebras by means of co-annihilators and IVRL-filters. To this end, we generalize the concept of co-annihilators and prove some theorems that determine more properties of triangle algebras.

In the present paper, we define co-annihilators and prove some of their elementary properties. Also, we consider the relationship between this subsets and IVRL-filters, so we study basic properties of co-annihilators. We define D^a and by it consider some properties of co-annihilators. Moreover, we study the influence of Rads on co-annihilators and prove $Rad(X^{\perp}) \subseteq Rad([X))$, where X is a subset of triangle algebra A. Finally, we will define CP-congruence on triangle algebras and give new characterizations of them. All these results will be an introduction for a future more detailed analysis on triangle algebras.

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2. Preliminaries

Definition 2.1. [6] A residuated lattice is an algebra $\mathcal{L} = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ with four binary operations and two constants 0,1 such that:
- $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.
- $*$ is commutative and associative, with 1 as neutral element, and
- $x * y \leq z$ iff $x \leq y \rightarrow z$, for all x, y and z in L.

The ordering $\leq$ and negation $\neg$ in a residuated lattice $\mathcal{L} = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ are defined as follows, for all x and y in L: $x \leq y$ iff $x \wedge y = x$ (or equivalently, iff $x \vee y = y$; or, also equivalently, iff $x \rightarrow y = 1$) and $\neg x = x \rightarrow 0$.

Lemma 2.2. [5, 3] *Let $\mathcal{L} = (L, \vee, \wedge, *, \rightarrow, 0, 1)$ be a residuated lattice. Then the following properties are valid, for all x, y and z in L:*
(1) $x \vee y \leq (x \rightarrow y) \rightarrow y$ *(in particular* $x \leq \neg\neg x$*),*
(2) $x * \vee_{i \in I} y_i = \vee_{i \in I}(x * y_i)$,
(3) $(\vee_{i \in I} y_i) \rightarrow x = \wedge_{i \in I}(y_i \rightarrow x)$,
(4) $(x \rightarrow y) * (y \rightarrow z) \leqslant (x \rightarrow z)$,
(5) *If* $x \leqslant y$*, then* $x * z \leqslant y * z, z \rightarrow x \leqslant z \rightarrow y$ *and* $y \rightarrow z \leqslant x \rightarrow z$,
(6) $(y \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$,
(7) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x * y) \rightarrow z$,
(8) $\vee_{i \in I}(y_i \rightarrow x) \leq (\wedge_{i \in I} y_i) \rightarrow x$,
(9) $x \rightarrow y \leqslant (y \rightarrow z) \rightarrow (x \rightarrow z)$,
(10) $\neg x \wedge \neg y \leq \neg(x \vee y)$.

Definition 2.3. [6] Given a lattice $\mathcal{A} = (A, \vee, \wedge)$, its triangularization $\mathbb{T}(\mathcal{A})$ is the structure $\mathbb{T}(\mathcal{A}) = (Int(\mathcal{A}), \vee, \wedge)$ defined by
- $Int(\mathcal{A}) = \{[x_1, x_2] : (x_1, x_2) \in A^2 \text{ and } x_1 \leq x_2\}$,
- $[x_1, x_2] \wedge [y_1, y_2] = [x_1 \wedge y_1, x_2 \wedge y_2]$,
- $[x_1, x_2] \vee [y_1, y_2] = [x_1 \vee y_1, x_2 \vee y_2]$.

The set $D_{\mathcal{A}} = \{[x, x] : x \in L\}$ is called the diagonal of $\mathbb{T}(\mathcal{A})$.

Definition 2.4. [6] An interval-valued residuated lattice (IVRL) is a residuated lattice $(Int(\mathcal{A}), \vee, \wedge, \odot, \rightarrow_{\odot}, [0, 0], [1, 1])$ on the triangularization $\mathbb{T}(\mathcal{A})$ of a bounded lattice $\mathcal{A}$, in which the diagonal $D_{\mathcal{A}}$ is closed under $\odot$ and $\rightarrow_{\odot}$, i.e. $[x, x] \odot [y, y] \in D_{\mathcal{A}}$ and $[x, x] \rightarrow_{\odot} [y, y] \in D_{\mathcal{A}}$, for all x, y in L.

In triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$, operator ν (necessity) and μ (possibility) are modal operators, and u (uncertainty, $u \neq 0, u \neq 1$) is a new constant. It turns out that triangle algebras are the equational representations of interval-valued residuated lattices (IVRLs).

Theorem 2.5. [6] *There is a one-to-one correspondence between the class of IVRLs and the class of triangle algebras. Every extended IVRL is a triangle algebra and conversely, every triangle algebra is isomorphic to an extended IVRL.*

Definition 2.6. [6] A triangle algebra is a structure $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, \nu, \mu, 0, u, 1)$ in which $(A, \vee, \wedge, *, \rightarrow, 0, 1)$ is a residuated lattice, ν and μ are unary operations on A, u a constant, and satisfying the following conditions:

$(T.1)\nu x \leq x$, $(T.1^{'})x \leq \mu x$,

$(T.2)\nu x \leq \nu\nu x$, $(T.2^{'})\mu\mu x \leq \mu x$,

$(T.3)\nu(x \wedge y) = \nu x \wedge \nu y$, $(T.3^{'})\mu(x \wedge y) = \mu x \wedge \mu y$,

$(T.4)\nu(x \vee y) = \nu x \vee \nu y$, $(T.4^{'})\mu(x \vee y) = \mu x \vee \mu y$,

$(T.5)\nu u = 0$, $(T.5^{'})\mu u = 1$,

$(T.6)\nu\mu x = \mu x$, $(T.6^{'})\mu\nu x = \nu x$,

$(T.7)\nu(x \to y) \leq \nu x \to \nu y$,

$(T.8)(\nu x \leftrightarrow \nu y) * (\mu x \leftrightarrow \mu y) \leq (x \leftrightarrow y)$,

$(T.9)\nu x \to \nu y \leq \nu(\nu x \to \nu y)$.

From now $(A, \vee, \wedge, \to, \nu, \mu, 0, u, 1)$ or simply A is a triangle algebra.

Definition 2.7. [8] A triangle algebra A is called an MTL-triangle algebra if it satisfies the following equation, for all $x, y \in A$:

$$(x \to y) \vee (y \to x) = 1 \text{ (prelinearity)}.$$

Definition 2.8. [5] An IVRL-filter of A is a non-empty subset F of A satisfying:
$(F.1)$ if $x \in F, y \in A$ and $x \leq y$, then $y \in F$,
$(F.2)$ if $x, y \in F$, then $x * y \in F$,
$(F.3)$ if $x \in F$, then $\nu x \in F$.

An alternative definition for an IVRL-filter F of a triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ is the following:

- $1 \in F$,
- for all x and y in A: if $x \in F$ and $x \to y \in F$, then $y \in F$.
- if $x \in F$, then $\nu x \in F$.

For all $x, y \in A$, we write $x \equiv_F y$ iff $x \to y$ and $y \to x$ are both in F.

$\equiv_F$ is always a congruence relation [5]. Note that $(F.3)$ is a necessary condition for this statement. Indeed, if $\equiv_F$ is a congruence relation on a triangle algebra $\mathcal{A} = (A, \vee, \wedge, *, \to, \nu, \mu, 0, u, 1)$ and $x \in F$, then $x \equiv_F 1$ and therefor $\nu x \equiv_F \nu 1 = 1$, which is equivalent with $\nu x \in F$.

Definition 2.9. [9] Let $S \subseteq A$, a nonempty subset of A, $a \in A$. Then $[S) = \{x \in A : s_1 * ... * s_n \leq \nu x$, for some $n \geq 1$ and $s_1, ..., s_n \in S\}$.

Definition 2.10. [8] Let F be an IVRL-filter of MTL-triangle algebra A. Then

$$Rad(F) = \{x \in A : \neg(\nu x^n) \to \nu x \in F, \text{ for all } n \in \mathbb{N}\}.$$

Definition 2.11. [8]Dense elements of a triangle algebra A is defined as $D_s(A) = \{a \in A \mid \neg\nu a = 0\}$.

Definition 2.12. [5] • An IVRL-extended prime filter of A is a filter F of A such that $\nu x \to \nu y \in F$ or $\nu y \to \nu x \in F$, for all $x, y \in A$.

- An IVRL-extended prime filter of the second kind is a filter F of A such that for all x and y in L: if $\nu(x \vee y) \in F$, then $\nu x \in F$ or $\nu y \in F$ (or both).

3. Co-annihilators in Triangle Algebras

Definition 3.1. Let X be a nonempty subset of A. Then we define

$$X^{\perp} = \{a \in A \mid \nu a \vee x = 1, \text{ for all } x \in X\}.$$

$X^{\perp}$ is called co-annihilator of X.

Clearly, $\{1\}^{\perp} = A$, $\{0\}^{\perp} = A^{\perp} = \{1\}$. We have $X^{\perp} = \bigcap\{\{x\}^{\perp} \mid x \in X\}$, where $X \subseteq A$ and $\bigcup\{a\}^{\perp} = A$, for all $a \in A$.

Example 3.2. Let $A = \{[0,0],[0,a],[0,b],[a,a],[b,b],[0,1],[a,1],[b,1],[1,1]\}$. Define $\odot$ and $\Rightarrow$ as follows:

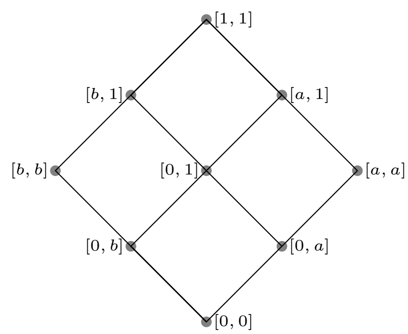
$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\Rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

And we define ν, μ, $*$ and $\to$ as follows:

$\nu[x_1,x_2] = [x_1,x_1]$, $\mu[x_1,x_2] = [x_2,x_2]$, $[x_1,x_2] * [y_1,y_2] = [x_1 \odot y_1, x_2 \odot y_2]$, $[x_1,x_2] \to [y_1,y_2] = [(x_1 \Rightarrow y_1) \wedge (x_2 \Rightarrow y_2), x_2 \Rightarrow y_2]$.

Then $(A, \vee, \wedge, *, \to, \nu, \mu, [0,0], [0,1], [1,1])$ is a triangle algebra with $[0,0]$ as smallest and $[1,1]$ as greatest element. Let $X = \{[a,1]\}$. Then $X^{\perp} = \{[b,b],[b,1],[1,1]\}$.



Proposition 3.3. *Let X be a nonempty subset of A. Then $X^{\perp}$ is an IVRL-filter of A. If $X \neq \{1\}$, then $X^{\perp}$ is proper.*

Proof. It is clear that $1 \in X^{\perp}$. Let $a, a \to b \in X^{\perp}$. Then $\nu(a \to b) \leq \nu a \to \nu b \leq \nu a \to (\nu b \vee x)$, $x \leq \nu b \vee x = 1 \to (\nu b \vee x) = (\nu a \vee x) \to (\nu b \vee x)$. So

$$\begin{aligned}
1 &= \nu(a \to b) \vee x \\
&\leq (\nu a \to \nu b) \vee x \\
&\leqslant [\nu a \to (\nu b \vee x)] \vee [(\nu a \vee x) \to (\nu b \vee x)] \\
&\leqslant [\nu a \wedge (\nu a \vee x)] \to (\nu b \vee x)] \\
&= \nu a \to (\nu b \vee x).
\end{aligned}$$

Thus $\nu a \leq \nu b \vee x$, so $1 = \nu a \vee x \leq (\nu b \vee x) \vee x = \nu b \vee x$. Hence $b \in X^{\perp}$. Since $\nu\nu a = \nu a$, if $a \in X^{\perp}$, then $\nu a \in X^{\perp}$. Let $X \neq \{1\}$ and X be a nonempty subset of A. Then there is an element $a \in X$ such that $a \neq 1$ and $0 \vee a = a \neq 1$. Therefore, $X^{\perp}$ is proper. □

Proposition 3.4. *Let $X, Y \subseteq A$. Then $(X \times Y)^{\perp} = X^{\perp} \times Y^{\perp}$.*

Proof. $\forall\ (x, y) \in X \times Y$, we have

$$\begin{aligned}(X \times Y)^{\perp} &= \{(a, b) \in A \times A \mid (\nu(a, b) \vee (x, y)) = (1, 1)\} \\ &= \{(a, b) \in A \times A \mid ((\nu a, \nu b) \vee (x, y)) = (1, 1)\} \\ &= \{(a, b) \in A \times A \mid ((\nu a \vee x) = 1, (\nu b \vee y) = 1))\} \\ &= X^{\perp} \times Y^{\perp}.\end{aligned}$$

□

Proposition 3.5. *Let F be an IVRL-filter of A and $X/F \subseteq A/F$. Then $(X/F)^{\perp} = X^{\perp}/F$.*

Proof. For all $[x] \in X/F$, we have

$$\begin{aligned}(X/F)^{\perp} &= \{[a] \in A/F \mid \nu[a] \vee [x] = [1]\} \\ &= \{[a] \in A/F \mid [\nu a \vee x] = [1]\} \\ &= X^{\perp}/F.\end{aligned}$$

□

In the following example we show $Rad([X]) \neq Rad(X^{\perp})$.

Example 3.6. In Example 3.2, let $X = \{[a, a], [a, 1]\}$. Then $[X) = \{[a, a], [a, 1], [1, 1]\}$, $Rad([X)) = \{[a, a], [a, 1], [1, 1]\}$. We have $X^{\perp} = \{[b, b], [b, 1], [1, 1]\}$, $Rad(X^{\perp}) = \{[b, b], [b, 1], [1, 1]\}$. So $Rad([X)) \neq Rad(X^{\perp})$.

For all $a \in A$, we denote

$$D^{a} = \{x \in A \mid x \to \nu a = \nu a, \nu a \to x = x\}.$$

Clearly, $\nu a \in D^{x}$ iff $\nu x \in D^{a}$. Also, we have $D^{\mu\nu a} = D^{\nu a} = D^{a}$, $D^{1} = A$.

Remark 3.7. For all, $x \in D_s(A)$ we have $\nu x \to 0 = 0$ so $D^{0} \subseteq D_s(A)$. Since $0 \to \nu x = 1 \neq \nu x$, then the converse is not true.

Let $x \vee y := [(x \to y) \to y] \wedge [(y \to x) \to x]$, for all $x, y \in A$.

Proposition 3.8. *For all $a \in A$, D^{a} is an IVRL-filter of A, $D^{a} = \{a\}^{\perp}$.*

Proof. Let $\nu a \vee x = 1$. Then $(\nu a \to x) \to x = 1$ and $(x \to \nu a) \to \nu a = 1$, so $\nu a \to x = x$ and $x \to \nu a = \nu a$. Thus

$$\begin{aligned}\{a\}^{\perp} &= \{x \in A \mid \nu a \vee x = 1\} \\ &= \{x \in A \mid x \to \nu a = \nu a, \nu a \to x = x\} \\ &= D^{a}.\end{aligned}$$

□

Definition 3.9. The order of $x \in A$, denoted by $ord(x)$ is the smallest $n \in \mathbb{N}$ such that $x^n = \underbrace{x * ... * x}_{n-times} = 0$. If there is no such n, thus $ord(x) = \infty$.

In the following proposition and example, the relationship between $ord(a)$ and D^a will be considered.

Proposition 3.10. *If $ord(a) < \infty$, then $D^a = \{1\}$.*

Proof. Assume $ord(a) = m < \infty$, $x \in D^a$. Then $a \in D^x$ which is an IVRL-filter. So $a^m \in D^x$, hence $0 \in D^x$ and so $\nu 0 \to x = 0 \to x = x$. Hence $x = 1$. □

In the following example we show that the converse of above proposition is not true.

Example 3.11. In Example 3.2, $D^{[0,a]} = \{[1, 1]\}$. But $ord([0, a]) = \infty$.

In the following proposition we consider the relationship between $X^\perp$ and D^x, for all $X \subseteq A$, $x \in A$.

Proposition 3.12. *Let X be a non empty subset of A and $x \in X$. Then $X^\perp = \cap_{x \in X} D^x = \cap_{x \in X}\{x\}^\perp$.*

Proof. $a \in X^\perp$ iff $\forall x \in X$: $\nu a \vee x = 1$ iff $\forall x \in X$: $a \in \{x\}^\perp$ iff $a \in \cap_{x \in X}\{x\}^\perp = \cap_{x \in X} D^x$. □

Proposition 3.13. *Let A be an MTL-triangle algebra and $X^\perp$ be an IVRL-extended prime filter of A and $a, b \in X$, then either $x \in D^{a \to b}$ or $x \in D^{b \to a}$, for all $x \in X$.*

Proof. Since $(a \to b) \vee (b \to a) = 1$ and $X^\perp$ is an IVRL-extended prime filter of A, either $a \to b \in X^\perp = \cap_{x \in X} D^x$ or $b \to a \in X^\perp = \cap_{x \in X} D^x$. So $x \in D^{a \to b}$ or $x \in D^{b \to a}$, for all $x \in X$. □

In the following example we show that the converse of above proposition is not true.

Example 3.14. in Example 3.2, if $X = \{[0, 1]\}$, then $X^\perp = \{[1, 1]\}$ and $X^\perp$ is not an IVRL-extended prime filter of A. But $x \in D^{[1,1]} = A$, for all $x \in X$.

Proposition 3.15. *Let $\emptyset \neq X \subseteq A$. Then $[X) \cap X^\perp = \{1\}$, in particular, $D^x \cap [X) = \{1\}$, for all $x \in A$. If F is an IVRL-filter of A, then $F \cap F^\perp = \{1\}$.*

Proof. Let $a \in$ [X)$\cap X^\perp$. Then $a \in X^\perp$ and $x \to \nu a = \nu a$, for all $x \in X$. Whence $a \in [x), \nu a \geq x_1 * x_2 * ... * x_n$, for some $x_1, ..., x_n \in X$. Hence

$$1 = x_1 \to (...x_{n-1} \to (x_n \to \nu a)...) = \nu a.$$

Second part follows by the fact $D^x = \{x\}^\perp$. Since F is an IVRL-filter and $a \in [F)$, then $a \in F$ as $\nu a \geq x_1 * x_2 * ... * x_n$, for some $x_1, ..., x_n \in F$ and $x_1 * ... * x_n \in F$. Hence $[F) = F$ and $F \cap F^\perp = [F) \cap F^\perp = \{1\}$. □

Proposition 3.16. *Let F be an IVRL-filter of A. Then $F^\perp$ is an IVRL-extended prime filter of the second kind of A iff F is linear, $F \neq \{1\}$.*

Proof. Let F be linear and $F \neq \{1\}$ and $\nu a \vee \nu b \in F^{\perp}$ but $\nu a \notin F^{\perp}, \nu b \notin F^{\perp}$. Then there exist $x, y \in F$ such that $\nu a \vee x \neq 1, \nu b \vee y \neq 1$. Set $z = x \wedge y$. Then $z \in F$ as F is an IVRL-filter of A, we have $\nu a \vee z \neq 1, \nu b \vee z \neq 1$. Since $z \leq \nu a \vee z, \nu b \vee z$, we conclude $\nu a \vee z, \nu b \vee z \in F$. As F is linear, we may assume $\nu b \vee z \leq \nu a \vee z$. Now

$$1 = (\nu a \vee \nu b) \vee z = \nu a \vee (\nu b \vee z) \leq \nu a \vee (\nu a \vee z) = \nu a \vee z,$$

which is a contradiction. Therefore, $\nu a \in F^{\perp}$ or $\nu b \in F^{\perp}$ and so F is an IVRL-extended prime filter of the second kind.

Conversely, let F be an IVRL-extended prime filter of the second kind. Then $F \neq \{1\}$ as otherwise we would have $F^{\perp} = A$. Let $a, b \in F$. Since $b \leq a \to b, a \leq b \to a$ and F is an IVRL-filter, $a \to b, b \to a \in F$. By Proposition 3.13, either $a \to b, b \to a \in D^{a \to b}$ or $a \to b, b \to a \in D^{b \to a}$. In the first case $1 = (b \to a) \vee (b \to a) = (b \to a)$, hence $b \leq a$. So F is linear. □

In the following theorem we give some properties of $X^{\perp}$.

Theorem 3.17. *Let $\emptyset \neq X, Y \subseteq A$. Then we have:*
(1) *If $X \subseteq Y$, then $Y^{\perp} \subseteq X^{\perp}$.*
(2) $X \subseteq X^{\perp\perp}$.
(3) $X^{\perp} = X^{\perp\perp\perp}$.
(4) $X^{\perp} = [X)^{\perp}$.
(5) *If $h : A \longrightarrow A$ is an endomorphism and $X \subseteq A$, then $h(X^{\perp}) \subseteq h(X)^{\perp}$.*
(6) $(\bigcup_{i \in I} X_i)^{\perp} = \bigcap_{i \in I} X_i^{\perp}$, *for $X_i \subseteq A$.*
(7) $\bigcap_{i \in I} X_i^{\perp} \subseteq (\bigcap_{i \in I} X_i)^{\perp}$, *for $X_i \subseteq A$.*
(8) *If $X \bigcap X^{\perp} \neq \emptyset$, then $\{1\} \subseteq X \bigcap X^{\perp}$.*
(9) $(X * Y)^{\perp} \subseteq X^{\perp} \cap Y^{\perp}$.

Proof. (1) Let $a \in \cap_{y \in Y} \{y\}^{\perp}$. Then for all $x \in X \subseteq Y$ we have $x \to \nu a = \nu a, \nu a \to x = x$. So $a \in \cap_{x \in X} \{x\}^{\perp}$. Hence $Y^{\perp} = \cap_{y \in Y} \{y\}^{\perp} \subseteq \cap_{x \in X} \{x\}^{\perp} = X^{\perp}$.

(2) $X^{\perp\perp} = \{a \in A \mid \nu a \vee x = 1 \text{ for all } x \in X^{\perp}\}$. If $b \in X$, then $\nu b \vee x = 1$, for all $\nu b \vee x = 1$, for all $x \in X^{\perp}$. So $b \in X^{\perp\perp}$.

(3) By (2), $X^{\perp} \subseteq X^{\perp\perp\perp}$. By (1) we have $X^{\perp\perp\perp} \subseteq X^{\perp}$. Thus (3) holds.

(4) We know $X \subseteq [X)$, so $[X)^{\perp} \subseteq X^{\perp}$. Conversely, let $y \in X^{\perp}$. Then for all, $x_i \in X, i = 1, ..., n$, $\nu y \vee x_i = 1$. By Induction on n we have $\nu y \vee (x_1 * \cdots * x_n) = 1$. For $n = 1$, the claim is true. If for $n = k$ the claim is true, we show that for $n = k + 1$ is true. So

$$\begin{aligned}
1 &= (\nu y \vee x) * (\nu y \vee x_{k+1}) \\
&= [\nu y * (\nu y \vee x_{k+1})] \vee [x * (\nu y \vee x_{k+1})] \\
&= \nu y \vee [(x * \nu y) \vee (x * x_{k+1})] \\
&\leq \nu y \vee [\nu y \vee (x_1 * \cdots * x_{k+1})] \\
&= \nu y \vee (x_1 * \cdots * x_{k+1}).
\end{aligned}$$

Hence, the claim holds for all natural numbers n. Let $a \in [X)$. Then $\nu a \geq x_1 * \cdots * x_n$, for some $x_1, \cdots, x_n \in X$. Thus $1 = \nu y \vee (x_1 * \cdots * x_n) \leq \nu y \vee \nu a$. And so $\nu y \vee \nu a = 1$, for all $a \in [X)$. Hence $y \in [X)^\perp$ and $X^\perp \subseteq [X)^\perp$.

(5) Let $X \subseteq A$, $h : A \longrightarrow A$ be endomorphism and $y \in h(X^\perp)$. So there exists $x \in X^\perp$ such that $y = h(x)$. Hence $\nu x \vee a = 1$, for all $a \in X$. Since h is endomorphism we have $\nu h(x) \vee h(a) = h(1)$, for all $a \in X$. Thus $y = h(x) \in h(X)^\perp$.

(6) Let $x \in \bigcap_{i \in I} X_i^\perp$. Then $(\forall i \in I)$ $x \in X_i^\perp$. So $(\forall i \in I)$ $(\forall a \in X_i)$, $\nu x \vee a = 1$. Thus $\nu x \vee a = 1$, for all $a \in \bigcup_{i \in I} X_i$; hence $x \in (\bigcup_{i \in I} X_i)^\perp$.

The converse is clear by part (1).

(7) Let $a \in \cap_{i \in I} X_i^\perp$, where $i \in I$. Then $a \in X_i^\perp$, for all $i \in I$. Thus $\nu a \vee x_i = 1$, for all $x_i \in X_i$, $i \in I$. So $\nu a \vee k = 1$, for all $k \in \bigcap_{i \in I} X_i$; hence $a \in (\bigcap_{i \in I} X_i)^\perp$.

(8) Let $X \bigcap X^\perp \neq \emptyset$. Since $1 \vee x = 1$, for all $x \in X$, we have $\{1\} \subseteq X \bigcap X^\perp$.

(9) Let $a \in (X * Y)^\perp$. Then $\nu a \vee (x * y) = 1$, for all $x \in X, y \in Y$. Since $1 = \nu a \vee (x * y) \leq \nu a \vee x$, so $\nu a \vee x = 1$ and so $a \in X^\perp$. Similarly, $a \in Y^\perp$. Hence $a \in X^\perp \cap Y^\perp$. □

In the following example we show that equality in Theorem 3.17 (1), (2), (7) does not hold generally.

Example 3.18. In Example 3.2,

- Let $X = \{[a, 1]\}, Y = \{[1, 1]\}$. Then $\{[b, b], [b, 1], [1, 1]\} = X^\perp \subseteq Y^\perp = A$, but $Y \nsubseteq X$. So the converse of part (i) is not true.
- If $X = \{[0, 1]\}$, then $X^\perp = \{[1, 1]\}, X^{\perp\perp} = A$. Thus $X^{\perp\perp} \nsubseteq X$ and so equality in (ii) does not hold.
- Let $X = \{[1, 1]\}, Y = \{[a, a], [a, 1], [1, 1]\}$. Then $X^\perp = A, Y^\perp = \{[b, b], [b, 1], [1, 1]\}$ and so $(X \cap Y)^\perp = A \nsubseteq X^\perp \cap Y^\perp = Y^\perp$.

Proposition 3.19. *Let F be a linear IVRL-filter of A and $1 \neq x \in F$, $x \vee \neg x = 1$. Then νx is the least element of F.*

Proof. Since $x \vee \neg x = 1$, By Lemma 2.2, $\neg x \wedge x = 0$, whence $\neg x \wedge x \leq \neg x \wedge \neg\neg x = 0$. If $a \in F$, then

$$a = a \vee 0 = a \vee (x \wedge \neg x) = (a \vee x) \wedge (a \vee \neg x). \tag{1}$$

By Proposition 3.16, $F^\perp$ is an IVRL-extended prime filter of A. Whence $x \vee \neg x = 1$, either $x \in F^\perp$ or $\neg x \in F^\perp$ and $x \vee \nu x = x \neq 1$, we have $\neg x \in F^\perp$. So $\nu \neg x \vee a = 1$, for all $a \in F$. By $(T.4)$ we have $\nu(\nu \neg x \vee a = 1) = \nu \neg x \vee \nu a = 1$. By (1), $(T.4)$ and $(T.3)$, we have $\nu a = (\nu a \vee \nu x) \wedge (\nu a \vee \nu \neg x)$. Thus $\nu a = \nu a \vee \nu x$, $\nu x \leq \nu a \leq a$ and the proof is complete. □

4. Co-annihilator Preserving Congruences

The family of the IVRL-extended prime filter of the second kind of A is denoted by $\chi(A)$. If $\varphi : A \longrightarrow \wp(\chi(A))$ is the map defined by $\varphi(a) = \{P \in \chi(A) \mid a \in P\}$, for all $a \in A$. Let $Y \subseteq A$. Consider the subset

$$\psi(Y) = \{P \in \chi(A) \mid Y \cap P \neq \emptyset\}.$$

It is easy to see that $\psi(Y) = \cup\{\varphi(a) \mid a \in Y\}$, for all $a \in A$.

Let θ be a congruence relation of A. We will $(a,b) \in \theta$ or $a \equiv_\theta b$. The equivalence class of an element $a \in A$ is denoted by $[a]_\theta = \{b \in A \mid a \equiv_\theta b\}$. The canonical map with respect to θ is the function $q : A \longrightarrow A/\theta$ defined by $q(a) = [a]_\theta$.

Definition 4.1. Let θ be a congruence relation of A. We say that θ is a congruence preserving co-annihilator, or CP-congruence of A, if (CP) for each pair $a, b \in A, a \equiv_\theta b$ implies that each all $x \in \{a\}^\perp$ there exists $y \in \{b\}^\perp$ such that $x \equiv_\theta y$.

Example 4.2. In Example 3.2,

$$\theta = \{([b,b],[b,b]),([b,1],[b,1]),([1,1],[1,1]),([b,1],[b,b]),([b,b],[b,1]),([1,1],[b,b]),\\ ([b,b],[1,1]),([1,1],[b,1]),([b,1],[1,1])\}.$$

is a congruence relation of A. Clearly, $[b,1] \equiv_\theta [b,b]$ as $[b,1]^\perp = \{[a,a],[a,1],[1,1]\} = [b,b]^\perp$ then θ is a CP-congruence of A.

If F is an IVRL-filter of A, then $\equiv_F$ is not CP-congruence necessity:

Example 4.3. In Example 3.2, let $F = \{[a,a],[a,1],[1,1]\}$. Then

$$\equiv_F = \theta =\\ \{([1,1],[1,1]),([a,a],[a,a]),([a,1],[a,1]),([a,a],[a,1]),([a,1],[a,a]),([a,1],[1,1]),\\ ([1,1],[a,1]),([a,a],[1,1]),([1,1],[a,a])\}.$$

Clearly, $[1,1] \equiv_F [a,1]$. we have $\{[a,1]\}^\perp = \{[b,b],[b,1],[1,1]\}$ and $[1,1]^\perp = A$. So θ is not a CP-congruence.

To indicate that the pair (a,b) satisfies the condition for each $x \in \{a\}^\perp$ there exists $y \in \{b\}^\perp$ such that $x \equiv_\theta y$ for above definition we will use the following notation:

$$(\{a\}^\perp, \{b\}^\perp) \in \overline{\theta}, \text{ or } \{a\}^\perp \equiv_{\overline{\theta}} \{b\}^\perp.$$

Thus a congruence relation θ is a CP-congruence if for all $a, b \in A, \{a\}^\perp \equiv_{\overline{\theta}} \{b\}^\perp$, whenever $a \equiv_\theta b$. Also, for every $Y \subseteq \chi(A)$ the set

$$\theta(Y) = \{(a,b) \in A^2 \mid \varphi(a) \cap Y = \varphi(b) \cap Y\},$$

is a congruence relation.

Clearly, $\nu a \in [a]_\theta$, for all $a \in A$.

Lemma 4.4. *Let $Y \subseteq \chi(A)$ and $a \equiv_{\theta(Y)} b$. Then the following conditions are equivalent:*

(i) $\{a\}^\perp \equiv_{\overline{\theta}(Y)} \{b\}^\perp$,

(ii) $\psi(\{a\}^\perp) \cap Y = \psi(\{b\}^\perp) \cap Y$.

Proof. $(i \Rightarrow ii)$ We prove that $\psi(\{a\}^\perp) \cap Y \subseteq \psi(\{b\}^\perp) \cap Y$. Let $P \in \psi(\{a\}^\perp) \cap Y = \bigcup\{\varphi(e) \mid e \in \{a\}^\perp\} \cap Y$. Then there exists $e \in \{a\}^\perp$ such that $e \in P$. Since $(\{a\}^\perp, \{b\}^\perp) \in \overline{\theta}(Y)$, there exists $f \in \{b\}^\perp$ such that $e \equiv_{\theta(Y)} f$, so $\varphi(e) \cap Y = \varphi(f) \cap Y$. Hence $P \in \varphi(e) \cap Y = \varphi(f) \cap Y \subseteq \psi(\{b\}^\perp) \cap Y$.

$(ii \Rightarrow i)$ Let (ii) hold. Then

$$\bigcup\{\varphi(e) \mid e \in \{a\}^\perp\} \cap Y = \bigcup\{\varphi(f) \mid f \in \{b\}^\perp\} \cap Y.$$

If $e \in \{a\}^{\perp}$. We need to prove that there exists $f \in \{b\}^{\perp}$ such that $(e, f) \in \theta$. We have:

$$\begin{aligned}\varphi(e) \cap Y &\subseteq \bigcup\{\varphi(x) \mid x \in \{a\}^{\perp}\} \cap Y \\ &= \psi(\{a\}^{\perp}) \cap Y = \psi(\{b\}^{\perp}) \cap Y \\ &= \bigcup\{\varphi(y) \mid y \in \{b\}^{\perp}\} \cap Y \\ &\subseteq \bigcup\{\varphi(y) \mid x \in \{b\}^{\perp}\}.\end{aligned}$$

Since $\varphi(e) \cap Y$ is closed, there are $f_1, f_2, ..., f_n \in \{b\}^{\perp}$ such that

$$\varphi(e) \cap Y \subseteq \varphi(f_1) \cup ... \cup \varphi(f_n) = \varphi(f_1 \vee ... \vee f_n) = \varphi(f).$$

Hence $\varphi(e) \cap Y = \varphi(e \vee f) \cap Y$. Let $h = e \vee f$. Then $e \equiv_{\theta(Y)} h$. Since $h \geq f$ and $f \in \{b\}^{\perp}$, we have $h \in \{b\}^{\perp}$. □

In MTL-triangle algebras we have:

Theorem 4.5. *Let θ be a congruence relation, $Y \subseteq \chi(A)$ and $\theta = \theta(Y)$. Then the following conditions are equivalent:*

(i) If $a \equiv_\theta 1$, then $\{a\}^{\perp} \equiv_\theta A$,

(ii) θ is an CP-congruence,

(iii) $[\{a\}^{\perp}]_\theta = \{[a]_\theta\}^{\perp}$, for all $a \in A$,

(iv) For all $a, b \in A$, if $a \vee b \equiv_\theta 1$, then there exists $c \in A$ such that $a \vee c = 1$ and $c \equiv_\theta b$.

Proof. $(i \Rightarrow ii)$ Let $a, b \in A$ be such that $a \equiv_\theta b$. By Lemma 4.4, we must prove $\psi(\{a\}^{\perp}) \cap Y = \psi(\{b\}^{\perp}) \cap Y$. Assume $P \in \chi(A)$ such that $\{a\}^{\perp} \cap P \neq \emptyset$ and $1 = a \vee t \equiv_\theta b \vee t$. Then $\{a \vee t\}^{\perp} \equiv_\theta A$, so

$$\psi(\{a \vee t\}^{\perp}) \cap Y = \psi(A) \cap Y = \chi(A) \cap Y = Y. \tag{2}$$

Also, we have

$$\varphi(t) \cap \psi(\{b\}^{\perp}) = \varphi(t) \cap \psi(\{b \vee t\}^{\perp}).$$

Then by (2), we have

$$\varphi(t) \cap \psi(\{b\}^{\perp}) \cap Y = \varphi(t) \cap \psi(\{b \vee t\}^{\perp}) \cap Y = \varphi(t) \cap Y.$$

Since $P \in \varphi(t) \cap Y, \{b\}^{\perp} \cap P \neq \emptyset$, then $\psi(\{a\}^{\perp}) \cap Y \subseteq \psi(\{b\}^{\perp}) \cap Y$. The other inclusion is similar.

$(ii \Rightarrow iii)$ We prove $[\{a\}^{\perp}]_\theta \subseteq \{[a]_\theta\}^{\perp}$, for all $a \in A$. Let $[x]_\theta \in [\{a\}^{\perp}]_\theta$. Then there exists $y \in \{a\}^{\perp}$ such that $[x]_\theta = [y]_\theta$. Since $\nu y \vee a = 1$,

$$[\nu y \vee a]_\theta = [\nu y]_\theta \vee [a]_\theta = [y]_\theta \vee [a]_\theta = [x]_\theta \vee [a]_\theta = [1]_\theta.$$

So $[x]_\theta \in \{[a]_\theta\}^{\perp}$.

Conversely, let $x \in A$ be such that $[x]_\theta \in \{[a]_\theta\}^{\perp}$ but $[x]_\theta \notin [\{a\}^{\perp}]_\theta$. Then $x \vee a \equiv_\theta 1$. Since $\{a\}^{\perp}$ is an IVRL-filter, there exists $P_\theta \in \chi(A/\theta)$ such that $[\{a\}^{\perp}] \cap P_\theta \neq \emptyset$ and $[x]_\theta \in P_\theta$. Since $\theta : A \longrightarrow A/\theta$ is surjective, $\{a\}^{\perp} \cap P = \emptyset$, where $P = q^{-1}(P_\theta) \in Y$. Thus θ is an CP-congruence and $x \vee a \equiv_\theta \{1\}^{\perp} = A$. Thus

$$\psi(\{x \vee a\}^{\perp}) \cap Y = \chi(A) \cap Y = Y.$$

So, $Y \subseteq \psi(\{x \vee a\}^{\perp})$ and $\{x \vee a\}^{\perp} \cap P \neq \emptyset$. Since $x \in P \subseteq M$, and $a \in M$, then $x \vee a \in M$, which is a contradiction. Therefore, $\{[a]_{\theta}\}^{\perp} \subseteq [\{a\}^{\perp}]_{\theta}$.

($iii \Rightarrow iv$) Let $a, b \in A$ be such that $a \vee b \equiv_{\theta} 1$. Then $[b]_{\theta} \in \{[a]_{\theta}\}^{\perp} = [\{a\}^{\perp}]_{\theta}$. So there exists $c \in \{a\}^{\perp}$ such that $c \equiv_{\theta} b$.

($iv \Rightarrow i$) Let $a, b \in A$ be such that $a \equiv_{\theta} 1$. We prove that for every $b \in A$ there exists $c \in \{a\}^{\perp}$ such that $b \equiv_{\theta} c$. Let $b \in A$. Then $a \vee b \equiv_{\theta} 1$. Thus, there exists $c \in A$ such that $a \vee c = 1$ and $c \equiv_{\theta} b$. □

Definition 4.6. A triangle algebra A have double co-annihilator property if for all $a \in A$, there exists $b \in A$ such that $a^{\perp\perp} = b^{\perp}$, where

$$a^{\perp\perp} = \{c \in A \mid \nu c \vee e = 1, \text{ for all } e \in a^{\perp}\}.$$

Example 4.7. Clearly, triangle algebra A in Example 3.2, have double co-annihilator property.

Theorem 4.8. *Let A have double co-annihilator property and θ be a CP-congruence. Then A/θ has double co-annihilator property.*

Proof. We need to prove that for $[a] \in A/\theta$ there exists $[b] \in A/\theta$ such that $[a]^{\perp\perp} = [b]^{\perp}$. Let $a \in A$. Since A is 1-triangle algebra, there exists $b \in A$ such that $a^{\perp\perp} = b^{\perp}$. First we prove that $[a^{\perp\perp}] \subseteq [a]^{\perp\perp}$. Let $[y] \in [a^{\perp\perp}]$. Then there exists $x \in a^{\perp\perp}$ such that $[y] = [x]$. Take an element $[k] \in [a]^{\perp}$. As $[a]^{\perp} = [a^{\perp}]$, there exists $t \in a^{\perp}$ such that $[k] = [t]$. We note that $x \vee t = 1$, because $x \in a^{\perp\perp}$ and $t \in a^{\perp}$. Then $[y] \vee [k] = [x] \vee [t] = [x \vee t] = [1]$. So, $[y] \in [a]^{\perp\perp}$. Hence, $[b]^{\perp} = [b^{\perp}] = [a^{\perp\perp}] \subseteq [a]^{\perp\perp}$.

Now, we prove that $[a]^{\perp\perp} \subseteq [b]^{\perp}$. Let $[x] \in [a]^{\perp\perp}$. As $a^{\perp\perp} = b^{\perp}, a^{\perp} = a^{\perp\perp\perp} = b^{\perp\perp}$, and taking into account that $b \in b^{\perp\perp}$, we have $\nu b \vee a = 1$. Since θ is congruence relation, $[\nu b \vee a] = [b] \vee [a] = [1]$. i.e. $[b] \in [a]^{\perp}$. As $[x] \in [a]^{\perp\perp}, [x] \vee [b] = [1]$, i.e $[x] \in [b]^{\perp}$. □

5. **Conclusion**

In this paper we introduced the notions of co-annihilator of a subset X in triangle algebra A. Moreover, we show that $X^{\perp}$ is an IVRI-filter of A. We investigated many important properties of the co-annihilators. Also, the special set D^{a} is defined and the relation between this set and co-annihilators are considered and some properties of them are given. Finally, we defined CP-congruence and some results of them are founded. We proved θ is an CP-congruence iff $a \equiv_{\theta} 1$, then $\{a\}^{\perp} \equiv_{\theta} A$.

In our future work, we will continue our study of algebraic properties of this special sets on triangle algebras, with the view to identify a classification for these structures.

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