

**THE CHAIN PROPERTIES AND LI-YORKE SENSITIVITY OF
ZADEH'S EXTENSION ON THE SPACE OF UPPER
SEMI-CONTINUOUS FUZZY SETS**

X. WU, L. WANG AND J. LIANG

ABSTRACT. Some characterizations on the chain recurrence, chain transitivity, chain mixing property, shadowing and h -shadowing for Zadeh's extension are obtained. Besides, it is proved that a dynamical system is spatiotemporally chaotic provided that the Zadeh's extension is Li-Yorke sensitive.

1. Introduction

A *dynamical system* is a pair (X, T) , where X is a nontrivial compact metric space with a metric d and $T : X \rightarrow X$ is a continuous surjection. A nonempty invariant closed subset $Y \subset X$ (i.e., $T(Y) \subset Y$) defines naturally a *subsystem* $(Y, T|_Y)$ of (X, T) . Throughout this paper, let $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$. For any $n \in \mathbb{N}$, let $(X^n, T^{(n)})$ be the n -fold product system $(\underbrace{X \times \dots \times X}_{n \text{ times}}, \underbrace{T \times \dots \times T}_{n \text{ times}})$.

Given a dynamical system (X, T) , one can naturally obtain some associated systems induced by (X, T) . The first one is $(M(X), T_M)$ on the space $M(X)$ consisting of all Borel probability measures with the Prohorov metric, which can be viewed as statistical states, representing imperfect knowledge of the system. Its topological parallelism is $(K(X), T_K)$ on the hyperspace $K(X)$ consisting of all nonempty closed subsets of X with the Hausdorff metric. And the third one is the Zadeh's extension $(\mathbb{F}_0(X), T_F)$ (more generally g -fuzzification $(\mathbb{F}_0(X), T_F^g)$ which was introduced by Kupka [16] to generalize Zadeh's extension) on the space $\mathbb{F}_0(X)$ consisting of all nonempty upper semi-continuous fuzzy sets with the level-wise metric induced by the extended Hausdorff metric. A systematic study on the connections between dynamical properties of (X, T) and its two induced systems $(K(X), T_K)$ and $(M(X), T_M)$ was initiated by Bauer and Sigmund in [5], and later has been widely developed by several authors. For more results on this topic, one is referred to [3, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 28, 29, 31, 32, 33, 34, 35, 36, 37, 38, 39] and references therein. For $n \in \mathbb{N}$, denote

$$K_n(X) = \{A \in K(X) : |A| \leq n\} \text{ and } K_\infty(X) = \cup_{n \in \mathbb{N}} K_n(X).$$

Definition 1.1. [8, 14] A dynamical system (X, T) is

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- (1) *exact* if for any nonempty open subset $U \subset X$, there exists $n \in \mathbb{Z}^+$ such that $T^n(U) = X$;
- (2) *transitive* if for any pair of nonempty open subsets $U, V \subset X$, there exists $n \in \mathbb{Z}^+$ such that $T^n(U) \cap V \neq \emptyset$;
- (3) *totally transitive* if T^n is transitive for any $n \in \mathbb{N}$;
- (4) *weakly mixing* if $(X^2, T^{(2)})$ is transitive;
- (5) *mildly mixing* if for every transitive system (Y, S) , $(X \times Y, T \times S)$ is transitive;
- (6) *topologically mixing* if for any pair of nonempty open subsets $U, V \subset X$, there exists $m \in \mathbb{Z}^+$ such that for any $n \geq m$, $T^n(U) \cap V \neq \emptyset$.

Banks [3] proved that (X, T) is weakly mixing if and only if $(K(X), T_K)$ is transitive, which is equivalent to the weakly mixing property of $(K(X), T_K)$ (also see [23]). Román-Flores and Chalco-Cano [26, 27] studied some chaotic properties (for example, transitivity, turbulence, sensitive dependence, periodic density) for the Zadeh's extension. Wang and Wei [30] studied the dynamics of Zadeh's extension in the hit-or-miss topology. Then, Kupka [15] investigated the relations between Devaney chaos in the original system and in the Zadeh's extension and proved the following:

Lemma 1.2. [15, Lemma 1, Remark 1, Theorem 1] *Let (X, T) be a dynamical system and $\lambda \in (0, 1]$. Then,*

- (1) *the set of piecewise constants is dense in $\mathbb{F}(X)$, $\mathbb{F}^{\geq \lambda}(X)$ and $\mathbb{F}^{\lambda}(X)$.*
- (2) *$(\mathbb{F}^{\lambda}(X), T_F|_{\mathbb{F}^{\lambda}(X)})$ is periodically dense in $\mathbb{F}^{\lambda}(X)$ if and only if $(K(X), T_K)$ is periodically dense in $K(X)$.*

Kupka [17] continued in studying chaotic properties (for example, Li-Yorke chaos, distributional chaos, ω -chaos, transitivity, total transitivity, exactness, sensitive dependence, weakly mixing, mildly mixing, topologically mixing) of g -fuzzification and showed that if the g -fuzzification $(\mathbb{F}^1(X), T_F^g)$ has the property P , then (X, T) also has the property P , where P denotes the following properties: exactness, sensitive dependence, weakly mixing, mildly mixing, or topologically mixing. Meanwhile, he asked that does the P -property of (X, T) imply the P -property of $(\mathbb{F}^1(X), T_F^g)$? To negatively answer this question, we [31] obtained a sufficient condition on $g \in D_m(I)^1$ to ensure that for every dynamical system (X, T) , its g -fuzzification $(\mathbb{F}^1(X), T_F^g)$ is not transitive (thus, not weakly mixing) and constructed a sensitive dynamical system whose g -fuzzification is not sensitive for any $g \in D_m(I)$.

Very recently, Fernández and Good [10] proved the following result:

Lemma 1.3. [10, Theorem 3.3, Theorem 3.4] *Let (X, T) be a dynamical system. Then, the following statements are equivalent:*

- (1) *(X, T) has shadowing;*
- (2) *$(K_\infty(X), T_K|_{K_\infty(X)})$ has finite shadowing;*
- (3) *$(K(X), T_K)$ has shadowing.*

¹ $D_m(I)$ is the set of all nondecreasing right-continuous functions $g : I \rightarrow I$ with $g(0) = 0$ and $g(1) = 1$.

Lemma 1.4. [10, Theorem 4.5, Theorem 4.6] *Let (X, T) be a dynamical system. Then, the following statements are equivalent:*

- (1) (X, T) has h -shadowing;
- (2) $(K_\infty(X), T_K|_{K_\infty(X)})$ has h -shadowing;
- (3) $(K(X), T_K)$ has h -shadowing.

Remark 1.5. Gómez-Rueda et al. [11] proved that if $(K_n(X), T_K|_{K_n(X)})$ has shadowing, then (X, T) has shadowing and obtained that if (X, T) has shadowing, then $(K_2(X), T_K|_{K_2(X)})$ has shadowing. They also constructed a dynamical system (X, T) for which T has shadowing but $(K_n(X), T_K|_{K_n(X)})$ does not have shadowing for any $n \geq 3$.

In this paper, we further investigate chain recurrence, chain transitivity, shadowing, h -shadowing, and Li-Yorke sensitivity of the Zadeh's extension through further developing the results in [9, 10, 13, 28, 41]. In particular, we prove that a dynamical system has chain recurrence (resp., chain mixing property, shadowing, h -shadowing) if and only if the Zadeh's extension has chain recurrence (resp., chain mixing property, finite shadowing, h -shadowing) and obtain that if the Zadeh's extension is Li-Yorke sensitive, then the dynamical system is spatiotemporally chaotic.

2. Basic Definitions and Notations

2.1. Hyperspace $K(X)$. For any $\delta > 0$ and any nonempty closed subset $A \subset X$, the δ -neighborhood of A is given by $B(A, \delta) = \{y \in X : \inf\{d(x, y) : x \in A\} < \delta\}$. In particular, when $A = \{x\}$, denote this by $B(x, \delta)$ instead of $B(\{x\}, \delta)$. Let $K(X)$ be the hyperspace on X , i.e., the space of all nonempty closed subsets of X with the Hausdorff metric d_H defined by

$$\begin{aligned} d_H(A_1, A_2) &= \max \left\{ \max_{x \in A_1} \min_{y \in A_2} d(x, y), \max_{y \in A_2} \min_{x \in A_1} d(x, y) \right\} \\ &= \inf \{ \varepsilon : A_1 \subset B(A_2, \varepsilon) \text{ and } A_2 \subset B(A_1, \varepsilon) \}, \quad \forall A_1, A_2 \in K(X). \end{aligned}$$

It is known that $(K(X), d_H)$ is also a compact metric space (see [12]).

A dynamical system (X, T) induces naturally a set-valued dynamical system $(K(X), T_K)$, where $T_K : K(X) \rightarrow K(X)$ is defined as $T_K(A) = T(A)$ for any $A \in K(X)$. For any finite collection A_1, \dots, A_n of nonempty subsets of X , take

$$\langle A_1, \dots, A_n \rangle = \left\{ A \in K(X) : A \subset \bigcup_{i=1}^n A_i, A \cap A_i \neq \emptyset \text{ for all } i = 1, \dots, n \right\}.$$

It follows from [12] that the topology on $K(X)$ given by the metric d_H is the same as the Vietoris or finite topology, which is generated by a basis consisting of all sets of the following form,

$$\langle U_1, \dots, U_n \rangle, \quad \text{where } U_1, \dots, U_n \text{ are an arbitrary finite collection of nonempty open subsets of } X.$$

Clearly, under this topology $K_\infty(X)$ is a dense invariant subset of $K(X)$.

2.2. Zadeh's Extension. Let $I = [0, 1]$. A *fuzzy set* A on a space X is a function $A : X \rightarrow I$. Given a fuzzy set A , its α -cuts (or α -level sets) $[A]_\alpha$ and *support* $\text{supp}(A)$ are defined respectively by

$$[A]_\alpha = \{x \in X : A(x) \geq \alpha\}, \quad \forall \alpha \in I,$$

and

$$\text{supp}(A) = \overline{\{x \in X : A(x) > 0\}}.$$

Let $\mathbb{F}(X)$ denote the set of all upper semicontinuous fuzzy sets defined on X and set

$$\mathbb{F}^{\geq \lambda}(X) = \{A \in \mathbb{F}(X) : A(x) \geq \lambda \text{ for some } x \in X\},$$

$$\mathbb{F}^{\leq \lambda}(X) = \{A \in \mathbb{F}(X) : \max A := \max\{A(x) : x \in X\} = \lambda\}.$$

For any $A \in \mathbb{F}(X)$, let $\xi = \sup\{A(x) : x \in X\}$. It is easy to see that there exists a sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $\xi - \frac{1}{n} < A(x_n) \leq \xi$. Assume that $\lim_{n \rightarrow \infty} x_n = z$. The upper semicontinuity of A implies that $\xi \geq A(z) \geq \limsup_{n \rightarrow \infty} A(x_n) = \xi$, i.e., $\max A = \sup\{A(x) : x \in X\}$. Especially, let $\mathbb{F}^1(X)$ denote the system of all normal fuzzy sets on X .

Define \emptyset_X as the empty fuzzy set ($\emptyset_X \equiv 0$) in X , and $\mathbb{F}_0(X)$ as the set of all nonempty upper semicontinuous fuzzy sets. Since the Hausdorff metric d_H is measured only between two nonempty closed subsets in X , one can consider the following extension of the Hausdorff metric:

$$d_H(\emptyset, \emptyset) = 0 \text{ and } d_H(\emptyset, A) = d_H(A, \emptyset) = \text{diam}(X), \quad \forall A \in K(X).$$

Under this Hausdorff metric, one can define a *levelwise metric* d_∞ on $\mathbb{F}(X)$ by

$$d_\infty(A, B) = \sup\{d_H([A]_\alpha, [B]_\alpha) : \alpha \in (0, 1]\}, \quad \forall A, B \in \mathbb{F}(X).$$

It is well known that the spaces $(\mathbb{F}(X), d_\infty)$ and $(\mathbb{F}^1(X), d_\infty)$ are complete, but not compact and not separable (see [16] and references therein).

A fuzzy set $A \in \mathbb{F}(X)$ is *piecewise constant* if there exist a finite number of sets $D_i \subset X$ such that $\bigcup D_i = X$ and $A|_{\text{int}D_i}$ is constant. In this case, a piecewise constant A can be represented by a strictly decreasing sequence of closed subsets $\{A_1, A_2, \dots, A_k\} \subset K(X)$ and a strictly increasing sequence of reals $\{\alpha_1, \alpha_2, \dots, \alpha_k = \max\{A(x) : x \in X\}\} \subset (0, 1]$ if

$$[A]_\alpha = A_{i+1}, \text{ whenever } \alpha \in (\alpha_i, \alpha_{i+1}].$$

Remark 2.1. Fix any two piecewise constants $A, B \in \mathbb{F}(X)$ which are represented by strictly decreasing sequences of closed subsets $\{A_1, A_2, \dots, A_k\}, \{B_1, B_2, \dots, B_s\} \subset K(X)$ and strictly increasing sequences of reals $\{\alpha_1, \alpha_2, \dots, \alpha_k\}, \{\beta_1, \beta_2, \dots, \beta_s\} \subset (0, 1]$ with

$$[A]_\alpha = A_{i+1}, \forall \alpha \in (\alpha_i, \alpha_{i+1}] \text{ and } [B]_\alpha = B_{j+1}, \forall \beta \in (\beta_j, \beta_{j+1}],$$

respectively. Arrange all reals $\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_s$ by the natural order ' $<$ ' and denote them by $\gamma_1, \gamma_2, \dots, \gamma_n$ ($n \leq k + s$). Then, it can be verified that for any $1 \leq t < n$, there exist $1 \leq i \leq k$ and $1 \leq j \leq s$ such that for any $\gamma \in (\gamma_t, \gamma_{t+1}]$,

$$[A]_\gamma = A_i \text{ and } [B]_\gamma = B_j.$$

This implies that there exist (not necessarily strictly) decreasing sequences of closed subsets $\{C_1, C_2, \dots, C_n\}$, $\{D_1, D_2, \dots, D_n\} \subset K(X)$ and a strictly increasing sequence of reals $\gamma_1, \gamma_2, \dots, \gamma_n \subset (0, 1]$ such that

$$[A]_\gamma = C_{i+1} \text{ and } B_\gamma = D_{i+1}, \text{ whenever } \gamma \in (\gamma_i, \gamma_{i+1}].$$

Zadeh's extension (also called *usual fuzzification*) of a dynamical system (X, T) is a map $T_F : \mathbb{F}(X) \rightarrow \mathbb{F}(X)$ defined by

$$T_F(A)(x) = \sup \{A(y) : y \in T^{-1}(x)\}, \forall A \in \mathbb{F}(X), \forall x \in X.$$

Remark 2.2. For any $A, B \in K(X)$, any $n \in \mathbb{N}$ and any $\lambda \in (0, 1]$,

$$d_H(T_K^n(A), T_K^n(B)) = d_\infty(T_F^n(\lambda \cdot \chi_A), T_F^n(\lambda \cdot \chi_B)). \tag{1}$$

This shows that for any fixed $\lambda \in (0, 1]$, the subsystem $(\mathbb{F}_{\lambda \cdot \chi} := \{\lambda \cdot \chi_A \in \mathbb{F}_0(X) : A \in K(X)\}, T_F|_{\mathbb{F}_{\lambda \cdot \chi}})$ is topologically conjugated to $(K(X), T_K)$. Meanwhile, it is not difficult to check that $(\mathbb{F}^{\lambda}(X), T_F|_{\mathbb{F}^{\lambda}(X)})$ is topologically conjugated to $(\mathbb{F}^1(X), T_F|_{\mathbb{F}^1(X)})$ for any $\lambda \in (0, 1]$.

3. Chain Properties and Shadowing for Zadeh's Extension

3.1. Chain Properties for Zadeh's Extension. For any $\delta > 0$, a δ -pseudo-orbit is a finite or infinite sequence $\{x_0, x_1, \dots\}$ such that $d(T(x_i), x_{i+1}) < \delta$ for any $i \geq 0$. A finite δ -pseudo-orbit $\Gamma = \{x_0, x_1, \dots, x_n\}$ is also called a δ -chain from x_0 to x_n of length $|\Gamma| = n$. A map is *chain transitive* if for any $\delta > 0$ and any two points $x, y \in X$, there exists a δ -chain from x to y . Chain transitivity is a natural generalization of transitivity. There is a surprising result [25] which shows that chains do not distinguish between total transitivity and mixing. Precisely speaking, if (X, T^n) is chain transitive for all $n \in \mathbb{N}$ then it is *chain mixing*, that is, for any $x, y \in X$ and any $\delta > 0$, there exists $N \in \mathbb{N}$ such that there exists a δ -chain from x to y of length n for every $n \geq N$. Recently, we [40] also proved that this also holds for iterated function systems.

A point $x \in X$ is *chain recurrent* if for every $\delta > 0$ there exists a δ -chain from x to x (see [7]). The set of all chain recurrent points is denoted $\text{CR}(T)$. By compactness, it is not difficult to check that $\text{CR}(T)$ is a closed set and $T(\text{CR}(T)) = \text{CR}(T)$. If $\text{CR}(T) = X$, then (X, T) is *chain recurrent*. The chain recurrence which was introduced by Conley [7] as a generalization of non-wandering property is a way of describing the recurrent phenomena of dynamical systems.

If $\Gamma_1 = \{x_0, x_1, \dots, x_n\}$ and $\Gamma_2 = \{y_0, y_1, \dots, y_m\}$ are two δ -chains such that $x_n = y_0$, then $\Gamma_1 + \Gamma_2$ denotes the *concatenation* of Γ_1 with Γ_2 ,

$$\Gamma_1 + \Gamma_2 = \{x_0, x_1, \dots, x_n = y_0, y_1, \dots, y_m\}.$$

Clearly, $\Gamma_1 + \Gamma_2$ is an δ -chain and $|\Gamma_1 + \Gamma_2| = |\Gamma_1| + |\Gamma_2|$. In particular, if Γ is an δ -chain from a point to itself and $m \in \mathbb{N}$, $m\Gamma$ denotes $\underbrace{\Gamma + \Gamma + \dots + \Gamma}_{m \text{ times}}$.

Lemma 3.1. [10, Lemma 3.1] *Let (X, T) be a dynamical system and Y be a dense invariant subset of X . Then, (X, T) has finite shadowing if and only if $(Y, T|_Y)$ has finite shadowing.*

Lemma 3.2. *Let (X, T) be a dynamical system and Y be a dense invariant subset of X . Then, (X, T) is chain recurrent (resp., chain transitive, chain mixing) if and only if $(Y, T|_Y)$ is chain recurrent (resp., chain transitive, chain mixing).*

Proof. The proof of this theorem is Similar to Lemma 3.1 in [10]. For completeness, we give the proof for chain recurrence.

(\implies). For any $\delta > 0$ and any $x \in Y$, as (X, T) is chain recurrent, there exists a $\frac{\delta}{3}$ -chain $\{x, x_1, \dots, x_{n-1}, x_n = x\}$. The uniform continuity of (X, T) implies that there exists $0 < \varepsilon < \frac{\delta}{3}$ such that for any $y, z \in X$ with $d(y, z) < \varepsilon$, $d(T(y), T(z)) < \frac{\delta}{3}$. As Y is dense in X , then for any $1 \leq i \leq n-1$, there exists $x'_i \in Y$ such that $d(x'_i, x_i) < \varepsilon$. This implies that

$$\begin{aligned} d(T(x'_i), x'_{i+1}) &\leq d(T(x'_i), T(x_i)) + d(T(x_i), x_{i+1}) + d(x_{i+1}, x'_{i+1}) \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \varepsilon < \delta. \end{aligned}$$

Therefore, $\{x, x'_1, \dots, x'_{n-1}, x\}$ is a δ -chain in Y , i.e., $(Y, T|_Y)$ is chain recurrent.

(\impliedby). For any $\delta > 0$ and any $x \in X$, there exists $0 < \varepsilon < \frac{\delta}{2}$ such that for any $y, z \in X$ with $d(y, z) < \varepsilon$, $d(T(y), T(z)) < \frac{\delta}{2}$. As Y is dense in X , there exists $y \in Y$ such that $d(x, y) < \varepsilon$. The chain recurrence of $(Y, T|_Y)$ implies that there a $\frac{\delta}{2}$ -chain $\{y, y_1, \dots, y_{n-1}, y\}$ in Y . Meanwhile, $d(T(x), y_1) \leq d(T(x), T(y)) + d(T(y), y_1) < \delta$ and $d(T(y_{n-1}), x) \leq d(T(y_{n-1}), y) + d(y, x) < \delta$. This implies that $\{x, y_1, \dots, y_{n-1}, x\}$ is a δ -chain from x to x . Therefore, (X, T) is chain recurrent. \square

Theorem 3.3. *Let (X, T) be a dynamical system. Then, the following statements are equivalent:*

- (1) (X, T) is chain recurrent;
- (2) $(K(X), T_K)$ is chain recurrent;
- (3) $(\mathbb{F}_0(X), T_F)$ is chain recurrent;
- (4) $(\mathbb{F}^1(X), T_F|_{\mathbb{F}^1(X)})$ is chain recurrent.

Proof. Applying [13, Proposition 4.1, Proposition 4.2, Proposition 4.6] yields that (3) \implies (2) \implies (1). It follows immediately from the definition of chain recurrence that (4) \iff (3).

(1) \implies (2). Applying Lemma 3.2, it suffices to prove that $(K_\infty(X), T_K|_{K_\infty(X)})$ is chain recurrent. Given any $A \in K_\infty(X)$ and any $\delta > 0$, without loss of generality, assume that $A = \{x_1, x_2, \dots, x_n\}$. As (X, T) is chain recurrent, then for any $1 \leq i \leq n$, there exists a δ -chain Γ_i from x_i to x_i . Take a sequence $\Gamma(i) = |\Gamma_1| \cdots |\Gamma_{i-1}| |\Gamma_{i+1}| \cdots |\Gamma_n| \Gamma_i$ which is denoted by $\{z_0^{(i)}, z_1^{(i)}, \dots, z_{|\Gamma_1| \cdots |\Gamma_n|}^{(i)}\}$ and choose $\{A_0, A_1, \dots, A_{|\Gamma_1| \cdots |\Gamma_n|}\} \subset K_\infty(X)$ with $A_k = \{z_k^{(1)}, \dots, z_k^{(n)}\}$ for any $0 \leq k \leq |\Gamma_1| \cdots |\Gamma_n|$. It is not difficult to verify that $\{A_0, A_1, \dots, A_{|\Gamma_1| \cdots |\Gamma_n|}\}$ is a δ -chain from A to A . Therefore, $(K_\infty(X), T_K)$ is chain recurrent.

(2) \implies (3). Given any piecewise constant $A \in \mathbb{F}_0(X)$ which is represented by strictly decreasing sequences of closed subsets $\{A_1, A_2, \dots, A_k\} \subset K(X)$ and strictly increasing sequences of reals $\{\alpha_1, \alpha_2, \dots, \alpha_k\} \subset (0, 1]$ with $[A]_\alpha = A_{i+1}$, $\forall \alpha \in$

$(\alpha_i, \alpha_{i+1}]$ and any $\delta > 0$, as $(K(X), T_K)$ is chain recurrent, it follows that for any $1 \leq i \leq k$, there exists a δ -chain from A_i to A_i , denoted by $\{A_0^{(i)}, A_1^{(i)}, \dots, A_{n_i}^{(i)}\}$. Take $B_0 = A$ and $B_l \in \mathbb{F}_0(X)$ with $[B_l]_\alpha = A_{l - \lfloor \frac{l}{n_{i+1}} \rfloor}^{(i+1)}$, $\forall \alpha \in (\alpha_i, \alpha_{i+1}]$ for any $l \in \mathbb{N}$. It can be verified that $\{B_0, B_1, \dots, B_{n_1 \dots n_k}\}$ is a δ -chain from A to A . This, together with Lemma 3.2, implies that $(\mathbb{F}_0(X), T_F)$ is chain recurrent. \square

Remark 3.4. It follows from Theorem 3.3 that the converses of [13, Proposition 4.1, Proposition 4.2, Theorem 4.7] hold trivially. Theorem 3.3 also implies that the assumption of connectedness of [13, Corollary 4.9] can be removed.

Theorem 3.5. *Let (X, T) be a dynamical system and Y be an invariant closed subset of $\mathbb{F}_0(X)$. If $(Y, T_F|_Y)$ is chain transitive, then for any $A, B \in Y$, $\max A = \max B$, i.e., there exists $\lambda \in [0, 1]$ such that $Y \subset \mathbb{F}^\lambda(X)$.*

Proof. For any $A, B \in Y$ and any $0 < \delta < \frac{\text{diam}(X)}{2}$, as $(Y, T_F|_Y)$ is chain transitive, there exists a δ -chain $\{A_0, A_1, \dots, A_n\}$ from A to B . Then, for any $0 \leq i < n$ and any $\alpha \in (0, 1]$,

$$\begin{aligned} d_H(T([A_i]_\alpha), [A_{i+1}]_\alpha) &= d_H([T_F(A_i)]_\alpha, [A_{i+1}]_\alpha) \\ &\leq d_\infty(T_F(A_i), A_{i+1}) < \delta < \frac{\text{diam}(X)}{2}. \end{aligned} \quad (2)$$

This implies that $\max A_i = \max A_{i+1}$.

In fact, suppose that $\xi := \max A_i \neq \eta := \max A_{i+1}$, then

$$d_H(T([A_i]_{\max\{\xi, \eta\}}), [A_{i+1}]_{\max\{\xi, \eta\}}) = \text{diam}(X),$$

which contradicts (2).

Thus, $\max A = \max A_0 = \max A_n = \max B$. \square

Corollary 3.6. $(\mathbb{F}_0(X), T_F)$ is not chain transitive.

Remark 3.7. Corollary 3.6 implies that the assumptions of [13, Proposition 4.3, Lemma 4.4] do not hold.

Lemma 3.8. [13, Theorem 4.8] *A dynamical system (X, T) is chain mixing if and only if $(K(X), T_K)$ is chain mixing.*

Theorem 3.9. *Let (X, T) be a dynamical system. Then, the following statements are equivalent:*

- (1) (X, T) is chain mixing;
- (2) $(K(X), T_K)$ is chain transitive;
- (3) $(K(X), T_K)$ is chain mixing;
- (4) $(\mathbb{F}^{-1}(X), T_F|_{\mathbb{F}^{-1}(X)})$ is chain transitive;
- (5) $(\mathbb{F}^{-1}(X), T_F|_{\mathbb{F}^{-1}(X)})$ is chain mixing.

Proof. It follows from [9, Theorem 3, A1, A2 and F2] that T is chain mixing if and only if $(K_\infty(X), T_K|_{K_\infty(X)})$ is chain transitive if and only if $(K(X), T_K)$ is chain transitive. This, together with Lemma 3.2 and [13, Theorem 4.8], implies that (1) \iff (2) \iff (3). (5) \implies (4) holds trivially.

(4) \implies (2). For any $A, B \in K(X)$ and any $0 < \delta < \frac{\text{diam}(X)}{2}$, noting that $\chi_A, \chi_B \in \mathbb{F}^{-1}(X)$, as $(\mathbb{F}^{-1}(X), T_F|_{\mathbb{F}^{-1}(X)})$ is chain transitive, then there exists a δ -chain $\{F_0, F_1, \dots, F_n\}$ from χ_A to χ_B . Take a sequence $\Gamma = \{[F_0]_1, [F_1]_1, \dots, [F_n]_1\}$. According to the proof of Theorem 3.5, it is easy to see that $\{[F_0]_1, [F_1]_1, \dots, [F_n]_1\} \subset K(X)$. Meanwhile, it can be verified that for any $0 \leq i < n$,

$$d_H(T_K([F_i]_1), [F_{i+1}]_1) = d_H([T_F(F_i)]_1, [F_{i+1}]_1) \leq d_\infty(T_F(F_i), F_{i+1}) < \delta,$$

and

$$[F_0]_1 = A, \quad [F_n]_1 = B,$$

i.e., Γ is a δ -chain from A to B . Thus, $(K(X), T_K)$ is chain transitive.

(3) \implies (5). For any two piecewise constants $A, B \in \mathbb{F}^{-1}(X)$ which are represented by decreasing sequences of closed subsets $\{A_1, A_2, \dots, A_k\}, \{B_1, B_2, \dots, B_k\} \subset K(X)$ and a strictly increasing sequence of reals $\{\alpha_1, \alpha_2, \dots, \alpha_k = 1\} \subset [0, 1]$ such that

$$[A]_\alpha = A_{i+1}, \quad [B]_\alpha = B_{i+1}, \quad \text{whenever } \alpha \in (\alpha_i, \alpha_{i+1}],$$

and any $\delta > 0$, as $(K(X), T_K)$ is chain mixing, there exists $N \in \mathbb{N}$ such that for any $1 \leq i \leq k$ and any $n \geq N$, there exists a δ -chain $\{C_0^{(i)}, C_1^{(i)}, \dots, C_n^{(i)}\} \subset K(X)$ from A_i to B_i of length n . For any $0 \leq j \leq n$, take a fuzzy set $F_j \in \mathbb{F}^{-1}(X)$ with

$$[F_j]_\alpha = \cup_{l=i+1}^k C_j^{(l)}, \quad \text{whenever } \alpha \in (\alpha_i, \alpha_{i+1}].$$

It is not difficult to verify the following:

- (i) each F_j is a piecewise constant;
- (ii) $F_0 = A, F_n = B$;
- (iii) $\{F_0, F_1, \dots, F_n\}$ is a δ -chain of T_F .

This, together with Lemma 3.2, implying that $(\mathbb{F}^{-1}(X), T_F|_{\mathbb{F}^{-1}(X)})$ is chain mixing, because the set of piecewise constants is dense in $\mathbb{F}^{-1}(X)$ (see Lemma 1.2). \square

3.2. Shadowing for Zadeh's Extension. We say that a dynamical system (X, T) has *shadowing* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for every δ -pseudo-orbit $\{x_n\}_{n=0}^\infty$ of T , there exists $z \in X$ such that $\{x_n\}_{n=0}^\infty$ is ε -shadowed by z , i.e., $d(T^n(z), x_n) < \varepsilon$ for all $n \in \mathbb{Z}^+$ (see [2]). In the case that only finite pseudo-orbits are shadowed, then we say that (X, T) has *finite shadowing*. By compactness of X , it can be verified that (X, T) has shadowing if and only if it has finite shadowing.

Theorem 3.10. *A dynamical system (X, T) has shadowing if and only if $(\mathbb{F}^{-1}(X), T_F|_{\mathbb{F}^{-1}(X)})$ has finite shadowing.*

Proof. (\Leftarrow). As $(\mathbb{F}^{-1}(X), T_F|_{\mathbb{F}^{-1}(X)})$ has finite shadowing, noting that $\{\chi_{\{x\}} : x \in X\} \subset \mathbb{F}^{-1}(X)$, it can be verified that (X, T) has finite shadowing. This implies that (X, T) has shadowing by the compactness of X .

(\Rightarrow). For any $\varepsilon > 0$, as (X, T) has shadowing, then there exists $0 < \delta < \frac{\text{diam}(X)}{2}$ such that every δ -pseudo-orbit of T is $\frac{\varepsilon}{2}$ -shadowed by some point in X . For every finite δ -pseudo-orbit $\{A_0, A_1, \dots, A_n\} \subset \mathbb{F}^{-1}(X)$ of piecewise constants, applying Remark 2.1 implies that there exist decreasing sequences of closed subsets

$\{A_1^{(j)}, A_2^{(j)}, \dots, A_m^{(j)}\} \subset K(X)$ ($0 \leq j \leq n$), and a strictly increasing sequence of reals $\{\alpha_1, \alpha_2, \dots, \alpha_m = 1\} \subset (0, 1]$ such that for any $0 \leq j \leq n$,

$$[A_j]_\alpha = A_{i+1}^{(j)}, \text{ whenever } \alpha \in (\alpha_i, \alpha_{i+1}].$$

It is easy to verify that every $\{A_i^0, A_i^{(1)}, \dots, A_i^{(n)}\}$ ($1 \leq i \leq m$) is a δ -chain. Then there exists $F_i \in K(X)$ such that for any $0 \leq k \leq n$, $d_H(T_K^k(F_i), A_i^{(k)}) < \frac{\varepsilon}{2}$. Take a fuzzy set $F \in \mathbb{F}^1(X)$ with

$$[F]_\alpha = \cup_{l=i+1}^k F_l, \text{ whenever } \alpha \in (\alpha_i, \alpha_{i+1}].$$

Then it can be verified that $\{A_0, A_1, \dots, A_n\} \subset \mathbb{F}^1(X)$ can be ε -shadowed by F . Therefore, $(\mathbb{F}^1(X), T_F|_{\mathbb{F}^1(X)})$ has finite shadowing by applying Lemma 1.2 and Lemma 3.1. \square

Remark 3.11. As $\mathbb{F}^1(X)$ is not compact, we do not know whether the shadowing of (X, T) implies that $(\mathbb{F}^1(X), T_F|_{\mathbb{F}^1(X)})$ has shadowing.

Definition 3.12. [4] A dynamical system (X, T) has *h-shadowing* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for every finite δ -pseudo-orbit $\Gamma = \{x_0, x_1, \dots, x_n\}$, there exists $z \in X$ such that $d(T^i(z), x_i) < \varepsilon$ for any $0 \leq i \leq n$ and $T^n(x) = x_n$.

Lemma 3.13. [10, Lemma 4.4] *Let (X, T) be a dynamical system and Y be a dense invariant subset of X . If $(Y, T|_Y)$ has h-shadowing, then (X, T) has h-shadowing.*

Theorem 3.14. *Let (X, T) be a dynamical system. Then, the following statements are equivalent:*

- (1) (X, T) has h-shadowing;
- (2) $(K(X), T_K)$ has h-shadowing;
- (3) $(\mathbb{F}^1(X), T_F|_{\mathbb{F}^1(X)})$ has h-shadowing.

Proof. Similarly to the proof of Theorem 3.10, applying Lemma 3.13 and Lemma 1.4, it is not difficult to verify that this is true. \square

4. Li-Yorke Sensitivity for Zadeh's Extension

In [6], Blanchard et al. introduced the concept of spatiotemporal chaos. A dynamical system (X, T) is called *spatiotemporally chaotic* (also called *chaotic dependence on initial conditions* in [28]) if any neighbourhood of any $x \in X$ contains a point y such that (x, y) is a Li-Yorke pair, i.e., $\liminf_{n \rightarrow \infty} d(T^n(x), T^n(y)) = 0$ and $\limsup_{n \rightarrow \infty} d(T^n(x), T^n(y)) > 0$. Later, Akin and Kolyada [1] introduced the concept of Li-Yorke sensitivity which links the Li-Yorke chaos with the notion of sensitivity and proved that the weakly mixing property implies Li-Yorke sensitivity. Recall [1] that a dynamical system (X, T) is *Li-Yorke sensitive* if there exists $\varepsilon > 0$ (Li-Yorke sensitive constant) such that any neighbourhood of any $x \in X$ contains a point y satisfying

$$\liminf_{n \rightarrow \infty} d(T^n(x), T^n(y)) = 0 \text{ and } \limsup_{n \rightarrow \infty} d(T^n(x), T^n(y)) > \varepsilon.$$

Recently, We [37] studied the sensitivity of $(K(X), T_K)$ in Furstenberg families and proved that \mathcal{F} -sensitivity of $(K(X), T_K)$ implies that of (X, T) , and the converse is also true if the Furstenberg family \mathcal{F} is a filter. In [28, Proposition 2.6], Sharma and Nagar proved that (X, T) is spatiotemporally chaotic provided that $(K(X), T_K)$ is Li-Yorke sensitive. Similarly to this, here we shall show that (X, T) is spatiotemporally chaotic if its Zadeh's extension $(\mathbb{F}_0(X), T_F)$ is Li-Yorke sensitive (see Theorem 4.1).

Theorem 4.1. *Let (X, T) be a dynamical system. If $(\mathbb{F}_0(X), T_F)$ is Li-Yorke sensitive, then (X, T) is spatiotemporally chaotic.*

Proof. Let $\varepsilon > 0$ be a Li-Yorke sensitive constant of T_F . For any $x \in X$ and any $\delta > 0$, it suffices to prove that there exists $z \in B(x, \delta)$ such that

$$\liminf_{n \rightarrow \infty} d(T^n(x), T^n(z)) = 0 \text{ and } \limsup_{n \rightarrow \infty} d(T^n(x), T^n(z)) > 0.$$

Noting that $\chi_{\{x\}} \in \mathbb{F}_0(X)$, as T_F is Li-Yorke sensitive, there exists $A \in B(\chi_{\{x\}}, \delta)$ such that

$$\liminf_{n \rightarrow \infty} d_\infty(T_F^n(\chi_{\{x\}}), T_F^n(A)) = 0, \quad (3)$$

and

$$\limsup_{n \rightarrow \infty} d_\infty(T_F^n(\chi_{\{x\}}), T_F^n(A)) > \varepsilon. \quad (4)$$

Applying (3) implies that there exists $n_1 \in \mathbb{N}$ such that for any $\lambda \in (0, 1]$,

$$d_H(T^{n_1}(\{x\}), T^{n_1}([A]_\lambda)) = \sup\{d(T^{n_1}(x), T^{n_1}(y)) : y \in [A]_\lambda\} < \frac{1}{2}.$$

Meanwhile, applying (4) implies that there exist $m_1 \in \mathbb{N}$, $\lambda \in (0, 1]$ and $x_1 \in [A]_\lambda$ such that

$$d(T^{m_1}(x), T^{m_1}(x_1)) > \varepsilon.$$

Clearly, $d(x_1, x) < \delta$. It follows from (3) that $\liminf_{n \rightarrow \infty} d(T^n(x), T^n(x_1)) = 0$. If (x, x_1) is a Li-Yorke pair, we are done. Otherwise, $\lim_{n \rightarrow \infty} d(T^n(x), T^n(x_1)) = 0$. Then, take $0 < \delta_1 < \frac{1}{2}$ satisfying the following conditions:

- (a₁) $\overline{B(x_1, \delta_1)} \subset B(x, \delta)$;
- (b₁) for any $y \in B(x_1, \delta_1)$, $d(T^{m_1}(x), T^{m_1}(y)) > \varepsilon$;
- (c₁) for any $y \in B(x_1, \delta_1)$, $d(T^{n_1}(x), T^{n_1}(y)) < \frac{1}{2}$.

As T_F is Li-Yorke sensitive, there exists $A_1 \in B(\chi_{\{x_1\}}, \delta_1)$ such that

$$\liminf_{n \rightarrow \infty} d_\infty(T_F^n(\chi_{\{x_1\}}), T_F^n(A_1)) = 0,$$

and

$$\limsup_{n \rightarrow \infty} d_\infty(T_F^n(\chi_{\{x_1\}}), T_F^n(A_1)) > \varepsilon.$$

This, together with $\lim_{n \rightarrow \infty} d(T^n(x), T^n(x_1)) = 0$, implies that

$$\liminf_{n \rightarrow \infty} d_\infty(T_F^n(\chi_{\{x\}}), T_F^n(A_1)) = 0, \quad (5)$$

and

$$\limsup_{n \rightarrow \infty} d_\infty(T_F^n(\chi_{\{x\}}), T_F^n(A_1)) > \varepsilon. \quad (6)$$

Similarly, there exist $x_2 \in B(x_1, \delta_1)$, $n_2 > n_1$ and $m_2 > m_1$ such that

$$d(T^{n_2}(x), T^{n_2}(x_2)) < \frac{1}{4},$$

$$d(T^{m_2}(x), T^{m_2}(x_2)) > \varepsilon,$$

and

$$\liminf_{n \rightarrow \infty} d(T^n(x), T^n(x_2)) = 0.$$

If (x, x_2) is a Li-Yorke pair, we are done. Otherwise, $\lim_{n \rightarrow \infty} d(T^n(x), T^n(x_2)) = 0$. Then, take $0 < \delta_2 < \frac{1}{2^2}$ satisfying the following conditions:

- (a₂) $\overline{B(x_2, \delta_2)} \subset B(x_1, \delta_1)$;
- (b₂) for any $y \in B(x_2, \delta_2)$, $d(T^{m_2}(x), T^{m_2}(y)) > \varepsilon$;
- (c₂) for any $y \in B(x_2, \delta_2)$, $d(T^{n_2}(x), T^{n_2}(y)) < \frac{1}{2^2}$.

Proceeding inductively, we either get a point $x_n \in B(x, \delta)$ such that (x, x_n) is a Li-Yorke pair or we get $\{x_k\}_{k=1}^{\infty} \subset X$, $n_1 < n_2 < \dots < n_k < n_{k+1} < \dots$, $m_1 < m_2 < \dots < m_k < m_{k+1} < \dots$ and $0 < \delta_k < \frac{1}{k}$ such that for any $k \in \mathbb{N}$,

- (a_k) $\overline{B(x_k, \delta_k)} \subset B(x_{k-1}, \delta_{k-1})$, where $x_0 = x$, $\delta_0 = \delta$;
- (b_k) for any $y \in B(x_k, \delta_k)$, $d(T^{m_k}(x), T^{m_k}(y)) > \varepsilon$;
- (c_k) for any $y \in B(x_k, \delta_k)$, $d(T^{n_k}(x), T^{n_k}(y)) < \frac{1}{2^k}$.

Choose $z \in \bigcap_{k=1}^{\infty} B(x_k, \delta_k)$. It is no difficult to verify that $\liminf_{n \rightarrow \infty} d(T^n(x), T^n(z)) = 0$ and $\limsup_{n \rightarrow \infty} d(T^n(x), T^n(z)) \geq \varepsilon$. Thus, T is spatiotemporally chaotic. \square

Similarly, it can be verified that the following theorem holds.

Theorem 4.2. *Let (X, T) be a dynamical system. If $(\mathbb{F}_0(X), T_F)$ is Li-Yorke sensitive, then $(K(X), T_K)$ is spatiotemporally chaotic.*

Remark 4.3. We believe that the converses of Theorem 4.1 and 4.2 do not hold. However, we can not obtain some proper examples. Meanwhile, the converse of [28, Proposition 2.6] which claims that the Li-Yorke sensitivity of $(K(X), T_K)$ implies that (X, T) is spatiotemporally chaotic is also open.

5. Conclusion

We firstly obtain a few equivalent characterizations of the chain recurrence, chain transitivity, chain mixing property, shadowing and h-shadowing for Zadehs extension. Then, we prove that that If $(\mathbb{F}_0(X), T_F)$ is Li-Yorke sensitive, then both (X, T) and $(K(X), T_K)$ are spatiotemporally chaotic.

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XINXING WU, SCHOOL OF SCIENCES, SOUTHWEST PETROLEUM UNIVERSITY, CHENGDU, SICHUAN, 610500, PEOPLE'S REPUBLIC OF CHINA
E-mail address: wuxinxing5201314@163.com

LIDONG WANG*, ZHUHAI COLLEGE OF JILIN UNIVERSITY, ZHUHAI, GUANGDONG, 519041, PEOPLES REPUBLIC OF CHINA
E-mail address: wld0707@126.com

JIANHUA LIANG, SCHOOL OF MATHEMATICAL SCIENCES, DALIAN UNIVERSITY OF TECHNOLOGY, LIAONING, DALIAN, 116024, PEOPLE'S REPUBLIC OF CHINA
E-mail address: liangjianhua995@163.com

*CORRESPONDING AUTHOR