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Relationships between completeness of fuzzy quasi-uniform spaces

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Abstract

In this paper, we give a kind of Cauchy 1-completeness in probabilistic quasi-uniform spaces by using 1-filters. Utilizing the relationships among probabilistic quasi-uniformities, classical quasi-uniformities and Hutton $[0,1]$-quasi-uniformities, we show the relationships between their completeness. In fuzzy quasi-metric spaces, we establish the relationships between the completeness of induced probabilistic quasi-uniform spaces and both completeness of induced classical quasi-uniform spaces and induced Hutton $[0,1]$-quasi-uniform spaces.

Keywords: probabilistic quasi-uniformity; classical quasi-uniformity; Hutton $[0,1]$-quasi-uniformity; 1-filter; Cauchy 1-completeness.

1 Introduction

The theory of uniform structures is an important area of analysis and topology because it provides an appropriate context to link metrics with general topological structures. Quasi-uniformity is a uniformity structure which does not satisfy the symmetric condition. With the development of fuzzy topology, many mathematical structures have been generalized to the fuzzy case, such as fuzzy convergence structures [19, 20] and fuzzy convex structures [21, 22, 26, 27]. For uniformities, many researchers put forward various lattice-valued (quasi-)uniformities and obtain a series of interesting results: see e.g. Höhle's probabilistic (quasi-)uniformity [14], Lowen's (quasi-)uniformity [17], Hutton's L-(quasi-)uniformity [12, 31], Shi's pointwise (quasi-)uniformity [25, 31] and J. Gutiérrez García's L-uniformity [4]. In [4], J. Gutiérrez García studied the relationships between the different notions of fuzzy (quasi-)uniformities. It is worth mentioning that Zhang [32] studied a comparison of various types of uniformities in fuzzy topology and then analyzed the relationships between several notions of lattice-valued (quasi-)uniformities in [33].

The completeness discussed by means of filters theory is an important content in uniform spaces. Lowen in [17, 18] studied the completeness of fuzzy uniform spaces based on prefilters. Höhle studied $\top$-completeness of probabilistic uniform spaces based on $\top$-filters in [15]. J. Gutiérrez García and M. A. De Prada Vicente in [5] studied the completeness of Hutton $[0,1]$-quasi-uniform spaces based on tight and stratified L-filters. In this paper, with the help of the idea of J. L. Sieber and W. J. Pervin [23], we propose a kind of completeness of probabilistic quasi-uniform spaces, which is called Cauchy 1-completeness based on 1-filters in the unit interval $[0,1]$. Inspired by the relationships between various types of lattice-valued (quasi-)uniformities, we want to discuss relationships between completeness of probabilistic quasi-uniformities and both completeness of classical quasi-uniformities and Hutton $[0,1]$-quasi-uniformities. Fuzzy (quasi-)metric spaces draw much attention in fuzzy mathematics. The usual concept of fuzzy (quasi-)metric spaces can date back to George and Veeramani [7, 8], which slightly modified the definition given by Kramosil and Michalek [16] who adapted the concept of probabilistic metrics to the fuzzy setting. Furthermore, the completeness of fuzzy (quasi-)metric spaces also has studied in [9, 10, 11]. What's more, many authors associated to each fuzzy (quasi-)metric space a lattice-valued (quasi-)uniform space (such as [28, 29]). The paper is organized as follows. In section 2 we provide lattice theoretical environment and some concepts of lattice-valued quasi-uniformities used in this paper. Furthermore, we give a kind of Cauchy 1-completeness in probabilistic quasi-uniform spaces by using 1-filters. Section 3 and Section 4 are devoted to study the relationships between completeness of classical quasi-uniform spaces and

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probabilistic quasi-uniform spaces, and the relationships between completeness of probabilistic quasi-uniform spaces and Hutton $[0,1]$-quasi-uniform spaces. Finally, in section 5, in the framework of fuzzy quasi-metric spaces, we establish the relationships between completeness of induced classical quasi-uniform spaces and induced probabilistic quasi-uniform spaces and the relationships between the completeness of induced probabilistic quasi-uniform spaces and induced Hutton $[0,1]$-quasi-uniform spaces.

2 Preliminaries

In this paper, we use the unit interval $I=[0,1]$ as the true table although most of the results are also valid in complete residuated lattices.

2.1 Lattice theoretical preliminaries

Definition 2.1 ([24]). *A binary operation $*: I\times I\to I$ is called a (left-)continuous t-norm if it satisfies the following conditions:*

(1) $*$ is associative and commutative;

(2) 1 is the unit, i.e., $a*1=a$ for all $a\in I$;

(3) $a*b\leqslant c*d$ whenever $a\leqslant c$ and $b\leqslant d$;

(4) $*$ is (left-)continuous.

In this paper, we always assume $*$ is a left-continuous t-norm on I. We say that the left-continuous t-norm $*$ does not have nontrivial zero divisors, if $\alpha*\beta\neq 0$ whenever $\alpha,\beta\neq 0$.

If $*$ is a left-continuous t-norm, since the map $\alpha*(-): I\to I$ preserves arbitrary joins for each $\alpha\in I$, it has a right adjoint $\alpha\xrightarrow{*}(-): I\to I$ determined by the adjoint property $\alpha*\beta\leqslant\gamma\Leftrightarrow\beta\leqslant\alpha\xrightarrow{*}\gamma,\quad\alpha,\beta,\gamma\in I$. Hence the implication $\to$ is the binary operation on I given by $\alpha\xrightarrow{*}\gamma=\bigvee\{\beta\in I\mid\alpha*\beta\leqslant\gamma\},\quad\alpha,\gamma\in I$.

Three distinguished examples of left-continuous t-norm are $\wedge$, $\cdot$ and $*_{Ł}$ (the Łukasiewicz t-norm) which are given as

$$\alpha\wedge\beta=min\{\alpha,\beta\},\quad\alpha\cdot\beta=\alpha\beta\quad\text{and}\quad\alpha*_{Ł}\beta=max\{\alpha+\beta-1,0\};$$

$$\alpha\xrightarrow{\wedge}\beta=\begin{cases}1 & \text{if } \alpha\leqslant\beta,\\ \beta & \text{if } \beta<\alpha,\end{cases}\alpha\xrightarrow{\cdot}\beta=\begin{cases}1 & \text{if } \alpha\leqslant\beta\\ \dfrac{\beta}{\alpha} & \text{if } \beta<\alpha\end{cases}\text{ and }\alpha\xrightarrow{Ł}\beta=min\{1-\alpha+\beta,1\},$$

for all $\alpha,\beta\in I$. It is well-known and easy that $*\leqslant\wedge$ for each left-continuous t-norm $*$.

For a set X, a binary map $\mathcal{S}_X(-,-): I^X\times I^X\to I$ is defined by $\mathcal{S}_X(A,B)=\bigwedge\limits_{x\in X}\big(A(x)\to B(x)\big)$ for each $A,B\in I^X$, where $\mathcal{S}_X(A,B)$ can be interpreted as the degree to which A is a subset of B. It is called *fuzzy inclusion order* or *subsethood degree* of I-subsets. $\mathcal{S}_X(-,-)$ is also denoted as $\mathcal{S}(-,-)$.

2.2 Relationships among classical filters, 1-filters and I-filters

Below we collect some definitions and results regarding classical filters, 1-filters and I-filters for the unit interval I, that will be needed later on.

Definition 2.2 ([15]). *A nonempty subset $\mathbb{F}$ of I^X is called a 1-filter on X provided it satisfies the following properties:*

(IF1) If $A\in I^X$ such that $\bigvee\limits_{B\in\mathbb{F}}S(B,A)=1$, then $A\in\mathbb{F}$;

(IF2) $A_1\wedge A_2\in\mathbb{F}$ for all $A_1,A_2\in\mathbb{F}$;

(IF3) $\bigvee\limits_{x\in X}A(x)=1$ for all $A\in\mathbb{F}$.

The set of all 1-filters on X is denoted by $\mathbb{F}_I(X)$. For a 1-filter on X, the axiom $(IF1)$ means $B\in\mathbb{F}$ whenever $A\in\mathbb{F}$ with $A\leqslant B$. For each $x\in X$, let $[x]\subseteq I^X$ be $[x]=\{A\in I^X\mid A(x)=1\}$. Then $[x]$ is a 1-filter and $[x]$ is usually called *the point 1-filter of x*.

Definition 2.3 ([15]). *A nonempty subset $\mathbb{B} \subseteq I^X$ is called a 1-filter base on the set X if it satisfies the following conditions:*

(B1) $\bigvee_{B\in\mathbb{B}} S(B, C\wedge D) = 1$ *for all* $C, D \in \mathbb{B}$;

(B2) $\bigvee_{x\in X} C(x) = 1$ *for all* $C \in \mathbb{B}$.

Definition 2.4 ([13]). *Let X be a set. A map $\upsilon : I^X \to I$ is called an I-filter on X if it satisfies the following properties:*

(F0) $\upsilon(1_X) = 1$;

(F1) If $f_1,\ f_2 \in I^X$, $f_1 \leqslant f_2$ *then* $\upsilon(f_1) \leqslant \upsilon(f_2)$;

(F2) $\upsilon(f_1)\wedge\upsilon(f_2) \leqslant \upsilon(f_1\wedge f_2)$ *for each* $f_1,\ f_2 \in I^X$;

(F3) $\upsilon(1_\emptyset) = 0$.

An I-filter υ is said to be stratified if it satisfies the following additional axiom:

(F4) $\alpha * \upsilon(f) \leqslant \upsilon(\alpha * f)$ *for all* $\alpha \in I$ *and* $f \in I^X$.

An I-filter υ is said to be tight if it satisfies the following important axiom:

(F5) $\upsilon(\alpha * 1_X) = \alpha$ *for all* $\alpha \in I$.

From [30], we have the following results between classical filters and 1-filters.

Lemma 2.5. *Let $\mathcal{F}_I(X)$ be the set of all classical filter on X and $\mathbb{F}_I(X)$ be the set of all 1-filter on X. The following statements hold:*

(1)The order-preserving mapping $\omega : \mathcal{F}_I(X) \to \mathbb{F}_I(X)$ is given by $\omega(\mathcal{F}) = \{A \in I^X \mid \bigvee_{F\in\mathcal{F}}\bigwedge_{x\in F} A(x) = 1\}$, for each $\mathcal{F} \in \mathcal{F}_I(X)$. Then $\omega(\mathcal{F})$ is a 1-filter.

(2)The order-preserving mappings $[\ ]$, $\iota : \mathbb{F}_I(X) \to \mathcal{F}_I(X)$ are respectively given by $[\mathbb{F}] = \{u \in 2^X \mid \chi_u \in \mathbb{F}\}$ and $\iota(\mathbb{F}) = \{\sigma_r(A) \mid A \in \mathbb{F}, r \in [0,1)\}$, for each $\mathbb{F} \in \mathbb{F}_I(X)$, where $\sigma_r(A) = \{x \in X \mid A(x) > r\}$. Then $[\mathbb{F}]$ and $\iota(\mathbb{F})$ are classical filters.

Lemma 2.6. *Let $\mathcal{F}$ be a classical filter on X and $\mathbb{F}$ be a 1-filter. Then*

(1) $\iota(\omega(\mathcal{F})) = \mathcal{F}$;

(2) $\omega(\iota(\mathbb{F})) \supseteq \mathbb{F}$;

(3) $[\omega(\mathcal{F})] = \mathcal{F}$;

(4) $\omega([\mathbb{F}]) \subseteq \mathbb{F}$.

The lemma state that these adjoint connections hold, i.e., $\iota \dashv \omega \dashv [\ \ ]$.

Definition 2.7. *$\mathbb{F}$ is a 1-filter. $\mathbb{F}$ is called an induced 1-filter if there exists a classical filter $\mathcal{F}$ such that $\mathbb{F} = \omega(\mathcal{F})$.*

It is easy to check the following results.

Lemma 2.8. *$\mathbb{F}$ is an induced 1-filter if and only if $\iota(\mathbb{F}) = [\mathbb{F}]$.*

Furthermore, $\omega(\iota(\mathbb{F})) = \mathbb{F}$.
Next, we will mention the relationship between 1-filters and stratified and tight I-filters.

Lemma 2.9 ([2]). *Every 1-filter $\mathbb{F}$ on X induces a stratified and tight I-filter $\upsilon_\mathbb{F}$ by $\upsilon_\mathbb{F}(f) = \bigvee\{\alpha \in I \mid \alpha \to f \in \mathbb{F}\}$, for all $f \in I^X$.*

Lemma 2.10 ([2]). *Every stratified and tight I-filter υ on X determines a 1-filter $\mathbb{F}_\upsilon$ by $\mathbb{F}_\upsilon = \{f \in I^X \mid \upsilon(f) = 1\}$.*

J. Gutiérrez García established a relationship between 1-filters and I-filters by using some properties of the characteristic value in [3].

2.3 $(X,\mathbb{U})$, $(X,\mathcal{U})$ and $(X,\mathfrak{U})$

In this part, we now describe briefly the definition and some properties about classical quasi-uniformity $\mathbb{U}$, probabilistic quasi-uniformity $\mathcal{U}$ and Hutton $[0,1]$-quasi-uniformity $\mathfrak{U}$.

Definition 2.11 ([4, 15])**.** *A nonempty subset $\mathcal{U} \subseteq I^{X\times X}$ is called a probabilistic quasi-uniformity on X, if it is a 1-filter and still satisfies the following conditions:*

(IU0) $U \in \mathcal{U}$ implies $U(x,x) = 1$ for all $x \in X$,

(IUC) $U \in \mathcal{U}$ implies that there exist $V \in \mathcal{U}$ such that $V \circ V \leqslant U$.

For a probabilistic quasi-uniform space $(X,\mathcal{U})$, $\mathbb{N}_x^{\mathcal{U}}$ is the 1-filter generated by the set $N_x^{\mathcal{U}} = \{U(-,x) \mid U \in \mathcal{U}\}$. Therefore, $\tau_{\mathcal{U}} = \{A \in I^X \mid \forall x \in X,\ A(x) \leqslant \bigvee_{U\in\mathcal{U}} S(U(-,x),A)\}$ is generated I-topology by $\mathbb{N}_x^{\mathcal{U}}$. Next, we will introduce the definition of Hutton $[0,1]$-quasi uniformities from J. Gutiérrez García [5]. Let X be a set and $(I,\leqslant)$ be a complete lattice. By $\mathcal{H}_I(X)$ we denote the collection of all maps $\phi : I^X \to I^X$ satisfying the following conditions:

(1) $\phi(a) \geqslant a$ for each $a \in I^X$ (Enlarging),

(2) $\phi(\bigvee_{j\in J} a_j) = \bigvee_{j\in J} \phi(a_j)$ for each $\{a_j\}_{j\in J} \subset I^X$ (Join-preserving) and $\phi(1_\emptyset) = 1_\emptyset$.

Remark 2.12. *As pointed out in* [5]*, each arbitrary join-preserving element $\phi \in (I^X)^{I^X}$ is completely determined by the collection of I-set $\{\phi(\alpha * 1_{\{x\}}) \mid \alpha \in (0,1],\ x \in X\}$.*

Definition 2.13 ([5])**.** *Let X be a set and $([0,1],\leqslant)$ be a complete lattice. A Hutton $[0,1]$-quasi-uniformity on X is a nonempty subset $\mathfrak{U}$ of $\mathcal{H}_I(X)$ such that*

(HU1) if $\phi_1 \in \mathfrak{U},\ \phi_1 \leqslant \phi_2$ and $\phi_2 \in \mathcal{H}_I(X)$ then $\phi_2 \in \mathfrak{U}$,

(HU2) if $\phi_1,\ \phi_2 \in \mathfrak{U}$, there exist $\phi \in \mathfrak{U}$ such that $\phi \leqslant \phi_1$ and $\phi \leqslant \phi_2$,

(HU3) if $\phi \in \mathfrak{U}$, there exist $\psi \in \mathfrak{U}$ such that $\psi \circ \psi \leqslant \phi$ (where $\circ$ denotes the usual composition of functions).

The pair $(X,\mathfrak{U})$ is called a Hutton $[0,1]$-quasi-uniform space such that X is a set and $\mathfrak{U}$ is a Hutton $[0,1]$-quasi-uniformity on X. A nonempty subset $\mathfrak{B}$ of $\mathfrak{U}$, is a base for $\mathfrak{U}$ if for each $\phi \in \mathfrak{U}$, there exists $\varphi \in \mathfrak{B}$ such that $\varphi \leqslant \phi$.

A Hutton $[0,1]$-quasi-uniformity $\mathfrak{U}$ is said to be stratified if it has a base $\mathfrak{B}$ which satisfies:

(HU4) if $\forall\varphi \in \mathfrak{B},\ \forall\alpha \in I,\ \forall x \in X,\ \alpha * \varphi(1_{\{x\}}) \leqslant \varphi(\alpha * 1_{\{x\}})$.

2.4 Cauchy 1-completeness of probabilistic quasi-uniformity

In this part, we use the idea of Sieber and Pervin in [23] to give the Cauchy 1-completeness of probabilistic quasi-uniform spaces by 1-filters used in this paper.

Definition 2.14. *Let $(X,\mathcal{U})$ be a probabilistic quasi-uniform space. A 1-filter $\mathbb{F}$ is called a Cauchy 1-filter if and only if for every $U \in \mathcal{U}$ there exists a point $x \in X$ such that $U(-,x) \in \mathbb{F}$.*

If $\mathbb{F}$ is an induced 1-filter, then $\mathbb{F}$ is called an *induced Cauchy* 1*-filter* if and only if for every $U \in \mathcal{U}$ there exists a point $x \in X$ such that $U(-,x) \in \mathbb{F}$. It is easy to know that $\mathbb{N}_x^{\mathcal{U}}$ is a Cauchy 1-filter and we call it *the neighborhood* 1*-filter of* x. If $\mathbb{N}_x^{\mathcal{U}} \subseteq \mathbb{F}$, we call $\mathbb{F}$ converges to x.

Definition 2.15. *A probabilistic quasi-uniform space $(X,\mathcal{U})$ will be said to be Cauchy 1-complete if and only if each Cauchy 1-filter converges.*

Furthermore, a probabilistic quasi-uniform space $(X,\mathcal{U})$ is said to be *induced Cauchy* 1*-complete* if each induced Cauchy 1-filter converges. Now we recall the definition about the completeness of classical quasi-uniform space $(X,\mathbb{U})$ in [23].

Definition 2.16 ([23])**.** *A filter $\mathcal{F}$ in a quasi-uniform space $(X,\mathbb{U})$ will be called a Cauchy filter if and only if for every $u \in \mathbb{U}$ there exists a point $z \in X$ such that $u[z] \in \mathcal{F}$, where $u[z] = \{y \in X \mid (y,z) \in u\}$.*

From general topology, we obtain that for a quasi-uniform space $(X,\mathbb{U})$, $\mathbf{N}_z^{\mathbb{U}}$, the neighborhood of z, is generated by the set $N_z^{\mathbb{U}} = \{u[z] \mid u \in \mathbb{U}\}$. If $\mathbf{N}_z^{\mathbb{U}} \subseteq \mathcal{F}$, we call $\mathcal{F}$ converges to z.

Definition 2.17 ([23])**.** *A quasi-uniform space $(X,\mathbb{U})$ will be said to be complete if and only if every Cauchy filter converges.*

3 Relationship between the completeness of $(X, \mathbb{U})$ and $(X, \mathcal{U})$

In this section, we will discuss the relationships between the completeness of classical quasi-uniform spaces and Cauchy 1-completeness of probabilistic quasi-uniform spaces. Next, we firstly introduce two functors Φ, Ψ about classical quasi-uniformities and probabilistic quasi-uniformities.

Proposition 3.1 ([6, 33])**.** *Let $(X, \mathbb{U})$ be a classical quasi-uniform space and $(X, \mathcal{U})$ be a probabilistic quasi-uniform space. Let $\Phi(\mathbb{U})$ be the probabilistic quasi-uniformity generated by $\{1_u \mid u \in \mathbb{U}\}$. Let $\Psi(\mathcal{U})$ be the classical quasi-uniformity generated by $\{\Psi(U) \mid U \in \mathcal{U}\}$, where $\Psi(U) = \{(x, y) \in X \times X \mid U(x, y) = 1\}$. Then:*

(1) $\Psi(\Phi(\mathbb{U})) = \mathbb{U}$*;*

(2) $\Phi(\Psi(\mathcal{U})) \subseteq \mathcal{U}$*;*

(3) Ψ *is a right adjoint of* Φ*.*

Lemma 3.2. *Let $(X, \mathbb{U})$ be a classical quasi-uniform space, $\mathcal{F}$ be a classical filter on X and $x_0 \in X$. Then:*

(1) $\mathcal{F}$ is a Cauchy filter in $(X, \mathbb{U})$ if and only if $\omega(\mathcal{F})$ is a Cauchy 1-filter in $(X, \Phi(\mathbb{U}))$.

(2) $\mathcal{F}$ converges to x_0 in $(X, \mathbb{U})$ if and only if $\omega(\mathcal{F})$ converges to x_0 in $(X, \Phi(\mathbb{U}))$.

Proof. (1) Necessity: Let $u \in \mathbb{U}$ and $1_u \in \Phi(\mathbb{U})$. Since $\mathcal{F}$ is a Cauchy filter in $(X, \mathbb{U})$, there exists $x \in X$ such that $u[x] \in \mathcal{F}$. Hence, $\bigvee_{F \in \mathcal{F}} \bigwedge_{y \in F} 1_u(y, x) \geqslant \bigwedge_{y \in u[x]} 1_u(y, x) = 1$. Therefore, $1_u(-, x) \in \omega(\mathcal{F})$.

Sufficiency: Let $u \in \mathbb{U}$. Then $1_u \in \Phi(\mathbb{U})$. Since $\omega(\mathcal{F})$ is a Cauchy 1-filter in $(X, \Phi(\mathbb{U}))$, there exists $x \in X$ such that $1_u(-, x) \in \omega(\mathcal{F})$. Then there is $\bigvee_{F \in \mathcal{F}} \bigwedge_{y \in F} 1_u(y, x) = 1$. So we can find $F_r \in \mathcal{F}$ satisfying $\bigwedge_{y \in F_r} 1_u(y, x) > r$ for all $r \in (0, 1)$. When $y \in F_r$, we have $(y, x) \in u$, namely, $y \in u[x]$. Hence, $F_r \subseteq u[x]$. Therefore, $u[x] \in \mathcal{F}$.

(2) Necessity: Let $u \in \mathbb{U}$, $1_u \in \Phi(\mathbb{U})$ and $1_u(-, x_0) \in \mathbb{N}_{x_0}^{\Phi(\mathbb{U})}$. Since $\mathcal{F}$ converges to x_0 in $(X, \mathbb{U})$, we have $\mathbf{N}_{x_0}^{\mathbb{U}} \subseteq \mathcal{F}$. So, for $u[x_0] \in \mathbf{N}_{x_0}^{\mathbb{U}}$, there is $u[x_0] \in \mathcal{F}$. Hence, $\bigvee_{F \in \mathcal{F}} \bigwedge_{y \in F} 1_u(y, x_0) \geqslant \bigwedge_{y \in u[x_0]} 1_u(y, x_0) = 1$. Therefore, $1_u(-, x_0) \in \omega(\mathcal{F})$.

Sufficiency: Let $u \in \mathbb{U}$. Then $1_u \in \Phi(\mathbb{U})$ and $u[x_0] \in \mathbf{N}_{x_0}^{\mathbb{U}}$. Since $\omega(\mathcal{F})$ converges to x_0 in $(X, \Phi(\mathbb{U}))$, we have $\mathbb{N}_{x_0}^{\Phi(\mathbb{U})} \subseteq \omega(\mathcal{F})$. Furthermore, when $1_u(-, x_0) \in \mathbb{N}_{x_0}^{\Phi(\mathbb{U})}$, we have $1_u(-, x_0) \in \omega(\mathcal{F})$. Then there is $\bigvee_{F \in \mathcal{F}} \bigwedge_{y \in F} 1_u(y, x_0) = 1$. So we can find $F_r \in \mathcal{F}$ satisfying $\bigwedge_{y \in F_r} 1_u(y, x_0) > r$ for all $r \in (0, 1)$. When $y \in F_r$, we have $(y, x_0) \in u$. Then there is $y \in u[x_0]$. Hence, $F_r \subseteq u[x_0]$. Therefore, $u[x_0] \in \mathcal{F}$. □

Lemma 3.3. *Let $(X, \mathcal{U})$ be a probabilistic quasi-uniform space, $\mathbb{F}$ be an induced 1-filter and $x_0 \in X$. Then:*

(1) $\mathbb{F}$ is a Cauchy 1-filter in $(X, \mathcal{U})$ if and only if $\iota(\mathbb{F})$ is a Cauchy filter in $(X, \Psi(\mathcal{U}))$;

(2) $\mathbb{F}$ converges to x_0 in $(X, \mathcal{U})$ if and only if $\iota(\mathbb{F})$ converges to x_0 in $(X, \Psi(\mathcal{U}))$.

Proof. (1) Sufficiency: Let $U \in \mathcal{U}$. Then $\Psi(U) \in \Psi(\mathcal{U})$. Since $\iota(\mathbb{F})$ is a Cauchy filter in $(X, \Psi(\mathcal{U}))$, there exists $x \in X$ such that $\Psi(U)[x] \in \iota(\mathbb{F}) = [\mathbb{F}]$. Let me denote $\Psi(U)[x] = A$. Furthermore, we have $\chi_A \in \mathbb{F}$. Hence,

$$\bigvee_{B \in \mathbb{F}} S(B, U(-, x)) \geqslant \bigwedge_{y \in A} (\chi_A(y) \to U(y, x)) = \bigwedge_{y \in A} U(y, x) = 1.$$

Therefore, $U(-, x) \in \mathbb{F}$.

Necessity: Let $u \in \Psi(\mathcal{U})$. Then there exists $U \in \mathcal{U}$ such that $\Psi(U) \subseteq u$. Since $\mathbb{F}$ is a Cauchy 1-filter in $(X, \mathcal{U})$, there is some $x \in X$ satisfying $U(-, x) \in \mathbb{F}$. On account of $\omega(\iota(\mathbb{F})) = \mathbb{F}$, it follows that $U(-, x) \in \omega(\iota(\mathbb{F}))$. Then there is $\bigvee_{G \in \iota(\mathbb{F})} \bigwedge_{y \in G} U(y, x) = 1$. So we can find $G_r \in \iota(\mathbb{F})$ such that $\bigwedge_{y \in G_r} U(y, x) > r$ for any $r \in [0, 1)$. When $y \in G_r$, we have $U(y, x) > r$ for any $r \in [0, 1)$. Hence, $y \in \Psi(U)[x]$, where $\Psi(U)[x] = \{y \in X \mid U(y, x) = 1\}$. Furthermore, there is $G_r \subseteq \Psi(U)[x] \subseteq u[x]$. Therefore, $u[x] \in \iota(\mathbb{F})$.

(2) Sufficiency: Let $U \in \mathcal{U}$, $\Psi(U) \in \Psi(\mathcal{U})$ and $U(-, x_0) \in \mathbb{N}_{x_0}^{\mathcal{U}}$. Since $\iota(\mathbb{F})$ converges to x_0 in $(X, \Psi(\mathcal{U}))$, we have $\mathbf{N}_{x_0}^{\Psi(\mathcal{U})} \subseteq \iota(\mathbb{F})$. Let me denote $\Psi(U)[x_0] = A$. For $A \in \mathbf{N}_{x_0}^{\Psi(\mathcal{U})}$, there is $A \in \iota(\mathbb{F})$. Furthermore, we obtain $\chi_A \in \mathbb{F}$. Hence,

$$\bigvee_{B \in \mathbb{F}} S(B, U(-, x_0)) \geqslant \bigwedge_{y \in A} (\chi_A(y) \to U(y, x_0)) = \bigwedge_{y \in A} U(y, x_0) = 1.$$

Therefore, $U(-,x_0)\in\mathbb{F}$.

Necessity: Let $U\in\mathcal{U}$ and $\Psi(U)[x_0]\in\mathbf{N}_{x_0}^{\Psi(\mathcal{U})}$. Since $\mathbb{F}$ converges to x_0 in $(X,\mathcal{U})$, we have $\mathbb{N}_{x_0}^{\mathcal{U}}\subseteq\mathbb{F}$. For $U(-,x_0)\in\mathbb{N}_{x_0}^{\mathcal{U}}$, there is $U(-,x_0)\in\mathbb{F}$. On account of $\omega(\iota(\mathbb{F}))=\mathbb{F}$, it follows that $U(-,x_0)\in\omega(\iota(\mathbb{F}))$. Then there is $\bigvee\limits_{G\in\iota(\mathbb{F})}\bigwedge\limits_{y\in G}U(y,x_0)=1$. So we can find $G_r\in\iota(\mathbb{F})$ such that $\bigwedge\limits_{y\in G_r}U(y,x_0)>r$ for any $r\in[0,1)$. When $y\in G_r$, we have $U(y,x_0)>r$ for any $r\in[0,1)$. Hence, $y\in\Psi(U)[x_0]$, where $\Psi(U)[x_0]=\{y\in X\mid U(y,x_0)=1\}$. Furthermore, there is $G_r\subseteq\Psi(U)[x_0]$. Therefore, $\Psi(U)[x_0]\in\iota(\mathbb{F})$. □

It is easy to check the following results by Lemma 3.2 and Lemma 3.3.

Theorem 3.4. *If $(X,\Phi(\mathbb{U}))$ is Cauchy 1-complete, then $(X,\mathbb{U})$ is complete.*

Corollary 3.5. *(1) $(X,\mathbb{U})$ is complete if and only if $(X,\Phi(\mathbb{U}))$ is induced Cauchy 1-complete;*

(2) if $(X,\Psi(\mathcal{U}))$ is complete, then $(X,\mathcal{U})$ is induced Cauchy 1-complete.

4 Relationship between the completeness of $(X,\mathcal{U})$ and $(X,\mathfrak{U})$

In this section, we will study the completeness of probabilistic quasi-uniformity $\mathcal{U}$ and Hutton $[0,1]$-quasi-uniformity $\mathfrak{U}$ through the following functors Λ and Υ. Gutiérrez García in [4, 6] has studied the functors and discussed the relationship between probabilistic quasi-uniformities and Hutton $[0,1]$-quasi-uniformities by use of them.

Proposition 4.1 ([4, 6])**.** *Let $(X,\mathcal{U})$ be a probabilistic quasi-uniform space and $(X,\mathfrak{U})$ be a Hutton $[0,1]$-quasi-uniform space. Let $\Lambda(\mathcal{U})$ be the Hutton $[0,1]$-quasi-uniformity generated by $\{\Lambda(U)\mid U\in\mathcal{U}\}$, where $[\Lambda(U)](a)(x)=\bigvee\limits_{y\in X}U(x,y)*a(y)$ for each $U\in\mathcal{U}$, $a\in I^X$ and $x\in X$. Let $\Upsilon(\mathfrak{U})$ be the probabilistic quasi-uniformity generated by $\{\Upsilon(\phi)\mid\phi\in\mathfrak{U}\}$, where $[\Upsilon(\phi)](x,y)=\bigwedge\limits_{\alpha\in I}\alpha\to[\phi(\alpha*1_{\{y\}})](x)$ for each $\phi\in\mathfrak{U}$ and $x,y\in X$. Then:*

(1) $\Upsilon(\Lambda(\mathcal{U}))=\mathcal{U}$;

(2) $\Lambda(\Upsilon(\mathfrak{U}))\subseteq\mathfrak{U}$;

(3) Υ is a right adjoint of Λ.

Since 0 is the zero element with respect to $*$, it follows that for each $\alpha\in I$ and $x\in X$, $[\Lambda(U)](\alpha*1_{\{x\}})=U(-,x)*\alpha=[\Lambda(U)](1_{\{x\}})*\alpha$. If Hutton $[0,1]$-quasi-uniformity $\mathfrak{U}$ is stratified, we can observe that $[\Upsilon(\phi)](-,x)=\phi(1_{\{x\}})$ for each $\phi\in\mathfrak{B}$ and $x\in X$, where $\mathfrak{B}$ is a base for $\mathfrak{U}$. First of all, we recall the completeness of Hutton $[0,1]$-quasi-uniform space $(X,\mathfrak{U})$ in [5].

Lemma 4.2 ([5])**.** *Let I-filter υ and Hutton $[0,1]$-quasi-uniformity $\mathfrak{U}$ on X be stratified and $p\in X$. Then:*

(1) υ converges to p if and only if $\forall\varphi\in\mathfrak{B}$, $\upsilon(\varphi(1_{\{p\}}))=1$;

(2) υ is a Cauchy I-filter if and only if $\forall\varphi\in\mathfrak{B},\exists p_\varphi\in X,\upsilon(\varphi(1_{\{p_\varphi\}}))=1$.

Where $\mathfrak{B}$ is a base for $\mathfrak{U}$.

Definition 4.3 ([5])**.** *A Hutton $[0,1]$-quasi-uniform space $(X,\mathfrak{U})$ is said to be complete if any stratified and tight Cauchy I-filter on X is convergent.*

Next, we will discuss the relationship between the completeness of probabilistic quasi-uniformities and Hutton [0,1]-quasi-uniformities.

Proposition 4.4. *Let $(X,\mathcal{U})$ be a probabilistic quasi-uniform space, $(X,\mathfrak{U})$ be a stratified Hutton $[0,1]$-quasi-uniform space, $\mathbb{F}$ be a 1-filter and υ be a stratified I-filter. Then:*

(1) $\mathbb{F}$ is a Cauchy 1-filter in $(X,\mathcal{U})$ if and only if $\upsilon_{\mathbb{F}}$ is a stratified Cauchy I-filter in $(X,\Lambda(\mathcal{U}))$;

(2) υ is a stratified Cauchy I-filter in $(X,\mathfrak{U})$ if and only if $\mathbb{F}_\upsilon$ is a Cauchy 1-filter in $(X,\Upsilon(\mathfrak{U}))$.

Proof. (1) Necessity: Let $U \in \mathcal{U}$ and $\Lambda(U)$ be a base element for $(X, \Lambda(\mathcal{U}))$. Since $\mathbb{F}$ is a Cauchy 1-filter in $(X, \mathcal{U})$, there exists $p_U \in X$ such that $U(-, p_U) \in \mathbb{F}$. Hence,

$$\upsilon_{\mathbb{F}}(\Lambda(U)(1_{\{p_U\}})) = \bigvee\{\alpha \in [0,1] \mid \alpha \to \Lambda(U)(1_{\{p_U\}}) \in \mathbb{F}\} = \bigvee\{\alpha \in [0,1] \mid \alpha \to U(-, p_U) \in \mathbb{F}\} \geqslant \{1 \in [0,1] \mid 1 \to U(-, p_U) \in \mathbb{F}\} = 1.$$

Therefore, $\upsilon_{\mathbb{F}}$ is a Cauchy I-filter on $(X, \Lambda(\mathcal{U}))$.

Sufficiency: Let $U \in \mathcal{U}$. Then $\Lambda(U)$ is a base element for $(X, \Lambda(\mathcal{U}))$. Since $\upsilon_{\mathbb{F}}$ is a stratified Cauchy I-filter in $(X, \Lambda(\mathcal{U}))$, there exists $p_U \in X$ such that $\upsilon_{\mathbb{F}}(\Lambda(U)(1_{\{p_U\}})) = 1$. Specifically,

$$\upsilon_{\mathbb{F}}(\Lambda(U)(1_{\{p_U\}})) = \bigvee\{\alpha \in [0,1] \mid \alpha \to \Lambda(U)(1_{\{p_U\}}) \in \mathbb{F}\} = \bigvee\{\alpha \in [0,1] \mid \alpha \to U(-, p_U) \in \mathbb{F}\} = 1.$$

So for any $\alpha \in [0,1]$, we can find $\beta_\alpha \in [0,1]$ satisfying $\beta_\alpha \to U(-, p_U) \in \mathbb{F}$ such that $\beta_\alpha \geqslant \alpha$. Hence,

$$\bigvee_{B \in \mathbb{F}} S(B, U(-, p_U)) \geqslant \bigvee_{\alpha \in [0,1]} S(\beta_\alpha \to U(-, p_U), U(-, p_U)) = \bigvee_{\alpha \in [0,1]} \bigwedge_{x \in X} ((\beta_\alpha \to U(x, p_U)) \to U(x, p_U)) \geqslant \bigvee_{\alpha \in [0,1]} \beta_\alpha \geqslant \bigvee_{\alpha \in [0,1]} \alpha = 1.$$

Therefore, $U(-, p_U) \in \mathbb{F}$.

(2) Necessity: Let $\phi \in \mathcal{B}$ and $\Upsilon(\phi) \in \Upsilon(\mathfrak{U})$, where $\mathfrak{B}$ is a base for $\mathfrak{U}$. Since υ is a stratified Cauchy I-filter in $(X, \mathfrak{U})$, there exists $p_\phi \in X$ such that $\upsilon(\phi(1_{\{p_\phi\}})) = 1$. An account of $\mathbb{F}_\upsilon = \{f \in I^X \mid \upsilon(f) = 1\}$, it follows that $\phi(1_{\{p_\phi\}}) \in \mathbb{F}_\upsilon$. Hence, $\Upsilon(\phi)(-, p_\phi) = \phi(1_{\{p_\phi\}}) \in \mathbb{F}_\upsilon$.

Sufficiency: Let $\phi \in \mathfrak{B}$, where $\mathfrak{B}$ is a base for $\mathfrak{U}$. Then $\Upsilon(\phi) \in \Upsilon(\mathfrak{U})$. Since $\mathbb{F}_\upsilon$ is a Cauchy 1-filter in $(X, \Upsilon(\mathfrak{U}))$, there exists $p \in X$ such that $\Upsilon(\phi)(-, p) \in \mathbb{F}_\upsilon$. Hence, $1 = \upsilon(\Upsilon(\phi)(-, p)) = \upsilon(\phi(1_{\{p\}}))$. □

Proposition 4.5. *Let $(X, \mathcal{U})$ be a probabilistic quasi-uniform space, $(X, \mathfrak{U})$ be a stratified Hutton $[0,1]$-quasi-uniform space, $\mathbb{F}$ be a 1-filter, υ be a stratified I-filter and $p \in X$. Then:*

(1) $\mathbb{F}$ converges to p in $(X, \mathcal{U})$ if and only if $\upsilon_{\mathbb{F}}$ converges to p in $(X, \Lambda(\mathcal{U}))$;

(2) υ converges to p in $(X, \mathfrak{U})$ if and only if $\mathbb{F}_\upsilon$ converges to p in $(X, \Upsilon(\mathfrak{U}))$.

Proof. (1) Necessity: Let $U \in \mathcal{U}$ and $\Lambda(U)$ be a base element for $(X, \Lambda(\mathcal{U}))$. Since $\mathbb{F}$ converges to p in $(X, \mathcal{U})$, we have $\mathbb{N}_p^{\mathcal{U}} \subseteq \mathbb{F}$. Then for any $U(-, p) \in \mathbb{N}_p^{\mathcal{U}}$, there is $U(-, p) \in \mathbb{F}$. Hence,

$$\upsilon_{\mathbb{F}}(\Lambda(U)(1_{\{p\}})) = \bigvee\{\alpha \in [0,1] \mid \alpha \to \Lambda(U)(1_{\{p\}}) \in \mathbb{F}\} = \bigvee\{\alpha \in [0,1] \mid \alpha \to U(-, p) \in \mathbb{F}\} \geqslant \{1 \in [0,1] \mid 1 \to U(-, p) \in \mathbb{F}\} = 1.$$

Therefore, $\upsilon_{\mathbb{F}}$ converges to p in $(X, \Lambda(\mathcal{U}))$.

Sufficiency: Let $U \in \mathcal{U}$ and $U(-, p) \in \mathbb{N}_p^{\mathcal{U}}$. Since $\upsilon_{\mathbb{F}}$ converges to p in $(X, \Lambda(\mathcal{U}))$, there exist $\Lambda(U)$ being a base element for $\Lambda(\mathcal{U})$ such that $\upsilon_{\mathbb{F}}(\Lambda(U)(1_{\{p\}})) = 1$. Specifically,

$$\upsilon_{\mathbb{F}}(\Lambda(U)(1_{\{p\}})) = \bigvee\{\alpha \in [0,1] \mid \alpha \to \Lambda(U)(1_{\{p\}}) \in \mathbb{F}\} = \bigvee\{\alpha \in [0,1] \mid \alpha \to U(-, p) \in \mathbb{F}\} = 1.$$

So for any $\alpha \in [0,1]$, we can find $\beta_\alpha \in [0,1]$ satisfying $\beta_\alpha \to U(-, p) \in \mathbb{F}$ such that $\beta_\alpha \geqslant \alpha$. Hence,

$$\bigvee_{B \in \mathbb{F}} S(B, U(-, p)) \geqslant \bigvee_{\alpha \in [0,1]} S(\beta_\alpha \to U(-, p), U(-, p)) = \bigvee_{\alpha \in [0,1]} \bigwedge_{x \in X} ((\beta_\alpha \to U(x, p)) \to U(x, p)) \geqslant \bigvee_{\alpha \in [0,1]} \beta_\alpha \geqslant \bigvee_{\alpha \in [0,1]} \alpha = 1.$$

Therefore, $U(-, p) \in \mathbb{F}$.

(2) Necessity: Let $\phi \in \mathcal{B}$ and $\Upsilon(\phi)(-, p) \in \mathbb{N}_p^{\Upsilon(\mathfrak{U})}$, where $\mathcal{B}$ is a base for $\mathfrak{U}$. Since υ converges to p in $(X, \mathfrak{U})$, we have $\upsilon(\phi(1_{\{p\}})) = 1$. Hence, $\upsilon(\Upsilon(\phi)(-, p)) = \upsilon(\phi(1_{\{p\}})) = 1$. Therefore, $\Upsilon(\phi)(-, p) \in \mathbb{F}_\upsilon$.

Sufficiency: Let $\phi \in \mathcal{B}$, where $\mathcal{B}$ is a base for $\mathfrak{U}$. Since $\mathbb{F}_\upsilon$ converges to p in $(X, \Upsilon(\mathfrak{U}))$, we have $\Upsilon(\phi)(-, p) \in \mathbb{N}_p^{\Upsilon(\mathfrak{U})} \subseteq \mathbb{F}_\upsilon$. Furthermore, we have $\upsilon(\Upsilon(\phi)(-, p)) = 1$. Therefore, $\upsilon(\phi(1_{\{p\}})) = \upsilon(\Upsilon(\phi)(-, p)) = 1$. □

Theorem 4.6. *Let $(X, \mathcal{U})$ be a probabilistic quasi-uniform space and $\Lambda(\mathcal{U})$ be Hutton [0,1]-quasi-uniformity induced by the mapping Λ. Then, $(X, \mathcal{U})$ is Cauchy 1-complete if and only if $(X, \Lambda(\mathcal{U}))$ is complete.*

Proof. Sufficiency: Suppose that $(X, \Lambda(\mathcal{U}))$ is complete, then $(X, \mathcal{U})$ is Cauchy 1-complete by Proposition 4.4(1) and Proposition 4.5(1).

Necessity: Let υ be a Cauchy I-filter on $(X, \Lambda(\mathcal{U}))$. Since $\Upsilon(\Lambda(\mathcal{U})) = \mathcal{U}$, $\mathbb{F}_\upsilon$ is a Cauchy filter on $(X, \Upsilon(\Lambda(\mathcal{U})))$ by Proposition 4.4(2). Since $(X, \Upsilon(\Lambda(\mathcal{U})))$ is Cauchy 1-complete, then $\mathbb{F}_\upsilon$ converges to p in $(X, \Upsilon(\Lambda(\mathcal{U})))$. By Proposition 4.5(2), we have $\upsilon_{\mathbb{F}_\upsilon}$ converges to p in $(X, \Lambda(\mathcal{U}))$. On account of $\upsilon_{\mathbb{F}_\upsilon} \leqslant \upsilon$, it follows that υ converges to p in $(X, \Lambda(\mathcal{U}))$. Hence, $(X, \Lambda(\mathcal{U}))$ is complete. □

Theorem 4.7. *Let $(X, \mathfrak{U})$ be a Hutton $[0,1]$-quasi-uniform space and $\Upsilon(\mathfrak{U})$ be probabilistic quasi-uniformity induced by the mapping Υ. Then, if $(X, \Upsilon(\mathfrak{U}))$ is Cauchy 1-complete, then $(X, \mathfrak{U})$ is complete.*

Proof. It is easy to check the result by Proposition 4.4(2) and Proposition 4.5(2). □

5 Completeness in fuzzy quasi-metric spaces

In this section, we will discuss the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced classical quasi-uniform spaces and the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced Hutton $[0,1]$-quasi-uniform space in fuzzy quasi-metric spaces. Let $(X, M, *)$ be a fuzzy quasi-metric space. From [7], we can associate $(X, M, *)$ with a classical quasi-uniformity $\mathbb{U}_M$ induced by the base $\{U_{\varepsilon,t} \mid \varepsilon \in (0,1), t > 0\}$, where $U_{\varepsilon,t} = \{(x,y) \mid M(x,y,t) > 1-\varepsilon\}$. We can also associate $(X, M, *)$ with a probabilistic quasi-uniformity $\mathcal{U}_M$ induced by the base $\{M(-,-,t) \mid t > 0\}$, i.e., $\mathcal{U}_M = \{U \in [0,1]^{X\times X} \mid \bigvee_{t>0} S(M(-,-,t), U) = 1\}$. From [1], We can associate $(X, M, *)$ with a Hutton $[0,1]$-quasi-uniformity $\mathfrak{U}_M$ induced by the base $\{\phi^M_{\varepsilon,t} \mid \varepsilon \in (0,1),\ \ t > 0\}$, where the mapping $\phi^M_{\varepsilon,t} : [0,1]^X \to [0,1]^X$ is defined by $\phi^M_{\varepsilon,t}(\alpha * 1_{\{x\}})(y) = \alpha * ((1-\varepsilon) \to M(x,y,t))$ for each $x, y \in X$ and $\alpha \in [0,1]$.

Lemma 5.1. *Let $(X, M, *)$ be a fuzzy quasi-metric space, $\mathcal{F}$ be a classical filter on X and $\mathbb{F}$ be a 1-filter. Then:*

(1) $\mathcal{F}$ is a Cauchy filter in $(X, \mathbb{U}_M)$ if and only if $\omega(\mathcal{F})$ is a Cauchy 1-filter in $(X, \mathcal{U}_M)$;

(2) If $\mathbb{F}$ is a Cauchy 1-filter in $(X, \mathcal{U}_M)$, then $\iota(\mathbb{F})$ is a Cauchy filter in $(X, \mathbb{U}_M)$.

Lemma 5.2. *Let $(X, M, *)$ be a fuzzy quasi-metric space, $\mathcal{F}$ be a classical filter on X, $\mathbb{F}$ be a 1-filter and $x_0 \in X$. Then:*

(1) $\mathcal{F}$ converges to x_0 in $(X, \mathbb{U}_M)$ if and only if $\omega(\mathcal{F})$ converges to x_0 in $(X, \mathcal{U}_M)$;

(2) If $\mathbb{F}$ converges to x_0 in $(X, \mathcal{U}_M)$, then $\iota(\mathbb{F})$ converges to x_0 in $(X, \mathbb{U}_M)$.

Corollary 5.3. *Let $(X, M, *)$ be a fuzzy quasi-metric space and $x_0 \in X$. If $\mathbb{F}$ is an induced 1-filter, Then:*

(1) $\mathbb{F}$ is a Cauchy 1-filter in $(X, \mathcal{U}_M)$ if and only if $\iota(\mathbb{F})$ is a Cauchy filter in $(X, \mathbb{U}_M)$;

(2) $\mathbb{F}$ converges to x_0 in $(X, \mathcal{U}_M)$ if and only if $\iota(\mathbb{F})$ converges to x_0 in $(X, \mathbb{U}_M)$.

The lemmas above can be similarly proved according to [30]. Through the lemmas above, we obtain the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced classical quasi-uniform spaces in fuzzy quasi-metric spaces.

Theorem 5.4. *If $(X, \mathcal{U}_M)$ is Cauchy 1-complete, then $(X, \mathbb{U}_M)$ is complete.*

Corollary 5.5. *$(X, \mathcal{U}_M)$ is induced Cauchy 1-complete if and only if $(X, \mathbb{U}_M)$ is complete.*

By the following proposition, we have the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced Hutton $[0,1]$-quasi-uniform spaces. According to the idea of Gutiérrez García [1] Corollary 16, we have the following results.

Proposition 5.6. *Let $(X, M, *)$ be a fuzzy quasi-metric space. Then:*

(1) $\mathfrak{U}_M = \Lambda(\mathcal{U}_M)$;

(2) $\mathcal{U}_M = \Upsilon(\mathfrak{U}_M)$.

Proof. (1) We know that the collection $\{\Lambda((1-\varepsilon) \to M(-,-,t)) \mid \varepsilon \in (0,1),\ \ t > 0\}$ is a basis for $\Lambda(\mathcal{U}_M)$. According to the definition of Λ, we have that $[\Lambda((1-\varepsilon) \to M(-,-,t))](1_{\{y\}})(x) = (1-\varepsilon) \to M(x,y,t)$ as desired.

(2) We know that the collection $\{\Upsilon(\phi^M_{\varepsilon,t}) \mid \varepsilon \in (0,1),\ \ t > 0\}$ is a basis for $\Upsilon(\mathfrak{U}_M)$. According to the definition of Υ, we have that $\Upsilon(\phi^M_{\varepsilon,t})(x,y) = \phi^M_{\varepsilon,t}(1_{\{y\}})(x) = (1-\varepsilon) \to M(x,y,t)$,for each $\varepsilon \in (0,1)$ and $t > 0$ as desired. □

Theorem 5.7. *Let $(X, M, *)$ be a fuzzy quasi-metric space, $\mathcal{U}_M$ be an induced probabilistic quasi-uniformity and $\mathfrak{U}_M$ be an induced Hutton [0,1]-quasi-uniformity. Then, $(X, \mathcal{U}_M)$ is Cauchy 1-complete if and only if $(X, \mathfrak{U}_M)$ is complete.*

6 Conclusions

In this paper, we use 1-filters to give a kind of Cauchy 1-completeness of probabilistic quasi-uniform spaces. We have established close relationships between the completeness of probabilistic quasi-uniform spaces and both completeness of classical quasi-uniform spaces and Hutton $[0,1]$-quasi-uniform spaces. In the framework of fuzzy quasi-metric spaces, we establish the relationship between completeness of induced classical quasi-uniform spaces and induced probabilistic quasi-uniform spaces and the relationship between the completeness of induced probabilistic quasi-uniform spaces and induced Hutton $[0,1]$-quasi-uniform spaces. These results illustrate the reasonableness of 1-filters in discussing the problem of completeness in probabilistic quasi-uniform spaces. Consequently, we have reasons to believe 1-filters are a good tool to study probabilistic quasi-uniform convergence spaces. In the future, we will investigate the relationship between the completeness of probabilistic quasi-uniform spaces and probabilistic quasi-uniform convergence spaces.

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