DEGREE OF F-IRRESOLUTE FUNCTION IN (L, M)-FUZZY TOPOLOGICAL SPACES

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ABSTRACT. In this paper, we present a new vision for studying **F**-open, **F**-continuous, and **F**-irresolute function in (L, M)-fuzzy topological spaces based on the implication operation and (L, M)-fuzzy **F**-open operator [3]. These kinds of functions are generalized with their elementary properties to (L, M)-fuzzy topological spaces setting based on graded concepts. Moreover, a systematic discussion of their relationship with the degree of **F**-compactness, **F**-connectedness, $\mathbf{F}T_1$, and $\mathbf{F}T_2$ is carried out.

1. Introduction

The concept of fuzzy topology was first formulated by Chang [2] as a completely new mathematical application for fuzzy sets theory [31]. Chang defined fuzzy topology as a collection of fuzzy sets which fulfills the same conditions as general topology while the topology itself still crisp. Later, Goguen [8] suggested to use a complete residuated lattice with possibly non-commutative truth function of conjunction instead the closed interval $[0, 1]$. Many researchers are interested in extending the Chang-Goguens approach. Accordingly, Höhle [9] and Ying [30] extended independently this approach to more general form which later known as fuzzifying topology. In 1985, Kubiak [12] and Šostak [24] defined independently a new extension to Chang-Goguens fuzzy topology. It realizes (L, M)-fuzzy topology on X as an M-fuzzy subset of L^X where L and M are appropriate lattices. Most early works mainly attempted to extend for (L, M)-fuzzy topology some basic properties and results of general topology and Chang-Goguens fuzzy topology (see [15, 11, 23, 14], for instance).

Various fuzzification of classical concepts related to topology evolved as further aspects and connections have been taken into account. For example, consider the important concept of category. A notion of fuzzy category was introduced in [27] and its study has been pursued for many years [13, 29] meaning that potential objects and morphisms have been endowed with a certain degree of the corresponding lattice. In 1999, Šostak [28] fuzzified some categories concerning to topology and algebra. The continuity degree considered as a significant example of a morphism degree in the fuzzy category. The obvious fuzzification of category led to the definition of many topological concepts endowed with degrees. Recently, Pang [17]

Received: ..... 2018; Revised: ...... 2018; Accepted: 2018

Key words and phrases: (L, M)-fuzzy topological space, **F**-open operator, **F**-continuous function, **F**-irresolute function.

defined the continuity, openness, and closeness degree of functions in L-fuzzifying topology. An important conception is due to Liang and Shi who approached the same degrees in (L, M)-fuzzy topology in the spirit of their pioneer paper [16] and suggested using the implication operation. Several properties in general topology generalized to (L, M)-fuzzy topology setting based on graded concepts. Recently, Shi [20] formulated a new technique for studying the semiopenness and preopenness in L-fuzzy topology by introducing L-fuzzy semiopen and L-fuzzy preopen operators based on implication operation. Also, several properties and results have been reformulated with the help of Shi's operators. Afterwards, Ghareeb [6, 3, 4, 5, 1, 7] presented other operators rely on Shi's operators and used it to discuss the related degree of continuity, connectedness, and compactness.

The main purpose of this paper is to define the degree of **F**-openness, **F**- continuity, and **F**-irresolutness for functions in (L, M)-fuzzy topology based on implication operation and (L, M)-fuzzy **F**-open operator [2]. Their primary properties will be introduced and discussed through graded concepts. Therefore, the function can be regard as **F**-open, **F**-continuous, and **F**-irresolute to some degree. The relationship with **F**-compactness, **F**-connectedness, $\mathbf{F}T_1$ and $\mathbf{F}T_2$ are discussed systematically.

2. Preliminaries

In the sequel, both L and M denote a completely distributive De Morgan algebras. X is a nonempty set. The smallest and greatest members in L and M are denoted by $\bot_L$, $\top_L$ and $\bot_M$, $\top_M$, respectively. For each r, $s \in L$, the element r is wedge below s in L [18], in symbols $r \ll s$, if for each subset $\mathcal{D} \subseteq L$, $\bigvee \mathcal{D} \geq s$ yields to $t \geq r$ for some $t \in \mathcal{D}$. We say the complete lattice L is completely distributive iff $s = \bigvee\{r \in L | r \ll s\}$ for any $s \in L$. A member $r \in L$ is said to be co-prime if $r \leq s \vee t$ yields to $r \leq s$ or $r \leq t$. The family of non-zero co-prime members in L is denoted by $J(L)$. L^X refers to the set of all L-subsets on X. $2^{\mathcal{U}}$ denotes to the collection of all finite sub-collections of $\mathcal{U} \subseteq L^X$. Evidently, L^X is a completely distributive DeMorgan algebra when it inherits the structure of the lattice L in a natural way, by defining $\bigvee$, $\bigwedge$, $\leq$ and $'$ pointwisely. It is evidently that $\{x_\lambda | \lambda \in J(L)\}$ is a collection of non-zero co-primes in L^X.

For any completely distributive De Morgan algebra L, there exists an implication operation $\mapsto : L \times L \longrightarrow L$ as the right adjoint for the meet operation $\wedge$ is defined by

$$r \mapsto s = \bigvee\{t \in L | r \wedge t \leq s\}.$$

Further, the operation $\leftrightarrow$ is given by

$$r \leftrightarrow s = (r \mapsto s) \wedge (s \mapsto r).$$

The properties of an implication operation listed in the following lemma.

Lemma 2.1. [10] *Let $(L, \bigvee, \bigwedge)$ be a completely distributive lattice and $\mapsto$ be the implication operation corresponding to $\wedge$. Then for all r, s, $t \in L$, $\{r_i\}_{i\in\Gamma}$, and $\{s_i\}_{i\in\Gamma} \subseteq L$, we have the following statements:*

(1) $(r \mapsto s) \geq t \Longleftrightarrow r \wedge t \leq s$.

(2) $r \leq s \Leftrightarrow r \mapsto s = \top_L$.
(3) $r \mapsto (s \mapsto t) = (r \wedge s) \mapsto t$.
(4) $(t \mapsto r) \wedge (r \mapsto s) \leq t \mapsto s$.
(5) $t \mapsto r \leq (r \mapsto s) \mapsto (t \mapsto s)$.
(6) $r \mapsto \bigwedge_{i\in\Gamma} r_i = \bigwedge_{i\in\Gamma} (r \mapsto r_i)$, *hence* $r \mapsto s \leq r \mapsto t$ *whenever* $s \leq t$.
(7) $\bigvee_{i\in\Gamma} r_i \mapsto s = \bigwedge_{i\in\Gamma} (r_i \mapsto s)$, *hence* $r \mapsto t \geq s \mapsto t$ *whenever* $r \leq s$.

Definition 2.2. [10, 12, 19, 24] For any non-empty set X, an (L, M)-fuzzy topology is defined as a function $\tau : L^X \longrightarrow M$ satisfies the following conditions:

($\mathcal{O}$1) $\tau(\top_{L^X}) = \tau(\perp_{L^X}) = \top_M$.
($\mathcal{O}$2) $\forall\ A_1,\ A_2 \in L^X$, $\tau(A_1 \wedge A_2) \geq \tau(A_1) \wedge \tau(A_2)$.
($\mathcal{O}$3) $\forall\ \{A_i\}_{i\in\Gamma} \subseteq L^X$, $\tau\left(\bigvee_{i\in\Gamma} A_i\right) \geq \bigwedge_{i\in\Gamma} \tau(A_i)$.

For any L-subset $A \in L^X$, $\tau(A)$ refers to the degree of openness of A. $\tau^*(A) = \tau(A')$ is the degree of closeness of A. The pair (X, τ) is said to be an (L, M)-fuzzy topological space. A function $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ is said to be continuous with respect to (L, M)-fuzzy topological spaces (X, τ) and (Y, σ) iff $\tau(\varphi^{\Leftarrow}(B)) \geq \sigma(B)$ for all $B \in L^Y$, where $\varphi^{\Leftarrow}$ is given by $\varphi^{\Leftarrow}(B)(x) = B(\varphi(x))$.

An (L, M)-fuzzy closure operators introduced by Shi [22] as follows.

Definition 2.3. [22] Let (X, τ) be an (L, M)-fuzzy topological space. The function $Cl^\tau : L^X \longrightarrow M^{J(L^X)}$ is called an (L, M)-fuzzy closure operator induced by τ, where

$$Cl^\tau(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq A_1 \geq A} (\tau(A_1'))',$$

for any $x_\lambda \in J(L^X)$ and $A \in L^X$.

Definition 2.4. [20] Let (X, τ) be an (L, M)-fuzzy topological space. The function $\mathcal{S} : L^X \longrightarrow M$ is called an (L, M)-fuzzy semiopen operator induced by τ, where

$$\mathcal{S}(A) = \bigvee_{A_1 \leq A} \left\{ \tau(A_1) \wedge \bigwedge_{x_\lambda \ll A} \bigwedge_{x_\lambda \not\leq A_2 \geq A_1} (\tau(A_2'))' \right\},$$

for any $A \in L^X$, where $\mathcal{S}(A)$ refers to the degree to which A is semiopen. $\mathcal{S}^*(A) = \mathcal{S}(A')$ is the degree to which A is semiclosed.

Based on Definition 2.3 and Definition 2.4, Shi [20] stated the next corollary.

Corollary 2.5. *Let* (X, τ) *be an* (L, M)*-fuzzy topological space. Then for all* $A \in L^X$*, we have*

$$\mathcal{S}(A) = \bigvee_{A_1 \leq A} \left\{ \tau(A_1) \wedge \bigwedge_{x_\lambda \ll A} Cl^\tau(A_1)(x_\lambda) \right\}.$$

Theorem 2.6. [20] *Let* (X, τ) *be an* (L, M)*-fuzzy topological space and* $\mathcal{S}$ *be the corresponding* (L, M)*-fuzzy semiopen operator. Then* $\tau(A) \leq \mathcal{S}(A)$ *for all* $A \in L^X$.

Theorem 2.7. [20] *Let (X,τ) be an (L,M)-fuzzy topological space and $\mathcal{S}$ be the corresponding (L,M)-fuzzy semiopen operator. Then*

$$\mathcal{S}\left(\bigvee_{i\in\Gamma} A_i\right) \geq \bigwedge_{i\in\Gamma} \mathcal{S}(A_i),$$

for each $\{A_i\}_{i\in\Gamma} \subseteq L^X$.

Definition 2.8. [3] Let (X,τ) be an (L,M)-fuzzy topological space. For any $A \in L^X$, the function $\mathscr{F} : L^X \longrightarrow M$ is called an (L,M)-fuzzy **F**-open operator induced by τ and $\mathcal{S}$, where

$$\mathscr{F}(A) = \bigvee_{B\leq A} \left\{\tau(B) \wedge \bigwedge_{x_\lambda \ll A} \bigwedge_{x_\lambda \not\leq C \geq B} (\mathcal{S}(C'))'\right\}.$$

$\mathscr{F}(A)$ refers to the degree to which A is **F**-open and $\mathscr{F}^*(A) = \mathscr{F}(A')$ refers to the degree to which A is **F**-closed.

Lemma 2.9. *Let $\mathscr{F} : L^X \longrightarrow M$ be an (L,M)-fuzzy **F**-open operator induced by (L,M)-fuzzy topology τ on X. Then $\mathscr{F}$ satisfies the following conditions:*

(1) $\mathscr{F}(\perp_{L^X}) = \mathscr{F}(\top_{L^X}) = \top_M$.

(2) $\mathscr{F}(\bigvee\limits_{i\in\Gamma} A_i) \geq \bigwedge\limits_{i\in\Gamma} \mathscr{F}(A_i)$ *for any $\{A_i\}_{i\in\Gamma} \subseteq L^X$.*

Proof. The proof of **(1)** is clear. To prove **(2)**, suppose that $r \in M$ and $r \ll \bigwedge_{i\in\Gamma} \mathscr{F}(A_i)$. Then for any $i \in \Gamma$, there exists $B_i \leq A_i$ with

$$r \ll \tau(B_i) \quad \text{and} \quad r \ll \bigwedge_{x_\lambda \ll A_i} \bigwedge_{x_\lambda \not\leq C \geq B_i} (\mathcal{S}(C'))'.$$

Hence

$$r \leq \bigwedge_{i\in\Gamma} \tau(B_i) \leq \tau\left(\bigvee_{i\in\Gamma} B_i\right) \quad \text{and} \quad r \leq \bigwedge_{i\in\Gamma} \bigwedge_{x_\lambda \ll A_i} \bigwedge_{x_\lambda \not\leq C \geq B_i} (\mathcal{S}(C'))'.$$

Since $\left\{x_\lambda : x_\lambda \ll \bigvee_{i\in\Gamma} A_i\right\} = \bigcup_{i\in\Gamma} \left\{x_\lambda : x_\lambda \ll A_i\right\}$, we have

$$\begin{aligned}
\mathscr{F}\left(\bigvee_{i\in\Gamma} A_i\right) &= \bigvee_{B\leq \bigvee_{i\in\Gamma} A_i} \left\{\tau(B) \wedge \bigwedge_{x_\lambda \ll \bigvee_{i\in\Gamma} A_i} \bigwedge_{x_\lambda \not\leq C \geq B} (\mathcal{S}(C'))'\right\} \\
&\geq \tau\left(\bigvee_{i\in\Gamma} B_i\right) \wedge \bigwedge_{i\in\Gamma} \bigwedge_{x_\lambda \ll A_i} \bigwedge_{x_\lambda \not\leq C \geq \bigvee_{i\in\Gamma} B_i} (\mathcal{S}(C'))' \\
&\geq \tau\left(\bigvee_{i\in\Gamma} B_i\right) \wedge \bigwedge_{i\in\Gamma} \bigwedge_{x_\lambda \ll A_i} \bigwedge_{x_\lambda \not\leq C \geq B_i} (\mathcal{S}(C'))' \\
&\geq r.
\end{aligned}$$

From this, we conclude that $\mathscr{F}\left(\bigvee_{i\in\Gamma} A_i\right) \geq \bigwedge_{i\in\Gamma} \mathscr{F}(A_i)$. □

Definition 2.10. [3] Let $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ be a function between two (L, M)-fuzzy topological spaces (X, τ) and (Y, σ). Then:

(1) φ is (L, M)-fuzzy **F**-continuous function iff $\sigma(B) \leq \mathscr{F}_1(\varphi^{\Leftarrow}(B))$ holds for any $B \in L^Y$.

(2) φ is (L, M)-fuzzy **F**-irresolute iff $\mathscr{F}_2(B) \leq \mathscr{F}_1(\varphi^{\Leftarrow}(B))$ holds for any $B \in L^Y$.

Corollary 2.11. *Let $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ be a function between two (L, M)-fuzzy topological spaces (X, τ) and (Y, σ). Then:*

(1) *φ is (L, M)-fuzzy **F**-continuous iff $\sigma^*(B) \leq \mathscr{F}_1^*(\varphi^{\Leftarrow}(B))$ for any $B \in L^Y$.*

(2) *φ is (L, M)-fuzzy **F**-irresolute iff $\mathscr{F}_2^*(B) \leq \mathscr{F}_1^*(\varphi^{\Leftarrow}(B))$ for any $B \in L^Y$.*

Proof. Clear. □

Definition 2.12. A function $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ between two (L, M)-fuzzy topological spaces (X, τ) and (Y, σ) is called an (L, M)-fuzzy **F**-open function iff $\tau(A) \leq \mathscr{F}_2(\varphi^{\Rightarrow}(A))$ for any $A \in L^X$.

Definition 2.13. An (L, M)-fuzzy quasi **F**-neighborhood system on X is a collection $\mathbf{F}Q = \{\mathbf{F}Q_{x_\lambda} | x_\lambda \in J(L^X)\}$ of functions $\{\mathbf{F}Q_{x_\lambda} : L^X \longrightarrow M\}$ which satisfying the following axioms:

(1) $FQ_{x_\lambda}(\perp_{L^X}) = \perp_M$.

(2) $FQ_{x_\lambda}(A) \neq \perp_M$, then $x_\lambda \not\leq A'$.

(3) $FQ_{x_\lambda}(A_1 \wedge A_2) \leq FQ_{x_\lambda}(A_1) \wedge FQ_{x_\lambda}(A_2)$.

(4) $FQ_{x_\lambda}(A) = \bigvee_{x_\lambda \not\leq A_1 \geq A'} \bigwedge_{y_\mu \not\leq A_1'} FQ_{y_\mu}(A_1)$.

Definition 2.14. [4] An (L, M)-fuzzy **F**-closure operator on X is a function $\mathbf{F}C : L^X \longrightarrow M^{J(L^X)}$ which satisfying the following axioms:

(1) $\mathbf{F}C(A)(x_\lambda) = \bigwedge_{\mu \ll \lambda} \mathbf{F}C(A)(x_\mu)$ for any $x_\lambda \in J(L^X)$.

(2) $\mathbf{F}C(\perp_{L^X})(x_\lambda) = \perp_M$ for any $x_\lambda \in J(L^X)$.

(3) $\mathbf{F}C(A)(x_\lambda) = \top_M$ for any $x_\lambda \leq A$.

(4) for any $r \in M_\perp$, $(\mathbf{F}C(\bigvee(\mathbf{F}C(A))_{[r]}))_{[r]} \subset (\mathbf{F}C(A))_{[r]}$.

$\mathbf{F}C(A)(x_\lambda)$ refers to the degree to which x_λ belongs to the **F**-closure of A.

From [22] and Lemma 2.9, we can easily prove Theorems 2.15, 2.16 and 2.19.

Theorem 2.15. *Let $\mathscr{F}$ be the (L, M)-fuzzy **F**-open operator on X and let $\mathbf{F}C^{\mathscr{F}}$ be the corresponding (L, M)-fuzzy **F**-closure operator. Then for any $x_\lambda \in J(L^X)$ and $A \in L^X$, we have*

$$\mathbf{F}C^{\mathscr{F}}(A)(x_\lambda) = \bigwedge_{x_\lambda \not\leq A_1 \geq A} (\mathscr{F}(A_1'))'.$$

Theorem 2.16. *Let (X, τ) be an (L, M)-fuzzy topological space, $\mathscr{F}$ be the corresponding (L, M)-fuzzy **F**-open operator, and $\mathbf{F}Q^{\mathscr{F}} = \{\mathbf{F}Q_{x_\lambda} | x_\lambda \in J(L^X)\}$ be the (L, M)-fuzzy quasi **F**-neighborhood system induced by $\mathscr{F}$. Define the function*

$\mathbf{F}C^{\mathscr{F}} : L^X \longrightarrow M^{J(L^X)}$ *by* $\mathbf{F}C^{\mathscr{F}}(A)(x_\lambda) = (\mathbf{F}Q_{x_\lambda}(A'))'$. *Then* $\mathbf{F}C^{\mathcal{F}}$ *is an* (L, M)-*fuzzy* $\mathbf{F}$-*closure operator on* X.

Definition 2.17. An (L, M)-fuzzy $\mathbf{F}$-neighborhood system on X is the collection $\mathbf{F}N = \{\mathbf{F}N_{x_\lambda} | x_\lambda \in J(L^X)\}$ of functions $\{\mathbf{F}N_{x_\lambda} : L^X \longrightarrow M\}$ which satisfying the following axioms:

(1) $\mathbf{F}N_{x_\lambda}(\perp_{L^X}) = \perp_M$.
(2) $\mathbf{F}N_{x_\lambda}(A) \neq \perp_M$, then $x_\lambda \nleq A$.
(3) $\mathbf{F}N_{x_\lambda}(A_1 \wedge A_2) \leq \mathbf{F}N_{x_\lambda}(A_1) \wedge \mathbf{F}N_{x_\lambda}(A_2)$.
(4) $\mathbf{F}N_{x_\lambda}(A) = \bigvee_{x_\lambda \leq A_1 \geq A} \bigwedge_{y_\mu \ll A_1} \mathbf{F}N_{y_\mu}(A_1)$.

Definition 2.18. An (L, M)-fuzzy $\mathbf{F}$-interior operator on X is a function $\mathbf{F}I : L^X \longrightarrow M^{J(L^X)}$ which satisfying the following axioms:

(1) $\mathbf{F}I(A)(x_\lambda) = \bigwedge_{\mu \ll \lambda} \mathbf{F}I(A)(x_\mu)$, for any $x_\lambda \in J(L^X)$.
(2) $\mathbf{F}I(\top_{L^X})(x_\lambda) = \top_M$ for any $x_\lambda \in J(L^X)$.
(3) $\mathbf{F}I(A)(x_\lambda) = \perp_M$ for any $x_\lambda \nleq A$.
(4) For all $r \in M_\perp$, $(\mathbf{F}I(A))^{(r)} \subseteq (\mathbf{F}I(\bigvee(\mathbf{F}I(A))^{(r)}))^{(r)}$.

$\mathbf{F}I(A)(x_\lambda)$ refers to the degree to which x_λ belongs to the $\mathbf{F}$-interior of A.

Theorem 2.19. *Let* (X, τ) *be an* (L, M)-*fuzzy topological space and* $\mathbf{F}N^\tau = \{\mathbf{F}N^\tau_{x_\lambda} | x_\lambda \in J(L^X)\}$ *be the corresponding* (L, M)-*fuzzy* $\mathbf{F}$-*neighborhood system. Define the functions* $\mathbf{F}I^\tau : L^X \longrightarrow M^{J(L^X)}$ *by* $\mathbf{F}I^\tau(A)(x_\lambda) = \mathbf{F}N^\tau_{x_\lambda}(A)$. *Then* $\mathbf{F}I^\tau$ *is an* (L, M)-*fuzzy* $\mathbf{F}$-*interior operator on* X.

Definition 2.20. Let (X, τ) be an (L, M)-fuzzy topological space and $A \in L^X$. Define

$$\mathbf{Fcon}(A) = \bigwedge_{\substack{A_1, A_2 \in L^X \{\perp_{L^X}\}, \\ A = A_1 \vee A_2}} \left\{ \bigvee_{x_\lambda \leq A_1} \mathbf{F}C(A_2)(x_\lambda) \vee \bigvee_{y_\mu \leq A_2} \mathbf{F}C(A_1)(y_\mu) \right\}.$$

Then $\mathbf{Fcon}(A)$ is said to be the $\mathbf{F}$-connectedness degree of A.

Theorem 2.21. *Let* (X, τ) *be an* (L, M)-*fuzzy topological space and* $A \in L^X$. *Then*

$$\mathbf{Fcon}(A) = \bigwedge_{\substack{A \wedge A_1 \neq \perp_{L^X},\ A \wedge A_2 \neq \perp_{L^X}, \\ A \wedge A_1 \wedge A_2 \neq \perp_{L^X},\ A \leq A_1 \vee A_2}} \left\{ (\mathscr{F}(A_1'))' \vee (\mathscr{F}(F_2'))' \right\}.$$

Proof. Straightforward. □

Definition 2.22. [25, 26] An L-fuzzy inclusion on X is a function $\tilde{\subset} : L^X \times L^X \longrightarrow L$ is given by the equality $\tilde{\subset}(A_1, A_2) = \bigwedge_{x \in X}(A_1'(x) \vee A_2(x))$. It is customary to denote the L-fuzzy inclusion $\tilde{\subset}(A_1, A_2)$ by the symbol $[A_1 \tilde{\subset} A_2]$.

Lemma 2.23. [21] *Let* $\varphi : X \longrightarrow Y$ *be a function. Then for any* $\mathcal{B} \subseteq L^Y$, *we have*

$$\bigwedge_{y \in Y} \left\{ \varphi^{\Rightarrow}(A)'(y) \vee \bigvee_{B \in \mathcal{B}} B(y) \right\} = \bigwedge_{x \in X} \left\{ B'(x) \vee \bigvee_{B \in \mathcal{B}} \varphi^{\Leftarrow}(B)(x) \right\}.$$

In [3], Ghareeb introduced the notion of **F**-compactness in L-fuzzy topology based on L-fuzzy **F**-open operator. In the following definitions, we define the degree of **F**-compactness, $\mathbf{F}T_1$, and $\mathbf{F}T_2$ in (L, M)-fuzzy topological spaces with the help of (L, M)-fuzzy **F**-open operator and the implication operation.

Definition 2.24. Let (X, τ) be an (L, M)-fuzzy topological space and $L = M$. For any $A \in L^X$, let

$$\begin{aligned}\mathbf{FCM}_\tau(A) &= \bigwedge_{\mathcal{U}\subseteq L^X} \left\{\mathscr{F}(\mathcal{U}) \mapsto \left(\left[A \tilde{\subset} \bigvee \mathcal{U}\right] \mapsto \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[A \tilde{\subset} \bigvee \mathcal{V}\right]\right)\right\}\\ &= \bigwedge_{\mathcal{U}\subseteq L^X} \left\{ \bigwedge_{A_1\in\mathcal{U}} \mathscr{F}(A_1) \mapsto \left\{ \bigwedge_{x\in X} \left(A' \vee \bigvee_{A_1\in\mathcal{U}} A_1\right)(x)\right.\right.\\ &\left.\left.\mapsto \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \bigwedge_{x\in X} \left(A' \vee \bigvee_{A_1\in\mathcal{V}} A_1\right)(x)\right\}\right\}.\end{aligned}$$

Then $\mathbf{FCM}_\tau(A)$ is called the **F**-compactness degree of A with respect to τ.

Definition 2.25. Let (X, τ) be an (L, M)-fuzzy topological space. Then:

(1) The degree $\mathbf{F}T_1(X, \tau)$ to which (X, τ) is $\mathbf{F}T_1$ is defined by:

$$\mathbf{F}T_1(X, \tau) = \bigwedge_{a_1 \not\leq a_2} \bigvee_{a_1 \not\leq A \geq a_2} \mathscr{F}(A').$$

(2) The degree $\mathbf{F}T_2(X, \tau)$ to which (X, τ) is $\mathbf{F}T_2$ is defined by:

$$\mathbf{F}T_2(X, \tau) = \bigwedge_{a_1 \not\leq a_2} \bigvee \left\{\mathscr{F}(A_1') \wedge \mathscr{F}(A_2) | a_1 \not\leq A_1 \geq A_2 \geq a_2\right\}.$$

3. Degree of F-openness, F-continuity and F-irresolutness for functions in (L, M)-fuzzy topological spaces

In this section, we will introduce the notions of **F**-openness, **F**-continuity, and **F**-irresolutness degree for functions in (L, M)-fuzzy topological spaces. Further, we will prove that this notions can be characterized by (L, M)-fuzzy quasi **F**-neighborhood systems, (L, M)-fuzzy **F**-neighborhood systems, (L, M)-fuzzy **F**- interior operators and (L, M)-fuzzy **F**-closure operators.

Definition 3.1. Let $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ be a function between two (L, M)-fuzzy topological spaces. Then:

(1) The degree $\mathbf{Fc}(\varphi)$ to which φ is **F**-continuous is defined by

$$\mathbf{Fc}(\varphi) = \bigwedge_{B\in L^Y} \left\{\sigma(B) \mapsto \mathscr{F}_1(\varphi^{\Leftarrow}(B))\right\}.$$

(2) The degree $\mathbf{Fo}(\varphi)$ to which φ is **F**-open is defined by

$$\mathbf{Fo}(\varphi) = \bigwedge_{A\in L^X} \left\{\mathscr{F}_1(A) \mapsto \mathscr{F}_2(\varphi^{\Rightarrow}(A))\right\}.$$

(3) The degree $\mathbf{Fi}(\varphi)$ to which φ is $\mathbf{F}$-irresolute is defined by

$$\mathbf{Fi}(\varphi) = \bigwedge_{B\in L^Y} \left\{ \mathscr{F}_2(B) \mapsto \mathscr{F}_1(\varphi^{\Leftarrow}(B)) \right\}.$$

Definition 3.2. For a bijective function $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ between two (L, M)-fuzzy topological spaces (X, τ) and (Y, σ), the degree $\mathbf{F\text{-}Hom}(\varphi)$ to which φ is $\mathbf{F}$-homomorphism is given by

$$\mathbf{F\text{-}Hom}(\varphi) = \mathbf{Fi}(\varphi) \wedge \mathbf{Fo}(\varphi).$$

Remark 3.3. **(1)** Based on Lemma 2.1 **(2)**, $\mathbf{Fc}(\varphi) = \top_M$ implies to $\mathscr{F}_1(\varphi^{\Leftarrow}(B)) \geq \sigma(B)$ for any $B \in L^Y$. This is exactly the definition of $\mathbf{F}$-continuous function. Similarly, for the cases $\mathbf{Fo}(\varphi) = \top_M$ and $\mathbf{Fi}(\varphi) = \top_M$, then **(2)** and **(3)** in Definition 3.1 are precisely the $\mathbf{F}$-open and $\mathbf{F}$-irresolute functions definition as in the sense of Ghareeb [2].

(2) For the identity function $i : (X, \tau) \longrightarrow (X, \tau)$, we have $\mathbf{Fi}(i) = \mathbf{Fo}(i) = \mathbf{F\text{-}Hom}(i) = \top_M$.

Based on Definition 3.1 and Corollary 2.11, we have the following corollary.

Corollary 3.4. *Let $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ be a function between two (L, M)-fuzzy topological spaces. Then:*

(1) The degree $\mathbf{Fc}(\varphi)$ to which φ is $\mathbf{F}$-continuous is characterized by

$$\mathbf{Fc}(\varphi) = \bigwedge_{B\in L^Y} \left\{ \sigma^*(B) \mapsto \mathscr{F}_1^*(\varphi^{\Leftarrow}(B)) \right\}.$$

(2) The degree $\mathbf{Fi}(\varphi)$ to which φ is $\mathbf{F}$-irresolute is characterized by

$$\mathbf{Fi}(\varphi) = \bigwedge_{B\in L^Y} \left\{ \mathscr{F}_2^*(B) \mapsto \mathscr{F}_1^*(\varphi^{\Leftarrow}(B)) \right\}.$$

Definition 3.5. For any function $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ between two (L, M)-fuzzy topological spaces, the degree $\mathbf{Fcl}(\varphi)$ to which φ is an $\mathbf{F}$-closed function is given by

$$\mathbf{Fcl}(\varphi) = \bigwedge_{A\in L^X} \left\{ \mathscr{F}_1^*(A) \mapsto \mathscr{F}_2^*(\varphi^{\Rightarrow}(A)) \right\}.$$

Theorem 3.6. *Let (X, τ), (Y, σ) and (Z, δ) be (L, M)-fuzzy topological spaces, $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ be two functions. Then:*

(1) $\mathbf{Fi}(\varphi) \wedge \mathbf{Fi}(\psi) \leq \mathbf{Fi}(\psi \circ \varphi)$.

(2) $\mathbf{Fo}(\varphi) \wedge \mathbf{Fo}(\psi) \leq \mathbf{Fi}(\psi \circ \varphi)$.

(3) $\mathbf{Fcl}(\varphi) \wedge \mathbf{Fi}(\psi) \leq \mathbf{Fcl}(\psi \circ \varphi)$.

Proof. Since the proof of **(2)** and **(3)** is clear, we only prove **(1)**. By using Definition 3.1 and Lemma 2.1 **(4)**, we obtain

$$\mathbf{Fi}(\varphi) \wedge \mathbf{Fi}(\psi) = \bigwedge_{B\in L^Y} \left\{ \mathscr{F}_2(B) \mapsto \mathscr{F}_1(\varphi^{\Leftarrow}(B)) \right\} \wedge \bigwedge_{C\in L^Z} \left\{ \mathscr{F}_3(C) \mapsto \mathscr{F}_2(\psi^{\Leftarrow}(C)) \right\}$$

$$
\begin{aligned}
&\leq \bigwedge_{C\in L^Z} \left\{ \mathscr{F}_2(\psi^{\Leftarrow}(C)) \mapsto \mathscr{F}_1(\varphi^{\Leftarrow}(\psi^{\Leftarrow}(C))) \right\} \\
&\quad \wedge \bigwedge_{C\in L^Z} \left\{ \mathscr{F}_3(C) \mapsto \mathscr{F}_2(\psi^{\Leftarrow}(C)) \right\} \\
&= \bigwedge_{C\in L^Z} \left\{ \Big(\mathscr{F}_2(\psi^{\Leftarrow}(C)) \mapsto \mathscr{F}_1((\psi \circ \psi)^{\Leftarrow}(C)) \Big) \right. \\
&\quad \left. \wedge \Big(\mathscr{F}_3(C) \mapsto \mathscr{F}_2(\psi^{\Leftarrow}(C)) \Big) \right\} \\
&\leq \bigwedge_{C\in L^Z} \left\{ \mathscr{F}_3(\psi^{\Leftarrow}(C)) \mapsto \mathscr{F}_1((\psi \circ \varphi)^{\Leftarrow}(C)) \right\} \\
&= \mathbf{Fi}(\psi \circ \varphi).
\end{aligned}
$$

□

By using Definition 3.2 and Theorem 3.6, we have the following corollary.

Corollary 3.7. *Let (X, τ), (Y, σ) and (Z, δ) be (L, M)-fuzzy topological spaces, $\varphi : X \longrightarrow Y$ and $\psi : Y \longrightarrow Z$ be two bijective functions. Then* $\mathbf{F}\text{-}\mathbf{Hom}(\varphi) \wedge \mathbf{F}\text{-}\mathbf{Hom}(\psi) \leq \mathbf{F}\text{-}\mathbf{Hom}(\psi \circ \varphi)$.

Theorem 3.8. *Let (X, τ), (Y, σ) and (Z, δ) be (L, M)-fuzzy topological spaces and $\psi : Y \longrightarrow Z$ be a surjective function. Then:*

- **(1)** $\mathbf{Fo}(\psi \circ \varphi) \wedge \mathbf{Fi}(\varphi) \leq \mathbf{Fo}(\psi)$.
- **(2)** $\mathbf{Fcl}(\psi \circ \varphi) \wedge \mathbf{Fi}(\varphi) \leq \mathbf{Fcl}(\psi)$.

Proof. **(1)** Since φ is a surjective function, we have $(\psi \circ \varphi)^{\Rightarrow}(\varphi^{\Leftarrow}(B)) = \psi^{\Rightarrow}(B)$ for each $B \in L^Y$. Based on Lemma 2.1 **(4)**, we get

$$
\begin{aligned}
\mathbf{Fo}(\psi \circ \varphi) \wedge \mathbf{Fi}(\varphi) &= \bigwedge_{A\in L^X} \left\{ \mathscr{F}_1(A) \mapsto \mathscr{F}_3((\psi \circ \varphi)^{\Rightarrow}(A)) \right\} \\
&\quad \wedge \bigwedge_{B\in L^Y} \left\{ \mathscr{F}_2(B) \mapsto \mathscr{F}_1(\varphi^{\Leftarrow}(B)) \right\} \\
&\leq \bigwedge_{B\in L^Y} \left\{ \mathscr{F}_1(\varphi^{\Leftarrow}(B)) \mapsto \mathscr{F}_3((\psi \circ \varphi)^{\Rightarrow}(\varphi^{\Leftarrow}(B))) \right\} \\
&\quad \wedge \bigwedge_{B\in L^Y} \left\{ \mathscr{F}_2(B) \mapsto \mathscr{F}_1(\psi^{\Leftarrow}(B)) \right\} \\
&= \bigwedge_{B\in L^Y} \left\{ \Big(\mathscr{F}_1(\varphi^{\Leftarrow}(B)) \mapsto \mathscr{F}_3(\psi^{\Rightarrow}(B)) \Big) \right. \\
&\quad \left. \wedge \Big(\mathscr{F}_2(B) \mapsto \mathscr{F}_1(\varphi^{\Leftarrow}(B)) \Big) \right\}
\end{aligned}
$$

$$\leq \bigwedge_{B\in L^Y} \left\{\mathscr{F}_2(B) \mapsto \mathscr{F}_3(\psi^{\Rightarrow}(B))\right\}$$
$$= \mathbf{Fo}(\psi).$$

Analogously, we can prove **(2)**. □

Similarly, the following theorem is true.

Theorem 3.9. *Let (X,τ), (Y,σ) and (Z,δ) be (L,M)-fuzzy topological spaces, $\varphi : X \longrightarrow Y$ be an injective function and $\psi : Y \longrightarrow Z$ be any function. Then*

(1) $\mathbf{Fo}(\psi\circ\varphi) \wedge \mathbf{Fi}(\psi) \leq \mathbf{Fo}(\varphi)$.
(2) $\mathbf{Fcl}(\psi\circ\varphi) \wedge \mathbf{Fi}(\psi) \leq \mathbf{Fcl}(\varphi)$.

Theorem 3.10. *Let (X,τ) and (Y,σ) be two (L,M)-fuzzy topological spaces and $\varphi : X \longrightarrow Y$ is a bijective function. Then*

(1) $\mathbf{Fi}(\varphi) = \bigwedge_{A\in L^X} \left\{\mathscr{F}_2(\varphi^{\Rightarrow}(A)) \mapsto \mathscr{F}_1(A)\right\}$.
(2) $\mathbf{Fo}(\varphi) = \bigwedge_{B\in L^Y} \left\{\mathscr{F}_1(\varphi^{\Leftarrow}(B)) \mapsto \mathscr{F}_2(B)\right\}$.
(3) $\mathbf{Fi}(\varphi^{-1}) = \mathbf{Fo}(\varphi) = \mathbf{Fcl}(\varphi)$.

Proof. The proof of **(2)** is similar to **(1)**, we only prove **(1)** and **(3)**.

(1) Since φ is a bijective function, we have $\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(A)) = A$ for any $A \in L^X$, and $\varphi^{\Rightarrow}(\varphi^{\Leftarrow}(B)) = B$ for any $B \in L^Y$. It follows that

$$\bigwedge_{A\in L^X} \left\{\mathscr{F}_2(\varphi^{\Rightarrow}(A)) \mapsto \mathscr{F}_1(A)\right\} = \bigwedge_{A\in L^X} \left\{\mathscr{F}_2(\varphi^{\Rightarrow}(A)) \mapsto \mathscr{F}_1(\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(A)))\right\}$$
$$\geq \bigwedge_{B\in L^Y} \left\{\mathscr{F}_2(B) \mapsto \mathscr{F}_1(\varphi^{\Leftarrow}(B))\right\}$$
$$= \bigwedge_{B\in L^Y} \left\{\mathscr{F}_2(\varphi^{\Rightarrow}(\varphi^{\Leftarrow}(B))) \mapsto \mathscr{F}_1(\varphi^{\Leftarrow}(B))\right\}$$
$$\geq \bigwedge_{A\in L^X} \left\{\mathscr{F}_2(\varphi^{\Rightarrow}(A)) \mapsto \mathscr{F}_1(A)\right\}.$$

Hence

$$\mathbf{Fi}(\varphi) = \bigwedge_{B\in L^Y} \left\{\mathscr{F}_2(B) \mapsto \mathscr{F}_1(\varphi^{\Leftarrow}(B))\right\}$$
$$= \bigwedge_{A\in L^X} \left\{\mathscr{F}_2(\varphi^{\Rightarrow}(A)) \mapsto \mathscr{F}_1(A)\right\}.$$

(3) Since φ is a bijective function, we get $(\varphi^{-1})^{\Leftarrow}(A) = \varphi^{\Rightarrow}(A)$ and $\varphi^{\Rightarrow}(A') = \varphi^{\Rightarrow}(A)'$ for any $A \in L^X$. Therefore

$$\mathbf{Fi}(\varphi^{-1}) = \bigwedge_{A\in L^X} \left\{\mathscr{F}_1(A) \mapsto \mathscr{F}_2((\varphi^{-1})^{\Leftarrow}(A))\right\}$$

$$= \bigwedge_{A\in L^X} \left\{ \mathscr{F}_1(A) \mapsto \mathscr{F}_2(\varphi^{\Rightarrow}(A)) \right\}$$
$$= \mathbf{Fo}(\varphi).$$

and

$$\begin{aligned}
\mathbf{Fo}(\varphi^{-1}) &= \bigwedge_{A\in L^X} \left\{ \mathscr{F}_1(A) \mapsto \mathscr{F}_2(\varphi^{\Rightarrow}(A)) \right\} \\
&= \bigwedge_{A\in L^X} \left\{ \mathscr{F}_1(A') \mapsto \mathscr{F}_2(\varphi^{\Rightarrow}(A')) \right\} \\
&= \bigwedge_{A\in L^X} \left\{ \mathscr{F}_1(A') \mapsto \mathscr{F}_2(\varphi^{\Rightarrow}(A)') \right\} \\
&= \mathbf{Fcl}(\varphi).
\end{aligned}$$

The proof is completed. □

Corollary 3.11. *Let $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ be a bijective function between (L, M)-fuzzy topological spaces (X, τ) and (Y, σ), then*

(1) $\mathbf{F}\text{-}\mathbf{Hom}(\varphi) = \mathbf{Fi}(\varphi) \wedge \mathbf{Fi}(\varphi^{-1}) = \mathbf{Fi}(\varphi) \wedge \mathbf{Fcl}(\varphi)$.

(2) $\mathbf{F}\text{-}\mathbf{Hom}(\varphi) = \bigwedge_{A\in L^X} \left\{ \mathscr{F}_2(\varphi^{\Rightarrow}(A)) \leftrightarrow \mathscr{F}_1(A) \right\}$.

(3) $\mathbf{F}\text{-}\mathbf{Hom}(\varphi) = \bigwedge_{B\in L^Y} \left\{ \mathscr{F}_1(\varphi^{\Leftarrow}(B)) \leftrightarrow \mathscr{F}_2(B) \right\}$.

In the following, we characterize **F**-irresolutness and **F**-openness degree with the help of (L, M)-fuzzy quasi **F**-neighborhood systems, (L, M)-fuzzy **F**-closure operators, and (L, M)-fuzzy **F**-interior operators.

Corollary 3.12. *Let $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ be a function between two (L, M)-fuzzy topological spaces. Then*

(1) $\mathbf{Fi}(\varphi) = \bigwedge_{B\in L^Y} \bigwedge_{x_\lambda\in J(L^X)} \left\{ FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(B) \mapsto FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B)) \right\}$.

(2) $\mathbf{Fi}(\varphi) = \bigwedge_{B\in L^Y} \bigwedge_{x_\lambda\in J(L^X)} \left\{ FN^{\mathscr{F}_2}_{\varphi(x)_\lambda}(B) \mapsto FN^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B)) \right\}$.

(3) $\mathbf{Fi}(\varphi) = \bigwedge_{B\in L^Y} \bigwedge_{x_\lambda\in J(L^X)} \left\{ FI^{\mathscr{F}_2}(B)(\varphi(x)_\lambda) \mapsto FI^{\mathscr{F}_1}(\varphi^{\Leftarrow}(B))(x_\lambda) \right\}$.

(4) $\mathbf{Fi}(\varphi) = \bigwedge_{B\in L^Y} \bigwedge_{x_\lambda\in J(L^X)} \left\{ FCl^{\mathscr{F}_2}(B)(\varphi(x)_\lambda)' \mapsto FCl^{\mathscr{F}_1}(\varphi^{\Leftarrow}(B))(x_\lambda)' \right\}$.

Proof. **(1)** Since $FQ^{\mathscr{F}_1}_{x_\lambda}(A) = \bigvee_{x_\lambda \not\leq A'_1 > A'} \mathscr{F}(A_1)$ for any $A \in L^X$, and for all $x_\lambda \in J(L^X)$ and B_1, $B_2 \in L^Y$, we have $\varphi(x)_\lambda \not\leq B'_1 \geq B'_2 \Rightarrow x_\lambda \not\leq \varphi^{\Leftarrow}(B_1)' \geq \varphi^{\Leftarrow}(B_2)$. Therefore, we have

$$\bigwedge_{B_2\in L^Y}\bigwedge_{x_\lambda\in J(L^X)}\left\{FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(B_2)\mapsto FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B_2))\right\}$$
$$=\bigwedge_{B_2\in L^Y}\bigwedge_{x_\lambda\in J(L^X)}\left\{\bigvee_{\varphi(x)_\lambda\not\leq B_1{}'\geq B_2'}\mathscr{F}_2(B_1)\mapsto\bigvee_{x_\lambda\not\leq A_1{}'\geq\varphi^{\Leftarrow}(B)'}\mathscr{F}_1(A_1)\right\}$$
$$\geq\bigwedge_{B_2\in L^Y}\bigwedge_{x_\lambda\in J(L^X)}\left\{\bigvee_{\varphi(x)_\lambda\not\leq B_1{}'\geq B_2'}\mathscr{F}_2(B_1)\mapsto\bigvee_{x_\lambda\not\leq\varphi^{\Leftarrow}(B_3)'\geq\varphi^{\Leftarrow}(B_2)'}\mathscr{F}_1(\varphi^{\Leftarrow}(B_3))\right\}$$
$$\geq\bigwedge_{B_2\in L^Y}\bigwedge_{x_\lambda\in J(L^X)}\bigwedge_{\varphi(x)_\lambda\not\leq B_1{}'\geq B_2{}'}\left\{\mathscr{F}_2(B_1)\mapsto\mathscr{F}_1(\varphi^{\Leftarrow}(B_1))\right\}$$
$$\geq\bigwedge_{B_2\in L^Y}\left\{\mathscr{F}_2(B_2)\mapsto\mathscr{F}_1(\varphi^{\Leftarrow}(B_2))\right\}=\mathbf{Fi}(\varphi).$$

Since $\mathscr{F}(A)=\bigwedge_{x_\lambda\not\leq A'}FQ^{\mathscr{F}_1}_{x_\lambda}(A)$ for any $A\in L^X$ and $x_\lambda\not\leq\varphi^{\Leftarrow}(B)'\Rightarrow\varphi(x)_\lambda\not\leq B'$ for any $x_\lambda\in J(L^X)$ and $B\in L^Y$, we have

$$\mathbf{Fi}(\varphi)=\bigwedge_{B\in L^Y}\left\{\mathscr{F}_2(B)\mapsto\mathscr{F}_1(\varphi^{\Leftarrow}(B))\right\}$$
$$=\bigwedge_{B\in L^Y}\left\{\bigwedge_{y_\mu\not\leq B'}FQ^{\mathscr{F}_2}_{y_\mu}(B)\mapsto\bigwedge_{x_\lambda\not\leq\varphi^{\Leftarrow}(B)'}FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B))\right\}$$
$$\geq\bigwedge_{B\in L^Y}\left\{\bigwedge_{\varphi(x)_\lambda\not\leq B'}FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(B)\mapsto\bigwedge_{x_\lambda\not\leq\varphi^{\Leftarrow}(B)'}FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B))\right\}$$
$$\geq\bigwedge_{B\in L^Y}\bigwedge_{x_\lambda\not\leq\varphi^{\Leftarrow}(B)'}\left\{FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(B)\mapsto FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B))\right\}$$
$$\geq\bigwedge_{B\in L^Y}\bigwedge_{x_\lambda\in J(L^X)}\left\{FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(B)\mapsto FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B))\right\}.$$

which completes the proof of **(1)**. Similarly, we can prove **(2)**, **(3)** and **(4)** from Theorem 2.16 and Theorem 2.19. □

Theorem 3.13. *Let $\varphi:(X,\tau)\longrightarrow(Y,\sigma)$ be a function between two (L,M)-fuzzy topological spaces. Then*

(1) $\mathbf{Fo}(\varphi)=\bigwedge_{A\in L^X}\bigwedge_{x_\lambda\in J(L^X)}\left\{FQ^{\mathscr{F}_1}_{x_\lambda}(A)\mapsto FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A))\right\}.$

(2) $\mathbf{Fo}(\varphi)=\bigwedge_{A\in L^X}\bigwedge_{x_\lambda\in J(L^X)}\left\{FN^{\mathscr{F}_1}_{x_\lambda}(A)\mapsto FN^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A))\right\}.$

(3) $\mathbf{Fo}(\varphi)=\bigwedge_{A\in L^X}\bigwedge_{x_\lambda\in J(L^X)}\left\{FI^{\mathscr{F}_1}(A)(x_\lambda)\mapsto FI^{\mathscr{F}_2}(\varphi^{\Rightarrow}(A))(\varphi(x)_\lambda)\right\}.$

(4) $\mathbf{Fo}(\varphi) = \bigwedge_{A\in L^X} \bigwedge_{x_\lambda\in J(L^X)} \left\{ FCl^{\mathscr{F}_1}(A')(x_\lambda)' \mapsto FCl^{\mathscr{F}_2}(\varphi^{\Rightarrow}(A)')(\varphi(x)_\lambda)' \right\}.$

(5) $\mathbf{Fo}(\varphi) = \bigwedge_{B\in L^Y} \bigwedge_{x_\lambda\in J(L^X)} \left\{ FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B)) \mapsto FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(B) \right\}.$

(6) $\mathbf{Fo}(\varphi) = \bigwedge_{B\in L^Y} \bigwedge_{x_\lambda\in J(L^X)} \left\{ FN^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B)) \mapsto FN^{\mathscr{F}_2}_{\varphi(x)_\lambda}(B) \right\}.$

(7) $\mathbf{Fo}(\varphi) = \bigwedge_{B\in L^Y} \bigwedge_{x_\lambda\in J(L^X)} \left\{ FI^{\mathscr{F}_1}(\varphi^{\Leftarrow}(B))(x_\lambda) \mapsto FI^{\mathscr{F}_2}(B)(\varphi(x)_\lambda) \right\}.$

(8) $\mathbf{Fo}(\varphi) = \bigwedge_{B\in L^Y} \bigwedge_{x_\lambda\in J(L^X)} \left\{ FCl^{\mathscr{F}_1}(\varphi^{\Leftarrow}(B)')(x_\lambda)' \mapsto FCl^{\mathscr{F}_2}(B')(\varphi(x)_\lambda)' \right\}.$

Proof. We prove only **(5)**. Firstly,

$$\begin{aligned}
&\bigwedge_{B\in L^Y} \bigwedge_{x_\lambda\in J(L^X)} \left\{ FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B)) \mapsto FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(B) \right\} \\
&= \bigwedge_{B\in L^Y} \bigwedge_{x_\lambda\in J(L^X)} \left\{ \bigvee_{x_\lambda \not\leq A_1{}' \geq \varphi^{\Leftarrow}(B)'} \mathscr{F}_1(A_1) \mapsto \bigvee_{\varphi(x)_\lambda \not\leq B_1{}' \geq B'} \mathscr{F}_2(B_1) \right\} \\
&\geq \bigwedge_{B\in L^Y} \bigwedge_{x_\lambda\in J(L^X)} \left\{ \bigvee_{x_\lambda \not\leq A_1{}' \geq \varphi^{\Leftarrow}(B)'} \mathscr{F}_1(A_1) \mapsto \bigvee_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A_2)' \geq B'} \mathscr{F}_2(\varphi^{\Rightarrow}(A_2)) \right\} \\
&\geq \bigwedge_{B\in L^Y} \bigwedge_{x_\lambda\in J(L^X)} \left\{ \bigvee_{x_\lambda \not\leq A_1{}' \geq \varphi^{\Leftarrow}(B)'} \mathscr{F}_1(A_1) \mapsto \bigvee_{x_\lambda \not\leq A_2' \geq \varphi^{\Leftarrow}(B)'} \mathscr{F}_2(\varphi^{\Rightarrow}(A_2)) \right\} \\
&= \bigwedge_{B\in L^Y} \bigwedge_{x_\lambda\in J(L^X)} \bigwedge_{x_\lambda \not\leq A_1{}' \geq \varphi^{\Leftarrow}(B)'} \left\{ \mathscr{F}_1(A_1) \mapsto \bigvee_{x_\lambda \not\leq A_2' \geq \varphi^{\Leftarrow}(B)'} \mathscr{F}_2(\varphi^{\Rightarrow}(A_2)) \right\} \\
&\geq \bigwedge_{B\in L^Y} \bigwedge_{x_\lambda\in J(L^X)} \bigwedge_{x_\lambda \not\leq A_1{}' \geq \varphi^{\Leftarrow}(B)'} \left\{ \mathscr{F}_1(A_1) \mapsto \mathscr{F}_2(\varphi^{\Rightarrow}(A_1)) \right\} \\
&\geq \bigwedge_{A\in L^X} \left\{ \mathscr{F}_1(A) \mapsto \mathscr{F}_2(\varphi^{\Rightarrow}(A)) \right\} = \mathbf{Fo}(\varphi).
\end{aligned}$$

Thus we have to prove

$$\bigwedge_{y_\mu \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{y_\mu}(\varphi^{\Rightarrow}(A)) = \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)).$$

for any $A \in L^X$. It is evident that

$$\bigwedge_{y_\mu \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{y_\mu}(\varphi^{\Rightarrow}(A)) \leq \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)).$$

Now we prove

$$\bigwedge_{y_\mu \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{y_\mu}(\varphi^{\Rightarrow}(A)) \geq \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)).$$

For any $y_\mu \in J(L^Y)$ with $y_\mu \not\leq \varphi^{\Rightarrow}(A)'$, we have $\mu \not\leq (\varphi^{\Rightarrow}(A)(y))' = \bigwedge_{\varphi(x)=y} A(x)'$. Then there exists $x \in X$ such that $\varphi(x) = y$ and $\mu \leq A(x)'$. Then $\mu \not\leq \bigwedge_{\varphi(x)=\varphi(z)} A(z)' = \varphi^{\Rightarrow}(A)(\varphi(x))'$. Hence $\varphi(x)_\mu \leq \varphi^{\Rightarrow}(A)'$. Since

$$\bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)) \leq FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)) = FQ^{\mathscr{F}_2}_{y_\mu}(\varphi^{\Rightarrow}(A)).$$

we get

$$\bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)) \leq \bigwedge_{y_\mu \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{y_\mu}(\varphi^{\Rightarrow}(A)).$$

Hence

$$\bigwedge_{y_\mu \not\leq \varphi^{\Rightarrow}(A))'} FQ^{\mathscr{F}_2}_{y_\mu}(\varphi^{\Rightarrow}(A)) = \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)).$$

To prove the following fact:

$$\bigwedge_{A \in L^X} \left\{ \bigwedge_{x_\lambda \not\leq A'} FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(A))) \mapsto \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)) \right\}$$

$$\geq \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B)) \mapsto FQ^{\mathscr{F}_1}_{\varphi(x)_\lambda}(B) \right\}.$$

Assume that $r \in M$ such that

$$r \ll \bigwedge_{B \in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B)) \mapsto FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(B) \right\}.$$

Then $r \leq FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B)) \mapsto FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(B)$ for each $B \in L^Y$ and $x_\lambda \in J(L^X)$. Based on Lemma 2.1 **(1)**, we have $r \wedge FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B)) \leq FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(B)$. For any $A \in L^X$ and $\varphi(x)_\lambda \leq \varphi^{\Rightarrow}(A)'$, we have $\lambda \leq \varphi^{\Rightarrow}(A)(\varphi(x))' = \bigwedge_{\varphi(x)=\varphi(z)} A(z)'$. Then there exists $z \in X$ such that $\varphi(z) = \varphi(x)$ and $\lambda \not\leq A(z)'$. It follows that $z_\lambda \not\leq A'$. From

$$\begin{aligned} r \wedge \bigwedge_{x_\lambda \not\leq A'} FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(A))) &\leq r \wedge FQ^{\mathscr{F}_1}_{z_\lambda}(\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(A))) \\ &\leq FQ^{\mathscr{F}_2}_{\varphi(z)_\lambda}(\varphi^{\Rightarrow}(A)) = FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)), \end{aligned}$$

we have

$$r \wedge \bigwedge_{x_\lambda \not\leq A'} FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(A))) \leq \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)).$$

Based on Lemma 2.1 **(1)**, we have

$$r \leq \bigwedge_{x_\lambda \not\leq A'} FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(A))) \mapsto \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)).$$

Hence

$$r \leq \bigwedge_{A\in L^X} \left\{ \bigwedge_{x_\lambda \not\leq A'} FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(A))) \leq \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)) \right\}.$$

Since r is arbitrary, we obtain

$$\bigwedge_{A\in L^X} \left\{ \bigwedge_{x_\lambda \not\leq A'} FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(A))) \leq \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)) \right\}$$

$$\geq \bigwedge_{B\in L^Y} \bigwedge_{x_\lambda \in J(L^X)} \left\{ \bigwedge_{x_\lambda \not\leq A'} FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(A)) \leq \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)) \right\}.$$

But $A \leq \varphi^{\Leftarrow}(\varphi^{\Rightarrow}(A))$ for any $A \in L^X$, then

$$\begin{aligned}
\mathbf{Fo}(\varphi) &= \bigwedge_{A\in L^X} \left\{ \mathscr{F}_1(A) \mapsto \mathscr{F}_2(\varphi^{\Rightarrow}(A)) \right\} \\
&= \bigwedge_{A\in L^X} \left\{ \bigwedge_{x_\lambda \not\leq A'} \left\{ FQ^{\mathscr{F}_1}_{x_\lambda}(A) \mapsto \bigwedge_{y_\mu \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{y_\mu}(\varphi^{\Rightarrow}(A)) \right\} \right. \\
&= \bigwedge_{A\in L^X} \left\{ \bigwedge_{x_\lambda \not\leq A'} FQ^{\mathscr{F}_1}_{x_\lambda}(A) \mapsto \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)) \right\} \\
&\geq \bigwedge_{A\in L^X} \left\{ \bigwedge_{x_\lambda \not\leq A'} FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(\varphi^{\Rightarrow}(A))) \mapsto \bigwedge_{\varphi(x)_\lambda \not\leq \varphi^{\Rightarrow}(A)'} FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(\varphi^{\Rightarrow}(A)) \right\} \\
&\geq \bigwedge_{B\in L^Y} \left\{ \bigwedge_{x_\lambda \in J(L^X)} FQ^{\mathscr{F}_1}_{x_\lambda}(\varphi^{\Leftarrow}(B) \mapsto FQ^{\mathscr{F}_2}_{\varphi(x)_\lambda}(B)) \right\},
\end{aligned}$$

which completes the proof. □

4. The application of F-continuity and F-irresoluteness degree in F-compactness, F-connectedness, FT_1, and FT_2 degree

In this section we verify the relationship between **F**-continuity and **F**- irresoluteness degree with **F**-compactness, **F**-connectedness, **F**T_1, and **F**T_2 degree.

Theorem 4.1. *Let $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ be a function between two (L, M)-fuzzy topological spaces and $L = M$. Then*

$$\mathbf{FCM}_{\mathscr{F}_1}(A) \wedge \mathbf{Fi}(\varphi) \leq \mathbf{FCM}_{\mathscr{F}_2}(\varphi^{\Rightarrow}(A))$$

for any $A \in L^X$.

Proof. Assume that $r \in M$ such that $r \ll \mathbf{FCM}_{\mathscr{F}_1}(A) \wedge \mathbf{Fi}(\varphi)$. Then

$$r \ll \mathbf{Fi}(\varphi) = \bigwedge_{B\in L^Y} \left\{ \mathscr{F}_2(B) \mapsto \mathscr{F}_1(\varphi^{\Leftarrow}(B)) \right\},$$

and

$$\begin{aligned} r &\lll \mathbf{FCM}_{\mathscr{F}_1}(A) \\ &= \bigwedge_{\mathcal{A}\in L^X} \left\{ \left(\bigwedge_{A_1\in\mathcal{A}} \mathscr{F}_1(A_1) \wedge \bigwedge_{x\in X} \left(A' \vee \bigvee_{A_1\in\mathcal{A}} A_1 \right)(x) \right) \right. \\ &\left. \mapsto \bigvee_{\mathcal{V}\in 2^{(\mathcal{A})}} \bigwedge_{x\in X} \left(A' \vee \bigvee_{A_1\in\mathcal{V}} A_1 \right)(x) \right\} \end{aligned}$$

Then for any $B \in L^Y$ and $\mathcal{A} \subseteq L^X$, we obtain $r \le \mathscr{F}_2(B) \mapsto \mathscr{F}_1(\varphi^{\Leftarrow}(B))$ and

$$r \le \left(\bigwedge_{A_1\in\mathcal{A}} \mathscr{F}_1(A_1) \wedge \bigwedge_{x\in X} \left(A' \vee \bigvee_{A_1\in\mathcal{A}} A_1 \right)(x) \right) \mapsto \bigvee_{\mathcal{V}\in 2^{(\mathcal{A})}} \bigwedge_{x\in X} \left(A' \vee \bigvee_{A_1\in\mathcal{V}} A_1 \right)(x).$$

By Lemma 2.1 **(1)**, we have $r \wedge \mathscr{F}_2(B) \le \mathscr{F}_1(\varphi^{\Leftarrow}(B))$ for any $B \in L^Y$, and

$$r \wedge \bigwedge_{W\in\mathcal{A}} \mathscr{F}_1(W) \wedge \bigwedge_{x\in X} \left(A' \vee \bigvee_{W\in\mathcal{A}} W \right)(x) \le \bigvee_{\mathcal{V}\in 2^{(\mathcal{A})}} \bigwedge_{x\in X} \left(A' \vee \bigvee_{W\in\mathcal{A}} W \right)(x).$$

To prove

$$\begin{aligned} r \le &\mathbf{FCM}_{\mathscr{F}_2}(\varphi^{\Rightarrow}(A)) \\ = &\bigwedge_{\mathcal{W}\in L^Y} \left\{ \left(\bigwedge_{B_1\in\mathcal{W}} \mathscr{F}_2(B_1) \wedge \bigwedge_{y\in Y} \left(\varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1\in\mathcal{W}} B_1 \right)(y) \right) \right. \\ &\left. \mapsto \bigvee_{\mathcal{D}\in 2^{(\mathcal{W})}} \bigwedge_{y\in Y} \left(\varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1\in\mathcal{W}} B_1 \right)(y) \right\}, \end{aligned}$$

for all $\mathcal{W} \subseteq L^Y$, let $\varphi^{\Leftarrow}(\mathcal{W}) = \{\varphi^{\Leftarrow}(B_1) | B_1 \in \mathcal{W}\} \subseteq L^X$. Then by Lemma 2.23, we have

$$\begin{aligned} &r \wedge \bigwedge_{B_1\in\mathcal{W}} \mathscr{F}_2(B_1) \wedge \bigwedge_{y\in Y} \left(\varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1\in\mathcal{W}} B_1 \right)(y) \\ &\le r \wedge \bigwedge_{B_1\in\mathcal{W}} \mathscr{F}_1(\varphi^{\Leftarrow}(B_1)) \wedge \bigwedge_{y\in Y} \left(\varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1\in\mathcal{W}} B_1 \right)(y) \\ &= r \wedge \bigwedge_{B_1\in\mathcal{W}} \mathscr{F}_1(\varphi^{\Leftarrow}(B_1)) \bigwedge_{x\in X} \left(A' \vee \bigvee_{B_1\in\mathcal{W}} \varphi^{\Leftarrow}(B_1) \right)(x) \\ &= r \wedge \bigwedge_{A_1\in\varphi^{\Leftarrow}(\mathcal{W})} \mathscr{F}_1(A_1) \wedge \bigwedge_{x\in X} \left(A' \vee \bigvee_{A_1\in\varphi^{\Leftarrow}(\mathcal{W})} (A_1) \right)(x) \\ &\le \bigvee_{\mathcal{V}\in 2^{(\varphi^{\Leftarrow}(\mathcal{W}))}} \bigwedge_{x\in X} \left(A' \vee \bigvee_{A_1\in\mathcal{W}} (A_1))(x) \right. \\ &= \bigvee_{\mathcal{D}\in 2^{(\mathcal{W})}} \bigwedge_{x\in X} \left(A' \vee \bigvee_{B_1\in\mathcal{D}} \varphi^{\Leftarrow}(B_1) \right)(x) \end{aligned}$$

$$= \bigvee_{\mathcal{D}\in 2^{(\mathcal{W})}} \bigwedge_{y\in Y} \Big(\varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1\in\mathcal{D}} B_1\Big)(y).$$

By using Lemma 2.1 **(1)**, we know

$$r \leq \Big(\bigwedge_{B_1\in\mathcal{W}} \mathscr{F}_2(B_1) \wedge \bigwedge_{y\in Y} \Big(\varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1\in\mathcal{W}} B_1\Big)(y)\Big)$$
$$\mapsto \bigvee_{\mathcal{D}\in 2^{(\mathcal{W})}} \bigwedge_{y\in Y} \Big(\varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1\in\mathcal{D}} B_1\Big)(y).$$

Thus

$$r \leq \bigwedge_{\mathcal{W}\subseteq L^Y} \Big\{\Big(\bigwedge_{B_1\in\mathcal{W}} \mathscr{F}_2(B_1) \wedge \bigwedge_{y\in Y} \Big(\varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1\in\mathcal{W}} B_1\Big)(y)\Big)$$
$$\mapsto \bigvee_{\mathcal{D}\in 2^{(\mathcal{W})}} \bigwedge_{y\in Y} \Big(\varphi^{\Rightarrow}(A)' \vee \bigvee_{B_1\in\mathcal{D}} B_1\Big)(y)\Big\} = \mathbf{FCM}_{\mathscr{F}_2}(\varphi^{\Rightarrow}(A)).$$

Since r is arbitrary, we have $\mathbf{FCM}_{\mathscr{F}_1}(A) \wedge \mathbf{Fi}(\varphi) \leq \mathbf{FCM}_{\mathscr{F}_2}(\varphi^{\Rightarrow}(A))$. The proof is completed. □

From Theorem 4.1, the following corollary is true.

Corollary 4.2. *Let$\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ be a surjective function between the two (L, M)-fuzzy topological spaces (X, τ) and (Y, σ) and $L = M$. Then*

$$\mathbf{FCM}_{\mathscr{F}_1}(\top_{L^X}) \wedge \mathbf{Fi}(\varphi) \leq \mathbf{FCM}_{\mathscr{F}_2}(\top_{L^Y}).$$

In general topology, it is well known that $\varphi(A)$ is connected subset if A is connected and φ is continuous function. Now we extend this result to (L, M)-fuzzy topological space as follows.

Theorem 4.3. *Let $\varphi : X \longrightarrow Y$ be any function between two (L, M)-fuzzy topological spaces (X, τ) and (Y, σ). Then*

$$\mathbf{Fcon}_{\mathscr{F}_2}(\varphi^{\Rightarrow}(A))' \wedge \mathbf{Fi}(\varphi) \leq \mathbf{Fcon}_{\mathscr{F}_1}(A)',$$

for any $A \in L^X$.

Proof. Assume that $r \in M$ such that $r \ll \mathbf{Fcon}_{\mathscr{F}_2}(\varphi^{\Rightarrow}(A))' \wedge \mathbf{Fi}(\varphi)$. From Theorems 2.21 and 3.4 **(2)**, we have

$$r \ll \mathbf{Fcon}_{\mathscr{F}_2}(\varphi^{\Rightarrow}(A))' = \bigvee_{\substack{\varphi^{\Rightarrow}(A)\wedge B_1 \neq \perp_{L^Y},\\ \varphi^{\Rightarrow}(A)\wedge B_2 \neq \perp_{L^Y},\\ \varphi^{\Rightarrow}(A)\wedge B_1\wedge B_2 = \perp_{L^Y},\\ \varphi^{\Rightarrow}(A)\leq B_1\vee B_2}} \Big\{\mathscr{F}_2(B_1') \vee \mathscr{F}_2(B_2')\Big\},$$

and

$$r \lll \mathbf{Fi}(\varphi) = \bigwedge_{B_3\in L^Y} \Big\{\mathscr{F}_2(B_3') \mapsto \mathscr{F}_1(\varphi^{\Rightarrow}(B_3)')\Big\}.$$

Then there exist B_1, $B_2 \in L^Y$ such that $\varphi^{\Rightarrow}(A)\wedge B_1 \neq \bot_{L^Y}$, $\varphi^{\Rightarrow}(A)\wedge B_2 \neq \bot_{L^Y}$, $\varphi^{\Rightarrow}(A) \wedge B_1 \wedge B_2 = \bot_{L^Y}$, $\varphi^{\Rightarrow}(A) \leq B_1 \vee B_2$ with $r \leq \mathscr{F}_2(B_1') \wedge \mathscr{F}_2(B_2')$, and $r \leq \mathscr{F}_2(B_3') \mapsto \mathscr{F}_1(\varphi^{\Leftarrow}(B_3)')$ for any $B_3 \in L^Y$. It follows that there exist B_1, $B_2 \in L^Y$ such that $A \wedge \varphi^{\Leftarrow}(B_1) \neq \bot_{L^X}$, $A \wedge \varphi^{\Leftarrow}(B_2) \neq \bot_{L^X}$, $A \wedge \varphi^{\Leftarrow}(B_1) \wedge \varphi^{\Leftarrow}(B_2) = \bot_{L^X}$, $A \leq \varphi^{\Leftarrow}(B_1) \vee \varphi^{\Leftarrow}(B_2)$ with

$$r \leq \mathscr{F}_2(B_1) \wedge \mathscr{F}_2(B_2), \tag{1}$$

and

$$r \wedge \mathscr{F}_2(B_3') \leq \mathscr{F}_1(\varphi^{\Leftarrow}(B_3)'), \tag{2}$$

for any $B_3 \in L^Y$. From Eqs. (1) and (2), we have

$$\begin{aligned} r &= r \wedge \mathscr{F}_2(B_1') \wedge \mathscr{F}_2(B_2') \leq \mathscr{F}_1(\varphi^{\Leftarrow}(B_1)') \wedge \mathscr{F}_1(\varphi^{\Leftarrow}(B_1)') \\ &\leq \bigvee_{\substack{A\wedge A_1 \neq \bot_{L^X},\ A\wedge A_2 \neq \bot_{L^X},\\ A\wedge A_1 \wedge A_2 \neq \bot_{L^X},\ A \leq A_1 \vee A_2}} \Big\{ \mathscr{F}_1(A_1') \vee \mathscr{F}_1(A_2') \Big\} \\ &= \mathbf{Fcon}_{\mathscr{F}_1}(A)'. \end{aligned}$$

Since r is arbitrary, we have $\mathbf{Fcon}_{\mathscr{F}_2}(\varphi^{\Rightarrow}(A))' \wedge \mathbf{Fi}(\varphi) \leq \mathbf{Fcon}_{\mathscr{F}_1}(A)'$.

□

It is easy to prove that $\mathbf{F}T_1$ and $\mathbf{F}T_2$ in our sense are preserved by **F**- homeomorphisms. In the remainder of this section, we generalize this result into the (L, M)-fuzzy topology setting.

Lemma 4.4. *Let $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ be a bijective function between two (L, M)-fuzzy topological spaces (X, τ) and (Y, σ). Then*

(1) $\mathbf{F}T_1(X, \tau) \wedge \mathbf{Fo}(\varphi) \leq \mathbf{F}T_1(Y, \sigma)$.
(2) $\mathbf{F}T_2(X, \tau) \wedge \mathbf{Fo}(\varphi) \leq \mathbf{F}T_2(Y, \sigma)$.

Proof. We only prove (1) and the proof of (2) is similar. Assume that $r \in M$ with

$$\begin{aligned} r &\ll \mathbf{F}T_1(X, \tau) \wedge \mathbf{Fo}(\varphi) \\ &= \bigwedge_{a \not\leq b} \bigvee_{a \not\leq A \geq b} \mathscr{F}_1(A') \wedge \bigwedge_{A_1 \in L^X} \Big(\mathscr{F}_1(A_1) \mapsto \mathscr{F}_2(\varphi^{\Rightarrow}(A_1)) \Big). \end{aligned}$$

Thus for each a_1, $a_2 \in J(L^X)$ such that $a_1 \not\leq a_2$, there exists $A \in L^X$ with $a_1 \not\leq A \geq a_2$ and $r \leq \mathscr{F}_1(A')$. For $A_1 \in L^X$, $r \leq \mathscr{F}_1(A_1) \mapsto \mathscr{F}_2(\varphi^{\Rightarrow}(A_1))$. From Lemma 2.1 (1), we obtain $r \wedge \mathscr{F}_1(A_1) \leq \mathscr{F}_2(\varphi^{\Rightarrow}(A_1))$. To verify

$$r \leq \mathbf{F}T_1(Y, \sigma) = \bigwedge_{b_1 \not\leq b_2} \bigvee_{b_1 \not\leq B \geq b_2} \mathscr{F}_2(B'),$$

let b_1, $b_2 \in J(L^Y)$ such that $b_1 \not\leq b_2$. From the bijectivity of the function φ, there exist a_1, $a_2 \in J(L^X)$ such that $a_1 \not\leq a_2$ with $b_1 = \varphi^{\Rightarrow}(a_1)$ and $b_2 = \varphi^{\Leftarrow}(a_2)$. Since

$a_1 \not\leq a_2$, there exists $A \in L^X$ such that $a_1 \not\leq A \geq a_2$ with $r \leq \mathscr{F}_1(A')$. Thus $b_1 = \varphi^{\Leftarrow}(a_1) \not\leq \varphi^{\Leftarrow}(A) \geq \varphi^{\Rightarrow}(a_2) = b_2$. Since φ is a bijective function, we know

$$r = r \wedge \mathscr{F}_1(A') \leq \mathscr{F}_2(\varphi^{\Rightarrow}(A')) = \mathscr{F}_2(\varphi^{\Rightarrow}(A)').$$

Therefore

$$r \leq \bigwedge_{b_1 \not\leq b_2} \bigvee_{b_1 \not\leq B \geq b_2} \mathscr{F}_2(B') = \mathbf{F}T_1(Y, \sigma).$$

Since r is arbitrary, we proved that $\mathbf{F}T_1(X, \tau) \wedge \mathbf{Fo}(\varphi) \leq \mathbf{F}T_1(Y, \sigma)$. This complete the proof. □

From Lemma 4.4 and Definition 3.2 (1), we have the following theorem.

Theorem 4.5. *Let $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ be a bijective function between two (L, M)-fuzzy topological spaces (X, τ) and (Y, σ). Then*

(1) $\mathbf{F}T_1(X, \tau) \wedge \mathbf{F}\text{-}\mathbf{Hom}(\varphi) \leq \mathbf{F}T_1\ (Y, \sigma)$, $\mathbf{F}T_1\ (Y, \sigma) \wedge \mathbf{F}\text{-}\mathbf{Hom}\ (\varphi) \leq \mathbf{F}T_1\ (X, \tau)$.

(2) $\mathbf{F}T_2\ (X, \tau) \wedge \mathbf{F}\text{-}\mathbf{Hom}\ (\varphi) \leq \mathbf{F}T_2\ (Y, \sigma)$, $\mathbf{F}T_2\ (Y, \sigma) \wedge \mathbf{F}\text{-}\mathbf{Hom}\ (\varphi) \leq \mathbf{F}T_2\ (X, \tau)$.

From Theorem 4.5, the following corollary is true.

Corollary 4.6. *Let $\varphi : (X, \tau) \longrightarrow (Y, \sigma)$ be a bijective function between two (L, M)-fuzzy topological spaces (X, τ) and (Y, σ). Then*

(1) $\mathbf{F}T_1(X, \tau) \wedge \mathbf{F}\text{-}\mathbf{Hom}(\varphi) = \mathbf{F}T_1(Y, \sigma) \wedge \mathbf{F}\text{-}\mathbf{Hom}(\varphi)$.

(2) $\mathbf{F}T_2(X, \tau) \wedge \mathbf{F}\text{-}\mathbf{Hom}(\varphi) = \mathbf{F}T_2(Y, \sigma) \wedge \mathbf{F}\text{-}\mathbf{Hom}(\varphi)$.

5. **Conclusion**

This paper introduced a new description of (L, M)-fuzzy topological spaces based on sub-varieties of new functions degree. We defined and discussed new degrees of weak forms of functions in (L, M)-fuzzy topology with the help of (L, M)-fuzzy **F**-open operator and implication operation. The properties of **F**-openness, **F**-continuity, and **F**-irresolutness degree of functions in (L, M)-fuzzy topology are investigated. Further, we proved that any function can be regard as **F**-open, **F**-continuous, and **F**-irresolute to some degree. Moreover, its relationship with **F**-compactness, **F**-connectedness, $\mathbf{F}T_1$, and $\mathbf{F}T_2$ degree have been constructed and analyzed systematically.

Acknowledgements. The authors would like to thank the editors and anonymous reviewers for their careful reading and constructive comments in improving this paper.

References

[1] Wadei F. Al-Omeri, O. H. Khalil, and A. Ghareeb, *Degree of (L, M)-fuzzy semi-precontinuous and (L, M)-fuzzy semi-preirresolute functions*, Demonstratio Mathematica, **51(1)** (2018), 182–197.

[2] C. L. Chang, *Fuzzy topological spaces*, Journal of Mathematical Analysis and Applications, **24 (1)** (1968), 182–190.

[3] A. Ghareeb, *A new form of* **F***-compactness in* L*-fuzzy topological spaces*, Mathematical and Computer Modelling, **54 (9)**(2011), 2544–2550.

[4] A. Ghareeb, *Preconnectedness degree of* L*-fuzzy topological spaces*, International Journal of Fuzzy Logic and Intelligent Systems, **11(1)** (2011), 54–58.

[5] A. Ghareeb, L*-fuzzy semi-preopen operator in* L*-fuzzy topological spaces*, Neural Computing and Applications, **21(1)** (2012), 87–92.

[6] A. Ghareeb and Wadei F. Al-Omeri, *New degrees for functions in* (L, M)*-fuzzy topological spaces based on* (L, M)*-fuzzy semiopen and* (L, M)*-fuzzy preopen operators*, Journal of Intelligent & Fuzzy Systems, DOI:10.3233/JIFS-18251, (2018).

[7] A. Ghareeb and F.-G. Shi, SP*-compactness and* SP*-connectedness degree in* L*-fuzzy pretopological spaces*, Journal of Intelligent and Fuzzy Systems, **31**(2016), 1435–1445.

[8] J. A. Goguen, *The fuzzy Tychonoff theorem*, Journal of Mathematical Analysis and Applications, **43(3)** (1973), 734–742.

[9] U. Höhle, *Probabilistic metrization of fuzzy uniformities*, Fuzzy Sets and Systems, **8(1)** (1982), 63–69.

[10] U. Höhle and A. P. Šostak, *Axiomatic Foundations of Fixed-Basis Fuzzy Topology*, In: Höhle U., Rodabaugh S.E. (eds) Mathematics of Fuzzy Sets. The Handbooks of Fuzzy Sets Series, Springer, Boston, MA, **3**(1999), 123–272.

[11] Q. Jin and L. Li, *One-axiom characterizations on lattice-valued closure (interior) operators*, Journal of Intelligent and Fuzzy Systems, **31(3)** (2016), 1679–1688.

[12] T. Kubiak, *On fuzzy topologies*, Ph.D. Thesis, Adam Mickiewicz, Poznan, Poland, 1985.

[13] T. Kubiak and A. P. Šostak, *A fuzzification of the category of* M*-valued* L*-topological spaces*, Applied General Topology, **5 (2)**(2004), 137–154.

[14] H. Lai and D. Zhang, *Fuzzy topological spaces with conical neighborhood systems*, Fuzzy Sets and Systems, **320** (2018), 87–104.

[15] L. Li, Q. Jin, and K. Hu, *Lattice-valued convergence associated with CNS spaces*, DOI: 10.1016/j.fss.2018.05.023, (2018).

[16] C.-Y. Liang and F.-G. Shi, *Degree of continuity for mappings of* (L, M)*-fuzzy topological spaces*, Journal of Intelligent and Fuzzy Systems, 27 (5)(2014), 2665–2677.

[17] B. Pang, *Degrees of continuous mappings, open mappings, and closed mappings in* L*-fuzzifying topological spaces*, Journal of Intelligent and Fuzzy Systems, **27 (2)** (2014), 805–816.

[18] G. N. Raney, A*Subdirect-union representation for completely distributive complete lattices*, Proc. Amer. Math. Soc., **4**(1953), 518–522.

[19] S. E. Rodabaugh, *Categorical Foundations of Variable-Basis Fuzzy Topology*. In: Hhle U., Rodabaugh S.E. (eds) Mathematics of Fuzzy Sets. The Handbooks of Fuzzy Sets Series, Springer, Boston, MA, **3**(1999), 273388.

[20] F.-G. Shi, L*-fuzzy semiopenness and* L*-fuzzy preopenness*. J. Nonlinear Sci. Appl., In Press.

[21] F.-G. Shi, *A new definition of fuzzy compactness*, Fuzzy Sets and Systems, **158 (13)** (2007), 1486–1495.

[22] F.-G. Shi, L*-fuzzy interiors and* L*-fuzzy closures*, Fuzzy Sets and Systems, **160 (9)** (2009), 1218–1232.

[23] Y. Shi and F.-G. Shi, *Characterizations of* L*-topologies*, Journal of Intelligent and Fuzzy Systems, **34(1)** (2018), 613–623.

[24] A. P. Šostak, On a fuzzy topological structure, Rendiconti Circolo Matematico Palermo, **11(Suppl. Ser. II)**(1985), 89–103.

[25] A. P. Šostak, *On compactness and connectedness degrees of fuzzy sets in fuzzy topological spaces*, General Topology and its Relations to Modern Analysis and Algebra, Heldermann Verlag, Berlin, (1988), 519–532.

[26] A. P. Šostak, *Two decades of fuzzy topology: basic ideas, notions, and results*, Russian Mathematical Surveys, **44 (6)** (1989), 1–25.

[27] A. P. Šostak, *Towards the concept of a fuzzy category*, Acta Univ Latv (Ser Math), **562**(1991), 85–94.

[28] A. P. Šostak, *Fuzzy categories related to algebra and topology*, Tatra Mount Math Publ, **16**(1999), 159–185.

[29] A. P. Šostak, *L-valued categories: generalities and examples related to algebra and topology*, In: Categorical Structures and Their Applications (eds., W. Gähler and G. Preuss), World Scientific, (2004), 291–312..

[30] Mingsheng Ying, *A new approach for fuzzy topology (I)*, Fuzzy Sets and Systems, **39 (3)** (1991), 303–321.

[31] L.A. Zadeh, *Fuzzy sets*, Information and Control, **8 (3)** (1965), 338–353.

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