2-TUPLE INTUITIONISTIC LINGUISTIC AGGREGATION OPERATORS IN MULTIPLE ATTRIBUTE DECISION MAKING

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Abstract. In this paper, we investigate the multiple attribute decision making (MADM) problems with 2-tuple intuitionistic linguistic information. Then, we utilize arithmetic and geometric operations to develop some 2-tuple intuitionistic linguistic aggregation operators. The prominent characteristic of these proposed operators are studied. Then, we have utilized these operators to develop some approaches to solve the 2-tuple intuitionistic linguistic MADM problems. Finally, a practical example for enterprise resource planning (ERP) system selection is given to verify the developed approach and to demonstrate its practicality and effectiveness.

1. Introduction

The 2-tuple linguistic representation model, characterized by a linguistic term and a numeric value, was proposed by Herrera and Martínez [17] based on the basis of the concept of symbolic translation. It has exact characteristic in linguistic information processing and can effectively avoid information loss and distortion which occur formerly in the linguistic computing process. In many practical decision-making problems, experts have a preference to show their opinions according to their experience and knowledge, and it is more suitable to provide assessments by means of linguistic terms rather than numerical ones [14, 15], due to the complexity of the objects and the vagueness of human thinking. Therefore, many researchers have investigated linguistic multiple attribute decision-making (MADM) problems or multiple attribute group decision-making (MAGDM) problems and proposed lots of methods to deal with linguistic evaluation information [3, 18, 27, 36, 20]. Herrera and Martínez [17] show 2-tuple linguistic information processing manner can effectively avoid the loss and distortion of information. Herrera, Herrera-Viedma [14] developed some 2-tuple arithmetic aggregating operators. Herrera et al. [20] presented the group decision making(GDM) model for managing non-homogeneous information processing. Herrera-Viedma et al. [19] developed the consensus support system with multi-granular linguistic preference relations. Liao et al. [25] used linguistic information processing model to select an ERP system. Herrera et al. [16] proposed a fuzzy linguistic methodology to deal with unbalanced linguistic

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term sets. Wang [38] presented a 2-tuple fuzzy linguistic evaluation model for selecting appropriate agile manufacturing system. Tai and Chen [33] developed the intellectual capital evaluation model linguistic variable. Fan et al. [10] evaluated knowledge management capability of organizations by using a fuzzy linguistic method. Wei [40] extended TOPSIS method to MAGDM with 2-tuple linguistic information. Wei [34] proposed ET-WG and ET-OWG operators for MAGDM with 2-tuple linguistic information. Fan and Liu [11] developed the multi-granularity uncertain linguistic GDM model. Chang and Wen [5] developed a novel efficient approach for DFMEA combining 2-tuple and the OWA operator. Jiang and Wei [22] proposed some 2-tuple linguistic Bonferroni mean operators. Xu et al. [56] developed some methods to deal with unacceptable incomplete 2-tuple fuzzy linguistic preference relations in GDM. Liu et al. [27] proposed the dependent interval 2-tuple linguistic aggregation operators for MAGDM. Dong and Herrera-Viedma [9] proposed the consistency-driven automatic methodology to set interval numerical scales of 2-tuple linguistic term sets. Wang et al. [37] developed the interval 2-tuple linguistic MGDM method. Qin and Liu [31] proposed the 2-tuple linguistic Muirhead mean operators for MAGDM. Zhang et al. [66] developed the consensus reaching model for 2-tuple linguistic MAGDM with incomplete weight information. Recently, other alternative fuzzy linguistic approaches recently presented that could be used to address the linguistic modelling in decision making as multi-granular fuzzy linguistic approach [23], discrete fuzzy number based linguistic approach [30] and personalized linguistic approach [52] .

Atanassov [1, 2] introduced the concept of intuitionistic fuzzy set (IFS) characterized by a membership function and a non-membership function, which is a generalization of the concept of fuzzy set [65] whose basic component is only a membership function. Xu [59] developed the intuitionistic fuzzy weighted averaging (IFWA) operator, intuitionistic fuzzy ordered weighted averaging (IFOWA) operator and the intuitionistic fuzzy hybrid aggregation (IFHA) operator. Xu and Yager [62] developed some geometric aggregation operators with intuitionistic fuzzy information. Wei [39] utilized the maximizing deviation method for intuitionistic fuzzy MADM with incomplete weight information. Wei [42] proposed some induced geometric aggregation operators with intuitionistic fuzzy information. Wei [41, 43] developed the GRA method for intuitionistic fuzzy MADM with incomplete weight information. Li [24] developed the GOWA operator based approach to MADM using intuitionistic fuzzy sets. Zhao and Wei [68] proposed some intuitionistic fuzzy Einstein hybrid aggregation operators for MADM problems. Vahdani et al. [35] proposed a new design of the elimination and choice translating reality method for multi-criteria GDM in an intuitionistic fuzzy environment. Yu and Xu [64] proposed the prioritized intuitionistic fuzzy aggregation operators. Chen [7] developed the inclusion-based TOPSIS method with interval-valued intuitionistic fuzzy sets for multiple criteria GDM. Zhao and Xu [67] developed a new measure for calculating the correlation coefficients between IFSs, and proved its desirable axiomatic properties. Sirbiladze and Badagadze [32] developed the intuitionistic fuzzy probabilistic aggregation operators based on the Choquet integral.

Although, intuitionistic fuzzy set (IFS) theory has been successfully applied in some areas, the IFS is also characterized by the membership degree and the non-membership degree, whose sum is less than or equal to 1. However, all the above approaches are unsuitable to describe the membership degree and the non-membership degree of an element to a set by linguistic variables on the basis of the given linguistic term sets, which can reflect the decision maker's confidence level when they are making an evaluation. In order to overcome this limit, we shall propose the concept of 2-tuple intuitionistic linguistic set to solve this problem based on the IFSs [1, 2] and 2-tuple linguistic information processing model [14, 15]. Thus, how to aggregate these 2-tuple intuitionistic linguistic numbers is an interesting topic. To solve this issue, in this paper, we shall develop some 2-tuple intuitionistic linguistic information aggregation operators based on the traditional arithmetic and geometric operations [61, 60, 55]. In order to do so, the remainder of this paper is set out as follows. In the next section, we shall propose the concept of 2-tuple intuitionistic linguistic sets (2TILSs) on the basis of the IFSs and 2-tuple linguistic information processing model. In Section 3, we shall propose some 2-tuple intuitionistic linguistic arithmetic aggregation operators. In Section 4, we shall propose some 2-tuple intuitionistic linguistic geometric aggregation operators. In Section 5, based on these operators, we shall present some models for MADM problems with 2-tuple intuitionistic linguistic information. In Section 6, we shall present a numerical example for enterprise resource planning (ERP) system selection in order to illustrate the method proposed in this paper. Section 7 concludes the paper with some remarks.

2. Preliminaries

In the following, we introduced some basic concepts related to 2-tuple linguistic term sets and intuitionistic fuzzy sets. Then, we propose the concept and basic operations of the 2-tuple intuitionistic linguistic sets.

2.1. **2-tuple linguistic term sets.** Let $S = \{s_i | i = 1, 2, \cdots, t\}$ be a linguistic term set with odd cardinality. Any label, s_i represents a possible value for a linguistic variable, and it should satisfy the following characteristics [14, 18, 20, 15, **?**, 58]:

(1) The set is ordered: $s_i > s_j$, if $i > j$; (2) Max operator: $\max(s_i, s_j) = s_i$, if $s_i \geq s_j$; (3) Min operator: $\min(s_i, s_j) = s_i$, if $s_i \leq s_j$.

For example, S can be defined as:

$S = \{s_1 = \mathit{extremely\,poor}, s_2 = \mathit{very\,poor}, s_3 = \mathit{poor}, s_4 = \mathit{medium}, s_5 = \mathit{good}, s_6 = \mathit{very\,good}, s_7 = \mathit{extremely\,good}\}$.

Herrera and Martinez [14, 15] developed the 2-tuple fuzzy linguistic representation model based on the concept of symbolic translation. It is used for representing the linguistic assessment information by means of a 2-tuple (s_i, α_i) , where s_i is a linguistic label from predefined linguistic term set S and α_i is the value of symbolic translation, and $\alpha_i \in [-0.5, 0.5)$.

Definition 2.1. Let β be the result of an aggregation of the indices of a set of labels assessed in a linguistic term set S , i.e., the result of a symbolic aggregation operation, $\beta \in [1, t]$, being t the cardinality of S. Let $i = \mathit{round}(\beta)$ and $\alpha = \beta - i$

be two values, such that, $i \in [1,t]$ and $\alpha \in [-0.5, 0.5)$ then α is called a symbolic translation [14, 15].

Definition 2.2. Let $S = \{s_1, s_2, \cdots, s_t\}$ be a linguistic term set and $\beta \in [1,t]$ is a number value representing the aggregation result of linguistic symbolic. Then the function Δ used to obtain the 2-tuple linguistic information equivalent to β is defined as:

$$\Delta : [1,t] \to S \times [-0.5, 0.5), \tag{1}$$

$$\Delta(\beta) = \begin{cases} s_i, i = \text{round}(\beta) \\ \alpha = \beta - i, \alpha \in [-0.5, 0.5), \end{cases} \tag{2}$$

where round$(\cdot)$ is the usual round operation, s_i has the closest index label to β and α is the value of the symbolic translation [14, 15].

Definition 2.3. Let $S = \{s_1, s_2, \cdots, s_t\}$ be a linguistic term set and (s_i, α_i) be a 2-tuple. There is always a function Δ^{-1} can be defined, such that, from a 2-tuple (s_i, α_i) it return its equivalent numerical value $\beta \in [1,t] \subset R$, which is [14, 15]

$$\Delta^{-1} : S \times [-0.5, 0.5) \to [1,t], \tag{3}$$

$$\Delta^{-1}(s_i, \alpha) = i + \alpha = \beta, \tag{4}$$

From Definitions 2.1 and 2.2, we can conclude that the conversion of a linguistic term into a linguistic 2-tuple consists of adding a value 0 as symbolic translation:

$$\Delta(s_i) = (s_i, 0). \tag{5}$$

2.2. **Intuitionistic fuzzy set.** Atanassov [1, 2] extended the fuzzy set to the IFS, shown as follows:

Definition 2.4. An IFS A in X is given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}, \tag{6}$$

Where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$, with the condition $0 \le \mu_A(x) + \nu_A(x) \le 1$,$\forall x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and non-membership degree of the element to the set A.

Definition 2.5. [1, 2] For each IFS A in X, if

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \forall x \in X, \tag{7}$$

Then $\pi_A(x)$ is called the degree of indeterminacy of x to A.

Definition 2.6. [6] Let $\tilde{a} = (\mu, \nu)$ be an intuitionistic fuzzy number(IFN), a score function S of an intuitionistic fuzzy value can be represented as follows:

$$S(\tilde{a}) = \mu - \nu, S(\tilde{a}) \in [-1, 1]. \tag{8}$$

Definition 2.7. [21] Let $\tilde{a} = (\mu, \nu)$ be an IFN, an accuracy function H of an IFN can be represented as follows:

$$H(\tilde{a}) = \mu + \nu, H(\tilde{a}) \in [0, 1], \tag{9}$$

to evaluate the degree of accuracy of the IFN $\tilde{a} = (\mu, \nu)$, where $H(\tilde{a}) \in [0, 1]$. The larger the value of $H(\tilde{a})$, the more the degree of accuracy of the IFN $\tilde{a}$ is.

Definition 2.8. [59] Let $\tilde{a_1} = (\mu_1, \nu_1)$ and $\tilde{a_2} = (\mu_2, \nu_2)$ be two IFNs, $S(\tilde{a_1}) = \mu_1 - \nu_1$ and $S(\tilde{a_2}) = \mu_2 - \nu_2$ be the scores of $\tilde{a}$ and $\tilde{b}$, respectively, and let $H(\tilde{a_1}) = \mu_1 + \nu_1$ and $H(\tilde{a_2}) = \mu_2 + \nu_2$ be the accuracy degrees of $\tilde{a}$ and $\tilde{b}$, respectively, then if $S(\tilde{a}) < S(\tilde{b})$, then $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} < \tilde{b}$; if $S(\tilde{a}) = S(\tilde{a})$, then

(1) if $H(\tilde{a}) = H(\tilde{b})$, then $\tilde{a}$ and $\tilde{b}$ represent the same information, denoted by $\tilde{a} = \tilde{b}$;

(2) if $H(\tilde{a}) < H(\tilde{b})$, $\tilde{a}$ is smaller than $\tilde{b}$, denoted by $\tilde{a} < \tilde{b}$.

Definition 2.9. [8] Let $\tilde{a_1} = (\mu_1, \nu_1)$, $\tilde{a_2} = (\mu_2, \nu_2)$, and $\tilde{a} = (\mu, \nu)$ be three IFNs, and some basic operations on them are defined as follows:

(1)$\tilde{a_1} \oplus \tilde{a_2} = (\mu_1 + \mu_2 - \mu_1\mu_2, \nu_1\nu_2)$;
(2)$\tilde{a_1} \otimes \tilde{a_2} = (\mu_1\mu_2, \nu_1 + \nu_2 - \nu_1\nu_2)$;
(3)$\lambda\tilde{a} = \left(1 - (1 - \mu)^\lambda, \nu^\lambda\right), \lambda > 0$;
(4)$(\tilde{a})^\lambda = \left(\mu^\lambda, 1 - (1 - \nu)^\lambda\right), \lambda > 0$;
(5)$\tilde{a}^c = (\nu, \mu)$.

For the intuitionistic fuzzy sets, the degree of membership and degree of non-membership is non-negative real number, however, intuitionistic fuzzy sets can't depict and deal with qualitative membership degree and qualitative non-membership degree, thus, we shall propose the concepts of 2-tuple intuitionistic fuzzy linguistic sets to solve this kind of issues.

2.3. **2-tuple intuitionistic linguistic sets.** In the following, we shall propose the concepts and basic operations of the 2-tuple intuitionistic linguistic sets on the basis of the IFSs [1, 2] and 2-tuple linguistic information processing model [14, 15].

Definition 2.10. A 2-tuple intuitionistic linguistic sets A in X is given

$$P = \{\langle s_{\mu(x)}, s_{\nu(x)}\rangle, x \in X\}, \tag{10}$$

where $s_{\mu(x)}$,$s_{\nu(x)} \in S$ with the condition $2 \le \mu(x)+\nu(x) \le t+1, \forall x \in X$. The numbers $s_{\mu(x)}$,$s_{\nu(x)}$ represent, respectively, the degree of membership and degree of non-membership of the element x to the set P. Then for $x \in X, s_{\pi(x)} = s_{t+1-\mu(x)-\nu(x)}$ could be called the degree of indeterminacy of the element x to the set P.

Then, we develop the 2-tuple intuitionistic linguistic representation model based on the concept of symbolic translation. It is used for representing the intuitionistic linguistic assessment information by means of a 2-tuple intuitionistic linguistic number $\langle(s_{\mu(x)}, \alpha), (s_{\nu(x)}, \beta)\rangle$, where $s_{\mu(x)}, s_{\nu(x)}$ are a linguistic labels from predefinedm linguistic term set S and α, β is the value of symbolic translation, and $\alpha, \beta \in [-0.5 - 0.5)$.

For convenience, we call $\tilde{p} = \langle(s_\mu, \alpha), (s_\nu, \beta)\rangle$ a 2-tuple intuitionistic linguistic number (2TILN), where $s_u, s_\nu \in S, \alpha, \beta \in [-0.5, 0.5), 2 \leq \triangle^{-1}(s_\mu, \alpha) + \triangle^{-1}(s_\nu, \beta) \leq t + 1$.

Definition 2.11. Let ϕ, φ be the results of an aggregation of the indices of a set of labels assessed in a linguistic term set S, i.e., the result of a symbolic aggregation operation, $\phi, \varphi \in [1, t], 2 \leq \phi + \varphi \leq t + 1$, being t the cardinality of S. Let $\mu = round(\phi), \nu = round(\varphi)$ and $\alpha = \phi - \mu, \beta = \varphi - \nu$ be four values, such that, $\phi, \varphi \in [1, t]$ and $\alpha, \beta \in [-0.5, 0.5)$ then α, β are called a symbolic translations.

Definition 2.12. Let $S = \{s_1, s_2, \cdots, s_t\}$ be a linguistic term set and $\phi, \varphi \in [1, t]$ are two number values representing the aggregation results of linguistic symbolic. Then the function Δ used to obtain the 2-tuple intuitionistic linguistic information equivalent to ϕ, φ are defined as:

$$\Delta : [1, t] \rightarrow S \times [-0.5, 0.5), \tag{11}$$

$$\Delta(\phi) = \begin{cases} s_\mu, \mu = round(\phi) \\ \alpha = \phi - \mu, \alpha \in [-0.5, 0.5), \end{cases} \tag{12}$$

$$\Delta(\varphi) = \begin{cases} s_\nu, \nu = round(\varphi) \\ \beta = \varphi - \nu, \beta \in [-0.5, 0.5), \end{cases} \tag{13}$$

where round(·) is the usual round operation, s_μ, s_ν have the closest index label to ϕ, φ and α, β are the values of the symbolic translation.

Definition 2.13. Let $S = \{s_1, s_2, \cdots, s_t\}$ be a linguistic term set and $\tilde{p} = \langle(s_\mu, \alpha), (s_\nu, \beta)\rangle$ be a 2-tuple intuitionistic linguistic number. There is always a function Δ^{-1} can be defined,such that, from a 2-tuple intuitionistic linguistic number $\langle(s_\mu, \alpha), (s_\nu, \beta)\rangle$ it return its equivalent numerical values $\phi, \varphi \in [1, t] \subset R$, which is

$$\Delta^{-1} : S \times [-0.5, 0.5) \rightarrow [1, t], \tag{14}$$

$$\Delta^{-1}(s_\mu, \alpha) = \mu + \alpha = \phi, \tag{15}$$

$$\Delta^{-1}(s_\nu, \beta) = \nu + \beta = \varphi. \tag{16}$$

From Definitions 2.11 and 2.12, we can conclude that the conversion of two linguistic terms into a 2-tuple intuitionistic linguistic number consists of adding a value 0 as symbolic translation:

$$\Delta\langle s_{\mu(x)}, s_{\nu(x)}\rangle = \langle(s_{\mu(x),0}), (s_{\nu(x),0})\rangle \tag{17}$$

Definition 2.14. Let $\tilde{p} = \langle(s_\mu, \alpha), (s_\nu, \beta)\rangle$ a 2-tuple intuitionistic linguistic number (2TILN), a score function S of a 2TILN can be represented as follows:

$$S(\tilde{p}) = \Delta\left(\frac{t + \Delta^{-1}(s_\mu, \alpha) - \Delta^{-1}(s_\nu, \beta)}{2}\right), \Delta^{-1}(S(\tilde{p})) \in [1, t] \tag{18}$$

Definition 2.15. Let $\tilde{p} = \langle(s_\mu, \alpha), (s_\nu, \beta)\rangle$ a 2-tuple intuitionistic linguistic number (2TILN), an accuracy function H of a 2TILN can be represented as follows:

$$H(\tilde{p}) = \Delta\left(\frac{\Delta^{-1}(s_\mu, \alpha) + \Delta^{-1}(s_\nu, \beta)}{2}\right), \Delta^{-1}(H(\tilde{p})) \in [1, t] \tag{19}$$

to evaluate the degree of accuracy of the 2TILN $\tilde{p} = \langle (s_\mu, \alpha), (s_\nu, \beta) \rangle$, $\Delta^{-1}(H(\tilde{p})) \in [1, t]$. The larger the value of $H(\tilde{p})$, the more the degree of accuracy of the $\tilde{p} = \langle (s_\mu, \alpha), (s_\nu, \beta) \rangle$.

Based on the score function S and the accuracy function H, in the following, we shall give an order relation between two 2TILNs, which is defined as follows:

Definition 2.16. Let $\tilde{p}_1 = \langle (s_{\mu_1}, \alpha_1), (s_{\nu_1}, \beta_1) \rangle$ and $\tilde{p}_2 = \langle (s_{\mu_2}, \alpha_2), (s_{\nu_2}, \beta_2) \rangle$ be two 2TILNs, $S(\tilde{p}_1) = \Delta\left(\frac{t+\Delta^{-1}(s_{\mu_1},\alpha_1)-\Delta^{-1}(s_{\nu_1},\beta_1)}{2}\right)$ be the scores of $\tilde{p}_1$ and $S(\tilde{p}_2) = \Delta\left(\frac{t+\Delta^{-1}(s_{\mu_2},\alpha_2)-\Delta^{-1}(s_{\nu_2},\beta_2)}{2}\right)$ be the scores of $\tilde{p}_2$, respectively, and let $H(\tilde{p}_1) = \Delta\left(\frac{\Delta^{-1}(s_{\mu_1},\alpha_1)+\Delta^{-1}(s_{\nu_1},\beta_1)}{2}\right)$ and $H(\tilde{p}_2) = \Delta\left(\frac{\Delta^{-1}(s_{\mu_2},\alpha_2)+\Delta^{-1}(s_{\nu_2},\beta_2)}{2}\right)$ be the accuracy degrees of $\tilde{p}_1$ and $\tilde{p}_2$, respectively, then if $S(\tilde{p}_1) < S(\tilde{p}_2)$, then $\tilde{p}_1$ is smaller than $\tilde{p}_2$, denoted by $\tilde{p}_1 < \tilde{p}_2$; if $S(\tilde{p}_1) = S(\tilde{p}_2)$, then

(1) if $H(\tilde{p}_1) = H(\tilde{p}_2)$, then $\tilde{p}_1$ and $\tilde{p}_2$ represent the same information, denoted by $\tilde{p}_1 = \tilde{p}_2$;

(2) if $H(\tilde{p}_1) < H(\tilde{p}_2)$, $\tilde{p}_1$ is smaller than $\tilde{p}_2$, denoted by $\tilde{p}_1 < \tilde{p}_2$.

Motivated by the operations of the 2-tuple linguistic information [14, 18] and Definition 2.9, in the following, we shall define some operational laws of 2TILNs.

Definition 2.17. Let $\tilde{p}_1 = \langle (s_{\mu_1}, \alpha_1), (s_{\nu_1}, \beta_1) \rangle$ and $\tilde{p}_2 = \langle (s_{\mu_2}, \alpha_2), (s_{\nu_2}, \beta_2) \rangle$ be two 2TILNs, then

$$\begin{aligned}
\tilde{p}_1 \oplus \tilde{p}_2 &= \Big\langle \Delta\Big(t\Big(\tfrac{\Delta^{-1}(s_{\mu_1},\alpha_1)}{t} + \tfrac{\Delta^{-1}(s_{\mu_2},\alpha_2)}{t} - \tfrac{\Delta^{-1}(s_{\mu_1},\alpha_1)}{t} \cdot \tfrac{\Delta^{-1}(s_{\mu_2},\alpha_2)}{t}\Big)\Big), \\
&\quad \Delta\Big(t\Big(\tfrac{\Delta^{-1}(s_{\nu_1},\beta_1)}{t} \cdot \tfrac{\Delta^{-1}(s_{\nu_2},\beta_2)}{t}\Big)\Big)\Big\rangle, \\
\tilde{p}_1 \otimes \tilde{p}_2 &= \Big\langle \Delta\Big(t\Big(\tfrac{\Delta^{-1}(s_{\mu_1},\alpha_1)}{t} \cdot \tfrac{\Delta^{-1}(s_{\mu_2},\alpha_2)}{t}\Big)\Big), \\
&\quad \Delta\Big(t\Big(\tfrac{\Delta^{-1}(s_{\nu_1},\beta_1)}{t} + \tfrac{\Delta^{-1}(s_{\nu_2},\beta_2)}{t} - \tfrac{\Delta^{-1}(s_{\nu_1},\beta_1)}{t} \cdot \tfrac{\Delta^{-1}(s_{\nu_2},\alpha_2)}{t}\Big)\Big)\Big\rangle,
\end{aligned}$$

$$\lambda\tilde{p}_1 = \Big\langle \Delta\Big(t\Big(1 - \Big(1 - \frac{\Delta^{-1}(s_{\mu_1}, \alpha_1)}{t}\Big)^{\lambda}\Big)\Big), \Delta\Big(t\Big(\frac{\Delta^{-1}(s_{\nu_1}, \beta_1)}{t}\Big)^{\lambda}\Big)\Big\rangle,$$

$$(\tilde{p}_1)^{\lambda} = \Big\langle \Delta\Big(t\Big(\frac{\Delta^{-1}(s_{\mu_1}, \alpha_1)}{t}\Big)^{\lambda}\Big), \Delta\Big(t\Big(1 - \Big(1 - \frac{\Delta^{-1}(s_{\nu_1}, \beta_1)}{t}\Big)^{\lambda}\Big)\Big)\Big\rangle.$$

Based on the Definition 2.17, we can derive the following Theorem 2.18 easily.

Theorem 2.18. *For any two 2TILNs $\tilde{p}_1 = \langle s_{\mu_1}, \alpha_1), (s_{\nu_1}, \beta_1) \rangle$ and $\tilde{p}_2 = \langle (s_{\mu_2}, \alpha_2), (s_{\nu_2}, \beta_2) \rangle$, it can be proved the calculation rules shown as follows:*

(1)$\tilde{p}_1 \oplus \tilde{p}_2 = \tilde{p}_2 \oplus \tilde{p}_1$,

(2)$\tilde{p}_1 \otimes \tilde{p}_2 = \tilde{p}_2 \otimes \tilde{p}_1$,

(3)$\lambda(\tilde{p}_1 \oplus \tilde{p}_2) = \lambda\tilde{p}_1 \oplus \lambda\tilde{p}_2, 0 \le \lambda \le 1$,

(4)$\lambda_1\tilde{p}_1 \oplus \lambda_2\tilde{p}_2 = (\lambda_1 \oplus \lambda_2)\tilde{p}_1, 0 \le \lambda_1, \lambda_2, \lambda_1 + \lambda_2 \le 1$,

(5)$\tilde{p}_1^{\lambda_1} \otimes \tilde{p}_1^{\lambda_2} = (\tilde{p}_1)^{\lambda_1+\lambda_2}, 0 \leq \lambda_1, \lambda_2, \lambda_1+\lambda_2 \leq 1,$

(6)$\tilde{p}_1^{\lambda_1} \otimes \tilde{p}_2^{\lambda_1} = (\tilde{p}_1 \otimes \tilde{p}_2)^{\lambda_1}, \lambda_1 \geq 0,$

(7)$\Big((\tilde{p}_1)^{\lambda_1}\Big)^{\lambda_2} = (\tilde{p}_1)^{\lambda_1\lambda_2}.$

3. 2-tuple intuitionistic linguistic arithmetic aggregation operators

In this section, we shall develop some arithmetic aggregation operators with 2-tuple intuitionistic linguistic information, such as 2-tuple intuitionistic linguistic weighted averaging (2TILWA) operator, 2-tuple intuitionistic linguistic ordered weighted averaging (2TILOWA) operator and 2-tuple intuitionistic linguistic hybrid average (2TILHA) operator.

Definition 3.1. Let $\tilde{p}_j = \langle (s_{\mu_j}, \alpha_j), (s_{\nu_j}, \beta_j)\rangle (j = 1, 2, \cdots, n)$ be a collection of 2TILNs, the 2-tuple intuitionistic linguistic weighted averaging (2TILWA) operator is a mapping $P^n \to P$ such that

$$2TILWA_\omega(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \mathop{\oplus}_{j=1}^{n} (\omega_j \tilde{p}_j) \tag{20}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be the weight vector of $\tilde{p}_j (j = 1, 2, \cdots, n)$, and $\omega_j > 0, \sum_{j=1}^{n} \omega_j = 1$.

Based on the Definition 2.17 and Theorem 2.18, we can get the following result:

Theorem 3.2. *The aggregated value by using 2TILWA operator is also a 2TILNs, where*

$$\begin{aligned} &2TILWA_\omega(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \oplus_{j=1}^{n} (\omega_j \tilde{p}_j) \\ &= \Big\langle \Delta\Big(t\Big(1 - \Pi_{j=1}^{n} \Big(1 - \tfrac{\Delta^{-1}(s_{\mu_j}, \alpha_j)}{t}\Big)^{\omega_j}\Big)\Big), \Delta\Big(t\,\Pi_{j=1}^{n} \Big(\tfrac{\Delta^{-1}(s_{\nu_j}, \beta_j)}{t}\Big)^{\omega_j}\Big)\Big\rangle, \end{aligned} \tag{21}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ *be the weight vector of* $\alpha_j (j = 1, 2, \cdots, n)$*, and* $\omega_j > 0$, $\sum_{j=1}^{n} \omega_j = 1$.

Proof. We prove equation (21) by mathematical induction on n.

(1) When $n = 2$, we have $2TILWA_\omega(\tilde{p}_1, \tilde{p}_2) = \omega_1\tilde{p}_1 \oplus \omega_2\tilde{p}_2$. By Definition 2.17 and Theorem 2.18 , we can see that both $\omega_1\tilde{p}_1$ and $\omega_2\tilde{p}_2$ are 2TILNs, and the value of $\omega_1\tilde{p}_1 \oplus \omega_2\tilde{p}_2$ is also a 2TILN. From the operational laws of 2-tuple intuitionistic linguistic number, we have

$$\omega_1\tilde{p}_1 = \Big\langle \Delta\Big(t\Big(1 - \Big(1 - \frac{\Delta^{-1}(s_{\mu_1}, \alpha_1)}{t}\Big)^{\omega_1}\Big)\Big), \Delta\Big(t\Big(\frac{\Delta^{-1}(s_{\nu_1}, \beta_1)}{t}\Big)^{\omega_1}\Big)\Big\rangle$$

$$\omega_2\tilde{p}_2 = \Big\langle \Delta\Big(t\Big(1 - \Big(1 - \frac{\Delta^{-1}(s_{\mu_2}, \alpha_2)}{t}\Big)^{\omega_2}\Big)\Big), \Delta\Big(t\Big(\frac{\Delta^{-1}(s_{\nu_2}, \beta_2)}{t}\Big)^{\omega_2}\Big)\Big\rangle$$

Then

$$2TILWA_{\omega}(\widetilde{p}_1,\widetilde{p}_2)=\omega_1\widetilde{p}_1\oplus\omega_2\widetilde{p}_2$$
$$=\left\langle\Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(s_{\mu_1},\alpha_1)}{t}\right)^{\omega_1}\left(1-\frac{\Delta^{-1}(s_{\mu_2},\alpha_2)}{t}\right)^{\omega_1}\right)\right),\right.$$
$$\left.\Delta\left(t\left(\frac{\Delta^{-1}(s_{v_1},\beta_1)}{t}\right)^{\omega_1}\left(\frac{\Delta^{-1}(s_{v_2},\beta_2)}{t}\right)^{\omega_2}\right)\right\rangle$$

(2) Suppose that $n=k$, equation (21) holds, i.e.,

$$2TILWA_{\omega}(\widetilde{p}_1,\widetilde{p}_2,\cdots,\widetilde{p}_k)=\omega_1\widetilde{p}_1\bigoplus\omega_2\widetilde{p}_2\bigoplus\cdots\omega_k\widetilde{p}_k$$
$$=\left\langle\Delta\left(t\left(1-\prod_{j=1}^{k}\left(1-\frac{\Delta^{-1}(s_{\mu_j},\alpha_j)}{t}\right)^{\omega_j}\right)\right),\Delta\left(t\prod_{j=1}^{k}\left(\frac{\Delta^{-1}(s_{v_j},\beta_j)}{t}\right)^{\omega_j}\right)\right\rangle$$

And the aggregated value is a 2TILN, then when $n=k+1$, by the operational laws of 2-tuple intuitionistic linguistic number, we have

$$2TILWA_{\omega}(\widetilde{p}_1,\widetilde{p}_2,\cdots,\widetilde{p}_k)=\omega_1\widetilde{p}_1\bigoplus\omega_2\widetilde{p}_2\bigoplus\cdots\omega_k\widetilde{p}_k\bigoplus\omega_{k+1}\widetilde{p}_{k+1}$$
$$=\left\langle\Delta\left(t\left(1-\prod_{j=1}^{k}\left(1-\frac{\Delta^{-1}(s_{\mu_j},\alpha_j)}{t}\right)^{\omega_j}\right)\right),\Delta\left(t\prod_{j=1}^{k}\left(\frac{\Delta^{-1}(s_{v_j},\beta_j)}{t}\right)^{\omega_j}\right)\right\rangle\bigoplus$$
$$\left\langle\Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(s_{\mu_{k+1}},\alpha_{k+1})}{t}\right)^{\omega_{k+1}}\right)\right),\Delta\left(t\left(\frac{\Delta^{-1}(s_{v_{k+1}},\beta_{k+1})}{t}\right)^{\omega_{k+1}}\right)\right\rangle$$
$$=\left\langle\Delta\left(t\left(1-\prod_{j=1}^{k+1}\left(1-\frac{\Delta^{-1}(s_{\mu_j},\alpha_j)}{t}\right)^{\omega_j}\right)\right),\Delta\left(t\prod_{j=1}^{k+1}\left(\frac{\Delta^{-1}(s_{v_j},\beta_j)}{t}\right)^{\omega_j}\right)\right\rangle$$

By which aggregated value is also a $2TILN$, Therefore, when $n=k+1$, equation (21) holds. Thus, by (1) and (2), we know that equation (21) holds for all n. The proof is completed.

□

It can be easily proved that the 2TILWA operator has the following properties.

Property. (Idempotency) If all $\widetilde{p}_j(j=1,2,\ldots,n)$ are equal, i.e. $\widetilde{p}_j=\widetilde{p}$ for all j, then

$$2TILWA_{\omega}(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)=\widetilde{p}. \tag{22}$$

Property. (Boundedness) Let $\widetilde{p}_j(j=1,2,\ldots,n)$ be a collection of 2TILNs, and let

$$\widetilde{p}^-=\min_j\widetilde{p}_j,\ \widetilde{p}^+=\max_j\widetilde{p}_j,$$

then

$$\widetilde{p}^-\leq 2TILWA_{\omega}(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)\leq\widetilde{p}^+. \tag{23}$$

Property. (Monotonicity) Let $\widetilde{p}_j(j=1,2,\ldots,n)$ and $\widetilde{p}_j'(j=1,2,\ldots,n)$ be two set of 2TILNs, if $\widetilde{p}_j\leq\widetilde{p}_j'$ for all j, then

$$2TILWA_{\omega}(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)\leq 2TILWA_{\omega}(\widetilde{p}_1',\widetilde{p}_2',\ldots,\widetilde{p}_n'). \tag{24}$$

Further, we give a 2-tuple intuitionistic linguistic ordered weighted averaging (2TILOWA) operator below:

Definition 3.3. Let $\widetilde{P}_j = \langle (s_{\mu_j}, \alpha_j), (s_{v_j}, \beta_j) \rangle$ $(j = 1, 2, \ldots, n)$ be a collection of 2TILNs, the 2-tuple intuitionistic linguistic ordered weighted averaging (2TILOWA) operator of dimension n is a mapping 2TILOWA: $P^n \to P$, that has an associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$. Furthermore,

$$\begin{aligned} & 2TILOWA_\omega(\widetilde{P}_1, \widetilde{P}_2, \ldots, \widetilde{P}_n) = \bigoplus_{j=1}^{n} \left(\omega_j \widetilde{p}_{\sigma(j)}\right) \\ = & \left\langle \Delta\left(t\left(1 - \prod_{j=1}^{n}\left(1 - \frac{\Delta^{-1}(s_{\mu_{\sigma(j)}}, \alpha_{\sigma(j)})}{t}\right)^{\omega_j}\right)\right), \right. \\ & \left. \Delta\left(t \prod_{j=1}^{n}\left(\frac{\Delta^{-1}(s_{v_{\sigma(j)}}, \beta_{\sigma(j)})}{t}\right)^{\omega_j}\right) \right\rangle \end{aligned} \tag{25}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$, such that $\widetilde{P}_{\sigma(j-1)} \geq \widetilde{P}_{\sigma(j)}$ for all $j = 2, \ldots, n$.

It can be easily proved that the 2TILOWA operator has the following properties.

Property. (Idempotency) If all $\widetilde{p}_j (j = 1, 2, \ldots, n)$ are equal, i.e. $\widetilde{p}_j = \widetilde{p}$ for all j, then

$$2TILOWA_\omega(\widetilde{p}_1, \widetilde{p}_2, \cdots, \widetilde{p}_n) = \widetilde{p}. \tag{26}$$

Property. (Boundedness) Let $\widetilde{p}_j (j = 1, 2, \ldots, n)$ be a collection of 2TILNs, and let

$$\widetilde{p}^- = \min_j \widetilde{p}_j, \ \widetilde{p}^+ = \max_j \widetilde{p}_j,$$

$$\widetilde{p}^- \leq 2TILOWA_\omega(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) \leq \widetilde{p}^+. \tag{27}$$

Property. (Monotonicity) Let $\widetilde{p}_j (j = 1, 2, \ldots, n)$ and $\widetilde{p}_j' (j = 1, 2, \ldots, n)$ be two set of 2TILNs, if $\widetilde{p}_j \leq \widetilde{p}_j'$ for all j, then

$$2TILOWA_\omega(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) \leq 2TILOWA_\omega(\widetilde{p}_1', \widetilde{p}_2', \ldots, \widetilde{p}_n'). \tag{28}$$

Property. (Commutativity) Let $\widetilde{p}_j (j = 1, 2, \ldots, n)$ and $\widetilde{p}_j' (j = 1, 2, \ldots, n)$ be two set of 2TILNs, for all j, then

$$2TILOWA_\omega(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = 2TILOWA_\omega(\widetilde{p}_1', \widetilde{p}_2', \ldots, \widetilde{p}_n'). \tag{29}$$

where $\widetilde{p}_j' (j = 1, 2, \ldots, n)$ is any permutation of $\widetilde{p}_j (j = 1, 2, \ldots, n)$.

From Definitions 3.1-3.3, we know that the 2TILWA operators only weights the 2-tuple intuitionistic linguistic number itself, while the 2TILOWA operators weights the ordered positions of the 2-tuple intuitionistic linguistic number instead of weighting the arguments itself. Therefore, the weights represent two different aspects in both the 2TILWA and 2TILOWA operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose the 2-tuple intuitionistic linguistic hybrid average (2TILHA) operator.

Definition 3.4. Let $\widetilde{p}_j = \left\langle \left(s_{\mu_j}, \alpha_j\right), \left(s_{v_j}, \beta_j\right)\right\rangle (j = 1, 2, \ldots, n)$ be a collection of 2TILNs, a 2-tuple intuitionistic linguistic hybrid average (2TILHA) operator is a mapping: $P^n \to P$, such that

$$2TILHA_{w,\omega}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \bigotimes_{j=1}^{n} \left(\dot{\widetilde{p}}_{\sigma(j)}\right)^{w_j}$$
$$= \left\langle \Delta\left(t \prod_{j=1}^{n} \left(\frac{\Delta^{-1}(\dot{s}_{\mu_{\sigma(j)}}, \dot{\alpha}_{\sigma(j)})}{t}\right)^{w_j}\right),\right.$$
$$\left.\Delta\left(t\left(1 - \prod_{j=1}^{n}\left(1 - \frac{\Delta^{-1}(\dot{s}_{\nu_{\sigma(j)}}, \dot{\beta}_{\sigma(j)})}{t}\right)^{w_j}\right)\right)\right\rangle \tag{30}$$

where $w = (w_1, w_2, \cdots, w_n)$ is the associated weighting vector, $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$, and $\dot{\widetilde{p}}_{\sigma(j)}$ is the j-th largest element of the 2-tuple intuitionistic linguistic arguments $\dot{\widetilde{p}}_{\sigma(j)} \left(\dot{\widetilde{p}}_{\sigma(j)} = (n\omega_j)\widetilde{p}_j, j = 1, 2, \cdots, n\right)$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ is the weighting vector of 2-tuple intuitionistic linguistic arguments $\widetilde{p}_j (j = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$, $\sum_{j=1}^{n} \omega_j = 1$ and n is the balancing coefficient. Especially, if $w = (1/n, 1/n, \cdots, 1/n)^T$, then 2TILHA is reduced to the 2-tuple intuitionistic linguistic weighted average (2TILWA) operator; if $\omega = (1/n, 1/n, \cdots, 1/n)^T$, then 2TILHA is reduced to the 2-tuple intuitionistic linguistic ordered weighted average (2TILOWA) operator.

4. 2-tuple intuitionistic linguistic geometric aggregation operators

In this section, we shall develop some geometric aggregation operators with 2-tuple intuitionistic linguistic information, such as 2-tuple intuitionistic linguistic weighted geometric (2TILWG) operator, 2-tuple intuitionistic linguistic ordered weighted geometric (2TILOWG) operator and 2-tuple intuitionistic linguistic hybrid geometric (2TILHG) operator.

Definition 4.1. Let $\tilde{p}_j = \langle(s_{\mu_j}, \alpha_j), (s_{\nu_j}, \beta_j)\rangle (j = 1, 2, \cdot\cdot\cdot, n)$ be a collection of 2TILNs, The 2-tuple intuitionistic linguistic weighted geometric (2TILWG) operator is a mapping $P^n \to P$ such that

$$2TILWG_\omega(\tilde{p}_1, \tilde{p}_2, \cdot\cdot\cdot, \tilde{p}_n) = \mathop{\oplus}_{j=1}^{n} (\tilde{p}_j)^{\omega_j} \tag{31}$$

where $\omega = (\omega_1, \omega_2, \cdot\cdot\cdot, \omega_n)^T$ be the weight vector of $\tilde{p}_j (j = 1, 2, \cdot\cdot\cdot, n)$, and $\omega_j > 0, \sum_{j=1}^{n} \omega_j = 1$.

Based on the Definition 2.17 and Theorem 2.18, we can get the following result:

Theorem 4.2. *The aggregated value by using 2TILWG operator is also a 2TILN, where*

$$2TILWG_\omega(\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_n) = \bigotimes_{j=1}^{n} (\tilde{p}_j)^{\omega_j}$$
$$= \left\langle \Delta\left(t \prod_{j=1}^{n} \left(\frac{\Delta^{-1}(s_{\mu_j}, \alpha_j)}{t}\right)^{\omega_j}\right), \Delta\left(t\left(1 - \prod_{j=1}^{n}\left(1 - \frac{\Delta^{-1}(s_{\nu_j}, \beta_j)}{t}\right)^{\omega_j}\right)\right)\right\rangle \tag{32}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ *be the weight vector of* $\alpha_j (j = 1, 2, \cdots, n)$*, and* $\omega_j > 0$, $\sum_{j=1}^{n} \omega_j = 1$.

Proof. We prove equation (32) by mathematical induction on n.

(1) When $n = 2$, we have $2TILWG_\omega(\tilde{p}_1, \tilde{p}_2) = (\tilde{p}_1)^{\omega_1} \bigotimes (\tilde{p}_2)^{\omega_2}$. By Definition 2.17 and Theorem 2.18 , we can see that both $(\tilde{p}_1)^{\omega_1}$ and $(\tilde{p}_2)^{\omega_2}$ are also 2TILNs. From the operational laws of 2-tuple intuitionistic linguistic number, we have

$$(\tilde{p}_1)^{\omega_1} = \left\langle \Delta\Big(t\Big(\frac{\Delta^{-1}(s_{\mu_1}, \alpha_1)}{t}\Big)^{\omega_1}\Big), \Delta\Big(t\Big(1 - \Big(1 - \frac{\Delta^{-1}(s_{\nu_1}, \beta_1)}{t}\Big)^{\omega_1}\Big)\Big)\right\rangle$$

$$(\tilde{p}_2)^{\omega_2} = \left\langle \Delta\Big(t\Big(\frac{\Delta^{-1}(s_{\mu_2}, \alpha_2)}{t}\Big)^{\omega_2}\Big), \Delta\Big(t\Big(1 - \Big(1 - \frac{\Delta^{-1}(s_{\nu_2}, \beta_2)}{t}\Big)^{\omega_2}\Big)\Big)\right\rangle$$

Then

$$\begin{aligned} & 2TILWG_\omega(\tilde{p}_1, \tilde{p}_2) = (\tilde{p}_1)^{\omega_1} \bigotimes (\tilde{p}_2)^{\omega_2} \\ = & \Big\langle \Delta\left(t\left(\tfrac{\Delta^{-1}(s_{\mu_1},\alpha_1)}{t}\right)^{\omega_1}\left(\tfrac{\Delta^{-1}(s_{\mu_2},\alpha_2)}{t}\right)^{\omega_2}\right), \\ & \Delta\left(t\left(1 - \left(1 - \tfrac{\Delta^{-1}(s_{\nu_1},\beta_1)}{t}\right)^{\omega_1}\left(1 - \tfrac{\Delta^{-1}(s_{\nu_2},\beta_2)}{t}\right)^{\omega_1}\right)\right)\Big\rangle. \end{aligned}$$

(2) Suppose that $n = k$, equation (32) holds, i.e.,

$$\begin{aligned} & 2TILWG_\omega(\widetilde{p}_1, \widetilde{p}_2, \cdots, \widetilde{p}_k) = (\widetilde{p}_1)^{\omega_1} \bigotimes (\widetilde{p}_2)^{\omega_2} \bigotimes \cdots (\widetilde{p}_k)^{\omega_k} \\ = & \left\langle \Delta\left(t\prod_{j=1}^{k}\left(\tfrac{\Delta^{-1}(s_{\mu_j},\alpha_j)}{t}\right)^{\omega_j}\right), \Delta\left(t\left(1 - \prod_{j=1}^{k}\left(1 - \tfrac{\Delta^{-1}(s_{\nu_j},\beta_j)}{t}\right)^{\omega_j}\right)\right)\right\rangle \end{aligned}$$

and the aggregated value is a 2TILN, then by the operational laws of 2-tuple intuitionistic linguistic number, we have

$$\begin{aligned} & 2TILWG_\omega(\widetilde{p}_1, \widetilde{p}_2, \cdots, \widetilde{p}_k) = (\widetilde{p}_1)^{\omega_1} \bigotimes (\widetilde{p}_2)^{\omega_2} \bigotimes \cdots (\widetilde{p}_k)^{\omega_k} \bigotimes (\widetilde{p}_{k+1})^{\omega_{k+1}} \\ = & \left\langle \Delta\left(t\prod_{j=1}^{k}\left(\tfrac{\Delta^{-1}(s_{\mu_j},\alpha_j)}{t}\right)^{\omega_j}\right), \Delta\left(t\left(1 - \prod_{j=1}^{k}\left(1 - \tfrac{\Delta^{-1}(s_{\nu_j},\beta_j)}{t}\right)^{\omega_j}\right)\right)\right\rangle \bigotimes \\ & \left\langle \Delta\left(t\left(\tfrac{\Delta^{-1}(s_{\mu_{k+1}},\alpha_{k+1})}{t}\right)^{\omega_{k+1}}\right), \Delta\left(t\left(1 - \left(1 - \tfrac{\Delta^{-1}(s_{\nu_{k+1}},\beta_{k+1})}{t}\right)^{\omega_{k+1}}\right)\right)\right\rangle \\ = & \left\langle \Delta\left(t\prod_{j=1}^{k+1}\left(\tfrac{\Delta^{-1}(s_{\mu_j},\alpha_j)}{t}\right)^{\omega_j}\right), \Delta\left(t\left(1 - \prod_{j=1}^{k+1}\left(1 - \tfrac{\Delta^{-1}(s_{\nu_j},\beta_j)}{t}\right)^{\omega_j}\right)\right)\right\rangle \end{aligned}$$

By which aggregated value is also a 2TILN, Therefore, when $n = k + 1$, equation (32) holds. Thus, by (1) and (2), we know that equation (32) holds for all n. The proof is completed.

□

Property. (Idempotency) if all $\widetilde{p}_j (j = 1, 2, \ldots, n)$ are equal, $\widetilde{p}_j = \widetilde{p}$ for all j, then

$$2TILWG_\omega(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \widetilde{p}. \tag{33}$$

Property. (Boundedness) Let $\widetilde{p}_j(j=1,2,\ldots,n)$ be a collection of 2TILNs, and let

$$\widetilde{p}^{-}=\min_j \widetilde{p}_j,\ \widetilde{p}^{+}=\max_j \widetilde{p}_j, \widetilde{p}^{-}\leq 2TILWG_\omega(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)\leq \widetilde{p}^{+}. \tag{34}$$

Property. (Monotonicity) Let $\widetilde{p}_j(j=1,2,\ldots,n)$ and $\widetilde{p}_j^{'}(j=1,2,\ldots,n)$ be two set of 2TILNs, if $\widetilde{p}_j\leq\widetilde{p}_j^{'}$ for all j, then

$$2TILWG_\omega(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)\leq 2TILWG_\omega(\widetilde{p}_1^{'},\widetilde{p}_2{'},\ldots,\widetilde{p}_n{'}). \tag{35}$$

Further, we give a 2-tuple intuitionistic linguistic ordered weighted geometric (2TILOWG) operator below:

Definition 4.3. Let $\widetilde{P}_j=\langle(r_j,\alpha_j),(\mu_j,\nu_j)\rangle\,(j=1,2,\ldots,n)$ be a collection of 2TILNs, the 2-tuple intuitionistic linguistic ordered weighted geometric (2TILOWG) operator of dimension n is a mapping 2TILOWG: $P^n\to P$, that has an associated weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ such that $\omega_j>0$ and $\sum\limits_{j=1}^{n}\omega_j=1$. Furthermore,

$$\begin{aligned}2TILOWG_\omega(\widetilde{P}_1,\widetilde{P}_2,\ldots,\widetilde{P}_n)&=\bigotimes_{j=1}^{n}\left(\widetilde{p}_{\sigma(j)}\right)^{\omega_j}\\ =\ &\left\langle\Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}(s_{\mu_{\sigma(j)}},\alpha_{\sigma(j)})}{t}\right)^{\omega_j}\right),\right.\\ &\left.\Delta\left(t\left(1-\prod_{j=1}^{n}\left(1-\frac{\Delta^{-1}(s_{\nu_{\sigma(j)}},\beta_{\sigma(j)})}{t}\right)^{\omega_j}\right)\right)\right\rangle\end{aligned} \tag{36}$$

where $(\sigma(1),\sigma(2),\ldots,\sigma(n))$ is a permutation of $(1,2,\ldots,n)$, such that $\widetilde{P}_{\sigma(j-1)}\geq\widetilde{P}_{\sigma(j)}$ for all $j=2,\ldots,n$.

It can be easily proved that the 2TILOWG operator has the following properties.

Property. (Idempotency) if all $\widetilde{p}_j(j=1,2,\ldots,n)$ are equal, i.e. $\widetilde{p}_j=\widetilde{p}$ for all j, then

$$2TILOWG_\omega(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)=\widetilde{p}. \tag{37}$$

Property. (Boundedness) Let $\widetilde{p}_j(j=1,2,\ldots,n)$ be a collection of 2TILNs, and let

$$\widetilde{p}^{-}=\min_j \widetilde{p}_j,\ \widetilde{p}^{+}=\max_j \widetilde{p}_j,\ \widetilde{p}^{-}\leq 2TILOWG_\omega(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)\leq \widetilde{p}^{+}. \tag{38}$$

Property. (Monotonicity) Let $\widetilde{p}_j(j=1,2,\ldots,n)$ and $\widetilde{p}_j^{'}(j=1,2,\ldots,n)$ be two set of 2TILNs, if $\widetilde{p}_j\leq\widetilde{p}_j^{'}$ for all j, then

$$2TILOWG_\omega(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)\leq 2TILOWG_\omega(\widetilde{p}_1^{'},\widetilde{p}_2{'},\ldots,\widetilde{p}_n{'}). \tag{39}$$

Property. (Commutativity) Let $\widetilde{p}_j(j=1,2,\ldots,n)$ and $\widetilde{p}_j^{'}(j=1,2,\ldots,n)$ be two set of 2TILNs, for all j, then

$$2TILOWG_\omega(\widetilde{p}_1,\widetilde{p}_2,\ldots,\widetilde{p}_n)=2TILOWG_\omega(\widetilde{p}_1^{'},\widetilde{p}_2{'},\ldots,\widetilde{p}_n{'}). \tag{40}$$

where $\widetilde{p}_j{'}(j=1,2,\ldots,n)$ is any permutation of $\widetilde{p}_j(j=1,2,\ldots,n)$.

From Definitions 21-22, we know that the 2TILWG operator only weights the 2-tuple intuitionistic linguistic number itself, while the 2TILOWG operator weights the ordered positions of the 2-tuple intuitionistic linguistic number instead of weighting the arguments itself. Therefore, the weights represent two different aspects in both the 2TILWG and 2TILOWG operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose the 2-tuple intuitionistic linguistic hybrid geometric (2TILHG) operator.

Definition 4.4. Let $\widetilde{p}_j = \left\langle \left(s_{\mu_j}, \alpha_j\right), \left(s_{v_j}, \beta_j\right) \right\rangle (j = 1, 2, \ldots, n)$ be a collection of 2TILNs, a 2-tuple intuitionistic linguistic hybrid geometric (2TILHG) operator is a mapping 2TILHG: $P^n \to P$, such that

$$2TILHG_{w,\omega}(\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_n) = \bigoplus_{j=1}^{n} \left(w_j \dot{\widetilde{p}}_{\sigma(j)}\right)$$
$$= \left\langle \Delta\left(t\left(1 - \prod_{j=1}^{n}\left(1 - \frac{\Delta^{-1}(\dot{s}_{\mu_{\sigma(j)}}, \dot{\alpha}_{\sigma(j)})}{t}\right)^{w_j}\right)\right), \right.$$
$$\left. \Delta\left(t \prod_{j=1}^{n}\left(\frac{\Delta^{-1}(\dot{s}_{v_{\sigma(j)}}, \dot{\beta}_{\sigma(j)})}{t}\right)^{w_j}\right) \right\rangle, \tag{41}$$

where $w = (w_1, w_2, \cdots, w_n)$ is the associated weighting vector, $w_j \in [0, 1]$, $\sum_{j=1}^{n} w_j = 1$, and $\dot{\widetilde{p}}_{\sigma(j)}$ is the j-th largest element of the 2-tuple intuitionistic linguistic arguments $\dot{\widetilde{p}}_{\sigma(j)} \left(\dot{\widetilde{p}}_{\sigma(j)} = (\widetilde{p}_j)^{(n\omega_j)}, j = 1, 2, \cdots, n\right)$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ is the weighting vector of 2-tuple intuitionistic linguistic arguments $\widetilde{p}_j (j = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$, $\sum_{j=1}^{n} \omega_j = 1$ and n is the balancing coefficient. Especially, if $w = (1/n, 1/n, \cdots, 1/n)^T$, then 2TILHG is reduced to the 2-tuple intuitionistic linguistic weighted geometric (2TILWG)operator; if $\omega = (1/n, 1/n, \cdots, 1/n)^T$, then 2TILHG is reduced to the 2-tuple intuitionistic linguistic ordered weighted geometric (2TILOWG) operator.

5. Models for MADM with 2-tuple intuitionistic linguistic information

Based the 2TILWA (2TILWG) operators, in this section, we shall propose the models for MADM with 2-tuple intuitionistic linguistic information. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a discrete set of alternatives, and $G = \{G_1, G_2, \ldots, G_n\}$ be the set of attributes, $\omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ is the weighting vector of the attribute $G_j (j = 1, 2, \ldots, n)$, where $\omega_j \in [0, 1]$, $\sum_{j=1}^{n} \omega_j = 1$. Suppose that $\widetilde{R} = (\widetilde{r}_{ij})_{m \times n} = \langle \left(s_{\mu_{ij}}, \alpha_{ij}\right), \left(s_{\nu_{ij}}, \beta_{ij}\right) \rangle_{m \times n}$ is the 2-tuple intuitionistic linguistic decision matrix, where $\widetilde{r}_{ij}$ take the form of the 2-tuple intuitionistic linguistic numbers, where $\left(s_{\mu_{ij}}, \alpha_{ij}\right)$ indicates the degree that the alternative A_i satisfies the attribute G_j given by the decision maker, $\left(s_{\nu_{ij}}, \beta_{ij}\right)$ indicates the degree that the alternative A_i doesnt satisfy the attribute G_j given by the decision maker, $s_{\mu_{ij}} \in S$, $s_{\nu_{ij}} \in S$, $\alpha_{ij}, \beta_{ij} \in [-0.5, 0.5)$, $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$.

In the following, we apply the 2TILWA (2TILWG) operator to the MADM problems with 2-tuple intuitionistic linguistic information.

Step 1: We utilize the decision information given in matrix $\widetilde{R}$, and the 2TILWA operator

$$\begin{aligned}\widetilde{p}_i &= 2TILWA(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \ldots, \widetilde{r}_{in}) = \bigoplus_{j=1}^{n} (\omega_j \widetilde{r}_{ij}) \\ &= \Big\langle \Delta\Big(t\Big(1 - \prod_{j=1}^{n} \Big(1 - \frac{\Delta^{-1}(s_{\mu_{ij}}, \alpha_{ij})}{t}\Big)^{\omega_j}\Big)\Big), \\ &\quad \Delta\Big(t \prod_{j=1}^{n} \Big(\frac{\Delta^{-1}(s_{\nu_{ij}}, \beta_{ij})}{t}\Big)^{\omega_j}\Big)\Big\rangle, \; i = 1, 2, \ldots, m. \end{aligned} \tag{42}$$

or

$$\begin{aligned}\widetilde{p}_i &= 2TILWG(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \ldots, \widetilde{r}_{in}) = \bigotimes_{j=1}^{n} (\widetilde{r}_{ij})^{\omega_j} \\ &= \Big\langle \Delta\Big(t \prod_{j=1}^{n} \Big(\frac{\Delta^{-1}(s_{\mu_{ij}}, \alpha_{ij})}{t}\Big)^{\omega_j}\Big), \\ &\quad \Delta\Big(t\Big(1 - \prod_{j=1}^{n} \Big(1 - \frac{\Delta^{-1}(s_{\nu_{ij}}, \beta_{ij})}{t}\Big)^{\omega_j}\Big)\Big), \Big\rangle, \; i = 1, 2, \ldots, m. \end{aligned} \tag{43}$$

to derive the overall preference values $\widetilde{p}_i (i = 1, 2, \ldots, m)$ of the alternative A_i .

Step 2: Calculate the scores $S(\widetilde{p}_i)(i = 1, 2, \ldots, m)$ of the overall 2-tuple intuitionistic linguistic numbers $\widetilde{p}_i (i = 1, 2, \ldots, m)$ to rank all the alternatives $A_i (i = 1, 2, \ldots, m)$ and then to select the best one(s). If there is no difference between two scores $S(\widetilde{p}_i)$ and $S(\widetilde{p}_j)$, then we need to calculate the accuracy degrees $H(\widetilde{p}_i)$ and $H(\widetilde{p}_j)$ of the overall 2-tuple intuitionistic linguistic numbers $\widetilde{p}_i$ and $\widetilde{p}_j$, respectively, and then rank the alternatives A_i and A_j in accordance with the accuracy degrees $H(\widetilde{p}_i)$ and $H(\widetilde{p}_j)$.

Step 3: Rank all the alternatives $A_i (i = 1, 2, \ldots, m)$ and select the best one(s) in accordance with $S(\widetilde{p}_i)(i = 1, 2, \ldots, m)$.

Step 4: End.

6. Numerical example

In this section, we utilize a practical MADM problem to illustrate the application of the developed approaches. Suppose an organization plans to implement enterprise resource planning (ERP) system (adapted from [25]). The first step is to form a project team that consists of CIO and two senior representatives from user departments. By collecting all possible information about ERP vendors and systems, project term choose five potential ERP systems $A_i (1 = 1, 2, \ldots, 5)$ as candidates. The company employs some external professional organizations (or experts) to aid this decision-making. The project team selects four attributes to evaluate the alternatives: (1) function and technology G_1, (2) strategic fitness G_2, (3) vendors ability G_3; (4) vendors reputation G_4. The five possible ERP systems are to be evaluated using the 2-tuple intuitionistic linguistic numbers $S = \{s_1 = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, s_4 = \text{medium}, s_5 = \text{good}, s_6 = \text{very good}, s_7 = \text{extremely good}$ by the decision makers under the above four attributes (whose weighting vector is $\omega = (0.2, 0.1, 0.3, 0.4)$), and construct the following matrix $\widetilde{R} = (\widetilde{r}_{ij})_{5\times 4}$ is shown in Table 1.

	G_1	G_2	G_3	G_4
A_1	$\langle (S_4, 0), (S_1, 0)\rangle$	$\langle (S_5, 0), (S_3, 0)\rangle$	$\langle (S_5, 0), (S_3, 0)\rangle$	$\langle (S_4, 0), (S_2, 0)\rangle$
A_2	$\langle (S_6, 0), (S_1, 0)\rangle$	$\langle (S_5, 0), (S_3, 0)\rangle$	$\langle (S_5, 0), (S_3, 0)\rangle$	$\langle (S_6, 0), (S_2, 0)\rangle$
A_3	$\langle (S_5, 0), (S_2, 0)\rangle$	$\langle (S_4, 0), (S_2, 0)\rangle$	$\langle (S_4, 0), (S_2, 0)\rangle$	$\langle (S_5, 0), (S_3, 0)\rangle$
A_4	$\langle (S_6, 0), (S_1, 0)\rangle$	$\langle (S_4, 0), (S_3, 0)\rangle$	$\langle (S_6, 0), (S_1, 0)\rangle$	$\langle (S_4, 0), (S_4, 0)\rangle$
A_5	$\langle (S_4, 0), (S_3, 0)\rangle$	$\langle (S_5, 0), (S_3, 0)\rangle$	$\langle (S_3, 0), (S_4, 0)\rangle$	$\langle (S_5, 0), (S_2, 0)\rangle$

TABLE 1. The 2-tuple intuitionistic linguistic decision matrix.

In the following, in order to select the most desirable ERP systems, we utilize the 2TILWA (2TILWG) operator to develop an approach to MADM problems with 2-tuple intuitionistic linguistic information, which can be described as following.

Step 1: According to Table 1, aggregate all 2-tuple intuitionistic linguistic numbers $\widetilde{r}_{ij}(j = 1, 2, \ldots, n)$ by using the 2TILWA (2TILWG) operator to derive the overall 2-tuple intuitionistic linguistic numbers $\widetilde{p}_i(i = 1, 2, 3, 4, 5)$ of the alternative A_i. The aggregating results are shown in Table 2.

	2TILWA	2TILWG
A_1	$\langle (S_4, 0.45), (S_2, 0.05)\rangle$	$\langle (S_4, 0.37), (S_2, 0.26)\rangle$
A_2	$\langle (S_6, -0.32), (S_2, 0.05)\rangle$	$\langle (S_6, -0.42), (S_2, 0.26)\rangle$
A_3	$\langle (S_5, 0.35), (S_2, 0.35)\rangle$	$\langle (S_5, -0.43), (S_2, 0.43)\rangle$
A_4	$\langle (S_5, 0.27), (S_2, -0.06)\rangle$	$\langle (S_5, -0.10), (S_3, -0.37)\rangle$
A_5	$\langle (S_4, 0.33), (S_3, -0.22)\rangle$	$\langle (S_4, 0.10), (S_3, -0.01)\rangle$

TABLE 2. The aggregating results of the ERP systems by the 2TILWA (2TILWG) operators.

Step 2: According to the aggregating results shown in Table 2 and the score functions of the ERP systems are shown in Table 3.

	2TILWA	2TILWG
A_1	$(S_5, -0.30)$	$(S_4, -0.44)$
A_2	$(S_5, 0.32)$	$(S_5, 0.16)$
A_3	$(S_5, -0.35)$	$(S_5, -0.43)$
A_4	$(S_5, 0.16)$	$(S_5, -0.37)$
A_5	$(S_4, 0.27)$	$(S_4, 0.06)$

TABLE 3. The score functions of the ERP systems.

Step 3: According to the score functions shown in Table 3 and the comparison formula of score functions, the ordering of the ERP systems are shown in Table 4. Note that '>' means 'preferred to'. As we can see, depending on the aggregation operators used, the ordering of the ERP systems is slightly different, but the best ERP system is A_2.

7. Conclusion

In this paper, we investigate the MADM problems with 2-tuple intuitionistic linguistic information. Then, we utilize arithmetic and geometric operations to develop some 2-tuple intuitionistic linguistic aggregation operators: 2-tuple intuitionistic linguistic weighted average (2TILWA) operator, 2-tuple intuitionistic linguistic weighted geometric (2TILWG) operator, 2-tuple intuitionistic linguistic ordered weighted average (2TILOWA) operator, 2-tuple intuitionistic linguistic ordered weighted geometric (2TILOWG) operator, 2-tuple intuitionistic linguistic hybrid average (2TILHA) operator and 2-tuple intuitionistic linguistic hybrid geometric (2TILHG) operator. The prominent characteristic of these proposed operators are studied. Then, we have utilized these operators to develop some approaches to solve the 2-tuple intuitionistic linguistic MADM problems. Finally, a practical example for enterprise resource planning (ERP) system selection is given to verify the developed approach and to demonstrate its practicality and effectiveness. In the future, the application of the proposed aggregating operators of 2TILSs needs to be explored in the decision making, risk analysis and many other fields under uncertain environment [50, 26, 29, 63, 45, 44, 28, 54] and usefulness of our operators in real problems or web applications as recommender systems, digital libraries and so on [4, 51, 52, 12, 13, 47, 48, 49, 46, 53].

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