FUZZY UNIVERSAL ALGEBRAS ON L-SETS

XIAOWEI WEI, YUELI YUE*

Abstract. This paper attempts to generalize universal algebras on classical sets to L-sets when L is a GL-quantale. Some basic notions of fuzzy universal algebra on an L-set are introduced, such as subalgebra, quotient algebra, homomorphism, congruence, and direct product etc. The properties of them are studied. L-valued power algebra is also introduced and it is shown there is an onto homomorphism from $P(A)/R^+$ to $P(A/R)$ for any congruence R on L-set A.

1. Introduction

Since Zadeh introduced the concept of fuzzy sets [29], many researchers have studied mathematical structures combined with fuzzy set theory, such as fuzzy topology [10], fuzzy rough sets [19, 28] and fuzzy convex structures [22, 23, 27], etc. Universal algebras [15] play important roles in mathematics and computer sciences. Also, many researchers provided a new vision for the development of universal algebra and began to study the theories of fuzzy algebras [4, 5, 6, 7, 9, 12, 17, 18, 21, 25]. For example, Murali studied fuzzy algebra associated with the given classical algebra by using Zadeh's extension principle. Bošnjak etc studied two kinds of algebras of fuzzy sets in [7] when L is a complete residuated lattice. Demirci introduced vague algebraic notions equipped with many valued fuzzy equivalence in [11, 12, 13]. Similar to Demirci's approach, considering the general structural algebraic notions, Bělohlávek and Vychodil [3, 4] studied universal algebras with fuzzy equalities and developed a fuzzy equational logic.

Here, we want to emphasize that the existed fuzzy universal algebraic theories are based on the classical sets. This is to say the underlying sets discussed are classical sets. Recently, Li and Yue in [19] studied fuzzy rough sets on L-sets. One natural question would be what is the theory of fuzzy universal algebraic theories on L-sets? Based on the idea of [19], We will give a basic theory of fuzzy universal algebra on an L-set in this paper.

This paper is organized as follows. In Section 2, we give the necessary lattice-theoretic background, notations, results at the reader's disposal, then we lift the L-valued relation on an L-set to its L-valued power set and study their relationship between them. In Section 3, we introduce the notion of fuzzy universal algebras as generalization of classical universal algebra and some isomorphism theorems. In Section 4, we study the power algebras on L-sets as generalization of Bošnjak's

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algebras of fuzzy sets by using the fuzzy inclusion order on L-valued power set, and show that there is an onto homomorphism from $P(A)/R^+$ to $P(A/R)$.

2. Preliminaries

In this section, we will recall some basic notions of GL-quantale, L-valued relations used in this paper.

Definition 2.1. [16] A quantale is a pair $(L, *)$, where L is a complete lattice with the top element 1 and the bottom element 0, and $*$ is a commutative semigroup operation on L such that

$$\alpha * (\bigvee_{j\in J} \beta_j) = \bigvee_{j\in J} \alpha * \beta_j,$$

for all $\alpha \in L$ and $\{\beta_j \mid j \in J\} \subseteq L$. $(L, *)$ is called unital if there exists an element e such that $e * \alpha = \alpha$ for all $\alpha \in L$.

For a given commutative quantale $(L, *)$, there is a binary operation $\rightarrow: L \times L \rightarrow L$, defined by

$$\alpha \rightarrow \beta = \bigvee\{\gamma \in L \mid \alpha * \gamma \leq \beta\},$$

called the implication (*operation*). Further, $*$ and $\rightarrow$ form an adjoint pair in the sense $\alpha * \gamma \leq \beta \Leftrightarrow \gamma \leq (\alpha \rightarrow \beta)$ for all $\alpha, \beta, \gamma \in L$.

Proposition 2.2. [16, 24] *For a commutative unital quantale $(L, *, e)$, where e is the unit element, the following condition are equivalent:*

(1) $\forall \alpha, \beta \in L$, $\alpha \leq \beta \Rightarrow \alpha = \beta * (\beta \rightarrow \alpha)$;
(2) $\forall \alpha, \beta, \gamma \in L$, $\alpha, \gamma \leq \beta \Rightarrow \gamma * (\beta \rightarrow \alpha) = \alpha * (\beta \rightarrow \gamma)$;
(3) $\forall \alpha, \beta \in L$, $\alpha \leq \beta \Rightarrow \exists\, \gamma \in L, \alpha = \beta * \gamma$;
(4) $\forall \alpha, \beta \in L$, $\alpha \wedge \beta = \alpha * (\alpha \rightarrow \beta)$.

In this case, the unit element e must be the top element 1 in L, i.e., $e = 1$.

A commutative unital quantale $(L, *)$ is said to be divisible if it satisfies one of the conditions $(1) - (4)$ in Proposition 2.2, in this case, $(L, *)$ is also called a GL-quantale.

In this paper, we always assume that $(L, *)$ is a GL-quantale. When $(L, *)$ is a GL-quantale, the underlying complete lattice L must be a complete Heyting algebra and $\alpha * (\beta \wedge \gamma) = (\alpha * \beta) \wedge (\alpha * \gamma)$ holds for $\alpha, \beta, \gamma \in L$.

An L-set A is a map from a set A_0 to L. A_0 is called the domain set of A and the valued $A(x)$ is interpreted as the degree to which the element x belongs to A.

Definition 2.3. [24] Let $A : A_0 \rightarrow L$ and $B : B_0 \rightarrow L$ be two L-sets. $R : A \rightharpoonup B$ is called an L-valued relation if $R : A_0 \times B_0 \rightarrow L$ satisfies $R(x, y) \leq A(x) \wedge B(y)$ for all $x \in A_0, y \in B_0$.

Definition 2.4. [24] Let A be an L-set and $R : A \rightharpoonup A$ be an L-valued relation.

(1) If $A(x) \leq R(x, x)$ for all $x \in A_0$, then R is called reflexive on A.
(2) If $R(x, y) * (A(y) \rightarrow R(y, z)) \leq R(x, z)$ for all $x, y, z \in A_0$, then R is called transitive on A.

(3) If $R(x,y) = R(y,x)$ for all $x, y \in A_0$, then R is called symmetric on A.
(4) If $(R(x,x) = R(x,y) = R(y,x) = R(y,y)) \Rightarrow (x = y)$, then R is called separated on A.

Definition 2.5. [24] Let A be an L-set and R be an L-valued relation on A.

(1) If R is reflexive and transitive, then R is called an L-valued preorder on A, the pair (A, R) is called an L-valued preordered set.
(2) If R is reflexive, symmetric and transitive, then R is called an L-valued equivalence on A, the pair (A, R) is called an L-valued set.
(3) If R is reflexive, separated and transitive, then R is called an L-valued partial order on A, the pair (A, R) is called an L-valued partial ordered set.

From [26], we know the powerset of an L-set $A : A_0 \to L$ is also an L-set $P(A) : P(A)_0 \to L$ given by $P(A)(f, \delta) = \delta$, where $P(A)_0$ is the set of all pairs (f, δ) satisfying $f(x) \le A(x) \wedge \delta$ for all $x \in A_0$. Define $S_A : P(A) \rightharpoonup P(A)$ as follows

$$S_A((f,\delta),(g,\varepsilon)) = \varepsilon \wedge \delta \wedge \bigwedge_{x \in A_0} ((\delta \to f(x)) \to g(x)),$$

for all $(f, \delta), (g, \varepsilon) \in P(A)_0$. Then it is easy to see that S_A is an L-valued partial order on $P(A)$. Furthermore, $S_A((f,\delta),(g,\varepsilon)) = \varepsilon \wedge (\delta * S_{A_0}(f,g))$, where $S_{A_0} : L^{A_0} \times L^{A_0} \to L$ is defined by

$$\forall f, g \in L^{A_0}, S_{A_0}(f,g) = \bigwedge_{x \in A_0} (f(x) \to g(x)).$$

Similar to those in [18], The following lemmas are valid.

Lemma 2.6. *Let $R : A \rightharpoonup B$ be an L-valued relation and define $\sigma(R) : B \rightharpoonup B$ by*

$$\sigma(R)(y_1, y_2) = S_A((R(-,y_1), B(y_1)), (R(-,y_2), B(y_2))).$$

Then $\sigma(R)$ is an L-valued preorder on B.

Lemma 2.7. *Let $R : A \rightharpoonup B$ be an L-valued relation. Then $\sigma(R)$ is the biggest L-valued relation $S : B \rightharpoonup B$ such that $S \bigodot R \le R$, where $S \bigodot R(x,z) = \bigvee_{y \in B_0} R(x,y) * (B(y) \to S(y,z))$.*

Lemma 2.8. *Let R be an L-valued relation on A. Then R is an L-valued preorder on A if and only if $\sigma(R) = R$.*

We slightly modify the definition of L-valued closure operator in [26] as the following definition instead of the original categorical language.

Definition 2.9. [26] Let $P(A)$ be the L-valued power set of A. The map $cl : P(A)_0 \to P(A)_0$ is called an L-valued closure operator on $P(A)$ if it has the following axioms:

(1) $cl : (P(A), S_A) \to (P(A), S_A)$ is an order-preserving map, i.e.,
  (i) $S_A((f,\delta),(g,\varepsilon)) \le S_A(cl(f,\delta), cl(g,\varepsilon))$;
  (ii) $P(A)(f,\delta) = P(A)(cl(f,\delta)) = \delta$ for all $(f,\delta),(g,\varepsilon) \in P(A)_0$;
(2) $(f,\delta) \le cl(f,\delta) = (cl(f),\delta)$ for all $(f,\delta) \in P(A)_0$;
(3) $cl(cl(f,\delta)) = cl(f,\delta)$ for all $(f,\delta) \in P(A)_0$.

Next, we begin to lift L-valued relation on L-set to L-valued power set and study the relationships between them.

Let $R : A \rightharpoonup A$ be an L-valued relation and define $R_l : P(A)_0 \to P(A)_0$ and $R_u : P(A)_0 \to P(A)_0$ by $R_l(f, \delta) = (R_l(f), \delta)$ and $R_u(f, \delta) = (R_u(f), \delta)$, respectively, where $R_l(f)(y) = \bigvee_{x\in A_0} f(x) * (A(x) \to R(y, x))$, and $R_u(f)(y) = \bigvee_{x\in A_0} f(x) * (A(x) \to R(x, y))$. Here, $R_l(f, \delta)$ is usually called the lower set of $(f, \delta))$ and $R_u(f, \delta)$ is called the upper set of $(f, \delta))$.

Theorem 2.10. *If $R : A \rightharpoonup A$ is an L-valued relation on A, then R is an L-valued preorder if and only if $R_u : P(A)_0 \to P(A)_0$ is an L-valued closure operator on $P(A)$.*

Proof. Necessity: Suppose that R is an L-valued preorder on A. Firstly, we show that R_u is order-preserving. For $(f, \delta), (g, \varepsilon) \in P(A)_0$, we have

$$
\begin{aligned}
&S_A(R_u(f, \delta), R_u(g, \varepsilon)) = S_A((R_u(f), \delta), (R_u(g), \varepsilon)) = \varepsilon \wedge (\delta * S_{A_0}(R_u(f), R_u(g)))\\
&= \varepsilon \wedge (\delta * \bigwedge_{y\in A_0} (R_u(f)(y) \to R_u(g)(y)))\\
&= \varepsilon \wedge (\delta * \bigwedge_{y\in A_0} (\bigvee_{x\in A_0} f(x) * (A(x) \to R(x, y)) \to \bigvee_{x\in A_0} g(x) * (A(x) \to R(x, y))))\\
&\geq \varepsilon \wedge (\delta * \bigwedge_{x\in A_0} \bigwedge_{y\in A_0} (f(x) * (A(x) \to R(x, y)) \to g(x) * (A(x) \to R(x, y))))\\
&\geq \varepsilon \wedge (\delta * \bigwedge_{x\in A_0} \bigwedge_{y\in A_0} (f(x) \to g(x))) = \varepsilon \wedge (\delta * \bigwedge_{x\in A_0} (f(x) \to g(x)))\\
&= S_A((f, \delta), (g, \varepsilon)).
\end{aligned}
$$

(cl2) Since R is an L-valued preorder on A, then $A(x) = R(x, x)$ for all $x \in A_0$. And then

$$R_u(f)(y) = \bigvee_{x\in A_0} f(x) * (A(x) \to R(x, y)) \geq f(y) * (A(y) \to R(y, y)) = f(y).$$

This is to say $(f, \delta) \leq (R_u(f), \delta)$, as desired.
(cl3) We want to prove $R_u(R_u(f)) = R_u(f)$ for all $(f, \delta) \in P(A)_0$. By (cl2), we only need to prove $R_u(R_u(f)) \leq R_u(f)$. In fact

$$
\begin{aligned}
R_u(R_u(f))(y) &= \bigvee_{x\in A_0} R_u(f)(x) * (A(x) \to R(x, y))\\
&= \bigvee_{x\in A_0} (\bigvee_{z\in A_0} f(z) * (A(z) \to R(z, x)) * (A(x) \to R(x, y)))\\
&\leq \bigvee_{z\in A_0} f(z) * (\bigvee_{x\in A_0} (A(z) \to (R(z, x) * (A(x) \to R(x, y)))))\\
&\leq \bigvee_{z\in A_0} f(z) * (\bigvee_{x\in A_0} (A(z) \to R(z, y)))\\
&= R_u(f)(y).
\end{aligned}
$$

Therefore, R_u is an L-valued closure operator on $P(A)$.

Sufficiency: Firstly, we show R is a reflexive L-valued relation. Let $x \in A_0$ and define $A_x : A_0 \to L$ by

$$A_x(z) = \begin{cases} A(x), & x = z, \\ 0, & others. \end{cases}$$

Then $(A_x, A(x)) \in P(A)_0$. Hence, by $R_u(A_x) \geq A_x$, it follows that $R_u(A_x)(x) \geq A_x(x)$, i.e., $A(x) * (A(x) \to R(x,x)) \geq A(x)$. Therefore, $A(x) \leq R(x,x)$.

Next, we will prove that R is a transitive L-valued relation. Let $x, y, z \in A_0$. Then we have

$$\begin{aligned} & R_u(R_u(A_x))(y) \\ = & \bigvee_{z \in A_0} R_u(A_x)(z) * (A(z) \to R(z,y)) \\ = & \bigvee_{z \in A_0} \bigvee_{w \in A_0} A_x(w) * (A(w) \to R(w,z)) * (A(z) \to R(z,y)) \\ = & \bigvee_{z \in A_0} A(x) * (A(x) \to R(x,z)) * (A(z) \to R(z,y)) \\ = & \bigvee_{z \in A_0} R(x,z) * (A(z) \to R(z,y)). \end{aligned}$$

On account of $R_u(R_u(A_x))(y) = R_u(A_x)(y) = R(x,y)$, it follows that

$$\bigvee_{z \in A_0} R(x,z) * (A(z) \to R(z,y)) = R(x,y).$$

Therefore, $R(x,z) * (A(z) \to R(z,y)) \leq R(x,y)$. □

Similarly, we have the following Theorem 2.11.

Theorem 2.11. *If $R : A \rightharpoonup A$ is an L-valued relation on A, then R is an L-valued preorder if and only if $R_l : P(A)_0 \to P(A)_0$ is an L-valued closure operator on $P(A)$.*

From [8, 14], for an L-valued relation $R : A \rightharpoonup A$, we can define $R^{\rightarrow} : P(A) \rightharpoonup P(A)$, $R^{\leftarrow} : P(A) \rightharpoonup P(A)$ and $R^{+} : P(A) \rightharpoonup P(A)$, respectively, as follows:

$$R^{\rightarrow}((f,\delta),(g,\varepsilon)) = S_A((f,\delta),(R_l(g),\varepsilon)),$$
$$R^{\leftarrow}((f,\delta),(g,\varepsilon)) = S_A((g,\varepsilon),(R_u(f),\delta))$$

and

$$R^{+}((f,\delta),(g,\varepsilon)) = R^{\rightarrow}((f,\delta),(g,\varepsilon)) \wedge R^{\leftarrow}((f,\delta),(g,\varepsilon)),$$

for all $(f,\delta),(g,\varepsilon) \in P(A)_0$.

Theorem 2.12. *The following statements are equivalent:*

(1) $R : A \rightharpoonup A$ is an L-valued preorder on A;
(2) $R^{\rightarrow}$ is an L-valued preorder on $P(A)$;
(3) $R^{\leftarrow}$ is an L-valued preorder on $P(A)$.

Proof. We only prove the (1) $\Leftrightarrow$ (2), and (1) $\Leftrightarrow$ (3) can be proved similarly.

(1)$\Rightarrow$(2): Since $R : A \rightharpoonup A$ is an L-valued preorder, it follows that $f \le R_l(f)$ and $R_l(f) = R_l(R_l(f))$ for all $(f, \delta) \in P(A)_0$ from Theorem 2.11. Then

$$R^{\rightarrow}((f,\delta),(f,\delta)) = S_A((f,\delta),(R_l(f),\delta)) = \delta = P(A)(f,\delta).$$

For $(f,\delta),(g,\varepsilon),(h,\eta) \in P(A)_0$, we have

$$\begin{aligned}
&R^{\rightarrow}((f,\delta),(g,\varepsilon)) * (\varepsilon \to R^{\rightarrow}((g,\varepsilon),(h,\eta)))\\
&= S_A((f,\delta),(R_l(g),\varepsilon)) * (\varepsilon \to S_A((g,\varepsilon),(R_l(h),\eta)))\\
&\le S_A((f,\delta),(R_l(g),\varepsilon)) * (\varepsilon \to S_A((R_l(g),\varepsilon),(R_l(R_l(h)),\eta)))\\
&= S_A((f,\delta),(R_l(g),\varepsilon)) * (\varepsilon \to S_A((R_l(g),\varepsilon),(R_l(h),\eta)))\\
&\le S_A((f,\delta),(R_l(h),\eta))\\
&= R^{\rightarrow}((f,\delta),(h,\eta)).
\end{aligned}$$

Hence, $R^{\rightarrow}$ is an L-valued preorder on $P(A)$.

(2)$\Rightarrow$(1): Let $x, y, z \in A_0$. Then $(A_x, A(x))$, $(A_y, A(y))$ and $(A_z, A(z)) \in P(A)_0$. It is easy to check that

$$R(x,y) = R^{\rightarrow}((A_x, A(x)),(A_y, A(y))).$$

Since $R^{\rightarrow}$ is an L-valued preorder, so $R : A \rightharpoonup A$ is an L-valued preorder on A. □

Similar to Lemma 2.8, it is easy to show R is an L-valued preorder on A if and only if $R^{\rightarrow} = \sigma(R^{\rightarrow})$.

Remark 2.13. For all $(f,\delta),(g,\eta) \in P(A)_0$, on one hand, we have

$$\begin{aligned}
&\sigma(R^{\rightarrow})((f,\delta),(g,\eta))\\
&= S_{P(A)}((R^{\rightarrow}(\,\cdot\,,(f,\delta)),\delta),(R^{\rightarrow}(\,\cdot\,,(g,\eta)),\eta))\\
&= \delta \wedge \eta \wedge \bigwedge_{(h,r)\in P(A)_0} [(\delta \to S_A((h,r),(R_l(f),\delta))) \to S_A((h,r),(R_l(g),\eta))]\\
&\le \delta \wedge \eta \wedge [(\delta \to S_A((R_l(f),\delta),(R_l(f),\delta)) \to S_A((R_l(f),\delta),(R_l(g),\eta)]\\
&= S_A((R_l(f),\delta),(R_l(g),\eta)).
\end{aligned}$$

On the other hand, for all $(h,r) \in P(A)_0$, we know

$$S_A((R_l(f),\delta),(R_l(g),\eta)) \le (\delta \to S_A((h,r),(R_l(f),\delta))) \to S_A((h,r),(R_l(g),\eta)).$$

Therefore,

$$\sigma(R^{\rightarrow})((f,\delta),(g,\eta)) = S_A((R_l(f),\delta),(R_l(g),\eta)).$$

3. Universal algebra on L-sets

In this section, we will introduce the notion of fuzzy universal algebra on an L-set and study the properties of it. For two L-sets A and B, $\alpha : A \to B$ is called a degree-preserving map if $\alpha : A_0 \to B_0$ is a map and satisfies $A(x) = B(\alpha(x))$ for all $x \in A_0$. For an L-set A and a nonnegative integer n, a n-ary mapping is any degree-preserving mapping $f : A^n \to A$, where $A^n : A_0^n \to L$ is defined by

$$A^n(x_1, x_2, \cdots, x_n) = A(x_1) = \cdots = A(x_n),$$

and $A_0^n = \{(x_1, x_2, \cdots, x_n) \in (A_0)^n \mid A(x_1) = A(x_2) = \cdots = A(x_n)\}$. A n-ary mapping $f : A^n \to A$ is also called a n-ary operation on A.

Definition 3.1. Let $A : A_0 \to L$ be an L-set with $A(x) \neq 0$ for all $x \in A_0$ and F consist of some operations on A indexed by the type τ. The pair $\mathcal{A} =< A, F >$ is called a fuzzy universal algebra on A.

When $A = 1_{A_0}$, where $1_{A_0} : A_0 \to L$ is defined by $1_{A_0}(x) = 1$ for all $x \in A_0$. Then, fuzzy universal algebra $< A, F >$ on A will reduce to the classical universal algebra on A_0. Hence, fuzzy universal algebra on an L-set could be seen as a generalization of universal algebra on classical set. The theories of fuzzy universal algebras on L-sets would be meaningful and important.

Example 3.2. Let $(L, *, e) = ([0, 1], \wedge, 1)$ and $A : A_0 \to [0, 1]$ be defined by $A(x) = \frac{1}{2}$ for all $x \in A_0$. We define the binary, unary and nullary operation in the following way:
binary operation: $\cdot : A \times A \to L$ is given by $\cdot(x_1, x_2) = x_1 \cdot x_2$;
unary operation: $-1 : A \to A$ is defined by $-1(x) = x^{-1}$;
nullary operation: $1 : A^0 \to A$, $1(\emptyset) = 1 \in A_0$, where A^0 is the $[0, 1]$-set $A^0 : \{\emptyset\} \to [0, 1]$ defined by $A^0(\emptyset) = \frac{1}{2}$.
If $\cdot, -1$ and 1 also satisfy the following conditions:
(i) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, (ii) $x \cdot 1 = x = 1 \cdot x$, (iii) $x \cdot x^{-1} = x^{-1} \cdot x = 1$.
Then, $(A, \cdot, -1, 1)$ could be understood as a group on L-set A.

Definition 3.3. Let $\mathcal{A} =< A, F_A >$ and $\mathcal{B} =< B, F_B >$ be the same type of the fuzzy universal algebras. If a degree-preserving map $\varphi : A \to B$ satisfies

$$\varphi(f(x_1, x_2, \cdots, x_n)) = f(\varphi(x_1), \varphi(x_2), \cdots, \varphi(x_n)),$$

for all $(x_1, x_2, \cdots, x_n) \in A_0^n$, $f \in F_A$, then φ is called a homomorphism from $\mathcal{A}$ to $\mathcal{B}$.

If the homomorphism φ is injective, then φ is called an embedding. And if the homomorphism φ is a bijection, then φ is called an isomorphism.

As is well-known, algebra is often considered as the study of those properties of algebras which are invariant under isomorphism, and such properties are called algebraic properties. Thus from an algebraic point of view, isomorphism algebras can be regarded as equal or the same, as they would have the same algebraic structures.

Definition 3.4. Let R be an L-valued relation on L-set A and $\mathcal{A} =< A, F >$ be a fuzzy universal algebra. R is called compatible with $\mathcal{A}$ if

$$\begin{aligned} &R(x_1, y_1) * (A(x_2) \to R(x_2, y_2)) * \cdots * (A(x_n) \to R(x_n, y_n)) \\ &\leq R(\phi(x_1, x_2, \cdots, x_n), \phi(y_1, y_2, \cdots, y_n)), \end{aligned}$$

for all $\phi \in F$ and $(x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n) \in A_0^n$.

A compatible L-valued equivalence on A is called a congruence.

Remark 3.5. When $A = 1_{A_0}$ and $\widetilde{R}$ is the characteristic function of $R \subseteq A_0 \times A_0$, which is defined by

$$\widetilde{R}(x,y) = \begin{cases} 1, & (x,y) \in R, \\ 0, & others, \end{cases}$$

then $\widetilde{R}$ is an L-valued relation on A. So $\widetilde{R}$ is a congruence with $< A, F >$ is equivalent to R is a congruence on $< A_0, \mathcal{F} >$. So Definition 3.4 can be seen as a generalization of the concept of congruence in classical universal algebra.

There are several important methods of constructing new algebras from given ones. Three of the most fundamental are the formation of subalgebras, quotient algebras and direct product algebras. These will occupy the next part of this section.

Definition 3.6. Let $\mathcal{A} =< A, F_A >$ and $\mathcal{B} =< B, F_B >$ be the same type algebras, if $B = A \mid B_0$ (*A restriction on* B_0), and for all $f^B \in F_B$, there is a $f^A \in F_A$ such that $f^B = f^A \mid B_0$, then $\mathcal{B}$ is called a subalgebra of $\mathcal{A}$.

Remarks 3.7. (1) If $\varphi : \mathcal{A} \to \mathcal{B}$ and $\alpha : \mathcal{B} \to \mathcal{C}$ are homomorphisms, then $\alpha \circ \varphi$ is a homomorphism from $\mathcal{A}$ to $\mathcal{C}$.

(2) If $\mathcal{A} =< A, F_A >$ and $\mathcal{C}$ are two fuzzy universal algebras. $\mathcal{B} =< B, F_B >$ is a subalgebra of $\mathcal{A}$, R is a congruence on A, $\varphi : \mathcal{A} \to \mathcal{C}$ is a homomorphism, then R_B (R restriction on B_0) is a congruence on B and $\varphi_B : \mathcal{B} \to \mathcal{C}$ is a homomorphism (φ_B is the φ restriction on B_0).

Similar to [20], now we construct fuzzy quotient algebra. Let $\mathcal{A} =< A, F >$ be a fuzzy universal algebra and R be a congruence on A. Define $A/R : (A/R)_0 \to L$ by $(A/R)(x/R) = A(x)$, where $(A/R)_0 = \{x/R \mid x \in A_0\}$ and $x/R = \{\ y \in A_0 \mid R(x,x) = R(y,x) = R(x,y) = R(y,y)\ \}$. For $f \in F$, define $f/R : (A/R)^n \to A/R$ by $(f/R)(x_1/R, \cdots, x_n/R) = f(x_1, \cdots, x_n)/R$. It is routine to check the definition of A/R and f/R are meaningful. $< A/R, F/R >$ is called a fuzzy quotient algebra.

We can also construct the other kind of fuzzy universal algebra in the following way: Define $\overline{A} : (\overline{A})_0 \to L$ by $\overline{A}([x]_R) = A(x)$, where $(\overline{A})_0 = \{[x]_R : A_0 \to L \mid x \in A_0\}$ and $[x]_R(y) = R(x,y)$. For $\phi \in F$, define $\overline{\phi} : (\overline{A})^n \to \overline{A}$ by $\overline{\phi}([x_1]_R, \cdots, [x_n]_R) = [\phi(x_1, \cdots, x_n)]_R$. Then $< \overline{A}, \overline{F} >$ is a fuzzy universal algebra.

There is an isomorphism h from $< \overline{A}, \overline{F} >$ to $< A/R, F/R >$ with $h([x]_R) = x/R$.

Definition 3.8. Let $\mathcal{A} =< A, F_A >$ and $\mathcal{B} =< B, F_B >$ be two fuzzy universal algebras, and $\alpha : \mathcal{A} \to \mathcal{B}$ be a homomorphism. The kernel of α, written by ker(α): $A \rightharpoonup A$, is defined by

$$ker(\alpha)(x,y) = \begin{cases} A(x), & \alpha(x) = \alpha(y), \\ 0, & \alpha(x) \neq \alpha(y), \end{cases}$$

for all $(x,y) \in A_0^2$.

From the above definition, it is easy to see $ker(\alpha)$ is a congruence on A. In order to understand fuzzy universal algebras more well, we will talk about some isomorphism theorems and the readers can prove them routinely.

Theorem 3.9. *(First Isomorphism Theorem) Suppose $\mathcal{A}$ and $\mathcal{B}$ are the same type of fuzzy universal algebras. $\alpha : \mathcal{A} \to \mathcal{B}$ is a homomorphism onto $\mathcal{B}$, and υ is a natural homomorphism from $\mathcal{A}$ to $\mathcal{A}/ker(\alpha)$, then there is an isomorphism $\beta : \mathcal{A}/ker(\alpha) \to \mathcal{B}$ defined by $\alpha = \beta \circ \upsilon$.*

Let $\mathcal{A}$ be a fuzzy universal algebra. θ, ϕ are congruences on A with $\theta \leq \phi$. Define $A/\theta : (A/\theta)_0 \to L$ by $(A/\theta)(x/\theta) = A(x)$ where $(A/\theta)_0 = \{x/\theta \mid x \in A_0\}$, and define $\phi/\theta : (A/\theta) \rightharpoonup (A/\theta)$ by $(\phi/\theta)(x/\theta, y/\theta) = \phi(x, y)$ for all $x/\theta, y/\theta \in (A/\theta)_0$. It is easy to show ϕ/θ is well-meaning and is a congruence on A/θ.

Theorem 3.10. *(Second Isomorphism Theorem) Let $\mathcal{A}$ be a fuzzy universal algebra. θ, ϕ is a congruence on A with $\theta \leq \phi$. Then the map $\alpha : (\mathcal{A}/\theta)/(\phi/\theta) \to \mathcal{A}/\phi$ defined by $\alpha((x/\theta)/(\phi/\theta)) = x/\phi$ is an isomorphism from $(\mathcal{A}/\theta)/(\phi/\theta)$ to $\mathcal{A}/\phi$.*

Let $\mathcal{B}$ be a fuzzy subalgebra of $\mathcal{A}$ and θ be a congruence on A. Define $B^\theta : (B^\theta)_0 \to L$ and $(B^\theta)_0 = \{x \in A_0 \mid B_0 \cap x/\theta \neq \varnothing\}$ by $(B^\theta)(x) = \bigvee_{y \in (x/\theta \bigcap B_0)} B(y)$ where $B^\theta = A|(B^\theta)_0$. We have that $< B^\theta, F >$ is a subalgebra of $\mathcal{B}$ and θ_{B^θ} is a congruence on B^θ.

Theorem 3.11. *(Third Isomorphism Theorem) Let $\mathcal{B}$ be a fuzzy subalgebra of $\mathcal{A}$ and θ be a congruence on A, then $B/\theta_B \cong B^\theta/\theta_{B^\theta}$.*

Then, we begin to talk about the direct product of fuzzy universal algebras.

Definition 3.12. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the same type of fuzzy universal algebras. Define the product $A_1 \times A_2$ to be the fuzzy universal algebra with

$$f^{A_1 \times A_2}((x_1, y_1), \cdots, (x_n, y_n)) = (f^{A_1}(x_1, \cdots, x_n), f^{A_2}(y_1, \cdots, y_n)),$$

where $(A_1 \times A_2) : (A_1 \times A_2)_0 \to L$ is given by $(A_1 \times A_2)(x, y) = A_1(x) = A_2(y)$ and $(A_1 \times A_2)_0 = \{(x, y) \mid A_1(x) = A_2(y)\}$.

Let map $\pi_i : A_1 \times A_2 \to A_i$ for $i = 1, 2$ and $\pi_i(x_1, x_2) = x_i$, called the projection map being the ith coordinate of $A_1 \times A_2$. Define $\Delta_{A_1 \times A_2} : (A_1 \times A_2)_0 \to L$ as follows:

$$\Delta_{A_1 \times A_2}(x, y) = \begin{cases} A_1(x), & x = y, \\ 0, & x \neq y, \end{cases}$$

for all $(x, y) \in (A_1 \times A_2)_0$.

Definition 3.13. Let $\mathcal{A}$ be a fuzzy universal algebra. A congruence R on A is called a factor congruence if there is a congruence R^* on A such that

$$R \wedge R^* = \Delta_{A \times A} \qquad (x/R) \cap (y/R^*) \neq \varnothing$$

for all $x, y \in A_0$.

Theorem 3.14. *Let $\mathcal{A}$ be a fuzzy universal algebra. If R and R^* are a pair of factor congruences on A, then $\mathcal{A} \cong (\mathcal{A}/R) \times (\mathcal{A}/R^*)$.*

4. L-valued power algebra

In this section, we study the L-valued power algebra. let $\mathcal{A} =< A, F >$ be a fuzzy universal algebra. For $\theta \in F$, $\theta : A^n \to A$ can be lifted to $\theta^+ : P(A)^n \to P(A)$ defined by

$$\theta^+((f_1, \delta), \cdots, (f_n, \delta)) = (\theta^{\to}(\bigotimes_{i=1}^{n} f_i), \delta),$$

where $\bigotimes_{i=1}^{n} f_i : A_0^n \to L$ is

$$(\bigotimes_{i=1}^{n} f_i)(x_1, \cdots, x_n) = \bigotimes_{i=1}^{n} f_i(x_i) = f_1(x_1) * f_2(x_2) * \cdots * f_n(x_n),$$

and $\theta^{\to}(\bigotimes_{i=1}^{n} f_i)(x) = \bigvee_{\theta(x_1,\cdots,x_n)=x} \bigotimes_{i=1}^{n} f_i(x_i)$. $P(\mathcal{A}) =< P(A), \{\theta^+ \mid \theta \in F\} >$ is called an L-valued power algebra of $< A, F >$.

Since the element in the underlying set of A^n is the point with the same height for all coordinates, we have to modify the transitivity of L-valued relation in order to study the relationship between fuzzy universal algebra and its power algebra more well. An L-valued relation is called weakly transitive on A if $R(x, y) * (A(y) \to R(y, z)) \leq R(x, z)$ for all $x, y, z \in A_0$ with $A(x) = A(y) = A(z)$. Then the L-valued preorder, L-valued partial ordered and L-valued equivalence on A reduce to the corresponding weakly L-valued preorder, weakly L-valued partial ordered and weakly L-valued equivalence on A. In this part, we assume that the transitivity is the weak transitivity. Similarly, the upper set and lower set in Section 2 can be modified by the following formulas:

$$\bar{R}_l(f)(y) = \bigvee_{x \in A_y} f(x) * (A(x) \to R(y, x))$$

and

$$\bar{R}_u(f)(y) = \bigvee_{x \in A_y} f(x) * (A(x) \to R(x, y)),$$

where $A_y = \{x \in A_0 \mid A(x) = A(y)\}$.

Correspondingly, for an L-valued relation $R : A \rightharpoonup A$, $R^{\to} : P(A) \rightharpoonup P(A)$, $R^{\leftarrow} : P(A) \rightharpoonup P(A)$ and $R^+ : P(A) \rightharpoonup P(A)$ are given as follows:

$$R^{\to}((f, \delta), (g, \varepsilon)) = S_A((f, \delta), (\bar{R}_l(g), \varepsilon)),$$
$$R^{\leftarrow}((f, \delta), (g, \varepsilon)) = S_A((g, \varepsilon), (\bar{R}_u(f), \delta))$$

and

$$R^+((f, \delta), (g, \varepsilon)) = R^{\to}((f, \delta), (g, \varepsilon)) \wedge R^{\leftarrow}((f, \delta), (g, \varepsilon)).$$

In this case, Theorem 2.10, Theorem 2.11, Theorem 2.12 and Remark 2.13 are still valid.

Theorem 4.1. *Let $\mathcal{A} =< A, F >$ be a fuzzy universal algebra and R be a weakly L-valued preorder on A. Then the following statements are equivalent:*

(1) R is compatible on A;
(2) $R^{\rightarrow}$ is compatible on $P(A)$;
(3) $R^{\leftarrow}$ is compatible on $P(A)$.

Proof. We only prove (1)⇔(2), and (1)⇔(3) can be proved similarly.

(1)⇒(2): Let $\theta \in F$, $(f_i, \delta), (g_i, \delta) \in P(A)_0$. On account of

$$\theta^{\rightarrow}(\bigotimes_{i=1}^{n} \bar{R}_l(g_i))(x) = \bigvee_{\theta(x_1,\cdots,x_n)=x} \bigotimes_{i=1}^{n} \bar{R}_l(g_i)(x_i)$$

$$= \bigvee_{\theta(x_1,\cdots,x_n)=x} \bigotimes_{i=1}^{n}(\bigvee_{y_i\in A_{x_i}} g_i(y_i) * (A(y_i) \rightarrow R(x_i, y_i)))$$

$$= \bigvee_{\theta(x_1,\cdots,x_n)=x} \bigvee_{(y_1,\cdots,y_n)\in A_0^n, y_i\in A_{x_i}} \bigotimes_{i=1}^{n} g_i(y_i) * \bigotimes_{i=1}^{n}(A(y_i) \rightarrow R(x_i, y_i))$$

$$\leq \bigvee_{\theta(x_1,\cdots,x_n)=x} \bigvee_{(y_1,\cdots,y_n)\in A_0^n, y_i\in A_{x_i}} \bigotimes_{i=1}^{n} g_i(y_i) * (A(y_1) \rightarrow (R(x_1, y_1)*$$

$$\bigotimes_{i=2}^{n}(A(x_i) \rightarrow R(x_i, y_i))))$$

$$\leq \bigvee_{\theta(x_1,\cdots,x_n)=x} \bigvee_{(y_1,\cdots,y_n)\in A_0^n, y_i\in A_{x_i}} \bigotimes_{i=1}^{n} g_i(y_i)*$$

$$(A(y_1) \rightarrow R(\theta(x_1, \cdots, x_n), \theta(y_1, \cdots, y_n))),$$

and

$$\bar{R}_l(\theta^{\rightarrow}(\bigotimes_{i=1}^{n} g_i))(x) = \bigvee_{y\in A_x} \theta^{\rightarrow}(\bigotimes_{i=1}^{n} g_i)(y) * (A(y) \rightarrow R(x, y))$$

$$= \bigvee_{y\in A_x} \bigvee_{\theta(y_1,\cdots,y_n)=y} \bigotimes_{i=1}^{n} g_i(y_i) * (A(y) \rightarrow R(x, y)),$$

we have $\bar{R}_l(\theta^{\rightarrow}(\bigotimes_{i=1}^{n} g_i)) \geq \theta^{\rightarrow}(\bigotimes_{i=1}^{n} \bar{R}_l(g_i))$. And then

$$R^{\rightarrow}(\theta^{+}((f_1, \delta), \cdots, (f_n, \delta)), \theta^{+}((g_1, \delta), \cdots, (g_n, \delta)))$$

$$= R^{\rightarrow}((\theta^{\rightarrow}(\bigotimes_{i=1}^{n} f_i), \delta), (\theta^{\rightarrow}(\bigotimes_{i=1}^{n} g_i), \delta))$$

$$= S_A((\theta^{\rightarrow}(\bigotimes_{i=1}^{n} f_i), \delta), (\bar{R}_l(\theta^{\rightarrow}(\bigotimes_{i=1}^{n} g_i)), \delta))$$

$$\geq S_A((\theta^{\rightarrow}(\bigotimes_{i=1}^{n} f_i), \delta), (\theta^{\rightarrow}(\bigotimes_{i=1}^{n} \bar{R}_l(g_i)), \delta)).$$

It follows that

$$R^{\rightarrow}((f_1,\delta),(g_1,\delta))*(\delta\rightarrow R^{\rightarrow}((f_2,\delta),(g_2,\delta)))*\cdots*(\delta\rightarrow R^{\rightarrow}((f_n,\delta),(g_n,\delta)))$$
$$=S_A((f_1,\delta),(\bar{R}_l(g_1),\delta))*(\delta\rightarrow S_A((f_2,\delta),(\bar{R}_l(g_2),\delta)))*\cdots*(\delta\rightarrow S_A((f_n,\delta),(\bar{R}_l(g_n),\delta)))$$
$$=\delta*S_{A_0}(f_1,\bar{R}_l(g_l))*(\delta\rightarrow\delta*S_{A_0}(f_2,\bar{R}_l(g_2)))*\cdots*(\delta\rightarrow\delta*S_{A_0}(f_n,\bar{R}_l(g_n)))$$
$$\leq\delta*\bigotimes_{i=1}^{n}S_{A_0}(f_i,\bar{R}_l(g_i))$$
$$\leq S_A((\theta^{\rightarrow}(\bigotimes_{i=1}^{n}f_i),\delta),(\theta^{\rightarrow}(\bigotimes_{i=1}^{n}\bar{R}_l(g_i)),\delta)).$$

So $R^{\rightarrow}$ is compatible on $P(A)$.

(2)⇒(1): Let $\theta\in F$, $(x_1,\cdots,x_n),(y_1,\cdots,y_n)\in A_0^n$ and $A(x_i)=A(y_j),\ i,j=1,\cdots,n$. Then

$$R(x_1,y_1)*(A(x_2)\rightarrow R(x_2,y_2))*\cdots*(A(x_n)\rightarrow R(x_n,y_n))$$
$$=R^{\rightarrow}((A_{x_1},A(x_1)),(A_{y_1},A(y_1)))*\cdots\cdots*(A(x_n)\rightarrow R^{\rightarrow}((A_{x_n},A(x_n)),(A_{y_n},A(y_n))))$$
$$\leq R^{\rightarrow}(\theta^{+}(((A_{x_1}),A(x_1)),\cdots,(A_{x_n},A(x_n))),\theta^{+}(((A_{y_1}),A(y_1)),\cdots,((A_{y_n}),A(y_n))))$$
$$=R^{\rightarrow}((\theta^{\rightarrow}(\bigotimes_{i=1}^{n}A_{x_i}),A(x_i)),(\theta^{\rightarrow}(\bigotimes_{i=1}^{n}A_{y_i}),A(y_i)))$$
$$=S_A((\theta^{\rightarrow}(\bigotimes_{i=1}^{n}A_{x_i}),A(x_i)),(\bar{R}_l(\theta^{\rightarrow}(\bigotimes_{i=1}^{n}A_{y_i}),A(y_i)))$$
$$\leq A(x_i)$$
$$=S_A((A_{x_i},A(x_i)),(A_{y_i},A(y_i)))$$
$$\leq S_A((A_{x_i},A(x_i)),(\bar{R}_l(A_{y_i}),A(y_i)))$$
$$=S_A((A_{\theta(x_1,\cdots,x_n)},A(\theta(x_1,\cdots,x_n))),(\bar{R}_l(A_{\theta(y_1,\cdots,y_n)}),A(\theta(y_1,\cdots,y_n)))$$
$$=R(\theta(x_1,\cdots,x_n),\theta(y_1,\cdots,y_n)).$$

Hence, R is compatible on A. □

Definition 4.2. Let $\mathcal{A}$ be a fuzzy universal algebra on A and R be an L-valued relation on A.

(1) R is called good on A if $\varepsilon(R)$ is a congruence on A, where $\varepsilon(R)=\sigma(R)\wedge\sigma(R)^{op}$;
(2) R is called Hoare good on A if $R^{\rightarrow}$ is good on $P(\mathcal{A})$;
(3) R is called Smyth good on A if $R^{\leftarrow}$ is good on $P(\mathcal{A})$.

Theorem 4.3. *Let $\mathcal{A}$ be a fuzzy universal algebra and R be a weakly L-valued preorder on A, then the following statements are equivalent when L is a completely Heyting algebra, i.e., $* = \wedge$:*

(1) R is compatible on A;
(2) R is Smyth good on A;
(3) R is Hoare good on A.

Proof. We only prove (1)⇔(2), and (1)⇔(3) can be proved similarly.

(1)⇒(2): Let R be a weakly L-valued preorder. Then $\varepsilon(R^{\leftarrow}) = R^{\leftarrow} \wedge (R^{\leftarrow})^{op}$. Since $R^{\leftarrow}$ and $(R^{\leftarrow})^{op}$ are still weakly L-valued preorder, it follows that $\varepsilon(R)$ is weakly L-valued preorder. The symmetry of $\varepsilon(R^{\leftarrow})$ is obvious. Because R is a compatible on A, then we know $R^{\leftarrow}$ is still compatible from Theorem 4.1. Next, we prove $\varepsilon(R^{\leftarrow})$ is compatible on $P(A)$.

For all $(f_i, \delta), (g_i, \delta) \in P(A)_0$, $1 \leq i \leq n$, then we have

$$
\begin{aligned}
&\varepsilon(R^{\leftarrow})((f_1,\delta),(g_1,\delta)) \wedge (\delta \to \varepsilon(R^{\leftarrow})((f_2,\delta),(g_2,\delta))) \wedge \cdots \\
&\ \wedge (\delta \to \varepsilon(R^{\leftarrow})((f_n,\delta),(g_n,\delta)))) \\
&= (R^{\leftarrow} \wedge (R^{\leftarrow})^{op})((f_1,\delta),(g_1,\delta)) \wedge (\delta \to ((R^{\leftarrow})^{op} \wedge R^{\leftarrow})((f_2,\delta),(g_2,\delta))) \wedge \cdots \\
&\ \wedge (\delta \to ((R^{\leftarrow})^{op} \wedge R^{\leftarrow}((f_n,\delta),(g_n,\delta)))) \\
&\leq [R^{\leftarrow}((f_1,\delta),(g_1,\delta)) \wedge (\delta \to R^{\leftarrow}((f_2,\delta),(g_2,\delta))) \wedge \cdots \\
&\wedge (\delta \to R^{\leftarrow}((f_n,\delta),(g_n,\delta))))] \wedge \\
&[(R^{\leftarrow})^{op}((f_1,\delta),(g_1,\delta)) \wedge (\delta \to (R^{\leftarrow})^{op}((f_2,\delta),(g_2,\delta))) \wedge \cdots \\
&\wedge (\delta \to (R^{\leftarrow})^{op}((f_n,\delta),(g_n,\delta))))] \\
&\leq (R^{\leftarrow})^{op}((\theta^{\rightarrow}(\bigwedge_{i=1}^{n} f_i),\delta),(\theta^{\rightarrow}(\bigwedge_{i=1}^{n} g_i),\delta)) \wedge R^{\leftarrow}((\theta^{\rightarrow}(\bigwedge_{i=1}^{n} f_i),\delta),(\theta^{\rightarrow}(\bigwedge_{i=1}^{n} g_i),\delta)) \\
&= \varepsilon(R^{\leftarrow})((\theta^{\rightarrow}(\bigwedge_{i=1}^{n} f_i),\delta),(\theta^{\rightarrow}(\bigwedge_{i=1}^{n} g_i),\delta)).
\end{aligned}
$$

This is to say R is Smyth good on A.

(2)⇒(1): Let $x_i, y_i \in A_0$ with $A(x_i) = A(y_j)$ for all $1 \leq i,j \leq n$ and define $A_{(x,y)} : A_0 \to L$ by

$$
A_{(x,y)}(z) = \begin{cases} A(x) \wedge A(y), & z = x, \\ A(y), & z = y, \\ 0, & others, \end{cases}
$$

then $(A_{(x,y)}, A(y)) \in P(A)_0$.

Considering

$$
\begin{aligned}
&R^{\leftarrow}((A_x, A(x)), (A_{(x,y)}, A(y))) = S_A((A_{(x,y)}, A(y))), (\bar{R_u}(A_x), A(x))) \\
&= A(x) \wedge (A(y) \wedge S_{A_0}(A_{(x,y)}, \bar{R_u}(A_x))) \\
&= A(x) \wedge (A(y) \wedge ((A(y) \to R(x,y)) \wedge ((A(x) \wedge A(y)) \to R(x,x))) \\
&= R(x,y),
\end{aligned}
$$

and

$$\begin{aligned}
&R^{\leftarrow}((A_{(x,y)}, A(y)), (A_x, A(x))) \\
&= S_A((A_x, A(x))), (\bar{R_u}(A_{(x,y)}), A(y))) \\
&= A(y) \wedge (A(x) \wedge S_{A_0}(A_x, \bar{R_u}(A_{(x,y)})) \\
&= A(y) \wedge (A(x) \wedge (A(x) \to \bar{R_u}(A_{(x,y)})(x))) \\
&= A(y) \wedge (A(x) \wedge (A(x) \to A(x) \wedge A(y))) \\
&= A(x) \wedge A(y),
\end{aligned}$$

we have the following formula:

$$\begin{aligned}
R(x,y) &= R(x,y) \wedge (A(x) \wedge A(y)) \\
&= R^{\leftarrow}((A_x, A(x)), (A_{(x,y)}, A(y))) \\
&\wedge R^{\leftarrow}((A_{(x,y)}, A(y)), (A_x, A(x))) \\
&= \varepsilon(R^{\leftarrow})((A_x, A(x)), (A_{(x,y)}, A(y))).
\end{aligned}$$

The following cans show that R is compatible on A:

$$\begin{aligned}
&R(x_1, y_1) \wedge (A(x_2) \to R(x_2, y_2)) \wedge \cdots \wedge (A(x_n) \to R(x_n, y_n)) \\
&= \varepsilon(R^{\leftarrow})((A_{x_1}, A(x_1)), (A_{(x_1,y_1)}, A(y_1)))\wedge \\
&(A(x_2) \to \varepsilon(R^{\leftarrow})((A_{x_2}, A(x_2)), (A_{(x_2,y_2)}, A(y_2))) \wedge \cdots \\
&\wedge (A(x_n) \to \varepsilon(R^{\leftarrow})((A_{x_n}, A(x_n)), (A_{(x_n,y_n)}, A(y_n))) \\
&\leq \varepsilon(R^{\leftarrow})((f^{\to}(\bigwedge_{i=1}^{n} A_{x_i}), A(x_i)), (f^{\to}(\bigwedge_{i=1}^{n} A_{(x_i,y_i)}), A(y_i))) \\
&\leq (R^{\leftarrow})((f^{\to}(\bigwedge_{i=1}^{n} A_{x_i}), A(x_i)), (f^{\to}(\bigwedge_{i=1}^{n} A_{(x_i,y_i)}), A(y_i))) \\
&= S_A((f^{\to}(\bigwedge_{i=1}^{n} A_{(x_i,y_i)}), A(y_i)), (\bar{R_u}(f^{\to}(\bigwedge_{i=1}^{n} A_{x_i})), A(x_i))) \\
&= A(x_i) \wedge (A(y_i) \wedge S_{A_0}(f^{\to}(\bigwedge_{i=1}^{n} A_{(x_i,y_i)}), \bar{R_u}(f^{\to}(\bigwedge_{i=1}^{n} A_{x_i})))) \\
&= A(x_i) \wedge (A(y_i) \wedge \bigwedge_{x \in A_0} (\bigvee_{f(w_1,\cdots,w_n)=x} \bigwedge_{i=1}^{n} A_{(x_i,y_i)}(w_i) \to \bar{R_u}(f^{\to}(\bigwedge_{i=1}^{n} A_{x_i})(x)))) \\
&\leq A(x_i) \wedge (A(y_i) \wedge (\bigwedge_{i=1}^{n} A(y_i) \to R(f(x_1, \cdots, x_n), f(y_1, \cdots, y_n)))) \\
&= A(x_i) \wedge (A(y_i) \wedge (A(y_i) \to R(f(x_1, \cdots, x_n), f(y_1, \cdots, y_n)))) \\
&= R(f(x_1, \cdots, x_n), f(y_1, \cdots, y_n))).
\end{aligned}$$

□

If R is a weakly L-valued equivalence on A, then $\bar{R}_l(f) = \bar{R}_u(f)$ for all $(f, \delta) \in P(A)_0$. It holds that

$$R^+ = R^{\rightarrow} \wedge R^{\leftarrow} = R^{\rightarrow} \wedge (R^{\rightarrow})^{op} = \varepsilon(R^{\rightarrow}).$$

Therefore, we have the following Theorem 4.4.

Theorem 4.4. *Let $\mathcal{A}$ be a fuzzy universal algebra and R be a weakly L-valued equivalence on A, if R is a congruence on A, then R^+ is a congruence on $P(A)$.*

Theorem 4.5. *Let $\mathcal{A}$ be a fuzzy universal algebra and R be a congruence on A, then there is an onto homomorphism from $P(A/R)$ to $P(A)/R^+$.*

Proof. Firstly, we should show the expression of the element in $(P(A)/R^+)_0$. For $(f, \delta) \in P(A)_0$. If $(g, \eta) \in [(f, \delta)]$, then

$$R^+((f, \delta), (f, \delta)) = R^+((f, \delta), (g, \eta)) = R^+((g, \eta), (f, \delta)) = R^+((g, \eta), (g, \eta)).$$

Hence, $\delta = \eta$, $f \leq \bar{R}_l(g)$ and $g \leq \bar{R}_l(f)$. Therefore,

$$[(f, \delta)] = \{(g, \delta) \in P(A)_0 \mid f \leq \bar{R}_l(g) \;\; and \;\; g \leq \bar{R}_l(f)\}.$$

Secondly, define $\theta : P(A/R) \to P(A)/R^+$ by $\theta((g, \eta)) = [(\hat{g}, \eta)]$, where $\hat{g} : A_0 \to L$ is defined by $\hat{g}(x) = g([x])$.

(1) θ is degree-preserving: $\forall\ (g, \eta) \in P(A/R)_0$, $P(A/R)((g, \eta)) = \eta$ and

$$P(A)/R^+(\theta((g, \eta))) = P(A)/R^+([(\hat{g}, \eta)]) = P(A)((\hat{g}, \eta)) = \eta.$$

Hence $P(A/R)((g, \eta)) = P(A)/R^+(\theta((g, \eta)))$.

(2) Surjective: Let $(f, \delta) \in P(A)_0$ and define $g_f : (A/R)_0 \to L$ by $g_f([x]) = \bar{R}_l(f)(x)$. In order to show this definition is well-defined, we need to prove that $\bar{R}_l(f)(x) = \bar{R}_l(f)(y)$ when $[x] = [y]$. In fact,

$$\bar{R}_l(f)(x) = \bigvee_{z \in A_x} f(z) * (A(z) \to R(x, z)),$$

and

$$\bar{R}_l(f)(y) = \bigvee_{z \in A_y} f(z) * (A(z) \to R(y, z)).$$

Since $[x] = [y]$, hence $A_x = A_y$, and we have

$$R(x, x) = R(x, y) = R(y, x) = R(y, y) = A(x) = A(y).$$

For $z \in A_x$,

$$\begin{aligned} f(z) * (A(z) \to R(x, z)) &= f(z) * (A(z) \to R(x, z)) * (A(x) \to R(x, y)) \\ &= (A(z) \to f(z)) * R(z, x) * (A(x) \to R(x, y)) \\ &\leq (A(z) \to f(z)) * R(z, y) \\ &= f(z) * (A(z) \to R(y, z)). \end{aligned}$$

Hence $\bar{R}_l(f)(x) \leq \bar{R}_l(f)(y)$. Symmetrically, $\bar{R}_l(f)(x) \geq \bar{R}_l(f)(y)$. Then $\bar{R}_l(f)(x) = \bar{R}_l(f)(y)$, thus

$$\theta((g_f, \delta)) = [(\hat{g_f}, \delta)] = [(\bar{R}_l(f), \delta)] = [(f, \delta)].$$

This is to say that θ is a surjective mapping.

(3) Finally, it remains to show that θ is a homomorphism. Let ϕ be a n-ary operation on A and $(g_i,\eta)\in P(A/R)_0$, for any $i=1,\cdots,n$. Then we have

$$\theta((\phi/R)^+((g_1,\eta),\cdots,(g_n,\eta)))=\theta((\phi/R)^{\rightarrow}(\bigotimes_{i=1}^{n}g_i),\eta)=[((\phi/R)^{\rightarrow}\widehat{(\bigotimes_{i=1}^{n}g_i)},\eta)],$$

and

$$\begin{aligned}\phi^+/R^+(\theta((g_1,\eta)),\cdots,\theta((g_n,\eta)))&=\phi^+/R^+([(\hat{g_1},\eta)],\cdots,[(\hat{g_n},\eta)])\\&=[\phi^+((\hat{g_1},\eta),\cdots,(\hat{g_n},\eta))]\\&=[(\phi^{\rightarrow}(\bigotimes_{i=1}^{n}\hat{g}_i,\eta))].\end{aligned}$$

Hence we need to show

$$[((\phi/R)^{\rightarrow}\widehat{(\bigotimes_{i=1}^{n}g_i)},\eta)]=[(\phi^{\rightarrow}(\bigotimes_{i=1}^{n}\hat{g}_i,\eta))].$$

It suffices to show that

$$(\phi/R)^{\rightarrow}\widehat{(\bigotimes_{i=1}^{n}g_i)}\leq\bar{R}_l(\phi^{\rightarrow}(\bigotimes_{i=1}^{n}\hat{g}_i)),$$

and

$$\phi^{\rightarrow}(\bigotimes_{i=1}^{n}\hat{g}_i)\leq\bar{R}_l((\phi/R)^{\rightarrow}\widehat{(\bigotimes_{i=1}^{n}g_i)}).$$

Let $x\in A_0$, we have the following two computations:

$$\begin{aligned}(\phi/R)^{\rightarrow}\widehat{(\bigotimes_{i=1}^{n}g_i)}(x)&=(\phi/R)^{\rightarrow}(\bigotimes_{i=1}^{n}g_i)([x])\\&=\bigvee_{(\phi/R)([x_1],\cdots,[x_n])=[x]}\bigotimes_{i=1}^{n}g_i([x_i])\\&=\bigvee_{\phi(x_1,\cdots,x_n)=[x]}\bigotimes_{i=1}^{n}g_i([x_i]),\end{aligned}$$

and

$$\begin{aligned}\phi^{\rightarrow}(\bigotimes_{i=1}^{n}\hat{g}_i)(x)&=\bigvee_{\phi(z_1,\cdots,z_n)=x}\bigotimes_{i=1}^{n}\hat{g}_i(z_i)\\&=\bigvee_{\phi(z_1,\cdots,z_n)=x}\bigotimes_{i=1}^{n}g_i([z_i]).\end{aligned}$$

It is easy to see that

$$\phi^{\rightarrow}(\bigotimes_{i=1}^{n}\hat{g}_i)(x) \leq (\phi/R)^{\rightarrow}(\widehat{\bigotimes_{i=1}^{n}} g_i)(x).$$

This is to say that

$$\phi^{\rightarrow}(\bigotimes_{i=1}^{n}\hat{g}_i) \leq (\phi/R)^{\rightarrow}(\widehat{\bigotimes_{i=1}^{n}} g_i).$$

On account of

$$(\phi/R)^{\rightarrow}(\widehat{\bigotimes_{i=1}^{n}} g_i) \leq \bar{R}_l((\phi/R)^{\rightarrow}(\widehat{\bigotimes_{i=1}^{n}} g_i)),$$

then

$$\phi^{\rightarrow}(\bigotimes_{i=1}^{n}\hat{g}_i) \leq \bar{R}_l((\phi/R)^{\rightarrow}(\widehat{\bigotimes_{i=1}^{n}} g_i)).$$

We still need to prove that

$$(\phi/R)^{\rightarrow}(\widehat{\bigotimes_{i=1}^{n}} g_i) \leq \bar{R}_l(\phi^{\rightarrow}(\bigotimes_{i=1}^{n}\hat{g}_i)).$$

$$\begin{aligned}\bar{R}_l(\phi^{\rightarrow}(\bigotimes_{i=1}^{n}\hat{g}_i))(x) &= \bigvee_{y\in A_x}\phi^{\rightarrow}(\bigotimes_{i=1}^{n}\hat{g}_i)(y) * (A(y) \rightarrow R(x,y)) \\ &= \bigvee_{y\in A_x}\bigvee_{\phi(z_1,\cdots,z_n)=y}\bigotimes_{i=1}^{n} g_i([z_i]) * (A(y) \rightarrow R(x,y)).\end{aligned}$$

For each $\phi(x_1,\cdots,x_n) \in [x]$, let $y^* = \phi(x_1,\cdots,x_n)$. Then

$$\begin{aligned}\bigotimes_{i=1}^{n} g_i([x_i]) &= \bigotimes_{i=1}^{n} g_i([x_i]) * (A(\phi(x_1,\cdots,x_n)) \rightarrow R(x,\phi(x_1,\cdots,x_n))) \\ &= \bigotimes_{i=1}^{n} g_i([x_i]) * (A(y^*) \rightarrow R(x,y^*)) \\ &\leq \bar{R}_l(\phi^{\rightarrow}(\bigotimes_{i=1}^{n}\hat{g}_i))(x).\end{aligned}$$

Hence,

$$(\phi/R)^{\rightarrow}(\widehat{\bigotimes_{i=1}^{n}} g_i) \leq \bar{R}_l(\phi^{\rightarrow}(\bigotimes_{i=1}^{n}\hat{g}_i)),$$

as desired. □

Theorem 4.6. *$P(A/R)/ker(\theta) \cong P(A)/R^+$, where θ is the onto homomorphism in Theorem 4.5.*

Remark 4.7. In classical universal algebras, we know that the power algebra of the quotient algebra is isomorphic to the quotient algebra of the power algebra from [1, 2]. In Theorem 4.5, we show that there is an onto homomorphism from $P(A/R)$ to $P(A)/R^+$. We don't know whether there is an isomorphism between $P(A/R)$ and $P(A)/R^+$, so we leave it as a question for readers.

5. Conclusion

In this paper, we study the basic theories of fuzzy universal algebra on an L-set when L is a GL-quantale. Some basic definitions and results of classical universal algebra can be generalized to those of fuzzy universal algebra on L-set. In Section 4, in order to study the theory of L-valued power algebra well, we must have the aid of weakly L-valued preorder on L-set. We want to know whether these are still effective for L-valued preorder and whether the definition of L-valued power algebra is proper. We leave them for future study.

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Xiaowei Wei, Department of Mathematics, Ocean University of China, 238 Songling Road, 266100, Qingdao, P.R.China
E-mail address: `2274495848@qq.com`

Yueli Yue*, Department of Mathematics, Ocean University of China, 238 Songling Road, 266100, Qingdao, P.R.China
E-mail address: `ylyue@ouc.edu.cn`

*Corresponding author