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Stratified (L, M)-semiuniform convergence tower spaces

J. Li[1]

[1] *Department of Mathematics, Henan Normal University, Xinxiang 453007, PR China, and School of Mathematics, Beijing Institute of Technology, Beijing 100081, PR China*

lijuan@htu.edu.cn

Abstract

The notion of stratified (L, M)-semiuniform convergence tower spaces is introduced, which extends the notions of probabilistic semiuniform convergence spaces and lattice-valued semiuniform convergence spaces. The resulting category is shown to be a strong topological universe. Besides, the relations between our category and that of stratified (L, M)-filter tower spaces are studied.

Keywords: Stratified L-filters, Stratified (L, M)-semiuniform convergence tower spaces, Lattice-valued semiuniform convergence spaces, Strong topological universe.

1 Introduction

In the theory of topological spaces, "uniform concepts" such as Cauchy filter, completeness , uniform continuous etc, cannot be described. Uniform spaces were introduced by Weil [26] in 1937 for describing "uniform concepts". But the category of uniform spaces and uniformly continuous mappings is not Cartesian closed. Cook and Fisher [3] generalized uniform spaces to uniform limit spaces, which was slightly modified by Wyler [27] in 1974. The resulting category is Cartesian closed. By omitting some axioms of uniform limit spaces, Preuss [23, 24] established the category of semiuniform convergence spaces and uniformly continuous mappings. The category of semiuniform convergence spaces is a strong topological universe including uniform spaces and topological spaces, and it is possible to study both topological and uniform aspects within this framework.

For the lattice-valued case, lattice-valued uniform convergence spaces are introduced in [19]. The category of lattice-valued uniform convergence spaces is a Cartesian closed supercategory of the category of lattice-valued uniform spaces [10]. In [4], the lattice context of these spaces was generalized from complete Heyting algebras to the case of enriched lattices. By making use of the lattice-valued inclusion order of stratified L-filters, Fang [6] proposed the concept of stratified L-ordered quasi-uniform limit structure. Fang [5] extended semiuniform convergence spaces to the lattice-valued case by relaxing the axioms of lattice-valued uniform convergence spaces, which was called stratified L-semiuniform convergence spaces. The category of stratified L-semiuniform convergence spaces is a strong topological universe when L is a completely distributive lattice.

In 1971, Frank [9] introduced the notion of a probabilistic topological space by using "θ-closure". For categorical consederation, some subsequent models were combinations of probabilistic ideas with notions of convergence spaces, filter spaces, and uniform convergence spaces [8, 9, 11, 20, 22, 25]. These probabilistic spaces were subsequently extended. Yang and Li [28] introduced stratified (L, M)-filter tower spaces, which extended the notions of probabilistic filter spaces and stratified (L, M)-filter spaces. Flores, Mohapatra and Richardson [7] proposed an alternative set of axioms for the study of lattice-valued convergence spaces [17, 18], which extended the notion of probabilistic convergence spaces.

This paper starts from this kind of extension idea and proposes the notion of stratified (L, M)-semiuniform convergence tower spaces as a kind of extension of probabilistic semiuniform convergence spaces and stratified L-semiuniform convergence spaces. The category of stratified (L, M)-semiuniform convergence tower spaces and uniformly continuous

Corresponding Author: J. Li

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mappings is shown to be a strong topological universe. Moreover, the relations between stratified (L, M)-semiuniform convergence spaces and stratified (L, M)-filter tower spaces are studied.

This paper is organized as follows. In Section 2, we give the necessary lattice-theoretic backgrounds, the notations and results about stratified L-filters as well as some concepts related to categorical theory. In Section 3, we show that the category of stratified (L, M)-semiuniform convergence tower spaces is a well-fibred topological category, and establish the relations between stratified L-semiuniform convergence spaces and stratified (L, L)-semiuniform convergence tower spaces. In Section 4, we show that the category of stratified (L, M)-semiuniform convergence tower spaces is Cartesian closed. Section 5 presents the extensionality of the category of stratified (L, M)-semiuniform convergence tower spaces. In Section 6, we show that the category of stratified (L, M)-semiuniform convergence tower spaces is closed under the formation of products of quotient mappings. Finally, we conclude that the category of stratified (L, M)-semiuniform convergence tower spaces is a strong topological universe. In Section 7, we investigate the relations between stratified (L, M)-semiuniform convergence tower spaces and stratified (L, M)-filter tower spaces.

2 Preliminaries

Throughout this paper, L(resp.,M) denotes a complete Heyting algebra, i.e., a complete lattice equipped with an implication $\to: L \times L \to L$ such that $a \wedge b \leqslant c \Leftrightarrow a \leqslant b \to c$ for all $a, b, c \in L$. The smallest element and the largest element in L (resp., M)are denoted by 0 and 1 respectively. For a nonempty set X, L^X denotes the set of all L-subsets on X. L^X is also a complete Heyting algebra, when it inherits the structure of the lattice L in a natural way. The smallest element and the largest element in L^X are denoted by 0_X and 1_X respectively. For each $a \in L$, we define the L-subset a_X by $a_X(x) = a$ for all $x \in X$.

Definition 2.1. [12, 17] *A mapping $\mathcal{F} : L^X \longrightarrow L$ is called a stratified L-filter on X if it satisfies:*

(LF1) $\mathcal{F}(0_X) = 0$, $\mathcal{F}(1_X) = 1$.
(LF2) $A \leqslant B \Longrightarrow \mathcal{F}(A) \leqslant \mathcal{F}(B)$.
(LF3) $\mathcal{F}(A) \wedge \mathcal{F}(B) \leqslant \mathcal{F}(A \wedge B)$.
(LF4) $a \wedge \mathcal{F}(A) \leqslant \mathcal{F}(a_X \wedge A)$.

The family of all stratified L-filters on X is denoted by $\mathcal{F}_L^s(X)$. On the set $\mathcal{F}_L^s(X)$, we define an order by $\mathcal{F} \leqslant \mathcal{G}$ if $\mathcal{F}(A) \leqslant \mathcal{G}(A)$ for all $A \in L^X$. Every nonempty family $\{\mathcal{F}_i\}_{i\in I}$ of stratified L-filters has an infimum $\bigwedge_{i\in I} \mathcal{F}_i$, which can be calculated as $\forall A \in L^X$, $(\bigwedge\limits_{i\in I} \mathcal{F}_i)(A) = \bigwedge\limits_{i\in I} \mathcal{F}_i(A)$. Let $f : X \longrightarrow Y$ be a mapping. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$, the mapping $f^{\Rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$, $A \mapsto \mathcal{F}(f^{\leftarrow}(A))$ is a stratified L-filter on Y and is called the image of $\mathcal{F}$ under f in [12]. For each $\mathcal{G} \in \mathcal{F}_L^s(Y)$, Jäger [17] proved that the mapping $f^{\Leftarrow}(\mathcal{G}) : L^X \longrightarrow L$ defined by $f^{\Leftarrow}(\mathcal{G}) = \bigvee\limits_{f^{\leftarrow}(B)\leqslant A} \mathcal{G}(B)$ is a stratified L-filter on X if and only if $f^{\leftarrow}(B) = 0_X$ implies $\mathcal{G}(B) = 0$. For a family $\{\mathcal{F}_i\}_{i\in I}$, $\forall i \in I$, $\mathcal{F}_i \in \mathcal{F}_L^s(X_i)$, Jäger [17] proposed that the product $\prod_i \mathcal{F}_i$ is defined by $\prod_i \mathcal{F}_i = \bigvee_i (Pr_{X_i})^{\Leftarrow}(\mathcal{F}_i)$. In particular, for $\mathcal{F} \in \mathcal{F}_L^s(X_1)$ and $\mathcal{G} \in \mathcal{F}_L^s(X_2)$, we have $(\mathcal{F} \times \mathcal{G})(A) = \bigvee\limits_{A_1\times A_2\leqslant A} \mathcal{F}(A_1) \wedge \mathcal{G}(A_2)$.

Example 2.2. [12, 17] *For each point $x \in X$, the mapping $[x] : L^X \longrightarrow L$, $A \mapsto A(x)$ is a stratified L-filter on X, called the point L-filter of x.*

For a stratified L-filter $\mathcal{F}$ on $X \times X$, a stratified L-filter $\mathcal{F}^{-1}$ [19] was defined by $\mathcal{F}^{-1}(A) = \mathcal{F}(A^{-1})$ for each $A \in L^{X\times X}$, where $A^{-1}(x, y) = A(y, x)$ for all $(x, y) \in X \times X$.

Lemma 2.3. [19] *If $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X \times X)$, then $\mathcal{F} \leqslant \mathcal{G} \Longrightarrow \mathcal{F}^{-1} \leqslant \mathcal{G}^{-1}$.*

Lemma 2.4. [19] *If $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X \times X)$, then $(\mathcal{F} \times \mathcal{G})^{-1} = \mathcal{G} \times \mathcal{F}$.*

For mappings $f : X \longrightarrow Z$ and $g : Y \longrightarrow W$, the product mapping $f \times g : X \times Y \longrightarrow Z \times W$ is defined by $(f \times g)(x, y) = (f(x), g(y))$, $\forall (x, y) \in X \times Y$. Furthermore, $(f \times g)^{\Rightarrow}(\mathcal{F} \times \mathcal{G}) = f^{\Rightarrow}(\mathcal{F}) \times g^{\Rightarrow}(\mathcal{G})$.

Lemma 2.5. [5] *Let $f : X \longrightarrow Y$ be a mapping. Then*
(1) $((f \times f)^{\Rightarrow}(\mathcal{F}))^{-1} = (f \times f)^{\Rightarrow}(\mathcal{F}^{-1})$ *for all* $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$.
(2) $(f \times f)^{\Leftarrow}(\mathcal{H})$ *exists if and only if* $(f \times f)^{\Leftarrow}(\mathcal{H}^{-1})$ *exists, further* $((f \times f)^{\Leftarrow}(\mathcal{H}))^{-1} = (f \times f)^{\Leftarrow}(\mathcal{H}^{-1})$ *for all* $\mathcal{H} \in \mathcal{F}_L^s(Y \times Y)$.

Definition 2.6. [1] *A category* $\mathbf{C}$ *is called a topological category over* $\mathbf{Set}$ *provided that for any set* X*, any class* J*, any family* $((X_j,\xi_j))_{j\in J}$ *of* $\mathbf{C}$*-objects and any family* $(f_j : X \longrightarrow X_j)_{j\in J}$ *of mappings, there exists a unique* $\mathbf{C}$*-structure* ξ *on* X *which is initial with respect to the source* $(f_j : X \longrightarrow (X_j,\xi_j))_{j\in J}$*. This means that for a* $\mathbf{C}$*-object* (Y,η)*, a mapping* $g:(Y,\eta)\longrightarrow(X,\xi)$ *is a* $\mathbf{C}$*-morphism if and only if for all* $j\in J$*,* $f_j\circ g:(Y,\eta)\longrightarrow(X_j,\xi_j)$ *is a* $\mathbf{C}$*-morphism.*

Definition 2.7. [1] *A subcategory* $\mathbf{A}$ *of* $\mathbf{B}$ *is said to be reflective in* $\mathbf{B}$ *if for each* $\mathbf{B}$*-object* B*, there exists an* $\mathbf{A}$*-object* C *and a* $\mathbf{B}$*-morphism* $f:B\longrightarrow C$ *such that for any* $\mathbf{B}$*-morphism* $g:B\longrightarrow A$ *from* B *to an* $\mathbf{A}$*-object* A*, there exists a unique* $\mathbf{A}$*-morphism* $h:C\longrightarrow A$ *with* $h\circ f=g$*.*

3 Stratified (L,M)-semiuniform convergence tower spaces

In this section, the definition of stratified (L,M)-semiuniform convergence tower spaces is introduced. It is shown that the category of stratified L-semiuniform convergence spaces is isomorphic to a reflective subcategory of our category when $L=M$.

Definition 3.1. *Let* X *be a non-void set and* $\bar{\mathcal{T}}=\{\mathcal{T}_\lambda\mid\lambda\in M\}$ *a nonempty family of subsets of* $\mathcal{F}_L^s(X\times X)$*. The pair* $(X,\bar{\mathcal{T}})$ *is called a stratified* (L,M)*-semiuniform convergence tower space if it satisfies*

(UCT1) *For all* $x\in X$*,* $\lambda\in M$*,* $[x]\times[x]\in\mathcal{T}_\lambda$*.*

(UCT2) $\mathcal{G}\in\mathcal{T}_\lambda$ *whenever* $\mathcal{F}\in\mathcal{T}_\lambda$ *and* $\mathcal{F}\leqslant\mathcal{G}$*.*

(UCT3) $\mathcal{F}^{-1}\in\mathcal{T}_\lambda$ *whenever* $\mathcal{F}\in\mathcal{T}_\lambda$*.*

(P1) $\mathcal{T}_\lambda\leqslant\mathcal{T}_\mu$ *whenever* $\mu\leqslant\lambda$*.*

(P2) $\mathcal{T}_0=\mathcal{F}_L^s(X\times X)$*.*

The pair $(X,\bar{\mathcal{T}})$ *is said to be left continuous if it satisfies* $\bigcap_{\nu\in A}\mathcal{T}_\nu=\mathcal{T}_{\vee A}$ *for any nonempty set* $A\subseteq M$*.*

A mapping $f:(X,\bar{\mathcal{T}}^X)\longrightarrow(Y,\bar{\mathcal{T}}^Y)$ *between two stratified* (L,M)*-semiuniform convergence tower spaces is called uniformly continuous if* $(f\times f)^{\Rightarrow}(\mathcal{F})\in\mathcal{T}_\lambda^Y$ *for all* $\mathcal{F}\in\mathcal{T}_\lambda^X$ *and for all* $\lambda\in M$*. The category of stratified* (L,M)*-semiuniform convergence tower spaces(resp., left continuous stratified* (L,M)*-semiuniform convergence tower spaces) and uniformly continuous mappings is denoted by* $\mathbf{S}(L,M)$-$\mathbf{SUConvTr}$ *(resp.,* $\mathbf{LC}$-$\mathbf{S}(L,M)$-$\mathbf{SUConvTr}$*).*

Remark 3.2. *A stratified* (L,M)*-semiuniform convergence tower space is not necessarily left continuous.*

Let $L=M=X=[0,1]$*. Define* $\bar{\mathcal{T}}=\{\mathcal{T}_\lambda\mid\lambda\in L\}$ *as follows:* $\mathcal{T}_1=\{\mathcal{F}\in\mathcal{F}_L^s(X\times X)\mid$ *there exists* $x\in X$ *such that* $[x]\times[x]\leqslant\mathcal{F}\}$*,* $\lambda\in[0,0.5)$*,* $\mathcal{T}_\lambda=\mathcal{F}_L^s(X\times X)$*, and* $\lambda\in[0.5,1)$*,* $\mathcal{T}_\lambda=\mathcal{T}_1$*. It is easily checked that* $(X,\bar{\mathcal{T}})$ *is a stratified* (L,M)*-semiuniform convergence tower space, but it is not left continuous.*

Example 3.3. *For* $L=\{0,1\}$ *and* $M=[0,1]$ *we can identify the notion of* $(2,M)$*-semiuniform convergence tower spaces with the notion of probabilistic semiuniform convergence spaces [22].*

Example 3.4. *Let* $L=\{0,1\}$ *and* $M=[0,\infty]$ *with the opposite order. Then a left continuous stratified* (L,M)*-semiuniform convergence tower space is a semi-approach uniform convergence space in the definition of Nauwelaerts*[21].

Example 3.5. *For* $L=M$ *we can identify the notion of left continuous* (L,L)*-semiuniform convergence tower spaces with the notion of lattice-valued semiuniform convergence spaces* [5] *(see Theorem 3.14) .*

Example 3.6. [13] *Let* (X,Λ) *be a lattice-valued uniform convergence space. Define* $\bar{\Lambda}=\{\Lambda_\lambda\mid\lambda\in L\}$*, where* $\Lambda_\lambda\subseteq\mathcal{F}_L^s(X\times X)$ *is called the* λ*-level structure defined by* $\mathcal{F}\in\Lambda_\lambda\Longleftrightarrow\Lambda(\mathcal{F})\geqslant\lambda$*. Then* $\bar{\Lambda}$ *satisfies* (UCT1) – (UCT3), (P1) *and* (P2)*. Therefore, the pair* $(X,\bar{\Lambda})$ *is a stratified* (L,L)*-semiuniform convergence tower space.*

Example 3.7. *Let* $L=\{0,1\}$ *and* $M=\triangle^+$ *be the frame of distance distribution functions with the pointwise minimum and supremum. Then a probabilistic uniform convergence space in the definition of Ahsanullah and Jäger*[2] *is a stratified* (L,M)*-semiuniform convergence tower space.*

Theorem 3.8. *The category* $\mathbf{S}(L,M)$-$\mathbf{SUConvTr}$ *is a well-fibred category over* $\mathbf{Set}$*.*

Proof. For the fibre-smallness of $\mathbf{S}(L,M)$-$\mathbf{SUConvTr}$, it is obvious that the class of all (L,M)-semiuniform convergence tower structures on a set X is a set. For the terminal separator property of $\mathbf{S}(L,M)$-$\mathbf{SUConvTr}$, let $X=\{x\}$. Then for each $\mathcal{F}\in\mathcal{F}_L^s(X\times X)$, $\mathcal{F}\geqslant[x]\times[x]$. By (UCT1) and (UCT2), we have $\mathcal{F}\in\mathcal{T}_\lambda\Longleftrightarrow\mathcal{F}\geqslant[x]\times[x]$. Hence, there is a unique stratified (L,M)-semiuniform convergence tower structure on $\{x\}$. □

Theorem 3.9. *The category* $\mathbf{S}(L,M)$-$\mathbf{SUConvTr}$ *is a topological category over* $\mathbf{Set}$*.*

Proof. Let $f_i : X \longrightarrow (X_i, \bar{\mathcal{T}}_i)$ be a mapping, where $(X_i, \bar{\mathcal{T}}_i)$ is a stratified (L, M)-semiuniform convergence tower space for all $i \in I$. For each $\lambda \in M$, define a set $\mathcal{T}_\lambda \subseteq \mathcal{F}_L^s(X \times X)$ by

$$\mathcal{T}_\lambda = \{\mathcal{F} \in \mathcal{F}_L^s(X \times X) \mid (f_i \times f_i)^{\Rightarrow}(\mathcal{F}) \in (\mathcal{T}_i)_\lambda, \forall i \in I\}.$$

It can be easily proved that $\bar{\mathcal{T}} = \{\mathcal{T}_\lambda \mid \lambda \in M\}$ is the initial structure of the source $(f_i : X \longrightarrow (X_i, \bar{\mathcal{T}}_i))_{i \in I}$. □

By Theorem 3.9 we know there exists a unique final structure with respect to a sink $\{f_i : (X_i, \bar{\mathcal{T}}_i) \longrightarrow X\}_{i \in I}$ in the category $\mathbf{S}(L, M)$-**SUConvTr**. Now we explore the concrete form of the final structure.

Proposition 3.10. *Let $\{f_i : (X_i, \bar{\mathcal{T}}_i) \longrightarrow X\}_{i \in I}$ be a family of mappings, where $(X_i, \bar{\mathcal{T}}_i)$ is a stratified (L, M)-semiuniform convergence tower space. Define $\bar{\mathcal{T}} = \{\mathcal{T}_\lambda \mid \lambda \in M\}$ by $\mathcal{T}_\lambda = \{\mathcal{G} \in \mathcal{F}_L^s(X \times X) \mid \exists i \in I, \mathcal{F} \in (\mathcal{T}_i)_\lambda$ such that $(f_i \times f_i)^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G}\} \bigcup \{\mathcal{H} \in \mathcal{F}_L^s(X \times X) \mid \exists x \in X,$ such that $\mathcal{H} \geqslant [x] \times [x]\}$. Then $\bar{\mathcal{T}}$ is the unique final structure with respect to the sink $\{f_i : (X_i, \bar{\mathcal{T}}_i) \longrightarrow X\}_{i \in I}$. Further, if the sink $\{f_i : (X_i, \bar{\mathcal{T}}_i) \longrightarrow X\}_{i \in I}$ is surjective, that is, $X = \bigcup_{i \in I} f_i[X_i]$, then $\mathcal{T}_\lambda = \{\mathcal{G} \in \mathcal{F}_L^s(X \times X) \mid \exists i \in I, \mathcal{F} \in (\mathcal{T}_i)_\lambda$ such that $(f_i \times f_i)^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G}\}$.*

Proof. The proof is routine and omitted. □

Let $(X, \bar{\mathcal{T}})$ be a stratified (L, M)-semiuniform convergence tower space. For a surjective mapping $f : (X, \bar{\mathcal{T}}) \longrightarrow Y$, we call $(Y, \bar{\xi})$ a quotient space of $(X, \bar{\mathcal{T}})$, f a quotient mapping, where $\bar{\xi} = \{\xi_\lambda \mid \lambda \in M\}$ is the final structure with respect to f. According to Proposition 3.10, $\xi_\lambda = \{\mathcal{G} \in \mathcal{F}_L^s(Y \times Y) \mid \exists \mathcal{F} \in \mathcal{T}_\lambda, (f \times f)^{\Rightarrow}(\mathcal{F}) \leqslant \mathcal{G}\}$.

Dually, we can define a subspace and an initial mapping.

Since the category $\mathbf{S}(L, M)$-**SUConvTr** is a topological category over **Set**, there is the product of stratified (L, M)-semiuniform convergence tower spaces in the category $\mathbf{S}(L, M)$-**SUConvTr**. We now give the definition of the product of stratified (L, M)-semiuniform convergence tower spaces.

Definition 3.11. *Let $\{(X_i, \bar{\mathcal{T}}_i)\}_{i \in I}$ be a family of stratified (L, M)-semiuniform convergence tower spaces and $\{p_j : \prod_{i \in I} X_i \longrightarrow (X_j, \bar{\mathcal{T}}_j)\}_{j \in I}$ be the source formed by the family of the projection mappings $\{p_j : \prod_{i \in I} X_i \longrightarrow X_j\}_{j \in I}$. The stratified (L, M)-semiuniform convergence tower structure on $X = \prod_{i \in I} X_i$, denoted by $\prod_{i \in I} \bar{\mathcal{T}}_i$, that is initial with respect to $\{p_j : X = \prod_{i \in I} X_i \longrightarrow (X_j, \bar{\mathcal{T}}_j)\}_{j \in I}$, is called the product stratified (L, M)-semiuniform convergence tower structure and the pair $(X, \prod_{i \in I} \bar{\mathcal{T}}_i)$ is called the product space briefly. Thus we have*

$$\left(\prod_{i \in I} \bar{\mathcal{T}}_i\right)_\lambda = \{\mathcal{F} \in \mathcal{F}_L^s(X \times X) \mid (p_i \times p_i)^{\Rightarrow}(\mathcal{F}) \in (\mathcal{T}_i)_\lambda, \forall i \in I\}.$$

For the product space of stratified (L, M)-semiuniform convergence tower spaces $(X, \bar{\mathcal{T}}^X)$ and $(Y, \bar{\mathcal{T}}^Y)$, we write $(X \times Y, \bar{\mathcal{T}}^X \times \bar{\mathcal{T}}^Y)$ for $(X, \bar{\mathcal{T}}^X) \times (Y, \bar{\mathcal{T}}^Y)$.

Theorem 3.12. LC-$\mathbf{S}(L, M)$-SUConvTr *is a reflective subcategory of* $\mathbf{S}(L, M)$-**SUConvTr**.

Proof. Let $(X, \bar{\mathcal{T}}) \in |\mathbf{S}(L, M)\text{-}\mathbf{SUConvTr}|$. Define $L\bar{\mathcal{T}} = \{(L\mathcal{T})_\lambda \mid \lambda \in M\}$ as follows:
$(L\mathcal{T})_\lambda = \{\mathcal{F} \in \mathcal{F}_L^s(X \times X) \mid$ there exists $A \subseteq M$, $\bigvee A = \lambda$ and $\mathcal{F} \in \mathcal{T}_\mu$ for each $\mu \in A\}$. Then it is easily checked that $(X, L\bar{\mathcal{T}}) \in |\mathbf{LC\text{-}S}(L, M)\text{-}\mathbf{SUConvTr}|$. Further, we show that $id_X : (X, \bar{\mathcal{T}}) \to (X, L\bar{\mathcal{T}})$ is the **LC-S**(L, M)-**SUConvTr**-reflection.

(1) Since $\mathcal{T}_\lambda \subseteq (L\mathcal{T})_\lambda$ for each $\lambda \in M$, $id_X : (X, \bar{\mathcal{T}}) \longrightarrow (X, L\bar{\mathcal{T}})$ is uniformly continuous.

(2) Assume that $f : (X, \bar{\mathcal{T}}) \longrightarrow (Y, \bar{\xi})$ is uniformly continuous, where $(Y, \bar{\xi}) \in |\mathbf{LC\text{-}S}(L, M)\text{-}\mathbf{SUConvTr}|$. Let $\mathcal{F} \in (L\mathcal{T})_\lambda$. Then there exists $A \subseteq M$ such that $\bigvee A = \lambda$ and $\mathcal{F} \in \mathcal{T}_\mu$ for each $\mu \in A$. Since $f : (X, \bar{\mathcal{T}}) \longrightarrow (Y, \bar{\xi})$ is uniformly continuous, $(f \times f)^{\Rightarrow}(\mathcal{F}) \in \xi_\mu$ for each $\mu \in A$. As $(Y, \bar{\xi})$ is left continuous, we have $(f \times f)^{\Rightarrow}(\mathcal{F}) \in \xi_\lambda$. This shows $f : (X, L\bar{\mathcal{T}}) \longrightarrow (Y, \bar{\xi})$ is uniformly continuous. □

Lattice-valued semiuniform convergence structures were introduced by Fang [5] as follows:

Definition 3.13. [5] *Let X be a non-void set. A mapping $T^X : \mathcal{F}_L^s(X \times X) \longrightarrow L$ is called a stratified L-semiuniform convergence structure on X if it fulfills*

(UC1) $T^X([x] \times [x]) = 1$ *for all* $x \in X$.

(UC2) $\mathcal{F} \leqslant \mathcal{G} \Longrightarrow T^X(\mathcal{F}) \leqslant T^X(\mathcal{G})$ *for all* $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X \times X)$.

(UC3) $T^X(\mathcal{F}) \leqslant T^X(\mathcal{F}^{-1})$ *for all* $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$.

The pair (X, T^X) is called a stratified L-semiuniform convergence space.

A mapping $f : (X, T^X) \longrightarrow (Y, T^Y)$ between two stratified L-semiuniform convergence spaces is called uniformly continuous if $T^X(\mathcal{F}) \leqslant T^Y((f \times f)^{\Rightarrow}(\mathcal{F}))$, for all $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$.

The category of stratified L-semiuniform convergence spaces and uniformly continuous mappings is denoted by **SL-SUConv**.

In the following, it is proved that Fang's category is isomorphic to a reflective subcategory of $\mathbf{S}(L, L)$-**SUConvTr**.

Define a mapping φ : $\mathbf{S}L$-**SUConv** $\longrightarrow$ **LC-** $\mathbf{S}(L, L)$-**SUConvTr** by $\varphi(f) = f$ and $\varphi(X, T) = (X, \bar{\mathcal{T}}_T)$, where $\bar{\mathcal{T}}_T = \{(\mathcal{T}_T)_\lambda \mid \lambda \in L\}$ and $\mathcal{F} \in (\mathcal{T}_T)_\lambda \Longleftrightarrow T(\mathcal{F}) \geqslant \lambda$. It is easily checked that $(X, \bar{\mathcal{T}}_T) \in |$**LC-S**$(L, L)$-**SUConvTr**$|$.

Conversely, define a mapping ψ : **LC-S**(L, L)-**SUConvTr** $\longrightarrow$ $\mathbf{S}L$-**SUConv** by $\psi(f) = f$ and $\psi(X, \bar{\mathcal{T}}) = (X, T_{\bar{\mathcal{T}}})$, where for each $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$, $T_{\bar{\mathcal{T}}}(\mathcal{F}) = \bigvee\{\lambda \in L \mid \mathcal{F} \in \mathcal{T}_\lambda\}$. It is easily checked that $(X, T_{\bar{\mathcal{T}}}) \in |$ $\mathbf{S}L$ -**SUConv** $|$.

Theorem 3.14. **LC-S**(L, L)-**SUConvTr** *is isomorphic to* $\mathbf{S}L$-**SUConv** .

Proof. Firstly, we prove that φ and ψ are functors. Assume that $f : (X, T^X) \longrightarrow (Y, T^Y)$ is uniformly continuous. If $\mathcal{F} \in (\mathcal{T}_{T^X})_\lambda$, then $T^X(\mathcal{F}) \geqslant \lambda$. Since $f : (X, T^X) \longrightarrow (Y, T^Y)$ is uniformly continuous, we have $T^Y((f \times f)^{\Rightarrow}(\mathcal{F})) \geqslant T^X(\mathcal{F}) \geqslant \lambda$. Thus $(f \times f)^{\Rightarrow}(\mathcal{F}) \in (\mathcal{T}_{T^Y})_\lambda$. Therefore, $f : (X, \bar{\mathcal{T}}_{T^X}) \longrightarrow (Y, \bar{\mathcal{T}}_{T^Y})$ is uniformly continuous and φ is a functor. Conversely, assume that $f : (X, \bar{\mathcal{T}}) \longrightarrow (Y, \bar{\xi})$ is uniformly continuous. Since $f : (X, \bar{\mathcal{T}}) \longrightarrow (Y, \bar{\xi})$ is uniformly continuous, for each $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$, we have $T_{\bar{\mathcal{T}}}(\mathcal{F}) = \bigvee\{\lambda \in L \mid \mathcal{F} \in \mathcal{T}_\lambda\} \leqslant \bigvee\{\lambda \in L \mid (f \times f)^{\Rightarrow}(\mathcal{F}) \in \xi_\lambda\} = T_{\bar{\xi}}((f \times f)^{\Rightarrow}(\mathcal{F}))$. Therefore, $f : (X, T_{\bar{\mathcal{T}}}) \longrightarrow (Y, T_{\bar{\xi}})$ is uniformly continuous and ψ is a functor.

It remains to show that $\varphi \circ \psi = id_{\mathbf{LC-S}(L,L)-\mathbf{SUConvTr}}$ and $\psi \circ \varphi = id_{\mathbf{S}L-\mathbf{SUConv}}$.

Let $(X, \bar{\mathcal{T}}) \in |$**LC-S**$(L, L)$-**SUConvTr**$|$. Then $\bar{\mathcal{T}}_{T_{\bar{\mathcal{T}}}} = \bar{\mathcal{T}}$. This follows from the fact: for each $\lambda \in L$, $\mathcal{F} \in (\mathcal{T}_{T_{\bar{\mathcal{T}}}})_\lambda \Longleftrightarrow T_{\bar{\mathcal{T}}}(\mathcal{F}) \geqslant \lambda \Longleftrightarrow \bigvee\{\mu \in L \mid \mathcal{F} \in \mathcal{T}_\mu\} \geqslant \lambda \Longleftrightarrow \mathcal{F} \in \mathcal{T}_\lambda$. This shows $\varphi \circ \psi = id_{\mathbf{LC-S}(L,L)-\mathbf{SUConvTr}}$.

Conversely, let $(X, T) \in |\mathbf{S}L$-**SUConv**$|$. Then $T_{\bar{\mathcal{T}}_T} = T$. This follows from the fact: for each $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$, $T_{\bar{\mathcal{T}}_T}(\mathcal{F}) = \bigvee\{\lambda \in L \mid \mathcal{F} \in (\mathcal{T}_T)_\lambda\} = \bigvee\{\lambda \in L \mid T(\mathcal{F}) \geqslant \lambda\} = T(\mathcal{F})$. This shows $\psi \circ \varphi = id_{\mathbf{S}L-\mathbf{SUConv}}$. □

4 Cartesian-closedness of $\mathbf{S}(L, M)$-SUConvTr

Recall a category $\mathbf{C}$ is called Cartesian-closed [1] provided that the following conditions are satisfied:
(1) For each pair (X, Y) of $\mathbf{C}$-objects there exists a product $X \times Y$ in $\mathbf{C}$,
(2) For each pair of $\mathbf{C}$-objects X and Y, there exists a $\mathbf{C}$-object Y^X (called power object) and a $\mathbf{C}$-morphism $ev_{X,Y}$: $Y^X \times X \longrightarrow Y$ (called evaluation morphism) such that for each C-object Z and each C-morphism $f : Z \times X \longrightarrow Y$, there exists a unique C-morphism $g : Z \longrightarrow Y^X$ such that $ev_{X,Y} \circ (g \times id_X) = f$.

Since $\mathbf{S}(L, M)$-**SUConvTr** is topological, the condition (1) is fulfilled. We now explore the concrete form of power objects in $\mathbf{S}(L, M)$-**SUConvTr**.

Given two stratified (L, M)-semiuniform convergence tower spaces $(X, \bar{\mathcal{T}})$ and $(Y, \bar{\xi})$. Put $[X, Y] = \{f \mid f : X \longrightarrow Y$ is uniformly continuous$\}$. Let $ev_{X,Y} : [X, Y] \times X \longrightarrow Y$ be the evaluation mapping such that $(f, x) \mapsto f(x)$ and $j : ([X, Y] \times [X, Y]) \times (X \times X) \longrightarrow ([X, Y] \times X) \times ([X, Y] \times X)$ be the canonical bijection such that $((f, g), (x, x')) \mapsto ((f, x), (g, x'))$.

For each $\lambda \in M$, define $\varepsilon_\lambda \subseteq \mathcal{F}_L^s([X, Y] \times [X, Y])$ as follows: $\varepsilon_\lambda = \{\mathcal{F} \in \mathcal{F}_L^s([X, Y] \times [X, Y]) \mid \mu \leqslant \lambda, ((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}(\mathcal{F} \times \mathcal{G}) \in \xi_\mu (\forall \mathcal{G} \in \mathcal{T}_\mu)\}$.

Lemma 4.1. [5] *Let* $f : X \longrightarrow Y$, $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$. *Then* $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}(([f] \times [f]) \times \mathcal{F}) \geqslant (f \times f)^{\Rightarrow}(\mathcal{F})$.

Lemma 4.2. [19] *Let* $\mathcal{H} \in \mathcal{F}_L^s((X \times Y) \times (X \times Y))$, $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$, $\mathcal{G} \in \mathcal{F}_L^s(Y \times Y)$, $P_X : X \times Y \longrightarrow X$, $P_Y : X \times Y \longrightarrow Y$ *be the projection mappings , and* $m : (X \times X) \times (Y \times Y) \longrightarrow (X \times Y) \times (X \times Y)$ *be the bijection with* $((x_1, x_2), (y_1, y_2)) \mapsto ((x_1, y_1), (x_2, y_2))$. *Then*

(1) $m^{\Rightarrow}((P_X \times P_X)^{\Rightarrow}(\mathcal{H}) \times (P_Y \times P_Y)^{\Rightarrow}(\mathcal{H})) \leqslant \mathcal{H}$.

(2) $(P_X \times P_X)^{\Rightarrow}(m^{\Rightarrow}(\mathcal{F} \times \mathcal{G})) \geqslant \mathcal{F}$ *and* $(P_Y \times P_Y)^{\Rightarrow}(m^{\Rightarrow}(\mathcal{F} \times \mathcal{G})) \geqslant \mathcal{G}$.

(3) $m^{\Rightarrow}(\mathcal{F}^{-1} \times \mathcal{G}) = (m^{\Rightarrow}(\mathcal{F} \times \mathcal{G}^{-1}))^{-1}$.

Corollary 4.3. *Let* $\mathcal{F} \in \mathcal{T}_\lambda$ *and* $\mathcal{G} \in \xi_\lambda$. *Then* $m^{\Rightarrow}(\mathcal{F} \times \mathcal{G}) \in (\bar{\mathcal{T}} \times \bar{\xi})_\lambda$.

Proposition 4.4. $([X, Y], \bar{\varepsilon})$ *is a stratified* (L, M)*-semiuniform convergence tower space.*

Proof. (UCT1) For all $f \in [X, Y]$ and $\lambda \in M$, let $\mu \leqslant \lambda$ and $\mathcal{G} \in \mathcal{T}_\mu$. Then $(f \times f)^{\Rightarrow}(\mathcal{G}) \in \xi_\mu$. By Lemma 4.1, we know $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}(([f] \times [f]) \times \mathcal{G}) \geqslant (f \times f)^{\Rightarrow}(\mathcal{G})$. Thus $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}(([f] \times [f]) \times \mathcal{G}) \in \xi_\mu$. This shows $[f] \times [f] \in \varepsilon_\lambda$.

(UCT2) is trivial.

(UCT3) Assume that $\mathcal{F} \in \varepsilon_\lambda$. Then for all $\mu \leqslant \lambda$, $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}(\mathcal{F} \times \mathcal{G}) \in \xi_\mu$ for each $\mathcal{G} \in \mathcal{T}_\mu$. Thus $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}(\mathcal{F}^{-1} \times \mathcal{G}) = (((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}(\mathcal{F} \times \mathcal{G}^{-1}))^{-1} \in \xi_\mu$. This shows $\mathcal{F}^{-1} \in \varepsilon_\lambda$.

(P_1) and (P_2) are trivial. □

Proposition 4.5. *Let* $(X, \bar{\mathcal{T}}), (Y, \bar{\xi})$ *be stratified* (L, M)*-semiuniform convergence tower spaces. Then the evaluation mapping* $ev_{X,Y} : [X, Y] \times X \longrightarrow Y$ *is uniformly continuous.*

Proof. Let $\mathcal{H} \in (\bar{\varepsilon} \times \bar{\mathcal{T}})_\lambda$. Then $(P_{[X,Y]} \times P_{[X,Y]})^{\Rightarrow}(\mathcal{H}) \in \varepsilon_\lambda$ and $(P_X \times P_X)^{\Rightarrow}(\mathcal{H}) \in \mathcal{T}_\lambda$. By construction of ε_λ, $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}((P_{[X,Y]} \times P_{[X,Y]})^{\Rightarrow}(\mathcal{H}) \times (P_X \times P_X)^{\Rightarrow}(\mathcal{H})) \in \xi_\lambda$. Since $j^{\Rightarrow}((P_{[X,Y]} \times P_{[X,Y]})^{\Rightarrow}(\mathcal{H}) \times (P_X \times P_X)^{\Rightarrow}(\mathcal{H})) \leqslant \mathcal{H}$, $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}((P_{[X,Y]} \times P_{[X,Y]})^{\Rightarrow}(\mathcal{H}) \times (P_X \times P_X)^{\Rightarrow}(\mathcal{H})) \leqslant (ev_{X,Y} \times ev_{X,Y})^{\Rightarrow}(\mathcal{H})$. This implies $(ev_{X,Y} \times ev_{X,Y})^{\Rightarrow}(\mathcal{H}) \in \xi_\lambda$. Therefore the evaluation mapping $ev_{X,Y} : [X,Y] \times X \longrightarrow Y$ is uniformly continuous. □

Let $f : Z \times X \longrightarrow Y$ be a mapping and $m_1 : (Z \times Z) \times (X \times X) \longrightarrow (Z \times X) \times (Z \times X)$ the canonical bijection. Define $f_* : Z \longrightarrow Y^X$ by $f_*(z)(x) = f(z,x)$.

Lemma 4.6. [19] *For each $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$ and $z \in Z$, it holds that $(f_*(z) \times f_*(z))^{\Rightarrow}(\mathcal{F}) \geqslant ((f \times f) \circ m_1)^{\Rightarrow}(([z] \times [z]) \times \mathcal{F})$.*

Lemma 4.7. [19] *Let $\mathcal{F} \in \mathcal{F}_L^s(Z \times Z)$ and $\mathcal{G} \in \mathcal{F}_L^s(X \times X)$. Then for each mapping $f : Z \times X \longrightarrow Y$, $((ev_{X,Y} \times ev_{X,Y}) \circ j)^{\Rightarrow}((f_* \times f_*)^{\Rightarrow}(\mathcal{F}) \times \mathcal{G}) = ((f \times f) \circ m_1)^{\Rightarrow}(\mathcal{F} \times \mathcal{G})$.*

By Lemmas 4.6, 4.7 and Corollary 4.3, we have the following Proposition.

Proposition 4.8. *Let $(X, \bar{\mathcal{T}}), (Y, \bar{\xi}), (Z, \bar{\eta})$ be stratified (L,M)-semiuniform convergence tower spaces. If $f : Z \times X \longrightarrow Y$ is uniformly continuous, then so is $f_* : Z \longrightarrow ([X,Y], \bar{\varepsilon})$.*

By Propositions 4.4, 4.5 and 4.8, we obtain the main result in this section.

Theorem 4.9. **S**(L,M)-**SUConvTr** *is Cartesian-closed.*

5 Extensionality of S(L,M)-SUConvTr

In this section, we will explore the extensionality of the category **S**(L,M)-**SUConvTr**. Recall in a topological category C, a partial morphism from Y to X is a C-morphism $f : Z \to X$ whose domain is a subobject of Y. A topological category C is called extensional provided that for every C-object X has a one-point extension X^*, in the sense that every C-object X can be embedded via the addition of a single point ∞ into a C-object X^* such that for every partial morphism $f : Z \to X$, the mapping $f^* : Y \longrightarrow X^*$ defined by

$$f^*(x) = \begin{cases} f(x), & x \in Z; \\ \infty, & x \notin Z. \end{cases}$$

is a C-morphism. In this section, i denotes the inclusion mapping on X. First we need some technical results of Fang [5].

Lemma 5.1. [5] *Let X be a non-void set. Put $X^* = X \bigcup \{\infty\}$, $\infty \notin X$. Define a mapping: $\inf_\infty : L^{X^* \times X^*} \longrightarrow L$ as follows:*

$$\inf_\infty(A) = \inf\{A(x,y) \mid (x,y) \in (\{\infty\} \times X^*) \bigcup (X^* \times \{\infty\})\}, A \in L^{X^* \times X^*}.$$

Then $\inf_\infty$ is a stratified L-filter on $X^* \times X^*$.

Lemma 5.2. [5] *Let $(X^*)^2 \setminus X^2$ denote the set $(\{\infty\} \times X^*) \bigcup (X^* \times \{\infty\})$. Then for each $\mathcal{H} \in \mathcal{F}_L^s(X^* \times X^*)$, $(i \times i)^{\Leftarrow}(\mathcal{H})$ does not exist if and only if $\mathcal{H}(1_{(X^*)^2 \setminus X^2}) \neq 0$. In general, let $Z \subseteq Y$ and $k : Z \longrightarrow Y$ be the inclusion mapping. Then $(k \times k)^{\Leftarrow}(\mathcal{H})$ does not exist if and only if $\mathcal{H}(1_{Y^2 \setminus Z^2}) \neq 0$ for each $\mathcal{H} \in \mathcal{F}_L^s(Y \times Y)$.*

Lemma 5.3. *[5] Let $\mathcal{F} \in \mathcal{F}_L^s(X \times X)$, then $(i \times i)^{\Leftarrow}((i \times i)^{\Rightarrow}(\mathcal{F}) \wedge inf_\infty) = \mathcal{F}$.*

Theorem 5.4. *Let $(X, \bar{\mathcal{T}})$ be a stratified (L,M)-semiuniform convergence tower space, define $\bar{\xi} = \{\xi_\lambda \mid \lambda \in M\}$ by $\xi_\lambda = \{\mathcal{G} \in \mathcal{F}_L^s(X^* \times X^*) \mid (i \times i)^{\Leftarrow}(\mathcal{G}) \in \mathcal{T}_\lambda, \mathcal{G}(1_{(X^*)^2 \setminus X^2}) = 0\} \bigcup \{\mathcal{G} \in \mathcal{F}_L^s(X^* \times X^*) \mid \mathcal{G}(1_{(X^*)^2 \setminus X^2}) \neq 0)\}$. Then $(X^*, \bar{\xi})$ is a stratified (L,M)-semiuniform convergence tower space.*

Proof. (UCT1) For each $x \in X$, since $[x] \times [x](1_{(X^*)^2 \setminus X^2}) = 0$ and $(i \times i)^{\Leftarrow}([x] \times [x]) = [x] \times [x] \in \mathcal{T}_\lambda$, $[x] \times [x] \in \xi_\lambda$. In addition, $([\infty] \times [\infty])(1_{(X^*)^2 \setminus X^2}) \neq 0$, so $[\infty] \times [\infty] \in \xi_\lambda$.

(UCT2) Suppose that $\mathcal{F} \in \xi_\lambda$ and $\mathcal{G} \geqslant \mathcal{F}$. If $\mathcal{G}(1_{(X^*)^2 \setminus X^2}) \neq 0$, then $\mathcal{G} \in \xi_\lambda$. Otherwise, we have $\mathcal{F}(1_{(X^*)^2 \setminus X^2}) = 0$, and thus $(i \times i)^{\Leftarrow}(\mathcal{F}) \in \mathcal{T}_\lambda$. Since $(i \times i)^{\Leftarrow}(\mathcal{G}) \geqslant (i \times i)^{\Leftarrow}(\mathcal{F})$, $(i \times i)^{\Leftarrow}(\mathcal{G}) \in \mathcal{T}_\lambda$, which means $\mathcal{G} \in \xi_\lambda$.

(UCT3) is proved from the facts that $\mathcal{F}(1_{(X^*)^2 \setminus X^2}) \neq 0 \Longleftrightarrow \mathcal{F}^{-1}(1_{(X^*)^2 \setminus X^2}) \neq 0$ and $(i \times i)^{\Leftarrow}(\mathcal{F}^{-1}) = ((i \times i)^{\Leftarrow}(\mathcal{F}))^{-1}$.

(P_1) and (P_2) are trivial.

□

Theorem 5.5. **S**(L, M)-**SUConvTr** *is extensional.*

Proof. Let $(X, \bar{\mathcal{T}})$ be a stratified (L, M)-semiuniform convergence tower space and $(X^*, \bar{\xi})$ be defined as above. By Theorem 5.4, $(X^*, \bar{\xi})$ is a stratified (L, M)-semiuniform convergence tower space. It suffice to show that $(X^*, \bar{\xi})$ is the one-point extension of $(X, \bar{\mathcal{T}})$. For this it suffices to prove:
(1) $(X, \bar{\mathcal{T}})$ is a subspace of $(X^*, \bar{\xi})$.
(2) $(X^*, \bar{\xi})$ is the one-point extension of $(X, \bar{\mathcal{T}})$.

(1) For each $\lambda \in M$, denote $\varepsilon_\lambda = \{\mathcal{F} \mid (i \times i)^{\Rightarrow}(\mathcal{F}) \in \xi_\lambda\}$. We need to prove $\varepsilon_\lambda = \mathcal{T}_\lambda$. Let $\mathcal{F} \in \mathcal{T}_\lambda$, since $(i\times i)^{\Leftarrow}((i\times i)^{\Rightarrow}(\mathcal{F})) = \mathcal{F}$, $(i\times i)^{\Rightarrow}(\mathcal{F}) \in \xi_\lambda$, which means $\mathcal{F} \in \varepsilon_\lambda$. Conversely, assume that $\mathcal{F} \in \varepsilon_\lambda$, then $(i\times i)^{\Rightarrow}(\mathcal{F}) \in \xi_\lambda$. Since $(i \times i)^{\Rightarrow}(\mathcal{F})(1_{(X^*)^2\setminus X^2}) = 0$, $(i \times i)^{\Leftarrow}((i \times i)^{\Rightarrow}(\mathcal{F})) = \mathcal{F} \in \mathcal{T}_\lambda$.

(2) Suppose that f is a partial morphism from $(Y, \bar{\mathcal{T}}^Y)$ to $(X, \bar{\mathcal{T}})$, i.e., there exists a stratified (L, M)-semiuniform convergence tower space $(Z, \bar{\mathcal{T}}^Z)$, which is a subspace of $(Y, \bar{\mathcal{T}}^Y)$ such that $f : (Z, \bar{\mathcal{T}}^Z) \longrightarrow (X, \bar{\mathcal{T}})$ is uniformly continuous. We now show f^* is uniformly continuous. Let $\mathcal{F} \in (\mathcal{T}^Y)_\lambda$.
Case 1 $(i \times i)^{\Leftarrow}(\mathcal{F})$ does not exist. Then $\mathcal{F}(1_{Y^2\setminus Z^2}) \neq 0$. However,
$(f^* \times f^*)^{\Rightarrow}(\mathcal{F})(1_{(X^*)^2\setminus X^2}) = \mathcal{F}(1_{Y^2\setminus Z^2}) \neq 0$. This means $(f^* \times f^*)^{\Rightarrow}(\mathcal{F}) \in \xi_\lambda$.
Case 2 $(i \times i)^{\Leftarrow}(\mathcal{F})$ exists. Since $(k \times k)^{\Rightarrow}((k \times k)^{\Leftarrow}(\mathcal{F})) \geqslant \mathcal{F} \in (\mathcal{T}^Y)_\lambda$ and $(Z, \bar{\mathcal{T}}^Z)$ is a subspace of $(Y, \bar{\mathcal{T}}^Y)$, $(k \times k)^{\Leftarrow}(\mathcal{F}) \in (\mathcal{T}^Z)_\lambda$. As f is uniformly continuous, $(f \times f)^{\Rightarrow}((k \times k)^{\Leftarrow}(\mathcal{F})) \in \mathcal{T}_\lambda$. Moreover, $(i \times i)^{\Leftarrow}((i \times i)^{\Rightarrow}((f \times f)^{\Rightarrow}((k\times k)^{\Leftarrow}(\mathcal{F}))\wedge \inf_\infty)) = (f\times f)^{\Rightarrow}((k\times k)^{\Leftarrow}(\mathcal{F})) \in \mathcal{T}_\lambda$. It follows that $(i\times i)^{\Rightarrow}((f\times f)^{\Rightarrow}((k\times k)^{\Leftarrow}(\mathcal{F}))\wedge \inf_\infty) \in \xi_\lambda$. Since $(i\times i)^{\Rightarrow}((f\times f)^{\Rightarrow}((k\times k)^{\Leftarrow}(\mathcal{F}))\wedge \inf_\infty) \leqslant (f^*\times f^*)^{\Rightarrow}(\mathcal{F})$ (See the proof of [[5], Theorem 7.5]), $(f^*\times f^*)^{\Rightarrow}(\mathcal{F}) \in \xi_\lambda$. This shows that f^* is uniformly continuous. □

6 Products of quotient mappings in S(L, M)-SUConvTr

In this section, we show that products of quotient mappings are quotient mappings, and conclude that **S**(L, M)-**SUConvTr** is a strong topological universe.

Lemma 6.1. [5] *Let $\{X_i\}_{i\in I}$ be a family of sets, $P_{X_i} : \prod_i X_i \longrightarrow X_i$ the projection mapping and $j : \prod_i X_i \times X_i \longrightarrow \prod_i X_i \times \prod_i X_i$ the bijection. Then*
(1) $\mathcal{G}_i \leqslant ((P_{X_i} \times P_{X_i}) \circ j)^{\Rightarrow}(\prod_i \mathcal{G}_i)$, $\mathcal{G}_i \in \mathcal{F}_L^s(X_i \times X_i)$, $\forall i \in I$.
(2) $j^{\Rightarrow}(\prod_i (P_{X_i} \times P_{X_i})^{\Rightarrow}(\mathcal{H})) \leqslant \mathcal{H}$ *for each* $\mathcal{H} \in \mathcal{F}_L^s(\prod_i X_i \times \prod_i X_i)$.

Lemma 6.2. [5] *Let $\{f_i : X_i \longrightarrow Y_i\}_{i\in I}$ be a family of surjective mappings and for all $i \in I$, $\mathcal{G}_i \in \mathcal{F}_L^s(X_i \times X_i)$. Then $((\prod_i f_i \times \prod_i f_i) \circ j)^{\Rightarrow}(\prod_i \mathcal{G}_i) \leqslant k^{\Rightarrow}(\prod_i (f_i \times f_i)^{\Rightarrow}(\mathcal{G}_i))$, where $k : \prod_i (Y_i \times Y_i) \longrightarrow \prod_i Y_i \times \prod_i Y_i$ is the bijection.*

Theorem 6.3. *Let $\{f_i : (X_i, \bar{\mathcal{T}_i}) \longrightarrow (Y_i, \bar{\xi_i})\}_{i\in I}$ be a family of quotient mappings, $(X, \bar{\mathcal{T}})$ the product of $\{(X_i, \bar{\mathcal{T}_i})\}_{i\in I}$ and $(Y, \bar{\xi})$ the product of $\{(Y_i, \bar{\xi_i})\}_{i\in I}$. Then $\prod_i f_i : (X, \bar{\mathcal{T}}) \longrightarrow (Y, \bar{\xi})$ is a quotient mapping.*

Proof. Obviously, $\prod_i f_i$ is surjective. Put $\varepsilon_\lambda = \{\mathcal{H} \in \mathcal{F}_L^s(Y \times Y) \mid \exists \mathcal{G} \in \mathcal{T}_\lambda, (\prod_i f_i \times \prod_i f_i)^{\Rightarrow}(\mathcal{G}) \leqslant \mathcal{H}$. It suffices to prove that $\varepsilon_\lambda = \xi_\lambda$ for all $\lambda \in M$.

Assume that $\mathcal{H} \in \varepsilon_\lambda$, then there exists $\mathcal{G} \in \mathcal{T}_\lambda$ such that $(\prod_i f_i \times \prod_i f_i)^{\Rightarrow}(\mathcal{G}) \leqslant \mathcal{H}$. Thus $(P_{Y_i} \times P_{Y_i})^{\Rightarrow}((\prod_i f_i \times \prod_i f_i)^{\Rightarrow}(\mathcal{G})) \leqslant (P_{Y_i}\times P_{Y_i})^{\Rightarrow}(\mathcal{H})$. Note that $(P_{Y_i}\times P_{Y_i})\circ(\prod_i f_i\times\prod_i f_i) = (f_i\times f_i)\circ(P_{X_i}\times P_{X_i})$, we have $(f_i\times f_i)^{\Rightarrow}((P_{X_i} \times P_{X_i})^{\Rightarrow}(\mathcal{G})) \leqslant (P_{Y_i} \times P_{Y_i})^{\Rightarrow}(\mathcal{H})$. Since f_i is a quotient mapping and $(P_{X_i} \times P_{X_i})^{\Rightarrow}(\mathcal{G}) \in (\mathcal{T}_i)_\lambda$, $(P_{Y_i} \times P_{Y_i})^{\Rightarrow}(\mathcal{H}) \in (\xi_i)_\lambda$. As $(Y, \bar{\xi})$ is the product of $\{(Y_i, \bar{\xi_i})\}_{i\in I}$, we obtain $\mathcal{H} \in \xi_\lambda$. Conversely, let $\mathcal{H} \in \xi_\lambda$, then $(P_{Y_i} \times P_{Y_i})^{\Rightarrow}(\mathcal{H}) \in (\xi_i)_\lambda$. Since f_i is a quotient mapping, there exists $\mathcal{G}_i \in (\mathcal{T}_i)_\lambda$ such that $(f_i \times f_i)^{\Rightarrow}(\mathcal{G}_i) \leqslant (P_{Y_i} \times P_{Y_i})^{\Rightarrow}(\mathcal{H})$. By $\mathcal{G}_i \leqslant ((P_{X_i} \times P_{X_i}) \circ j)^{\Rightarrow}(\prod_i \mathcal{G}_i)$, we have $((P_{X_i} \times P_{X_i}) \circ j)^{\Rightarrow}(\prod_i \mathcal{G}_i) \in (\mathcal{T}_i)_\lambda$. As $(X, \bar{\mathcal{T}})$ is the product of $\{(X_i, \bar{\mathcal{T}_i})\}_{i\in I}$, we have $j^{\Rightarrow}(\prod_i \mathcal{G}_i) \in \mathcal{T}_\lambda$. By Lemma 6.1 (2) and Lemma 6.2, the inequality $((\prod_i f_i \times \prod_i f_i) \circ j)^{\Rightarrow}(\prod_i \mathcal{G}_i)) \leqslant k^{\Rightarrow}(\prod_i (f_i \times f_i)^{\Rightarrow}(\mathcal{G}_i)) \leqslant k^{\Rightarrow}(\prod_i (P_{Y_i} \times P_{Y_i})^{\Rightarrow}(\mathcal{H})) \leqslant \mathcal{H}$ holds. Then we have $\mathcal{H} \in \varepsilon_\lambda$.

□

Recall that the following several convenient properties for a topological category C are proposed by Preuss in the book [24]:
(CP1) C is Cartesian closed.
(CP2) C is extensional.
(CP3) In C product of quotient mappings is a quotient mapping. Moreover, C is called
(1) strongly Cartesian closed provided that it fulfills (CP1) and (CP3).
(2) is a topological universe provided that it fulfills (CP1) and (CP2).
(3) is a strong topological universe provided that it fulfills (CP1), (CP2) and (CP3).
By Theorem 6.3, 5.5 and 4.9, we have the following Theorem.

Theorem 6.4. **S**(L, M)-**SUConvTr** *is a strong topological universe.*

7 The relations between S(L, M)-SUConvTr and S(L, M)-FilTr

In this section, we study the relations between stratified (L, M)-filter tower spaces and stratified (L, M)-semiuniform convergence tower spaces.

The concept of stratified (L, M)-filter tower spaces was introduced in [28] as follows:

Definition 7.1. *Let X be a nonempty set. If $\bar{\gamma} = \{\gamma_\lambda \mid \lambda \in M\}$, where, $\gamma_\lambda \subseteq \mathcal{F}_L^s(X)$, satisfies the following:*
(LMFT1) *For each $x \in X, \lambda \in M$, $[x] \in \gamma_\lambda$;*
(LMFT2) *$\mathcal{G} \in \gamma_\lambda$ whenever $\mathcal{F} \in \gamma_\lambda$ and $\mathcal{F} \leqslant \mathcal{G}$;*
(LFT1) *$\gamma_\lambda \leqslant \gamma_\mu$ whenever $\mu \leqslant \lambda$;*
(LFT2) *$\gamma_0 = \mathcal{F}_L^s(X)$,*
then the pair $(X, \bar{\gamma})$ is called a stratified (L, M)-filter tower space.

Definition 7.2. *A mapping $f : (X, \bar{\gamma}) \longrightarrow (Y, \bar{\eta})$ between two stratified (L, M)-filter tower spaces is called uniformly continuous if $f^{\Rightarrow}(\mathcal{F}) \in \eta_\lambda$ for all $\mathcal{F} \in \gamma_\lambda$ and for all $\lambda \in M$.*

The category of stratified (L, M)-filter tower spaces and uniformly continuous mappings is denoted by $\mathbf{S}(L, M)$-**FilTr**.

Lemma 7.3. *Let $(X, \bar{\mathcal{T}})$ be a stratified (L, M)-semiuniform convergence tower space. Then $(X, \bar{\gamma}_{\bar{\mathcal{T}}})$ is a stratified (L, M)-filter tower space, where $\bar{\gamma}_{\bar{\mathcal{T}}} = \{(\gamma_{\bar{\mathcal{T}}})_\lambda \mid \lambda \in M\}$, $(\gamma_{\bar{\mathcal{T}}})_\lambda = \{\mathcal{F} \mid \mathcal{F} \in \mathcal{F}_L^s(X), \mathcal{F} \times \mathcal{F} \in \mathcal{T}_\lambda\}$.*

Proof. (LMFT1) For each $x \in X$, $\lambda \in M$, since $[x] \times [x] \in \mathcal{T}_\lambda$, $[x] \in (\gamma_{\bar{\mathcal{T}}})_\lambda$.

(LMFT2) If $\mathcal{F} \in (\gamma_{\bar{\mathcal{T}}})_\lambda$ and $\mathcal{F} \leqslant \mathcal{G}$, then $\mathcal{F} \times \mathcal{F} \in \mathcal{T}_\lambda$ and $\mathcal{F} \times \mathcal{F} \leqslant \mathcal{G} \times \mathcal{G}$. By (UCT2), we have $\mathcal{G} \times \mathcal{G} \in \mathcal{T}_\lambda$. Thus $\mathcal{G} \in (\gamma_{\bar{\mathcal{T}}})_\lambda$.

(LMT1) and (LMT2) can be easily proved by (P1) and (P2) respectively. □

Lemma 7.4. *Let $(X, \bar{\gamma})$ be a stratified (L, M)-filter tower space. Then $(X, \bar{\mathcal{T}}\gamma)$ is a stratified (L, M)-semiuniform convergence tower space, where*

$$(\mathcal{T}_{\bar{\gamma}})_\lambda = \{\mathcal{F} \mid \mathcal{F} \in \mathcal{F}_L^s(X \times X), \exists \mathcal{H} \in \gamma_\lambda : \mathcal{F} \geqslant \mathcal{H} \times \mathcal{H}\}.$$

Proof. (UCT1) For each $x \in X$ and $\lambda \in M$, Since $[x] \in \gamma_\lambda$, $[x] \times [x] \in (\mathcal{T}_{\bar{\gamma}})_\lambda$.

(UCT2) If $\mathcal{F} \in (\mathcal{T}_{\bar{\gamma}})_\lambda$ and $\mathcal{F} \leqslant \mathcal{G}$, then there exists $\mathcal{H} \in \gamma_\lambda$ such that $\mathcal{H} \times \mathcal{H} \leqslant \mathcal{F} \leqslant \mathcal{G}$. So $\mathcal{G} \in (\mathcal{T}_{\bar{\gamma}})_\lambda$.

(UCT3) $\mathcal{F} \in (\mathcal{T}_{\bar{\gamma}})_\lambda \Longrightarrow \exists \mathcal{H} \in \gamma_\lambda : \mathcal{H} \times \mathcal{H} \leqslant \mathcal{F} \Longrightarrow (\mathcal{H} \times \mathcal{H})^{-1} \leqslant \mathcal{F}^{-1} \Longrightarrow \mathcal{H} \times \mathcal{H} \leqslant \mathcal{F}^{-1} \Longrightarrow \mathcal{F}^{-1} \in (\mathcal{T}_{\bar{\gamma}})_\lambda$.

(P1) and (P2) follow from (LMT1) and (LMT2) respectively. □

Define a mapping θ : $\mathbf{S}(L, M)$-**SUConvTr** $\longrightarrow$ $\mathbf{S}(L, M)$-**FilTr** by $(X, \bar{\mathcal{T}}) \longmapsto (X, \bar{\gamma}_{\bar{\mathcal{T}}})$ and $f \longmapsto f$, where $\bar{\gamma}_{\bar{\mathcal{T}}} = \{(\gamma_{\bar{\mathcal{T}}})_\lambda \mid \lambda \in M\}$, $(\gamma_{\bar{\mathcal{T}}})_\lambda = \{\mathcal{F} \mid \mathcal{F} \in \mathcal{F}_L^s(X), \mathcal{F} \times \mathcal{F} \in \mathcal{T}_\lambda\}$.

Conversely, define a mapping δ : $\mathbf{S}(L, M)$-**FilTr** $\longrightarrow$ $\mathbf{S}(L, M)$-**SUConvTr** by $(X, \bar{\gamma}) \longmapsto (X, \bar{\mathcal{T}}\gamma)$ and $f \longmapsto f$, where $\bar{\mathcal{T}}\gamma = \{(\mathcal{T}_{\bar{\gamma}})_\lambda \mid \lambda \in M\}$,

$$(\mathcal{T}_{\bar{\gamma}})_\lambda = \{\mathcal{F} \mid \mathcal{F} \in \mathcal{F}_L^s(X \times X), \exists \mathcal{H} \in \gamma_\lambda : \mathcal{F} \geqslant \mathcal{H} \times \mathcal{H}\}.$$

Theorem 7.5. (1) *θ and δ are functors.*
(2) *$\delta \circ \theta \leqslant id_{\mathbf{S}(L,M)-\mathbf{SUConvTr}}$ and $\theta \circ \delta = id_{\mathbf{S}(L,M)-\mathbf{FilTr}}$.*

Proof. (1) We first prove that θ is a functor. Assume that $f : (X, \bar{\mathcal{T}}^X) \longrightarrow (Y, \bar{\mathcal{T}}^Y)$ is uniformly continuous. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$ and $\lambda \in M$, if $\mathcal{F} \in (\gamma_{\bar{\mathcal{T}}^X})_\lambda$, then $\mathcal{F} \times \mathcal{F} \in \mathcal{T}_\lambda^X$. Since $f : (X, \bar{\mathcal{T}}^X) \longrightarrow (Y, \bar{\mathcal{T}}^Y)$ is uniformly continuous, we have $(f \times f)^{\Rightarrow}(\mathcal{F} \times \mathcal{F}) = f^{\Rightarrow}(\mathcal{F}) \times f^{\Rightarrow}(\mathcal{F}) \in \mathcal{T}_\lambda^Y$. This implies $f^{\Rightarrow}(\mathcal{F}) \in (\gamma_{\bar{\mathcal{T}}^Y})_\lambda$. Therefore $f : (X, \bar{\gamma}_{\bar{\mathcal{T}}^X}) \longrightarrow (Y, \bar{\gamma}_{\bar{\mathcal{T}}^Y})$ is uniformly continuous and θ is a functor.

We next prove that δ is a functor. If $f : (X, \bar{\gamma^X}) \longrightarrow (Y, \bar{\gamma^Y})$ is uniformly continuous, then $f : (X, \bar{\mathcal{T}}_{\bar{\gamma}^X}) \longrightarrow (Y, \bar{\mathcal{T}}_{\bar{\gamma}^Y})$ is uniformly continuous can be proved from the following: for each $\lambda \in M, \mathcal{F} \in (\mathcal{T}_{\bar{\gamma}})_\lambda \Longrightarrow \exists \mathcal{H} \in \gamma_\lambda : \mathcal{H} \times \mathcal{H} \leqslant \mathcal{F} \Longrightarrow (f \times f)^{\Rightarrow}(\mathcal{H} \times \mathcal{H}) \leqslant (f \times f)^{\Rightarrow}(\mathcal{F}) \Longrightarrow f^{\Rightarrow}(\mathcal{H}) \times f^{\Rightarrow}(\mathcal{H}) \leqslant (f \times f)^{\Rightarrow}(\mathcal{F}) \Longrightarrow (f \times f)^{\Rightarrow}(\mathcal{F}) \in (\mathcal{T}_{\bar{\gamma}})_\lambda$.

(2) Let $(X, \bar{\mathcal{T}}) \in \mid \mathbf{S}(L, M)$-**SUConvTr** $\mid$ and $(X, \bar{\gamma}) \in \mid \mathbf{S}(L, M)$-**FilTr** $\mid$. We now prove that $\bar{\mathcal{T}}_{\bar{\gamma}_{\bar{\mathcal{T}}}} \leqslant \bar{\mathcal{T}}$ and $\bar{\gamma}_{\bar{\mathcal{T}}_{\bar{\gamma}}} = \bar{\gamma}$. It is easily checked that $\bar{\mathcal{T}}_{\bar{\gamma}_{\bar{\mathcal{T}}}} \leqslant \bar{\mathcal{T}}$. For $\bar{\gamma}_{\bar{\mathcal{T}}_{\bar{\gamma}}} = \bar{\gamma}$, obviously, $\bar{\gamma}_{\bar{\mathcal{T}}_{\bar{\gamma}}} \geqslant \bar{\gamma}$. Conversely, $\mathcal{F} \in (\gamma_{\bar{\mathcal{T}}_{\bar{\gamma}}})_\lambda \Longrightarrow \mathcal{F} \times \mathcal{F} \in (\mathcal{T}_{\bar{\gamma}})_\lambda \Longrightarrow \exists \mathcal{H} \in \gamma_\lambda : \mathcal{F} \times \mathcal{F} \geqslant \mathcal{H} \times \mathcal{H} \Longrightarrow \forall A \in L^X, \mathcal{F}(A) = \mathcal{F} \times \mathcal{F}(A \times 1_X) \geqslant \mathcal{H} \times \mathcal{H}(A \times 1_X) = \mathcal{H}(A) \Longrightarrow \mathcal{F} \geqslant \mathcal{H} \Longrightarrow \mathcal{F} \in \gamma_\lambda$. Hence $\bar{\gamma}_{\bar{\mathcal{T}}_{\bar{\gamma}}} \leqslant \bar{\gamma}$. This implies $\delta \circ \theta \leqslant id_{\mathbf{S}(L,M)-\mathbf{SUConvTr}}$ and $\theta \circ \delta = id_{\mathbf{S}(L,M)-\mathbf{FilTr}}$ respectively. □

Corollary 7.6. $\mathbf{S}(L, M)$-**FilTr** *can be embedded in* $\mathbf{S}(L, M)$-**SUConvTr** *as a bicoreflective subcategory.*

8 Conclusions

We defined the notion of stratified (L, M)-semiuniform convergence tower spaces. The resulting category is a strong topological universe, hence, it is a suitable framework for studying probabilistic semiuniform convergence spaces and lattice-valued semiuniform convergence spaces. Moreover, the relations between stratified (L, M)-semiuniform convergence tower spaces and stratified (L, M)-filter tower spaces [28] are studied. It is shown that $\mathbf{S}(L, M)$-**FilTr** can be embedded in $\mathbf{S}(L, M)$-**SUConvTr** as a bicoreflective subcategory.

Recently, Jäger [14, 15, 16] extended the notion of a stratified L-filter and introduced the notion of a s-stratified LM-filter. This motivates us to extend our notion in this paper to the s-stratified LM-filter case in our future work.

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