

## MONOIDAL CLOSEDNESS OF $L$ -GENERALIZED CONVERGENCE SPACES

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**ABSTRACT.** In this paper, it is shown that the category of stratified  $L$ -generalized convergence spaces is monoidal closed if the underlying truth-value table  $L$  is a complete residuated lattice. In particular, if the underlying truth-value table  $L$  is a complete Heyting Algebra, the Cartesian closedness of the category is recaptured by our result.

### 1. Introduction

Since classical topology mainly deals with the category **Top** of topological spaces, it is inconvenient with respect to the formation of function spaces because **Top** isn't Cartesian-closed. So, convergence theory was proposed in favour of the existence of function spaces (Cf. [39]). Stratified  $[0, 1]$ -topological spaces (Cf. [16]) had been studied over several decades motivated by Zadeh [43] who introduced the concept of a fuzzy set in order to handle incomplete or inexact information, here the unit interval  $[0, 1]$  is considered as the truth-value table. As had been pointed out in [31], for stratified  $[0, 1]$ -topological spaces, the function space doesn't exist in general. [30, 31] consider fuzzy convergence spaces as a generalization of Choquet convergence spaces [4], and then the resulting category was shown to be Cartesian-closed. As Höhle [15] pointed out, the theory may turned out to be void in case of more general truth-value table, e.g. a complete residuated lattice [3], instead of the unit interval  $[0, 1]$ . In 2001, Jäger [17], by using the notion of  $L$ -filter defined in case  $L$  is a complete Heyting Algebra, developed the theory of stratified  $L$ -generalized convergence spaces and the resulting category  $SL$ -**GCS** of stratified  $L$ -generalized convergence spaces has the desired structural property of Cartesian-closedness and in 2010, with the same truth-value context, Fang [6] defined a subcategory  $SL$ -**OCS** of the  $SL$ -**GCS**, namely the category of stratified  $L$ -ordered convergence spaces, which is also Cartesian-closed. As a continuation of Jäger's works, the theory of these spaces was developed to a significant extent in recent years contributed by Craig and Jäger [5], Boustique and Richardson [2], Fang [7, 8, 9, 10, 11], Flores and others [12], Jäger [18, 20, 21, 22, 23], Jäger and Burton [19], Li [26, 27, 28, 29], Pang [34, 35, 36, 37, 38], Yao [40, 41], D.Orpen and G. Jäger [33] and others.

It is valued to note in case of the underlying lattice  $L$  being a complete Heyting Algebra, we see that Jäger and Fang all got nice conclusion. Regretfully, the idempotency about the meet operation  $\wedge$  of  $L$  is indispensable for Cartesian-closedness

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of the resulting categories. The requirement of idempotency is not very convenient because the semigroup operation in most of truth-value tables (e.g., complete  $MV$ -algebras, complete residuated lattices) is not idempotent.

Besides Cartesian-closedness, monoidal closedness in general is another property of fundamental importance in category theory [24, 25, 32] and monoidal categories are also frequently investigated as mathematical structures both in category theory and theoretical computer science. In addition, Cartesian-closedness is a special case of monoidal closedness. So it is more general to study monoidal closedness. Furthermore, we can also get the expected functional space if the category is monoidal closed. What's more, we remind the reader of the fact that a complete residuated lattice is itself a monoidal closed category. So, it is natural to research monoidal closedness of the category  $SL\text{-GCS}$  of stratified  $L$ -generalized convergence spaces when the underlying truth-value table is complete residuated lattice.

In this paper, let  $L = (L, *)$  be a complete residuated lattice, then we propose the notion of tensor product, denoted by  $\otimes$ , between two stratified  $L$ -generalized convergence spaces. Thus we can conclude that  $(SL\text{-GCS}, \otimes)$  is monoidal category. Further, we concern with closedness of the category  $SL\text{-GCS}$  of stratified  $L$ -generalized convergence spaces. As shall be shown, this category is monoidal closed indeed, and if the tensor operation on the underlying complete residuated lattice  $L$  is idempotent, equivalently  $L$  is a complete Heyting Algebra, the monoidal closedness degenerates into the Cartesian-closedness [17] of the category  $SL\text{-GCS}$  of stratified  $L$ -generalized convergence spaces.

The content is organized as follows. Section 2 recalls some basic definitions and results of complete residuated lattices and stratified  $L$ -filters. Section 3 focuses on the tensor product of stratified  $L$ -generalized convergence spaces, and we conclude that the class of all stratified  $L$ -generalized convergence spaces forms a symmetric monoidal category. The monoidal closedness of the category  $SL\text{-GCS}$  of stratified  $L$ -generalized convergence spaces is presented in Section 4.

## 2. Preliminaries

In this paper, the triple  $(L, \leq, *)$  denoted by  $L$  sometimes, is a complete residuated lattice, which means that  $(L, \leq)$  is complete lattice with the top element  $\top$  and bottom element  $\perp$  ( $\neq \top$ ), and a binary operation  $* : L \times L \rightarrow L$  satisfying the following axioms[3],

$$(CR1) \quad \alpha * \top = \alpha \text{ for all } \alpha \in L;$$

$$(CR2) \quad \alpha * \beta = \beta * \alpha \text{ for all } \alpha, \beta \in L;$$

$$(CR3) \quad \alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma \text{ for all } \alpha, \beta, \gamma \in L;$$

$$(CR4) \quad \alpha * \bigvee_{j \in J} \beta = \bigvee_{j \in J} \alpha * \beta \text{ for } \alpha \in L, \{\beta_j\}_{j \in J} \subseteq L.$$

From (CR4), there exists the binary operation  $\multimap : L \times L \rightarrow L$  corresponding to  $*$ , computed by

$$\alpha \longrightarrow \beta = \bigvee \{ \gamma \in L \mid \alpha * \gamma \leq \beta \} \text{ for all } \alpha, \beta \in L.$$

The binary operation  $\longrightarrow$  is called the implication about  $*$ . Further, for each  $\alpha \in L$ , the monotone mapping  $\alpha * (-) : L \rightarrow L$  is the left adjoint  $\alpha \longrightarrow (-) : L \rightarrow L$  in the sense that

$$\alpha * \beta \leq \gamma \Leftrightarrow \beta \leq \alpha \longrightarrow \gamma \text{ for all } \beta, \gamma \in L.$$

For  $B \subseteq L$ , write  $\bigvee B$  for the least upper bound of  $B$  and  $\bigwedge B$  for the greatest lower bound of  $B$ . In particular,  $\bigvee \emptyset = \perp$  and  $\bigwedge \emptyset = \top$ .

The implication plays a particular role in the sequel. We list some of its properties.

**Lemma 2.1** ([3, 16]). *Suppose that  $(L, \leq, *)$  is a complete residuated lattice and  $\longrightarrow$  is the implication corresponding to  $*$ . Then for all  $\alpha, \beta, \gamma, \delta \in L$ ,  $\{\alpha_j\}_{j \in J} \subseteq L$ , the following results hold:*

- (1)  $\top \longrightarrow \alpha = \alpha$ , and  $0 * \alpha = 0$ ;
- (2)  $\alpha \leq \beta$  if and only if  $\alpha \longrightarrow \beta = \top$ ;
- (3)  $(\alpha \longrightarrow \beta) * (\beta \longrightarrow \gamma) \leq (\alpha \longrightarrow \gamma)$ ;
- (4)  $\alpha \longrightarrow \bigwedge_{j \in J} \alpha_j = \bigwedge_{j \in J} (\alpha \longrightarrow \alpha_j)$ , hence  $(\alpha \longrightarrow \beta) \leq (\alpha \longrightarrow \gamma)$  whenever  $\beta \leq \gamma$ ;
- (5)  $(\bigvee_{j \in J} \alpha_j) \longrightarrow \beta = \bigwedge_{j \in J} (\alpha_j \longrightarrow \beta)$ , hence  $(\alpha \longrightarrow \gamma) \geq (\beta \longrightarrow \gamma)$  whenever  $\alpha \leq \beta$ ;
- (6)  $\alpha \longrightarrow (\beta \longrightarrow \gamma) = \beta \longrightarrow (\alpha \longrightarrow \gamma) = (\alpha * \beta) \longrightarrow \gamma$ ;
- (7)  $\alpha * (\alpha \longrightarrow \beta) \leq \beta$ ;
- (8)  $(\alpha \longrightarrow \beta) * (\gamma \longrightarrow \delta) \leq \alpha * \gamma \longrightarrow \beta * \delta$  and  $(\alpha \wedge \gamma) * (\beta \wedge \delta) \leq (\alpha * \beta) \wedge (\gamma * \delta)$ .

An  $L$ -subset on a set  $X$  is a map from  $X$  to  $L$ , and the family of all  $L$ -subsets on  $X$  will be denoted by  $L^X$ , called the  $L$ -power set of  $X$ . By  $\perp_X$  and  $\top_X$ , we denote the constant  $L$ -subsets on  $X$  taking the value  $\perp$  and  $\top$ , respectively. We don't distinguish an element  $\alpha \in L$  and the constant function  $\alpha : X \rightarrow L$  such that  $\alpha(x) = \alpha$  for all  $x \in X$ . Further, a subset  $A$  of  $X$  is identified with its *characteristic function*  $\chi_A : X \rightarrow \{\perp, \top\}$  defined by  $\chi_A(x) = \top$  if  $x \in A$  and  $= \perp$  otherwise. All algebraic operations on  $L$  can be extended to the  $L$ -power set  $L^X$  pointwise. That is, for all  $a, b \in L^X$ ,  $\alpha \in L$  and  $x \in X$ ,

- (1)  $(a \vee b)(x) = a(x) \vee b(x)$ ,
- (2)  $(a \wedge b)(x) = a(x) \wedge b(x)$ ,
- (3)  $(a * b)(x) = a(x) * b(x)$ ,
- (4)  $(a \longrightarrow b)(x) = a(x) \longrightarrow b(x)$ .

Obviously,  $(L^X, \leq)$  (here,  $a \leq b$  if and only if  $a(x) \leq b(x)$  for all  $x \in X$ ) is still a complete lattice and for a family  $\mathcal{A} \subseteq L^X$  of  $L$ -subsets, the infimum  $\bigwedge \mathcal{A}$  is given by  $(\bigwedge \mathcal{A})(x) = \bigwedge \{ f(x) \in L \mid f \in \mathcal{A} \}$  and the supremum  $\bigvee \mathcal{A}$  by  $(\bigvee \mathcal{A})(x) = \bigvee \{ f(x) \in L \mid f \in \mathcal{A} \}$  for all  $x \in X$ .

For a given set  $X$ , define a binary mapping  $S_X(-, -) : L^X \times L^X \rightarrow L$  by  $S_X(A, B) = \bigwedge_{x \in X} (A(x) \longrightarrow B(x))$  for each pair  $(A, B) \in L^X \times L^X$ , where " $\longrightarrow$ " is the implication corresponding to  $*$ . For all  $A, B \in L^X$ ,  $S_X(A, B)$  can be

interpreted as the degree to which  $A$  is a subset of  $B$ . It is called subsethood degree [14] or fuzzy inclusion order [42] of  $L$ -subsets.

Let  $\varphi : X \rightarrow Y$  be a map. Define  $\varphi^{\rightarrow} : L^X \rightarrow L^Y$  and  $\varphi^{\leftarrow} : L^Y \rightarrow L^X$  by  $\varphi^{\rightarrow}(f)(y) = \bigvee_{\varphi(x)=y} f(x)$  for  $f \in L^X$  and  $y \in Y$ , and  $\varphi^{\leftarrow}(g) = g \circ \varphi$  for  $g \in L^Y$ , respectively.

**Definition 2.2** ([16]). Let  $X$  be a nonempty set. A mapping  $\mathcal{F} : L^X \rightarrow L$  is called a stratified  $L$ -filter on  $X$  if it satisfies the following conditions:

- (F1)  $\mathcal{F}(\top_X) = \top$ ,  $\mathcal{F}(\perp_X) = \perp$ ;
- (F2)  $A \leq B$  means  $\mathcal{F}(A) \leq \mathcal{F}(B)$  for all  $A, B \in L^X$ ;
- (F3)  $\mathcal{F}(A) * \mathcal{F}(B) \leq \mathcal{F}(A * B)$  for all  $A, B \in L^X$ ;
- (Fs)  $\alpha * \mathcal{F}(A) \leq \mathcal{F}(\alpha * A)$  for  $\alpha \in L, A \in L^X$ .

The set of all stratified  $L$ -filters on  $X$  is denoted by  $F_L^s(X)$ .

**Example 2.3.** Let  $X$  be a nonempty set.

- (1) Given a point  $x \in X$ , the mapping  $[x] : L^X \rightarrow L$  in [16], defined by for all  $A \in L^X$ ,  $[x](A) = A(x)$ , is a stratified  $L$ -filter on  $X$ , which is called the point  $L$ -filter of  $x$ .
- (2) The mapping  $\mathcal{F}_0 : L^X \rightarrow L$  in [16], defined by  $\mathcal{F}_0(A) = \bigwedge_{x \in X} A(x)$  for  $A \in L^X$ , is the smallest stratified  $L$ -filter with respect to the pointwise order “ $\leq$ ” on  $F_L^s(X)$ , i.e.,

$$\mathcal{F} \leq \mathcal{G} \Leftrightarrow \forall A \in L^X, \mathcal{F}(A) \leq \mathcal{G}(A).$$

The infimum of a collection of stratified  $L$ -filters  $\{\mathcal{F}_j\}_{j \in J}$  with respect to “ $\leq$ ” always exists and it is determined by  $(\bigwedge_{j \in J} \mathcal{F}_j)(A) = \bigwedge_{j \in J} \mathcal{F}_j(A)$  for each  $A \in L^X$ .

The supremum of two stratified  $L$ -filters does not always exist. For its existence, we introduce the following proposition.

**Proposition 2.4** ([13]). *For two stratified  $L$ -filters  $\mathcal{F}$  and  $\mathcal{G}$  on a set  $X$ , the supremum  $\mathcal{F} \vee \mathcal{G}$  exists if and only if  $\mathcal{F}(A) * \mathcal{G}(B) = \perp$  for all  $A, B \in L^X$  whenever  $A * B = \perp_X$ . In particular, if exists, then the supremum is computed by*

$$(\mathcal{F} \vee \mathcal{G})(C) = \bigvee \{\mathcal{F}(A) * \mathcal{G}(B) \mid A, B \in L^X \text{ with } A * B \leq C\}$$

for all  $C \in L^X$ .

**Remark 2.5** (Images and inverse images of stratified  $L$ -filters,[16]). Let  $\varphi : X \rightarrow Y$  be a mapping,  $\mathcal{F} \in F_L^s(X)$ , and  $\mathcal{G} \in F_L^s(Y)$ .

- (a) The mapping  $\varphi^{\Rightarrow}(\mathcal{F}) : L^Y \rightarrow L$  defined for each  $B \in L^Y$  by  $\varphi^{\Rightarrow}(\mathcal{F})(B) = \mathcal{F}(\varphi^{\leftarrow}(B)) = \mathcal{F}(B \circ \varphi)$ , is a stratified  $L$ -filter, called the image of  $\mathcal{F}$  under  $\varphi$ .
- (b) The mapping  $\varphi^{\Leftarrow}(\mathcal{G}) : L^X \rightarrow L$  defined for each  $A \in L^X$  by  $\varphi^{\Leftarrow}(\mathcal{G})(A) = \bigvee \{\mathcal{G}(B) \mid \varphi^{\leftarrow}(B) \leq A\}$ ,



$$\begin{array}{ccc}
((A \otimes B) \otimes C) \otimes D & \xrightarrow{a_{A \otimes B, C, D}} & (A \otimes B) \otimes (C \otimes D) \\
a_{ABC \otimes 1} \downarrow & & \downarrow a_{A, B, C \otimes D} \\
(A \otimes (B \otimes C)) \otimes D & & \\
a_{A, B \otimes C, D} \downarrow & & \\
A \otimes ((B \otimes C) \otimes D) & \xrightarrow{1 \otimes a_{BCD}} & A \otimes (B \otimes (C \otimes D))
\end{array}$$

FIGURE 1. Diagram of Definition 2.8

**Proposition 2.7.** *Let  $\varphi : X \rightarrow U$  and  $\psi : Y \rightarrow V$  be two mappings, and  $\mathcal{F} \in F_L^s(X)$ ,  $\mathcal{G} \in F_L^s(Y)$ . Then*

$$(\varphi \times \psi)^\Rightarrow(\mathcal{F} \otimes \mathcal{G}) = \varphi^\Rightarrow(\mathcal{F}) \otimes \psi^\Rightarrow(\mathcal{G}).$$

The reader is referred to the classical text [32] for ideas and results in category theory. We only list the definition of monoidal category and an adjunction between categories, and its characteristic theorem as follows.

**Definition 2.8** ([1]). A monoidal category  $\mathbb{C}$  consists in giving:

- (1) a category  $\mathbb{C}$ ;
- (2) a bifunctor  $\otimes : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ , called the tensor product - we write  $A \otimes B$  for the image under  $\otimes$  of the pair  $(A, B)$ ;
- (3) an object  $I \in \mathbb{C}$ , called the unit;
- (4) for every triple  $A, B, C$  of objects, an ‘‘associativity’’ isomorphism

$$a_{ABC} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C);$$

- (5) for every object  $A$ , a ‘‘left unit’’ isomorphism

$$l_A : I \otimes A \rightarrow A;$$

- (6) for every object  $A$ , a ‘‘right unit’’ isomorphism

$$r_A : A \otimes I \rightarrow A.$$

These data must satisfy the following requirements:

- (1) the isomorphisms  $a_{ABC}$  are natural in  $A, B, C$ ;
- (2) the isomorphisms  $l_A$  are natural in  $A$ ;
- (3) the isomorphisms  $r_A$  are natural in  $A$ ;
- (4) Figure 1 is commutative for every quadruple of objects  $A, B, C, D$  (associativity coherence);
- (5) Figure 2 is commutative for every pair  $A, B$  of objects (unit coherence).

**Definition 2.9** ([1]). A monoidal category  $\mathbb{C}$  is symmetric when an isomorphism

$$s_{AB} : A \otimes B \rightarrow B \otimes A$$

is given for every pair of  $A, B$  of objects. These isomorphisms must be such that:

$$\begin{array}{ccc}
 (A \otimes I) \otimes B & \xrightarrow{a_{AIB}} & A \otimes (I \otimes B) \\
 \searrow r_A \otimes 1 & & \downarrow 1 \otimes l_B \\
 & & A \otimes B
 \end{array}$$

FIGURE 2. Diagram of Definition 2.8

$$\begin{array}{ccc}
 (A \otimes B) \otimes C & \xrightarrow{s_{AB} \otimes 1} & (B \otimes A) \otimes C \\
 a_{BAC} \downarrow & & \downarrow a_{BAC} \\
 A \otimes (B \otimes C) & & B \otimes (A \otimes C) \\
 s_{A,B \otimes C} \downarrow & & \downarrow 1 \otimes s_{AC} \\
 (B \otimes C) \otimes A & \xrightarrow{a_{BCA}} & B \otimes (C \otimes A)
 \end{array}$$

FIGURE 3. Diagram of Definition 2.9

$$\begin{array}{ccc}
 A \otimes I & \xrightarrow{s_{AI}} & I \otimes A \\
 \searrow r_A & & \downarrow l_A \\
 & & A
 \end{array}$$

FIGURE 4. Diagram of Definition 2.9

- (1) the morphisms  $s_{AB}$  are natural in  $A, B$ ;
- (2) Figure 3 is commutative for every triple  $A, B, C$  of objects (associativity coherence);
- (3) Figure 4 is commutative for every object  $A$  (unit coherence);
- (4) Figure 5 is commutative (symmetry axiom) for every pair  $A, B$  of objects.

**Definition 2.10.** Let  $\mathbf{F} : \mathbb{C} \rightarrow \mathbb{D}$  and  $\mathbf{G} : \mathbb{D} \rightarrow \mathbb{C}$  be functors. If for each object  $C$  in  $\mathbb{C}$  and object  $D$  in  $\mathbb{D}$ , there exists a bijection

$$\varphi := \varphi_{C,D} : [\mathbf{F}C, D]_{\mathbb{D}} \rightarrow [C, \mathbf{G}D]_{\mathbb{C}}$$

such that  $\varphi$  is natural in  $C$  and  $D$ , then the triple  $\langle \mathbf{F}, \mathbf{G}, \varphi \rangle$  is said to be an adjunction from  $\mathbb{C}$  and  $\mathbb{D}$ . In this case, the functor  $\mathbf{F}$  is said to be a left adjoint for  $\mathbf{G}$  and  $\mathbf{G}$  is called a right adjoint for  $\mathbf{F}$ .

$$\begin{array}{ccc}
A \otimes B & \xrightarrow{s_{AB}} & B \otimes A \\
& \searrow & \downarrow s_{BA} \\
& & A \otimes B
\end{array}$$

FIGURE 5. Diagram of Definition 2.9

**Theorem 2.11.** *Let  $\mathbb{C}$  and  $\mathbb{D}$  be categories,  $\mathbf{F} : \mathbb{C} \rightarrow \mathbb{D}$  and  $\mathbf{G} : \mathbb{D} \rightarrow \mathbb{C}$  be functors.  $\mathbf{F}$  is a left adjoint for  $\mathbf{G}$  if and only if for each object  $D$  in  $\mathbb{D}$ , there exists a morphism  $\varepsilon_D : \mathbf{F} \circ \mathbf{G}D \rightarrow D$  in  $\mathbb{D}$  such that for each object  $A$  in  $\mathbb{C}$  and a morphism  $f : \mathbf{F}A \rightarrow D$  in  $\mathbb{D}$ , there exists a unique morphism  $\bar{f} : A \rightarrow \mathbf{G}D$  in  $\mathbb{C}$  such that  $f = \varepsilon_D \circ \mathbf{F}\bar{f}$ , in the other words, the pair  $(D, \varepsilon_D)$  is a universal morphism from  $\mathbf{F}$  to  $D$ .*

### 3. The tensor product of stratified $L$ -generalized convergences

In this section, we will focus on the monoidal property of  $SL$ -GCS.

**Definition 3.1** ([17]). Let  $X$  be a nonempty set. A mapping  $\lim$  from  $F_L^s(X)$  to  $L^X$  subject to the conditions

- (GL1)  $\lim[x](x) = \top$  for all  $x \in X$ ,
- (GL2)  $\mathcal{F} \leq \mathcal{G}$  means  $\lim \mathcal{F} \leq \lim \mathcal{G}$  for all  $\mathcal{F}, \mathcal{G} \in F_L^s(X)$ ,

is called a stratified  $L$ -generalized convergence on  $X$ , and then the pair  $(X, \lim)$  a stratified  $L$ -generalized convergence space (in [15], those spaces were called stratified  $L$ -fuzzy convergence space). A mapping  $\varphi : (X, \lim^X) \rightarrow (Y, \lim^Y)$  is called continuous if and only if for all  $\mathcal{F} \in F_L^s(X)$ , we have

$$\lim^X \mathcal{F} \leq \varphi^{\leftarrow}(\lim^Y \varphi^{\Rightarrow}(\mathcal{F})), \text{ or equivalently, } \varphi^{\rightarrow}(\lim^X \mathcal{F}) \leq \lim^Y \varphi^{\Rightarrow}(\mathcal{F}).$$

The class of all stratified  $L$ -generalized convergence spaces and continuous mappings forms a category, which is denoted by  $SL$ -GCS.

For a nonempty set  $X$ , Jäger in [2] defined an order  $\leq$  on the fibre w.r.t.  $X$ , i.e.,

$$\{\lim \mid \lim \text{ is a stratified } L\text{-generalized convergence on } X\},$$

$\lim_{lg}(X)$  for short, in the sense that  $\lim_1 \leq \lim_2$  for  $\lim_1, \lim_2 \in \lim_{lg}(X)$  if and only if the identity mapping

$$\text{id}_X : (X, \lim_2) \rightarrow (X, \lim_1)$$

is continuous, that is,  $\lim_2 \mathcal{F}(x) \leq \lim_1 \mathcal{F}(x)$  for all  $\mathcal{F} \in F_L^s(X)$  and  $x \in X$ . We also write  $(X, \lim_1) \leq (X, \lim_2)$  in this case. If  $\lim_1 \leq \lim_2$  for  $\lim_1, \lim_2 \in \lim_{lg}(X)$ , we say  $(X, \lim_2)$  is finer than  $(X, \lim_1)$ , or  $(X, \lim_1)$  is coarser than  $(X, \lim_2)$ , as usual.



**Example 3.2** ([18]). Let  $X$  be a nonempty set.

- (1) If we define  $\lim_{ind} \mathcal{F}(x) = \top$  for all  $\mathcal{F} \in F_L^s(X)$ ,  $x \in X$ , then  $\lim_{ind}$  is the coarsest stratified  $L$ -generalized convergence on  $X$ , called the indiscrete convergence on  $X$ .
- (2) We define the discrete stratified  $L$ -generalized convergence on  $X$  by  $\lim_l = \top$  whenever  $\mathcal{F} = [x]$  for some  $x \in X$ , and  $= \perp$ , otherwise. Then it is the finest stratified  $L$ -generalized convergence on  $X$ .

In particular, if  $X$  is a single point set, say  $X = \{\infty\}$ , then  $[\infty] \leq \mathcal{F}$  holds for any stratified  $L$ -filter  $\mathcal{F}$  on the single set  $\{\infty\}$ . So, as a consequence of (GL1) and (GL2), there exists a unique stratified  $L$ -generalized convergence on  $\{\infty\}$ , which will be denoted by  $\lim_\infty$ , and the corresponding space is denoted by  $\lim_\infty$  also.

Let  $\{\lim_j\}_{j \in J}$  be a nonempty family of stratified  $L$ -generalized convergences on a set  $X$ . It is easy to see that the supremum  $\sup_{j \in J} \lim_j$  in  $(\lim_{lg}(X), \leq)$  is determined by

$$\forall \mathcal{F} \in F_L^s(X), x \in X, \quad (\sup_{j \in J} \lim_j) \mathcal{F}(x) = \bigwedge_{j \in J} \lim_j \mathcal{F}(x).$$

Moreover, by Example 3.2,  $\lim_{ind}$  is the coarsest  $L$ -generalized convergence on  $X$ . Hence, we have

**Theorem 3.3.** *For a nonempty set  $X$ ,  $(\lim_{lg}(X), \leq)$  is a complete lattice.*

The following theorem is shown in [17] by Jäger, which could also confirm Proposition 3.3 is true.

**Theorem 3.4** ([17]).  *$SL\text{-GCS}$  is a topological category over  $\mathbf{SET}$ .*

Now we are in the position to define the tensor of two stratified  $L$ -generalized convergence spaces. At first, we need to explore a technical proposition, the proof of which is omitted.

**Proposition 3.5.** *Let  $(X, \lim^X)$  and  $(Y, \lim^Y)$  be two objects in  $SL\text{-GCS}$ . If we define a mapping from  $F_L^s(X \times Y)$  to  $L^{X \times Y}$ , denoted by  $\lim^X \otimes \lim^Y$ , as follows:*

$$(\lim^X \otimes \lim^Y)(\mathcal{H}) = \lim^X p_X^{\vec{\rightarrow}}(\mathcal{H}) \otimes \lim^Y p_Y^{\vec{\rightarrow}}(\mathcal{H})$$

for  $\mathcal{H} \in F_L^s(X \times Y)$ , here  $p_X : X \times Y \rightarrow X$  and  $p_Y : X \times Y \rightarrow Y$  are the projections. Then  $\lim^X \otimes \lim^Y$  is a stratified  $L$ -generalized convergence on  $X \times Y$ , called  $\otimes$  the tensor operation of stratified  $L$ -generalized convergence spaces in the following.

Following Proposition 3.5, we will call  $(X \times Y, \lim^X \otimes \lim^Y)$  the tensor space of  $(X, \lim^X)$  and  $(Y, \lim^Y)$ , denoted by  $(X, \lim^X) \otimes (Y, \lim^Y)$ . In order to confirm that the tensor operation of stratified  $L$ -generalized convergence spaces has the desired properties, such as associativity, symmetry, and has the unit in particular, we present some of results needed. The propositions listed below follow from the definition of tensor operation  $\otimes$ , directly.

**Proposition 3.6.** *Let  $(X, \lim^X)$  and  $(Y, \lim^Y)$  be two objects in  $SL\text{-GCS}$ . Then  $p_X : (X, \lim^X) \otimes (Y, \lim^Y) \rightarrow (X, \lim^X)$  and  $p_Y : (X, \lim^X) \otimes (Y, \lim^Y) \rightarrow (Y, \lim^Y)$  are continuous.*

Then by using Proposition 3.6, the associativity and symmetry of  $\otimes$  are obtained as described in the following proposition, and their routine proofs are omitted.

**Proposition 3.7.** *Let  $(X, \lim^X)$ ,  $(Y, \lim^Y)$  and  $(Z, \lim^Z)$  be objects in  $SL\text{-GCS}$ . Then the canonical mapping*

$$j_{(XY)Z} : ((X, \lim^X) \otimes (Y, \lim^Y)) \otimes (Z, \lim^Z) \rightarrow (X, \lim^X) \otimes ((Y, \lim^Y) \otimes (Z, \lim^Z)),$$

given by  $j_{(XY)Z}((x, y), z) = (x, (y, z))$  for  $x \in X, y \in Y, z \in Z$ , is an isomorphism in  $SL\text{-GCS}$ . In particular,

$$((X, \lim^X) \otimes (Y, \lim^Y)) \otimes (Z, \lim^Z) \cong (X, \lim^X) \otimes ((Y, \lim^Y) \otimes (Z, \lim^Z)).$$

**Proposition 3.8.** *Let  $(X, \lim^X)$  and  $(Y, \lim^Y)$  be two objects in  $SL\text{-GCS}$ . Then the canonical mapping*

$$S_{XY} : (X, \lim^X) \otimes (Y, \lim^Y) \rightarrow (Y, \lim^Y) \otimes (X, \lim^X),$$

given by  $S_{XY}(x, y) = (y, x)$  for  $x \in X$  and  $y \in Y$ , is an isomorphism in  $SL\text{-GCS}$ . In particular,

$$(X \times Y, \lim^X \otimes \lim^Y) \cong (Y \times X, \lim^Y \otimes \lim^X).$$

The unitary property of  $\otimes$  could be shown as follows.

**Proposition 3.9.** *If  $(X, \lim^X)$  is an object in  $SL\text{-GCS}$ , then  $(X, \lim^X) \otimes \lim_\infty \cong (X, \lim^X)$ .*

*Proof.* Clearly, the projection  $p_X : X \times \{\infty\} \rightarrow X$  is bijection and  $p_X : (X, \lim) \otimes \lim_\infty \rightarrow (X, \lim)$  is continuous. To confirm that  $p_X$  is an isomorphism from  $(X, \lim) \otimes \lim_\infty$  to  $(X, \lim)$ , it suffices to check its inverse mapping  $I_X : (X, \lim) \rightarrow (X, \lim) \otimes \lim_\infty$ , such that  $I_X(x) = (x, \infty)$  is continuous, which could be checked by

$$(\lim \otimes \lim_\infty) I_X^\rightarrow(\mathcal{F})(I_X(x)) = \lim p_X^\rightarrow(I_X^\rightarrow(\mathcal{F}))(x) * \top = \lim \mathcal{F}(x)$$

for  $\mathcal{F} \in F_L^s(X)$  and  $x \in X$ . □

Further, the tensor operation of stratified  $L$ -generalized convergence spaces is a bifunctor. In fact, we have

**Proposition 3.10.** *The tensor operation  $\otimes$  of stratified  $L$ -generalized convergence spaces from the product category  $SL\text{-GCS} \times SL\text{-GCS}$  to  $SL\text{-GCS}$  is a bifunctor.*

*Proof.* It suffices to show the product map

$$f \times g : X \times U \rightarrow Y \times V,$$

defined by  $f \times g(x, u) = (f(x), g(u))$  for all  $(x, u) \in X \times U$ , is continuous from  $(X \times U, \lim^X \otimes \lim^U)$  to  $(Y \times V, \lim^Y \otimes \lim^V)$  for two continuous mappings  $f :$

$(X, \lim^X) \rightarrow (Y, \lim^Y)$  and  $g : (U, \lim^U) \rightarrow (V, \lim^V)$ . For this, we take any stratified  $L$ -filter  $\mathcal{H}$  on  $X \times U$ . Then, by using the definition of  $\lim^Y \otimes \lim^V$ , we have

$$\begin{aligned} & \lim^Y \otimes \lim^V (f \times g)^{\Rightarrow}(\mathcal{H}) \\ &= \lim^Y p_Y^{\Rightarrow}((f \times g)^{\Rightarrow}(\mathcal{H})) \otimes \lim^V p_V^{\Rightarrow}((f \times g)^{\Rightarrow}(\mathcal{H})) \\ &= \lim^Y f^{\Rightarrow}(p_X^{\Rightarrow}(\mathcal{H})) \otimes \lim^V g^{\Rightarrow}(p_U^{\Rightarrow}(\mathcal{H})) \\ &\geq f^{\rightarrow}(\lim^X p_X^{\Rightarrow}(\mathcal{H})) \otimes g^{\rightarrow}(\lim^U p_U^{\Rightarrow}(\mathcal{H})) \\ &= (f \times g)^{\rightarrow}(\lim^X p_X^{\Rightarrow}(\mathcal{H}) \otimes \lim^U p_U^{\Rightarrow}(\mathcal{H})) \\ &= (f \times g)^{\rightarrow}((\lim^X \otimes \lim^U)\mathcal{H}), \end{aligned}$$

here,  $p_U : X \times U \rightarrow U$ ,  $p_X : X \times U \rightarrow X$ ,  $p_V : Y \times V \rightarrow V$  and  $p_Y : Y \times V \rightarrow Y$  are the projections, which means that  $f \times g : (X \times U, \lim^X \otimes \lim^U) \rightarrow (Y \times V, \lim^Y \otimes \lim^V)$  is continuous.  $\square$

**Corollary 3.11.** *Let  $(X, \lim^X)$  be an object in  $SL\text{-GCS}$ . Define  $\mathbf{F}_{\otimes} : SL\text{-GCS} \rightarrow SL\text{-GCS}$  by for each object  $(Y, \lim^Y)$  in  $SL\text{-GCS}$ ,*

$$\mathbf{F}_{\otimes}(Y, \lim^Y) = (Y, \lim^Y) \otimes (X, \lim^X)$$

and for each continuous mapping  $\varphi : (Y, \lim^Y) \rightarrow (Z, \lim^Z)$ ,

$$\mathbf{F}_{\otimes}(\varphi) = \varphi \times id_X : (Y, \lim^Y) \otimes (X, \lim^X) \rightarrow (Z, \lim^Z) \otimes (X, \lim^X).$$

Then  $\mathbf{F}_{\otimes} : SL\text{-GCS} \rightarrow SL\text{-GCS}$  is a functor.

By using Propositions 3.7-3.10, standard argument can be employed to show:

(a) The coherent isomorphism  $j_{(XY)Z}$  :

$$((X, \lim^X) \otimes (Y, \lim^Y)) \otimes (Z, \lim^Z) \rightarrow (X, \lim^X) \otimes ((Y, \lim^Y) \otimes (Z, \lim^Z))$$

is natural for three objects  $(X, \lim^X)$ ,  $(Y, \lim^Y)$ ,  $(Z, \lim^Z)$  in  $SL\text{-GCS}$ .

(b)  $s_{XY} : (X, \lim^X) \otimes (Y, \lim^Y) \rightarrow (Y, \lim^Y) \otimes (X, \lim^X)$  as a coherent isomorphism is natural for two objects  $(X, \lim^X)$ ,  $(Y, \lim^Y)$  in  $SL\text{-GCS}$ .

(c)  $p_X : (X, \lim) \otimes \lim_{\infty} \rightarrow (X, \lim)$  as a coherent isomorphism is natural for each object  $(X, \lim^X)$  in  $SL\text{-GCS}$ .

Following (a), (b) and (c) above, we obtain the main result in the section.

**Theorem 3.12.** *The category  $SL\text{-GCS}$  of all stratified  $L$ -generalized convergence spaces is a symmetric monoidal category with respect to the tensor operation  $\otimes$ .*

At the end of the section, we point out that Proposition 3.10 tells us that for each object  $(X, \lim^X)$  in  $SL\text{-GCS}$ , we obtain a functor

$$\mathbf{F}_{\otimes} : SL\text{-GCS} \rightarrow SL\text{-GCS}$$

determined by  $(X, \lim^X)$  in the sense of

$$\mathbf{F}_{\otimes}(Y, \lim^Y) = (Y, \lim^Y) \otimes (X, \lim^X)$$

for each object  $(Y, \lim^Y)$  in  $SL\text{-GCS}$ , and

$$\mathbf{F}_{\otimes}(\varphi) = \varphi \times id_X : (Y, \lim^Y) \otimes (X, \lim^X) \rightarrow (Z, \lim^Z) \otimes (X, \lim^X)$$

for each continuous mapping  $\varphi : (Y, \lim^Y) \rightarrow (Z, \lim^Z)$ .

In next section, we will show that there exists a functor from  $SL\text{-GCS}$  to  $SL\text{-GCS}$ , which is the right adjoint of the functor  $\mathbf{F}_\otimes : SL\text{-GCS} \rightarrow SL\text{-GCS}$  determined by  $(X, \lim^X)$ . In this way, we will confirm the symmetric monoidal category  $SL\text{-GCS}$  relative to the our tensor operation  $\otimes$  is  $\otimes$ -closed in the sense of the definition below.

**Definition 3.13** ([1]). A symmetric monoidal category  $\mathbb{C}$  is said to be  $\otimes$ -closed with respect to the tensor operation  $\otimes$  if for each object  $B$  in  $\mathbb{C}$ , the functor  $(-) \otimes B : \mathbb{C} \rightarrow \mathbb{C}$  has a right adjoint  $(-)^B : \mathbb{C} \rightarrow \mathbb{C}$ .

#### 4. $\otimes$ -closedness of $SL\text{-GCS}$

In the last section, we denote the set of morphisms from an object  $(X, \lim^X)$  to another object  $(Y, \lim^Y)$  in  $SL\text{-GCS}$  by  $[X, Y]_\otimes$ , i.e.,

$$[X, Y]_\otimes := \{\varphi : (X, \lim^X) \rightarrow (Y, \lim^Y) \mid \varphi \text{ is continuous}\}.$$

Then we construct a natural function space structure  $\otimes\text{-lim}$  on the set  $[X, Y]_\otimes$ , which assures that the function space is obtained. Following this, our goal is to look for a functor from  $SL\text{-GCS}$  to itself, which is a right adjoint of the functor  $\mathbf{F}_\otimes$  determined by a space  $(X, \lim^X)$  introduced at the end of Section 3. Finally, the monoidal closedness of the category  $SL\text{-GCS}$  of stratified  $L$ -generalized convergence spaces is explored.

**Lemma 4.1** ([17] for  $* = \wedge$ ). *If  $\varphi : X \rightarrow Y$  is a mapping, and  $A \in L^X$ , then it holds that  $\varphi^\rightarrow(A) = ev_{XY}(\top_{\{\varphi\}} \otimes A)$ .*

**Lemma 4.2.** *Let  $\varphi : X \rightarrow Y$  be a mapping and  $\mathcal{G} \in F_L^s(X)$ . Then*

$$\varphi^\rightarrow(\mathcal{G}) \leq ev_{XY}^\rightarrow([\varphi] \otimes \mathcal{G}).$$

*Proof.* This conclusion is a  $\otimes$ -version of Lemma 8.2 [17]. We include here a simple proof for the convenience of the reader.

For an  $L$ -subset  $B \in L^Y$ , we observe that

$$\begin{aligned} ev_{XY}^\rightarrow([\varphi] \otimes \mathcal{G})(B) &= ([\varphi] \otimes \mathcal{G})(B \circ ev_{XY}) \\ &= \bigvee \left\{ [\varphi](C) * \mathcal{G}(D) \mid C \in L^{[X, Y]_\otimes}, D \in L^X \text{ with } C \otimes D \leq ev_{XY}^\leftarrow(B) \right\} \\ &\geq \bigvee \left\{ [\varphi](\top_{\{\varphi\}}) * \mathcal{G}(D) \mid D \in L^X \text{ with } \top_{\{\varphi\}} \otimes D \leq ev_{XY}^\leftarrow(B) \right\} \\ &= \bigvee \left\{ \top * \mathcal{G}(D) \mid D \in L^X \text{ with } ev_{XY}^\rightarrow(\top_{\{\varphi\}} \otimes D) \leq B \right\} \\ &= \bigvee \left\{ \mathcal{G}(D) \mid D \in L^X \text{ with } \varphi^\rightarrow(D) \leq B \right\} \quad (\text{By Lemma 4.1}) \\ &= \bigvee \left\{ \mathcal{G}(D) \mid D \in L^X \text{ with } D \leq \varphi^\leftarrow(B) \right\} = \mathcal{G}(\varphi^\leftarrow(B)) \\ &= \varphi^\rightarrow(\mathcal{G})(B), \end{aligned}$$

that is to say  $ev_{XY}^\rightarrow([\varphi] \otimes \mathcal{G}) \geq \varphi^\rightarrow(\mathcal{G})$  and the proof is completed.  $\square$

When  $L$  is a frame, i.e.,  $*$  =  $\wedge$  in this paper, Jäger in [15] showed there is a natural function space structure on the set  $[X, Y]_\wedge$ , and from this he concluded that the category of all stratified  $L$ -generalized convergence spaces is Cartesian closed if the underlying lattice  $L$  is a frame. Following Jäger's work, for two objects  $(X, \lim^X), (Y, \lim^Y)$  in  $SL\text{-GCS}$ , we define a mapping

$$\otimes\text{-lim} : F_L^s([X, Y]_\otimes) \rightarrow L^{[X, Y]_\otimes}$$

by for each  $\mathcal{H} \in F_L^s([X, Y]_\otimes), \varphi \in [X, Y]_\otimes$ ,

$$\otimes\text{-lim} \mathcal{H}(\varphi) = \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X(\lim^X \mathcal{G}, \varphi \leftarrow (\lim^Y \text{ev}_{XY}(\mathcal{H} \otimes \mathcal{G})))$$

Then we conclude

**Lemma 4.3.** *Let  $(X, \lim^X), (Y, \lim^Y)$  be two stratified  $L$ -generalized convergence spaces. Then  $([X, Y]_\otimes, \otimes\text{-lim})$  is a stratified  $L$ -generalized convergence space.*

*Proof.* For the conclusion of the lemma, we have to check that the structure  $\otimes\text{-lim}$  satisfies the axioms (GL1)-(GL2). From the definition of  $\otimes\text{-lim}$ , it is routine to check it satisfies (GL2). In order to check (GL1), we take any element  $\varphi \in [X, Y]_\otimes$ . And then it holds that

$$\begin{aligned} \otimes\text{-lim}[\varphi](\varphi) &= \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X(\lim^X \mathcal{G}, \varphi \leftarrow (\lim^Y \text{ev}_{XY}([\varphi] \otimes \mathcal{G}))) \\ &\geq \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X(\varphi \leftarrow (\lim^Y \varphi \Rightarrow (\mathcal{G})), \varphi \leftarrow (\lim^Y \text{ev}_{XY}([\varphi] \otimes \mathcal{G}))) \\ &\hspace{15em} \text{(by using the continuity of } \varphi) \\ &\geq \bigwedge_{\mathcal{G} \in F_L^s(X)} S_Y(\lim^Y \varphi \Rightarrow (\mathcal{G}), \lim^Y \text{ev}_{XY}([\varphi] \otimes \mathcal{G})) \\ &\geq \top, \end{aligned}$$

where, the last inequality is true by Lemma 4.2 and the property of  $\lim^Y$  satisfying (GL2). Thus from this, we see (GL1) is satisfied by  $\otimes\text{-lim}$  and the proof is completed.  $\square$

Now, by Lemma 4.3 above, we realize that for each fixed object  $(X, \lim^X)$ , an object-mapping  $\mathbf{G}_\otimes$  from  $SL\text{-GCS}$  to itself is given by

$$\mathbf{G}_\otimes(Y, \lim^Y) = ([X, Y]_\otimes, \otimes\text{-lim}), \quad \forall (Y, \lim^Y) \in |SL\text{-GCS}|.$$

In order to show the object-mapping  $\mathbf{G}_\otimes$  from  $SL\text{-GCS}$  to itself is functor, let us define the morphism-action of  $\mathbf{G}_\otimes$  in the sense of

$$\forall \varphi \in [X, Y]_\otimes, \quad \mathbf{G}_\otimes(\Phi)(\varphi) = \Phi \circ \varphi$$

for each continuous mapping  $\Phi$  from  $(Y, \lim^Y)$  to  $(Z, \lim^Z)$  in  $SL\text{-GCS}$ . Since the continuity of  $\Phi : (Y, \lim^Y) \rightarrow (Z, \lim^Z)$  and  $\varphi : (X, \lim^X) \rightarrow (Y, \lim^Y)$  assures the continuity of the composition  $\Phi \circ \varphi$ , we can conclude that  $\mathbf{G}_\otimes(\Phi)$  is well-defined. The following Lemma 4.4 confirms that it is a functor indeed, and hence it will be called a power-functor determined by the space  $(X, \lim^X)$ .

**Lemma 4.4.** *Let  $(X, \lim^X)$  be an object in  $SL\text{-GCS}$ . Then*

$$\mathbf{G}_\otimes(\Phi) : ([X, Y]_\otimes, \otimes\text{-lim}_{XY}) \rightarrow ([X, Z]_\otimes, \otimes\text{-lim}_{XZ})$$

*is continuous for each continuous mapping  $\Phi : (Y, \lim^Y) \rightarrow (Z, \lim^Z)$ .*

*Proof.* Let  $\Phi : (Y, \lim^Y) \rightarrow (Z, \lim^Z)$  be a continuous mapping. In order to show  $\mathbf{G}_\otimes(\Phi)$  is continuous, let us take any stratified  $L$ -filter  $\mathcal{H} \in F_L^s([X, Y]_\otimes)$ . By using the definition of  $\otimes\text{-lim}_{XY}$  and  $\otimes\text{-lim}_{XZ}$ , we observe that for any  $\varphi \in [X, Y]_\otimes$ ,

$$\begin{aligned} & \mathbf{G}_\otimes(\Phi)^\leftarrow \left( \otimes\text{-lim}_{XZ} \mathbf{G}_\otimes(\Phi)^\Rightarrow(\mathcal{H}) \right) (\varphi) \\ &= \otimes\text{-lim}_{XZ} \mathbf{G}_\otimes(\Phi)^\Rightarrow(\mathcal{H}) \left( \Phi \circ \varphi \right) \\ &= \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X \left( \lim^X \mathcal{G}, (\Phi \circ \varphi)^\leftarrow (\lim^Z ev_{XZ}^\Rightarrow(\mathbf{G}_\otimes(\Phi)^\Rightarrow(\mathcal{H}) \otimes \mathcal{G})) \right) \\ &= \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X \left( \lim^X \mathcal{G}, \varphi^\leftarrow \circ \Phi^\leftarrow (\lim^Z ev_{XZ}^\Rightarrow(\mathbf{G}_\otimes(\Phi)^\Rightarrow(\mathcal{H}) \otimes \mathcal{G})) \right) := P. \end{aligned}$$

We now use Proposition 2.7 and  $\Phi \circ ev_{XY} = ev_{XZ} \circ (\mathbf{G}_\otimes(\Phi) \times id_X)$  to produce the following equality:

$$\begin{aligned} P &= \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X \left( \lim^X \mathcal{G}, \varphi^\leftarrow \circ \Phi^\leftarrow (\lim^Z ev_{XZ}^\Rightarrow \circ (\mathbf{G}_\otimes(\Phi) \times id_X)^\Rightarrow(\mathcal{H} \otimes \mathcal{G})) \right) \\ &= \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X \left( \lim^X \mathcal{G}, \varphi^\leftarrow \circ \Phi^\leftarrow (\lim^Z \Phi^\Rightarrow \circ ev_{XY}^\Rightarrow(\mathcal{H} \otimes \mathcal{G})) \right) := Q. \end{aligned}$$

Then we can further produce another inequality by using the continuity of  $\Phi$ :

$$Q \geq \bigwedge_{\mathcal{G} \in F_L^s(X)} S_X \left( \lim^X \mathcal{G}, \varphi^\leftarrow (\lim^Y ev_{XY}^\Rightarrow(\mathcal{H} \otimes \mathcal{G})) \right) = \otimes\text{-lim}_{XY} \mathcal{H}(\varphi),$$

which implies that  $\mathbf{G}_\otimes(\Phi) : ([X, Y]_\otimes, \otimes\text{-lim}_{XY}) \rightarrow ([X, Z]_\otimes, \otimes\text{-lim}_{XZ})$  is continuous, and the proof is completed.  $\square$

Let  $(X, \lim^X)$  be an object in  $SL\text{-GCS}$ . Recall that there exists a functor  $\mathbf{F}_\otimes : SL\text{-GCS} \rightarrow SL\text{-GCS}$  determined by the space  $(X, \lim^X)$ . By Theorem 2.11, in order to confirm the power-functor  $\mathbf{G}_\otimes : SL\text{-GCS} \rightarrow SL\text{-GCS}$  determined by the space  $(X, \lim^X)$  is a right adjoint of the functor  $\mathbf{F}_\otimes$ , we have to complete the following tasks:

- (i) to find a continuous map

$$\varepsilon_{(Y, \lim^Y)} : \mathbf{F}_\otimes \circ \mathbf{G}_\otimes(Y, \lim^Y) \rightarrow (Y, \lim^Y)$$

for each space  $(Y, \lim^Y)$ .

- (ii) for a continuous map  $\varphi : \mathbf{F}_\otimes(Z, \lim^Z) \rightarrow (Y, \lim^Y)$ , there is a unique continuous map  $\bar{\varphi} : (Z, \lim^Z) \rightarrow \mathbf{G}_\otimes(Y, \lim^Y)$  such that  $\varphi = \varepsilon_{(Y, \lim^Y)} \circ \mathbf{F}_\otimes(\bar{\varphi})$ .

For tasks (i) and (ii), we firstly, confirm that the evaluation mapping  $ev_{XY} : [X, Y]_{\otimes} \times X \rightarrow Y$  such that

$$\forall(\varphi, x) \in [X, Y]_{\otimes} \times X, \quad ev_{XY}(\varphi, x) = \varphi(x),$$

is a candidate of the  $\varepsilon_{(Y, \lim^Y)}$  by Proposition 4.5 below, and then complete the task (ii) after it.

**Proposition 4.5.** *Let  $(X, \lim^X)$  be an object in  $SL\text{-GCS}$ . Then for each object  $(Y, \lim^Y)$ , the evaluation mapping*

$$ev_{XY} : \mathbf{F}_{\otimes} \mathbf{G}_{\otimes}(Y, \lim^Y) = ([X, Y]_{\otimes}, \otimes\text{-lim}) \otimes (X, \lim^X) \rightarrow (Y, \lim^Y)$$

*is continuous.*

*Proof.* In order to show the continuity of  $ev_{XY}$ , let us take any stratified  $L$ -filter  $\mathcal{H} \in F_L^s([X, Y]_{\otimes} \times X)$ . Now, by means of the definition w.r.t.  $(\otimes\text{-lim}) \otimes \lim^X$ , we observe that for each  $(\varphi, x) \in [X, Y]_{\otimes} \times X$ ,

$$\begin{aligned} & ((\otimes\text{-lim}) \otimes \lim^X) \mathcal{H}(\varphi, x) \\ &= \otimes\text{-lim } p_{[X, Y]_{\otimes}}^{\vec{\rightarrow}}(\mathcal{H})(\varphi) * \lim^X p_X^{\vec{\rightarrow}}(\mathcal{H})(x) \\ &\leq S_X \left( \lim^X p_X^{\vec{\rightarrow}}(\mathcal{H}), \varphi^{\leftarrow}(\lim^Y ev_{XY}^{\vec{\rightarrow}}(p_{[X, Y]_{\otimes}}^{\vec{\rightarrow}}(\mathcal{H}) \otimes p_X^{\vec{\rightarrow}}(\mathcal{H}))) \right) * \\ & \hspace{15em} \lim^X p_X^{\vec{\rightarrow}}(\mathcal{H})(x) \\ &\leq \lim^Y ev_{XY}^{\vec{\rightarrow}}(p_{[X, Y]_{\otimes}}^{\vec{\rightarrow}}(\mathcal{H}) \otimes p_X^{\vec{\rightarrow}}(\mathcal{H}))(\varphi(x)) \\ &\leq \lim^Y ev_{XY}^{\vec{\rightarrow}}(\mathcal{H})(\varphi(x)) \quad (\text{since } p_{[X, Y]_{\otimes}}^{\vec{\rightarrow}}(\mathcal{H}) \otimes p_X^{\vec{\rightarrow}}(\mathcal{H}) \leq \mathcal{H} \text{ in general.}) \\ &= ev_{XY}^{\leftarrow}(\lim^Y ev_{XY}^{\vec{\rightarrow}}(\mathcal{H}))(\varphi, x), \end{aligned}$$

that is to say  $((\otimes\text{-lim}) \otimes \lim^X) \mathcal{H} \leq ev_{XY}^{\leftarrow}(\lim^Y ev_{XY}^{\vec{\rightarrow}}(\mathcal{H}))$  holds for each  $\mathcal{H} \in F_L^s([X, Y]_{\otimes} \times X)$ . From this observation, we conclude that  $ev_{XY}$  is continuous, so the proof is completed.  $\square$

Let  $\varphi : Z \times X \rightarrow Y$  be a mapping. For each  $z \in Z$ , there exists a mapping  $\varphi(z, -) : X \rightarrow Y$  such that  $\varphi(z, -)(x) = \varphi(z, x)$  for any  $x \in X$ . Using the similar method of Lemma 8.5 [17], we obtain the inequality

$$\varphi(z, -)^{\Rightarrow}(\mathcal{F}) \geq \varphi^{\Rightarrow}([z] \otimes \mathcal{F})$$

for a stratified  $L$ -filter  $\mathcal{F} \in F_L^s(X)$ . Then if the mapping

$$\varphi : (Z, \lim^Z) \otimes (X, \lim^X) \rightarrow (Y, \lim^Y)$$

is continuous, we could conclude that for each  $z \in Z$ ,  $\varphi(z, -) : (X, \lim^X) \rightarrow (Y, \lim^Y)$  is continuous, in fact, which can be proved as follows:

$$\begin{aligned} \lim^X \mathcal{F}(x) &= \lim^Z [z](z) * \lim^X \mathcal{F}(x) \\ &\leq (\lim^Z \otimes \lim^X)([z] \otimes \mathcal{F})(z, x) \\ &\leq \lim^Y \varphi^{\Rightarrow}([z] \otimes \mathcal{F})(\varphi(z, x)) \\ &\leq \lim^Y \varphi(z, -)^{\Rightarrow}(\mathcal{F})(\varphi(z, -)(x)), \end{aligned}$$

for every stratified  $L$ -filter  $\mathcal{F} \in F_L^s(X)$ . In this way, for every continuous mapping  $\varphi : (Z, \lim^Z) \otimes (X, \lim^X) \rightarrow (Y, \lim^Y)$ , there is a well-defined  $\bar{\varphi} : (Z, \lim^Z) \rightarrow ([X, Y]_{\otimes}, \otimes\text{-lim})$  such that  $\bar{\varphi}(z) = \varphi(z, -)$  for all  $z \in Z$ . Moreover, the following proposition confirms that it is continuous.

**Proposition 4.6** ([17] for  $* = \wedge$ ). *Let  $(X, \lim^X)$ ,  $(Y, \lim^Y)$  and  $(Z, \lim^Z)$  be objects in  $SL\text{-GCS}$ . If  $\varphi : (Z, \lim^Z) \otimes (X, \lim^X) \rightarrow (Y, \lim^Y)$  is continuous, then  $\bar{\varphi} : (Z, \lim^Z) \rightarrow ([X, Y]_{\otimes}, \otimes\text{-lim})$  is continuous.*

*Proof.* First of all, note that  $ev_{XY} \circ (\bar{\varphi} \times id_X) = \varphi$ . From this and Proposition 2.7, we have for stratified  $L$ -filters  $\mathcal{F} \in F_L^s(Z)$  and  $\mathcal{G} \in F_L^s(X)$ ,

$$\varphi^{\Rightarrow}(\mathcal{F} \otimes \mathcal{G}) = (ev_{XY} \circ (\bar{\varphi} \times id_X))^{\Rightarrow}(\mathcal{F} \otimes \mathcal{G}) = ev_{XY}^{\Rightarrow}(\bar{\varphi}^{\Rightarrow}(\mathcal{F}) \otimes \mathcal{G}).$$

Then by using the equality  $\varphi^{\Rightarrow}(\mathcal{F} \otimes \mathcal{G}) = ev_{XY}^{\Rightarrow}(\bar{\varphi}^{\Rightarrow}(\mathcal{F}) \otimes \mathcal{G})$  obtained above, we observe that for each  $z \in Z$

$$\begin{aligned} \lim^Z \mathcal{F}(z) &\leq S_X \left( \lim^X \mathcal{G}, \lim^Z \mathcal{F}(z) * \lim^X \mathcal{G} \right) \\ &= \bigwedge_{x \in X} \left( \lim^X \mathcal{G}(x) \longrightarrow \lim^Z \mathcal{F}(z) * \lim^X \mathcal{G}(x) \right) \\ &\leq \bigwedge_{x \in X} \left( \lim^X \mathcal{G}(x) \longrightarrow (\lim^Z \otimes \lim^X)(\mathcal{F} \otimes \mathcal{G})(z, x) \right) \\ &\leq \bigwedge_{x \in X} \left( \lim^X \mathcal{G}(x) \longrightarrow \lim^Y \varphi^{\Rightarrow}(\mathcal{F} \otimes \mathcal{G})(\varphi(z, x)) \right) \\ &\hspace{15em} \text{(By means of the continuity of } \varphi) \\ &= \bigwedge_{x \in X} \left( \lim^X \mathcal{G}(x) \longrightarrow \lim^Y ev_{XY}^{\Rightarrow}(\bar{\varphi}^{\Rightarrow}(\mathcal{F}) \otimes \mathcal{G})(\bar{\varphi}(z)(x)) \right) \\ &= \otimes\text{-lim } \bar{\varphi}^{\Rightarrow}(\mathcal{F})(\bar{\varphi}(z)), \end{aligned}$$

which implies  $\lim^Z \mathcal{F} \leq \bar{\varphi}^{\leftarrow}(\otimes\text{-lim } \bar{\varphi}^{\Rightarrow}(\mathcal{F}))$ . So, we confirm that

$$\bar{\varphi} : (Z, \lim^Z) \rightarrow ([X, Y]_{\otimes}, \otimes\text{-lim})$$

is continuous, finally.  $\square$

From Propositions 4.5, 4.6 as well as Theorem 2.11, we could conclude that for each object  $(X, \lim^X)$ , the power-functor  $\mathbf{G}_{\otimes} : SL\text{-GCS} \rightarrow SL\text{-GCS}$  determined by it has a left adjoint  $\mathbf{F}_{\otimes} : SL\text{-GCS} \rightarrow SL\text{-GCS}$ . Then according to Theorem 3.12 and Definition 3.13, we have the following main result in the paper.

**Theorem 4.7.** *The category  $SL\text{-GCS}$  of stratified  $L$ -generalized convergence spaces is  $\otimes$ -closed with respect to the tensor operation  $\otimes$ .*

If the underlying lattice  $L$  is a complete Heyting Algebra, we recapture the Cartesian-closedness of the category  $SL\text{-GCS}$  by Theorem 4.7. In details, we have

**Corollary 4.8** (Jäger [17]). *If  $L$  is a complete Heyting Algebra, then the category  $SL\text{-GCS}$  of stratified  $L$ -generalized convergence spaces is Cartesian closed.*



## 5. Conclusion

In this paper, the lattice valued environment  $L$  is complete residuated lattice with nonidempotent semigroup operation. By defining the tensor operation of stratified  $L$ -generalized convergence spaces, we have proved that the category of stratified  $L$ -generalized convergence spaces is monoidal category, further certified that the category of stratified  $L$ -generalized convergence spaces is monoidal closed, thus we can get the desired function space. In particular, if the lattice valued environment  $L$  is a complete Heyting Algebra, the Cartesian closedness of the category is recaptured by our result.

These results fill the blank in the research of convergence space about monoidal closedness, and the tensor operation of stratified  $L$ -generalized convergence spaces is a bifunctor, which has elicitation for defining the tensor operation of general space objects.

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