COMPLEX FUZZY H_v-SUBGROUPS OF AN H_v-GROUP

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Abstract. The concept of complex fuzzy sets is a generalization of ordinary fuzzy sets. In this paper, we introduce the concept of complex fuzzy subhypergroups (H_v-subgroups) as well as the concept of complex anti-fuzzy subhypergroups (H_v-subgroups). We investigate their properties and their relations with the traditional fuzzy (anti-fuzzy) subhypergroups (H_v-subgroups), and we prove some results in this respect.

1. Introduction

Algebraic hyperstructures represent a natural generalization of classical algebraic structures and they were introduced by Marty [5] in 1934 at the eighth Congress of Scandinavian Mathematicians. In classical algebraic structures, the composition of two elements is an element whereas in algebraic hyperstructures, the composition of two elements is a set. Since then, many different hyperstructures (hyperring, hyperalgebra, hyperrepresentation, ...) were widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics: geometry, topology, cryptography and code theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets, automata theory, economy, etc. (see [1]). The H_v-structures are generalized algebraic hyperstructures where in the axioms of the classical hyperstructures the equality is replaced by the non-empty intersection. They were introduced by Vougiouklis [11], also see [9, 10].

On the other hand, the fuzzy mathematics forms a branch of mathematics related to fuzzy set theory and fuzzy logic. It was introduced in 1965 after the publication of Zadeh (see [12]) as an extension of the classical notion of set, when he proposed the idea of a multi-valued logic, which extends the traditional concept of a bivalent logic, which becomes a particular case of the new theory. The fuzzy set theory is based on the principle called by Zadeh "the principle of incompatibility", that is "the closer a phenomenon is studied, the more indistinct its definition becomes". Fuzzy sets are sets whose elements have degrees of membership. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in

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the real unit interval $[0,1]$. Fuzzy sets generalize classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. Rosenfield [8] applied this concept to the theory of groups and introduced the concept of a fuzzy subgroup of a group. Since then, a host of mathematicians are engaged in fuzzifying various notions and results of abstract algebra. In [2], Davvaz introduced the concept of fuzzy subhypergroup (H_v-subgroups) of a hypergroup (H_v-group. A short review of the theory of fuzzy algebraic hyperstructures appears in [4].

As an extension of fuzzy sets, Raymot et al. [6, 7] introduced the concept of complex fuzzy sets in which the codomain of membership function was the unit disc of the complex plane. They introduced different fuzzy complex operations and relations.

The remainder part of our paper is constructed as follows: after an Introduction, in Section 2 we present some definitions and results about hyperstructures and traditional fuzzy subhyperstructures. In Section 3, we define complex fuzzy H_v-subgroups as well as complex anti-fuzzy H_v-subgroups, investigate their properties and present different examples on them.

2. **Hyperstructures and traditional fuzzy subhyperstructures**

In this section, we present some definitions and theorems related to hyperstructures and fuzzy subhyperstructures that are used throughout the paper.

Definition 2.1. [3] Let H be a non-empty set. Then, a mapping $\circ : H \times H \to \mathcal{P}^*(H)$ is called a *binary hyperoperation* on H, where $\mathcal{P}^*(H)$ is the family of all non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*.

In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define:

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \ x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

Definition 2.2. [3] A hypergroupoid $(H, \circ)$ is called a:

- *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$;
- *quasihypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$ (This condition is called the reproduction axiom);
- *hypergroup* if it is a semihypergroup and a quasihypergroup;
- *H_v-group* if it is a quasihypergroup and for every $x, y, z \in H$, we have $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$.

Definition 2.3. [3] Let $(H, \circ)$ be a hypergroup (or H_v-group) and $K \subseteq H$. Then $(K, \circ)$ is a subhypergroup (or H_v-subgroup) of $(H, \circ)$ if for all $a \in K$, we have that $a \circ K = K \circ a = K$.

Definition 2.4. [12] A fuzzy set, defined on a universe of discourse U is characterized by a membership function $\mu_A(x)$ that assigns any element a grade of membership in A. The fuzzy set may be represented by the set of ordered pairs

$$A = \{(x, \mu_A(x)) : x \in U\},$$

where $\mu_A(x) \in [0,1]$.

Definition 2.5. [4] Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a fuzzy subset of H with membership function $\mu_A(x) \in [0,1]$. Then A is a fuzzy subhypergroup (or H_v-subgroup) of H if the following conditions hold:

(1) $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) : z \in x \circ y\}$ for all $x, y \in H$;
(2) For all $x, a \in H$, there exists $y \in H$ such that $x \in a \circ y$ and $\min\{\mu_A(x), \mu_A(a)\} \leq \mu_A(y)$;
(3) For all $x, a \in H$, there exists $z \in H$ such that $x \in z \circ a$ and $\min\{\mu_A(x), \mu_A(a)\} \leq \mu_A(z)$.

Lemma 2.6. [2] *Let $(H, \circ)$ be a hypergroup (or H_v-group) and μ be a fuzzy subhypergroup (or H_v-subgroup) of H. Then*

$$\min\{\mu(x_1), \mu(x_2), \cdots, \mu(x_n)\} \leq \inf\{\mu(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}$$

for all $x_1, x_2, \cdots, x_n \in H$.

Definition 2.7. [4] Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a fuzzy subset of H with membership function $\mu_A(x)$. Then A is an anti-fuzzy subhypergroup (or H_v-subgroup) of H if the following conditions hold:

(1) $\sup\{\mu_A(z) : z \in x \circ y\} \leq \max\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in H$;
(2) For all $x, a \in H$, there exists $y \in H$ such that $x \in a \circ y$ and $\mu_A(y) \leq \max\{\mu_A(x), \mu_A(a)\}$;
(3) For all $x, a \in H$, there exists $z \in H$ such that $x \in z \circ a$ and $\mu_A(z) \leq \max\{\mu_A(x), \mu_A(a)\}$.

Lemma 2.8. [4] *Let $(H, \circ)$ be a hypergroup (or H_v-group) and μ be an anti-fuzzy subhypergroup (or H_v-subgroup) of H. Then*

$$\max\{\mu(x_1), \mu(x_2), \cdots, \mu(x_n)\} \geq \sup\{\mu(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}$$

for all $x_1, x_2, \cdots, x_n \in H$.

Theorem 2.9. [4] *Let $(H, \circ)$ be a hypergroup (or H_v-group) and μ be a fuzzy subset of H. Then μ is a fuzzy subhypergroup (or H_v-subgroup) of H if and only if its complement μ^c is an anti-fuzzy subhypergroup (or H_v-subgroup) of H. Here, $\mu^c(x) = 1 - \mu(x)$ for all $x \in H$.*

3. Complex fuzzy subhyperstructures

In this section, we use the concept of complex fuzzy subsets to define complex fuzzy (anti-fuzzy) subhypergroups. And we investigate their properties.

3.1. Complex fuzzy H_v-subgroups.

Definition 3.1. Let $A = \{(x, \mu_A(x)) : x \in U\}$ be a fuzzy set. Then the set $A_\pi = \{(x, 2\pi\mu_A(x)) : x \in U\}$ is said to be a π-fuzzy set.

Proposition 3.2. *Let $(H, \circ)$ be a hypergroup (or H_v-group). A π-fuzzy set A_π is a π-fuzzy subhypergroup (or H_v-subgroup) of H if and only if A is a fuzzy subhypergroup (or H_v-subgroup) of H.*

Proof. The proof is straightforward. □

Definition 3.3. A complex fuzzy set, defined on a universe of discourse U is characterized by a membership function $\mu_A(x)$ that assigns any element a complex-valued grade of membership in A. The complex fuzzy set may be represented by the set of ordered pairs

$$A = \{(x, \mu_A(x)) : x \in U\},$$

where $\mu_A(x) = r(x)e^{iw(x)}$, $i = \sqrt{-1}$, $r(x) \in [0, 1]$ and $w(x) \in [0, 2\pi]$.

Remark 3.4. By setting $w(x) = 0$ in the above definition, we return back to the traditional fuzzy set.

Definition 3.5. [7] Let $A = \{(x, \mu_A(x)) : x \in U\}$ and $B = \{(x, \mu_B(x)) : x \in U\}$ be two complex fuzzy sets of the same universe U with the membership functions $\mu_A(x) = r_A(x)e^{i\omega_A(x)}$ and $\mu_B(x) = r_B(x)e^{i\omega_B(x)}$, respectively. Then

- $\mu_{A\cap B}(x) = r_{A\cap B}(x)e^{i\omega_{A\cap B}(x)} = \min\{r_A(x), r_B(x)\}e^{i\min\{\omega_A(x),\omega_B(x)\}}$;
- $\mu_{A\cup B}(x) = r_{A\cup B}(x)e^{i\omega_{A\cup B}(x)} = \max\{r_A(x), r_B(x)\}e^{i\max\{\omega_A(x),\omega_B(x)\}}$;
- $\mu_{A^c}(x) = (1 - r_A(x))e^{i(2\pi-\omega_A(x))}$, where A^c denotes the complement of A.

Definition 3.6. Let $A = \{(x, \mu_A(x)) : x \in H\}$ and $B = \{(x, \mu_B(x)) : x \in H\}$ be complex fuzzy subsets of a non-void set H with membership functions $\mu_A(x) = r_A(x)e^{iw_A(x)}$ and $\mu_B(x) = r_B(x)e^{iw_B(x)}$ respectively. Then

(1) A complex fuzzy subset A is said to be homogeneous if for all $x, y \in H$, we have
$$r_A(x) \leq r_A(y) \text{ if and only if } w_A(x) \leq w_A(y).$$
(2) A complex fuzzy subset A is said to be homogeneous with B if for all $x, y \in H$, we have
$$r_A(x) \leq r_B(y) \text{ if and only if } w_A(x) \leq w_B(y).$$

Notation 3.7. Let $A = \{(x, \mu_A(x)) : x \in H\}$ and $B = \{(x, \mu_B(x)) : x \in H\}$ be complex fuzzy subsets of a non-void set H with membership functions $\mu_A(x) = r_A(x)e^{iw_A(x)}$ and $\mu_B(x) = r_B(x)e^{iw_B}(x)$ respectively. By $\mu_A(x) \leq \mu_B(x)$, we mean that $r_A(x) \leq r_B(x)$ and $w_A(x) \leq w_B(x)$.

Throughout this paper, all complex fuzzy sets are considered homogeneous.

Definition 3.8. Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex fuzzy subhypergroup (or H_v-subgroup) of H if the following conditions hold:

(1) $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) : z \in x \circ y\}$ for all $x, y \in H$;
(2) For all $x, a \in H$, there exists $y \in H$ such that $x \in a\circ y$ and $\min\{\mu_A(x), \mu_A(a)\} \leq \mu_A(y)$;
(3) For all $x, a \in H$, there exists $z \in H$ such that $x \in z\circ a$ and $\min\{\mu_A(x), \mu_A(a)\} \leq \mu_A(z)$.

Example 3.9. Let $H = \{a, b\}$ and define the hypergroup $(H, \circ)$ by the following table:

$\circ$	a	b
a	a	H
b	H	b

We define a complex fuzzy subset μ of H as follows: $\mu(a) = 0.5e^{i0}$ and $\mu(b) = 1e^{i\frac{\pi}{2}}$. Then μ is homogeneous complex fuzzy subhypergroup of H.

Theorem 3.10. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex fuzzy subhypergroup (or H_v-subgroup) of H if and only if r_A is a fuzzy subhypergroup (or H_v-subgroup) of H and w_A is a π-fuzzy subhypergroup (or H_v-subgroup) of H.*

Proof. Suppose that A is a complex fuzzy subhypergroup (or H_v-subgroup) of H. We need to prove that the conditions of Definition 2.5 are satisfied for r_A and w_A. For all $x, y \in H$, we have $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) : z \in x \circ y\}$. The latter and Notation 3.7 imply that $\inf\{r_A(z) : z \in x \circ y\} \geq \min\{r_A(x), r_A(y)\}$ and $\inf\{w_A(z) : z \in x \circ y\} \geq \min\{w_A(x), w_A(y)\}$. Let $a, x \in H$. Then there exist $y, z \in H$ such that $x \in a \circ y$, $x \in z \circ a$ and $\min\{\mu_A(a), \mu_A(x)\} \leq \mu_A(y)$, $\min\{\mu_A(a), \mu_A(x)\} \leq \mu_A(z)$. Notation 3.7 implies that the conditions 2 and 3 of Definition 2.5 are satisfied for both r_A and w_A.

Suppose that r_A is a fuzzy subhypergroup (or H_v-subgroup) of H and w_A is a π-fuzzy subhypergroup (or H_v-subgroup) of H. We need to prove that the conditions of Definition 3.8 are satisfied. For all $x, y \in H$, we have $\inf\{r_A(z) : z \in x \circ y\} \geq \min\{r_A(x), r_A(y)\}$ and $\inf\{w_A(z) : z \in x \circ y\} \geq \min\{w_A(x), w_A(y)\}$. The latter and Notation 3.7 imply that $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) : z \in x \circ y\}$. Let $a, x \in H$. Then there exist $y, z \in H$ such that $x \in a \circ y$, $x \in z \circ a$ and $\min\{r_A(a), r_A(x)\} \leq r_A(y)$, $\min\{r_A(a), r_A(x)\} \leq r_A(z)$, $\min\{w_A(a), w_A(x)\} \leq w_A(y)$, $\min\{w_A(a), w_A(x)\} \leq w_A(z)$. Notation 3.7 implies that the conditions 2 and 3 of Definition 3.8 are satisfied for μ_A. □

Lemma 3.11. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and μ be a (homogeneous) complex fuzzy subhypergroup (or H_v-subgroup) of H. Then*

$$\min\{\mu(x_1), \mu(x_2), \cdots, \mu(x_n)\} \leq \inf\{\mu(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}$$

for all $x_1, x_2, \cdots, x_n \in H$.

Proof. Let $x_1, x_2, \cdots, x_n \in H$ and $\mu(x) = r(x)e^{iw(x)}$. To prove the lemma, it suffices to show that

$$\min\{r(x_1), r(x_2), \cdots, r(x_n)\} \leq \inf\{r(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}$$

and

$$\min\{w(x_1), w(x_2), \cdots, w(x_n)\} \leq \inf\{w(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}.$$

Since μ is homogeneous, it suffices to show that

$$\min\{r(x_1), r(x_2), \cdots, r(x_n)\} \leq \inf\{r(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}.$$

Theorem 3.10 asserts that r is a fuzzy subhypergroup (or H_v-subgroup) of H. Lemma 2.6 completes the proof. □

Definition 3.12. Let $A = \{(x, \mu_A(x)) : x \in H\}$ be a (homogeneous) complex fuzzy subset of a non-void set H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Define the level subset, μ_t, of H as follows:

$$\mu_t = \{x \in H : \mu_A(x) \geq t\},$$

where $t = se^{i\theta}$, $s \in [0, 1]$ and $\theta \in [0, 2\pi]$.

Remark 3.13. Let $A = \{(x, \mu_A(x)) : x \in H\}$ be a (homogeneous) complex fuzzy subset of a non-void set H. Then the following are true:

(1) If $t_1 \leq t_2$ then $\mu_{t_2} \subseteq \mu_{t_1}$.
(2) $\mu_{0e^{0i}} = H$.

Theorem 3.14. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex fuzzy subhypergroup (or H_v-subgroup) of H if and only if for all $t = se^{i\theta}$, $s \in [0, 1]$ and $\theta \in [0, 2\pi]$, $\mu_t \neq \emptyset$ is a subhypergroup (or H_v-subgroup) of H.*

Proof. Let A be a complex fuzzy subhypergroup (or H_v-subgroup) of H and $x, y \in \mu_t \neq \emptyset$. Then for all $a \in x \circ y$, we have that $\mu_A(a) \geq \min\{\mu_A(x), \mu_A(y)\} \geq t$. Thus $a \in x \circ y \subseteq \mu_t$. Hence, for every $a \in \mu_t$, we have $a \circ \mu_t \subseteq \mu_t$. Now let $x \in \mu_t$ then by condition 2 of Definition 3.8, there exists $y \in H$ such that $x \in a \circ y$ and $t = \min\{\mu_A(a), \mu_A(x)\} \leq \mu_A(y)$. The latter implies that $y \in \mu_t$. We can use condition 3 of Definition 3.8 to get that $\mu_t \circ a \subseteq \mu_t$.

For the converse, assume that for all $t = se^{i\theta}$, $s \in [0, 1]$ and $\theta \in [0, 2\pi]$, $\mu_t \neq \emptyset$ is subhypergroup (or H_v-subgroup) of H. Let $t_0 = s_0e^{i\theta_0} = \min\{\mu_A(x), \mu_A(y)\}$. Then $s_0 = \min\{r_A(x), r_A(y)\}$ and $\theta_0 = \min\{w_A(x), w_A(y)\}$. Since $x, y \in \mu_{t_0}$ and μ_{t_0} is a subhypergroup (or H_v-subgroup) of H, it follows that $x \circ y \subseteq \mu_{t_0}$. Therefore, for every $a \in x \circ y$ we have that $\mu_A(a) \geq t_0 = \min\{\mu_A(x), \mu_A(y)\}$ and thus, condition 1 of Definition 3.8 is verified. We prove now condition 2 and condition 3 is done in a similar manner. For every $a, x \in H$, set $t_1 = s_1e^{i\theta_1} = \min\{\mu_A(x), \mu_A(a)\}$, then $x, a \in \mu_{t_1}$. Having μ_{t_1} a subhypergroup (or H_v-subgroup) of H implies that $a \circ \mu_{t_1} = \mu_{t_1}$. The latter implies that there exists $y \in \mu_{t_1}$ such that $x \in a \circ y$. Therefore, $\mu_A(y) \geq t_1 = \min\{\mu_A(a), \mu_A(x)\}$. □

Corollary 3.15. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subhypergroup (or H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. If $0e^{0i} \leq t_1 = s_1e^{i\theta_1} < t_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$, then $\mu_{t_1} = \mu_{t_2}$ if and only if there is no $x \in H$ such that $t_1 \leq \mu_A(x) < t_2$.*

Proof. Let $0e^{0i} \leq t_1 = s_1e^{i\theta_1} < t_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$ such that $\mu_{t_1} = \mu_{t_2}$. Suppose that there exists $x \in H$ such that $t_1 \leq \mu_A(x) < t_2$. Then $x \in \mu_{t_1} = \mu_{t_2}$. The latter implies that $\mu_A(x) \geq t_2$.

Since $0e^{0i} \leq t_1 = s_1 e^{i\theta_1} < t_2 = s_2 e^{i\theta_2} \leq 1e^{2\pi i}$, it follows by Remark 3.13 that $\mu_{t_2} \subseteq \mu_{t_1}$. To show that $\mu_{t_1} \subseteq \mu_{t_2}$, we let $x \in \mu_{t_1}$. Then $\mu_A(x) \geq t_1$. Since there is no $x \in H$ such that $t_1 \leq \mu_A(x) < t_2$, it follows that $\mu_A(x) \geq t_2$. □

Corollary 3.16. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subhypergroup (or H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. If the range of μ_A is the finite set $\{t_1, t_2, \cdots, t_n\}$ then the set $\{\mu_{t_i} : i = 1, 2, \cdots, n\}$ contains all the level subhypergroups (or H_v-subgroups) of H. Moreover, if $t_1 \geq t_2 \geq \cdots \geq t_n$ then all the level subhypergroups (or H_v-subgroups) of H form the chain: $\mu_{t_1} \subseteq \mu_{t_2} \subseteq \cdots \subseteq \mu_{t_n}$.*

Proof. Let $\mu_s \neq \emptyset$ be a level subhypergroup (or H_v-subgroup) of H such that $\mu_s \neq \mu_{t_i}$ for all $1 \leq i \leq n$. Let t_k be closest complex number to s. We have two cases for s: $s < t_k$ and $s > t_k$. We consider only the first case, the second is done in a similar manner. Since the range of μ_A is the finite set $\{t_1, t_2, \cdots, t_n\}$, it follows that there is no $x \in H$ such that $s \leq \mu_A(x) < t_k$. Using Corollary 3.15, we get contradiction. □

Proposition 3.17. *Let $(H, \circ)$ be the biset hypergroup, i.e., $x \circ y = \{x, y\}$ for all $x, y \in H$ and let μ be any homogeneous complex fuzzy subset of H. Then μ is a complex fuzzy subhypergroup of H.*

Proof. Let $t = se^{i\theta}$, $s \in [0, 1]$ and $\theta \in [0, 2\pi]$. Then, by Theorem 3.14, it suffices to show that $\mu_t \neq \emptyset$ is a subhypergroup of H. We have that $\mu_t \subseteq a \circ \mu_t$ as for all $x \in \mu_t$, $x \in a \circ x = \{a, x\}$. Moreover, It is clear that $a \circ \mu_t = \mu_t \circ a = \{x \circ a : x \in \mu_t\} = \{x, a\} \subseteq \mu_t$ for all $a \in \mu_t$. □

Proposition 3.18. *Let $(H, \circ)$ be the total hypergroup, i.e., $x \circ y = H$ for all $x, y \in H$ and let μ be any homogeneous complex fuzzy subset of H. Then μ is a complex fuzzy subhypergroup of H if and only if μ is a constant complex function.*

Proof. If μ is a constant complex function then it is clear that μ is a complex fuzzy subhypergroup of H.

Let μ is a complex fuzzy subhypergroup of H and suppose for contradiction that μ is not a constant complex function. Then we can find $x, y \in H$, $t = se^{i\theta}$, $s \in [0, 1]$ and $\theta \in [0, 2\pi]$ such that $\mu(x) < \mu(y) = t$. It is clear that x is not an element in $\mu_t \ni y$. Since $\mu_t \neq \emptyset$ is a subhypergroup of H, it follows that $H = y \circ y \subseteq \mu_t$. □

Proposition 3.19. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H. Then A is a complex fuzzy subhypergroup (H_v-subgroup) of H if and only if for every $t = se^{i\theta}$, $s \in [0, 1]$ and $\theta \in [0, 2\pi]$, the following conditions are satisfied:*

1. *$\mu_t \circ \mu_t \subseteq \mu_t$;*
2. *$a \circ (H - \mu_t) - (H - \mu_t) \subseteq a \circ \mu_t$, for all $a \in \mu_t$;*
3. *$(H - \mu_t) \circ a - (H - \mu_t) \subseteq \mu_t \circ a$, for all $a \in \mu_t$.*

Proof. Let A be a complex fuzzy subhypergroup (H_v-subgroup) of H. Then, by Theorem 3.14, μ_t is a subhypergroup (H_v-subgroup) of H, i.e., $a \circ \mu_t = \mu_t$ for all

$a \in \mu_t$. Thus, we get that $\mu_t \circ \mu_t \subseteq \mu_t$. We need to show that $a \circ (H - \mu_t) - (H - \mu_t) \subseteq a \circ \mu_t$. Let $z \in a \circ (H - \mu_t) - (H - \mu_t)$. Then z is not an element in $(H - \mu_t)$. This implies that $z \in \mu_t = a \circ \mu_t$. Condition 3 can be proved in a similar manner.

For the converse, suppose that the conditions 1 and 2 hold. Then, by Theorem 3.14, it suffices to show that μ_t is a subhypergroup (H_v-subgroup) of H, i.e., $a \circ \mu_t = \mu_t \circ a = \mu_t$ for all $a \in \mu_t$. Assume that there exists $x \in \mu_t$ such that x is not an element in $a \circ \mu_t$. The reproduction axiom of $(H, \circ)$ asserts that there exists $b \in H$ such that $x \in a \circ b$. We consider the following two cases for b:

- Case $b \in \mu_t$. We get that $x \in a \circ b \subseteq a \circ \mu_t$ which is a contradiction.
- Case b is not an element in μ_t. We get that $b \in H - \mu_t$. And having $x \in a \circ b$ implies that $x \in a \circ (H - \mu_t)$. Since $x \in \mu_t$, it follows that x is not in $H - \mu_t$. Thus, $x \in a \circ (H - \mu_t) - (H - \mu_t) \subseteq a \circ \mu_t$ which is a contradiction.

We can prove that $\mu_t \circ a = \mu_t$, by applying condition 3, in a similar manner. □

Proposition 3.20. *Let $(H, \circ)$ be a hypergroup (or H_v-group). Then every subhypergroup (or H_v-subgroup) of H is a level H_v-subgroup of a fuzzy subhypergroup (H_v-subgroup) of H.*

Proof. Let M be a subhypergroup (or H_v-subgroup) of H. For a fixed complex number $t_0 = se^{i\theta}$, $s \in]0, 1], \theta \in]0, 2\pi]$, the fuzzy subset μ is defined as follows:

$$\mu(x) = \begin{cases} t_0, & \text{if } x \in M; \\ 0e^{i0}, & \text{otherwise.} \end{cases}$$

We have $M = \mu_{t_0}$ and $\mu_t = \begin{cases} H, & \text{if } t = 0; \\ M, & \text{if } 0 < t \leq t_0; \\ \emptyset, & \text{otherwise.} \end{cases}$ is either the empty set or a subhypergroup (or H_v-subgroup) of H. Then, by Theorem 3.14, we get that μ is a fuzzy subhypergroup (H_v-subgroup) of H. □

Proposition 3.21. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subhypergroup (H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Define $\overline{\mu}$ as follows:*

$$\overline{\mu} = \{x \in H : \mu_A(x) = 1e^{2\pi i}\}.$$

Then $\overline{\mu}$ is empty or subhypergroup (H_v-subgroup) of H.

Proof. Let $x, y \in \overline{\mu} \neq \emptyset$. We want to show that $a \circ \overline{\mu} = \overline{\mu} = \overline{\mu} \circ a$ for all $a \in \overline{\mu}$. Let $x \in \overline{\mu}$ and $z \in a \circ x$. Having $\mu_A(z) \geq \min\{\mu_A(a), \mu_A(x)\} = 1e^{2\pi i}$ implies that $\mu_A(z) = 1e^{2\pi i}$ and thus $z \in a \circ x \subseteq \overline{\mu}$. For all $a, x \in \overline{\mu}$, there exists $y \in H$ such that $x \in a \circ y$ and $\mu_A(y) \geq \min\{\mu_A(a), \mu_A(x)\} = 1e^{2\pi i}$. The latter implies that $\mu_A(y) = 1e^{2\pi i}$ and thus $y \in \overline{\mu}$. □

Proposition 3.22. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subhypergroup (H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Define the support, supp(μ), of μ as follows:*

$$supp(\mu) = \{x \in H : \mu_A(x) > 0e^{0i}\}.$$

Then supp(μ) is empty or subhypergroup (H_v-subgroup) of H.

Proof. Let $x, y \in supp(\mu) \neq \emptyset$. We want to show that $a \circ supp(\mu) = supp(\mu) = supp(\mu) \circ a$ for all $a \in supp(\mu)$. Let $x \in supp(\mu)$ and $z \in a \circ x$. Having $\mu_A(z) \geq \min\{\mu_A(a), \mu_A(x)\} > 0e^{0i}$ implies that $\mu_A(z) > 0e^{0i}$ and thus $z \in a \circ x \subseteq supp(\mu)$. For all $a, x \in supp(\mu)$, there exists $y \in H$ such that $x \in a \circ y$ and $\mu_A(y) \geq \min\{\mu_A(a), \mu_A(x)\} > 0e^{0i}$. The latter implies that $\mu_A(y) > 0e^{0i}$ and thus $y \in supp(\mu)$. □

Definition 3.23. Let $A = \{(x, \mu_A(x) = r_A(x)e^{iw_A(x)}) : x \in H\}$ be a homogeneous complex fuzzy subset of a non-void set H. We define the complement of the complex fuzzy subset A of H as follows:

$$A^c = \{(x, \mu_{A^c}(x) = (1 - r_A)(x)e^{i(2\pi - w_A(x))}) : x \in H\}.$$

Next, we present some examples where μ and μ^c are complex fuzzy subhypergroups (which in general is not always valid).

Example 3.24. We consider $(H, \circ)$ defined in Example 3.9 with the complex fuzzy subset μ of H as: $\mu(a) = 0.5e^{i0}$ and $\mu(b) = 1e^{i\frac{\pi}{2}}$. We get $\mu(a) = 0.5e^{i2\pi}$ and $\mu(b) = 0e^{i\frac{3\pi}{2}}$.Then μ and μ^c are homogeneous complex fuzzy subhypergroups of H.

Example 3.25. Let $(H, \circ)$ be any hypergroup (H_v-group) with the complex fuzzy subset μ of H as: $\mu(x) = re^{i\theta}$ where $r \in [0, 1], \theta \in [0, 2\pi]$ are fixed real numbers. Then μ and μ^c are homogeneous complex fuzzy subhypergroups of H.

Remark 3.26. Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subhypergroup (H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A^c is not necessary a complex fuzzy subhypergroup (or H_v-subgroup) of H.

We illustrate Remark 3.26 by the following example.

Example 3.27. Let $H = \{0, 1, 2\}$ and define the H_v-group $(H, +)$ by the following table:

+	0	1	2
0	0	{1, 2}	2
1	{1, 2}	2	0
2	2	0	1

And define a complex fuzzy subset μ of H as: $\mu(0) = 0.2e^{i\pi}$ and $\mu(1) = \mu(2) = 0.1e^{i\frac{\pi}{2}}$. Having

$$\mu_t = \begin{cases} H, & \text{if } t \leq 0.1e^{i\frac{\pi}{2}}; \\ \{0\}, & \text{if } 0.1e^{i\frac{\pi}{2}} < t \leq 0.2e^{i\pi}; \\ \emptyset, & \text{otherwise.} \end{cases}$$

either an empty set or a subhypergroup of H implies that μ is homogeneous complex fuzzy subhypergroup of H.

Since $0.8e^{i\pi} = \mu^c(0) = \mu^c(1 + 2) < \min\{\mu^c(1), \mu^c(2)\} = 0.9^{i\frac{3\pi}{2}}$, it follows that μ^c is not a complex fuzzy H_v-subgroup of H.

3.2. **Complex anti-fuzzy H_v-subgroups.**

Definition 3.28. Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex anti-fuzzy subhypergroup (or H_v-subgroup) of H if the following conditions hold:

(1) $\sup\{\mu_A(z) : z \in x \circ y\} \leq \max\{\mu_A(x), \mu_A(y)\}$ for all $x, y \in H$,
(2) For all $x, a \in H$, there exists $y \in H$ such that $x \in a \circ y$ and $\mu_A(y) \leq \max\{\mu_A(x), \mu_A(a)\}$,
(3) For all $x, a \in H$, there exists $z \in H$ such that $x \in z \circ a$ and $\mu_A(z) \leq \max\{\mu_A(x), \mu_A(a)\}$.

Next, we present some examples on complex anti-fuzzy H_v-subgroups.

Example 3.29. We consider $(H, \circ)$ defined in Example 3.9 with the complex fuzzy subset μ of H as: $\mu(a) = 0.5e^{i0}$ and $\mu(b) = 1e^{i\frac{\pi}{2}}$. We get $\mu(a) = 0.5e^{i2\pi}$ and $\mu(b) = 0e^{i\frac{3\pi}{2}}$. Then μ is a homogeneous complex anti-fuzzy subhypergroup of H.

Example 3.30. Let $(H, \circ)$ be any hypergroup (H_v-group) with the complex fuzzy subset μ of H as: $\mu(x) = re^{i\theta}$ where $r \in [0, 1], \theta \in [0, 2\pi]$ are fixed real numbers. Then μ is a homogeneous complex anti-fuzzy subhypergroup of H.

Proposition 3.31. *Let $(H, \circ)$ be a hypergroup (or H_v-group). A π-fuzzy set A_π is a π-anti-fuzzy subhypergroup (or H_v-subgroup) of H if and only if A is an anti-fuzzy subhypergroup (or H_v-subgroup) of H.*

Proof. The proof is straightforward. □

Theorem 3.32. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex anti-fuzzy subhypergroup (or H_v-subgroup) of H if and only if r_A is an anti- fuzzy subhypergroup (or H_v-subgroup) of H and w_A is a π-anti-fuzzy subhypergroup (or H_v-subgroup) of H.*

Proof. Suppose that A is a complex anti-fuzzy subhypergroup (or H_v-subgroup) of H. We need to prove that the conditions of Definition 2.7 are satisfied for r_A and w_A. For all $x, y \in H$, we have $\sup\{\mu_A(z) : z \in x \circ y\} \leq \max\{\mu_A(x), \mu_A(y)\}$. The latter and Notation 3.7 imply that $\sup\{r_A(z) : z \in x \circ y\} \leq \max\{r_A(x), r_A(y)\}$ and $\sup\{w_A(z) : z \in x \circ y\} \leq \max\{w_A(x), w_A(y)\}$. Let $a, x \in H$. Then there exist $y, z \in H$ such that $x \in a \circ y$, $x \in z \circ a$ and $\max\{\mu_A(a), \mu_A(x)\} \geq \mu_A(y)$, $\max\{\mu_A(a), \mu_A(x)\} \geq \mu_A(z)$. Notation 3.7 implies that the conditions 2 and 3 of Definition 2.7 are satisfied for both r_A and w_A.

Suppose that r_A is an anti-fuzzy subhypergroup (or H_v-subgroup) of H and w_A is a π-anti-fuzzy subhypergroup (or H_v-subgroup) of H. We need to prove that the conditions of Definition 3.28 are satisfied. For all $x, y \in H$, we have $\sup\{r_A(z) : z \in x \circ y\} \leq \max\{r_A(x), r_A(y)\}$ and $\sup\{w_A(z) : z \in x \circ y\} \leq \max\{w_A(x), w_A(y)\}$. The latter and Notation 3.7 imply that $\sup\{\mu_A(z) : z \in x \circ y\} \leq \max\{\mu_A(x), \mu_A(y)\}$. Let $a, x \in H$. Then there exist $y, z \in H$ such that $x \in a \circ y$, $x \in z \circ a$ and $\max\{r_A(a), r_A(x)\} \geq r_A(y)$, $\max\{r_A(a), r_A(x)\} \geq r_A(z)$, $\max\{w_A(a), w_A(x)\} \geq$

$w_A(y)$, $\max\{w_A(a), w_A(x)\} \geq w_A(z)$. Notation 3.7 implies that the conditions 2 and 3 of Definition 3.28 are satisfied for μ_A. □

Lemma 3.33. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and μ be a (homogeneous) complex anti-fuzzy subhypergroup (or H_v-subgroup) of H. Then*

$$\max\{\mu(x_1), \mu(x_2), \cdots, \mu(x_n)\} \geq \sup\{\mu(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}$$

for all $x_1, x_2, \cdots, x_n \in H$.

Proof. Let $x_1, x_2, \cdots, x_n \in H$ and $\mu(x) = r(x)e^{iw(x)}$. To prove the lemma, it suffices to show that

$$\max\{r(x_1), r(x_2), \cdots, r(x_n)\} \geq \sup\{r(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}$$

and

$$\max\{w(x_1), w(x_2), \cdots, w(x_n)\} \geq \sup\{w(a) : a \in x_1 \circ (x_2 \circ (\cdots) \circ x_n) \cdots)\}.$$

Since μ is homogeneous, it suffices to show that

$$\max\{r(x_1), r(x_2), \cdots, r(x_n)\} \geq \sup\{r(a) : a \in x_1 \circ (x_2 \circ (\cdots, x_n) \cdots)\}.$$

Theorem 3.32 asserts that r is an anti-fuzzy subhypergroup (or H_v-subgroup) of H. Lemma 2.8 completes the proof. □

Definition 3.34. Let $A = \{(x, \mu_A(x)) : x \in H\}$ be a (homogeneous) complex fuzzy subsets of a non-void set H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Define the lower subset, μ_t, of H as follows:

$$\overline{\mu}_t = \{x \in H : \mu_A(x) \leq t\},$$

where $t = se^{i\theta}$, $s \in [0, 1]$ and $\theta \in [0, 2\pi]$.

Remark 3.35. Let $A = \{(x, \mu_A(x)) : x \in H\}$ be a (homogeneous) complex fuzzy subsets of a non-void set H. Then the following are true:

(1) If $t_1 \leq t_2$ then $\overline{\mu}_{t_1} \subseteq \overline{\mu}_{t_2}$.
(2) $\overline{\mu}_{1e^{2\pi i}} = H$.

Theorem 3.36. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex anti-fuzzy subhypergroup (or H_v-subgroup) of H if and only if for all $t = se^{i\theta}$, $s \in [0, 1]$ and $\theta \in [0, 2\pi]$, $\overline{\mu}_t \neq \emptyset$ is a subhypergroup (or H_v-subgroup) of H.*

Proof. The proof is similar to that of Theorem 3.14. □

Corollary 3.37. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex anti-fuzzy subhypergroup (or H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. If $0e^{0i} \leq t_1 = s_1e^{i\theta_1} < t_2 = s_2e^{i\theta_2} \leq 1e^{2\pi i}$, then $\overline{\mu}_{t_1} = \overline{\mu}_{t_2}$ if and only if there is no $x \in H$ such that $t_1 \leq \mu_A(x) < t_2$.*

Proof. Let $0e^{0i} \leq t_1 = s_1 e^{i\theta_1} < t_2 = s_2 e^{i\theta_2} \leq 1e^{2\pi i}$ such that $\overline{\mu}_{t_1} = \overline{\mu}_{t_2}$. Suppose that there exists $x \in H$ such that $t_1 \leq \mu_A(x) < t_2$. Then $x \in \mu_{t_1} = \mu_{t_2}$. The latter implies that $\mu_A(x) \leq t_1$.

Since $0e^{0i} \leq t_1 = s_1 e^{i\theta_1} < t_2 = s_2 e^{i\theta_2} \leq 1e^{2\pi i}$, it follows by Remark 3.35 that $\overline{\mu}_{t_1} \subseteq \overline{\mu}_{t_2}$. To show that $\overline{\mu}_{t_2} \subseteq \overline{\mu}_{t_1}$, we let $x \in \overline{\mu}_{t_2}$. Then $\mu_A(x) \leq t_2$. Since there is no $x \in H$ such that $t_1 \leq \mu_A(x) < t_2$, it follows that $\mu_A(x) \leq t_1$. □

Corollary 3.38. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex anti-fuzzy subhypergroup (or H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. If the range of μ_A is the finite set $\{t_1, t_2, \cdots, t_n\}$ then the set $\{\overline{\mu}_{t_i} : i = 1, 2, \cdots, n\}$ contains all the lower level subhypergroups (or H_v-subgroups) of H. Moreover, if $t_1 \leq t_2 \leq \cdots \leq t_n$ then all the lower level subhypergroups (or H_v-subgroups) of H form the chain: $\overline{\mu}_{t_1} \subseteq \overline{\mu}_{t_2} \subseteq \cdots \subseteq \overline{\mu}_{t_n}$.*

Proof. Let $\overline{\mu}_s \neq \emptyset$ be a lower level subhypergroup (or H_v-subgroup) of H such that $\overline{\mu}_s \neq \overline{\mu}_{t_i}$ for all $1 \leq i \leq n$. Let t_k be closest complex number to s. We have two cases for s: $s < t_k$ and $s > t_k$. We consider only the first case, the second is done in a similar manner. Since the range of μ_A is the finite set $\{t_1, t_2, \cdots, t_n\}$, it follows that there is no $x \in H$ such that $s < \mu_A(x) < t_k$. Using Corollary 3.37, we get contradiction. □

Proposition 3.39. *Let $(H, \circ)$ be the biset hypergroup, i.e., $x \circ y = \{x, y\}$ for all $x, y \in H$ and let μ be any homogeneous complex fuzzy subset of H. Then μ is a complex anti-fuzzy subhypergroup of H.*

Proof. The proof is similar to that of Proposition 3.17. □

Proposition 3.40. *Let $(H, \circ)$ be the total hypergroup, i.e., $x \circ y = H$ for all $x, y \in H$ and let μ be any homogeneous complex fuzzy subset of H. Then μ is a complex anti-fuzzy subhypergroup of H if and only if μ is a constant complex function.*

Proof. The proof is similar to that of Proposition 3.18. □

Proposition 3.41. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H. Then A is a complex anti-fuzzy subhypergroup (H_v-subgroup) of H if and only if for every $t = se^{i\theta}$, $s \in [0, 1]$ and $\theta \in [0, 2\pi]$, the following conditions are satisfied:*

(1) $\overline{\mu}_t \circ \overline{\mu}_t \subseteq \overline{\mu}_t$;
(2) $a \circ (H - \overline{\mu}_t) - (H - \overline{\mu}_t) \subseteq a \circ \overline{\mu}_t$, *for all* $a \in \overline{\mu}_t$;
(3) $(H - \overline{\mu}_t) \circ a - (H - \overline{\mu}_t) \subseteq \overline{\mu}_t \circ a$, *for all* $a \in \overline{\mu}_t$.

Proof. Let A be a complex fuzzy subhypergroup (H_v-subgroup) of H. Then, by Theorem 3.36, $\overline{\mu}_t$ is a subhypergroup (H_v-subgroup) of H, i.e., $a \circ \overline{\mu}_t = \overline{\mu}_t$ for all $a \in \overline{\mu}_t$. Thus, we get that $\overline{\mu}_t \circ \overline{\mu}_t \subseteq \overline{\mu}_t$. we need to show that $a \circ (H - \overline{\mu}_t) - (H - \overline{\mu}_t) \subseteq a \circ \overline{\mu}_t$. Let $z \in a \circ (H - \overline{\mu}_t) - (H - \overline{\mu}_t)$. Then z is not an element in $(H - \overline{\mu}_t)$. This implies that $z \in \overline{\mu}_t = a \circ \overline{\mu}_t$. Condition 3 can be proved in a similar manner.

For the converse, suppose that the conditions 1 and 2 hold. Then, by Theorem 3.36, it suffices to show that $\overline{\mu}_t$ is a subhypergroup (H_v-subgroup) of H, i.e., $a \circ \overline{\mu}_t =$

$\overline{\mu}_t \circ a = \overline{\mu}_t$ for all $a \in \overline{\mu}_t$. Assume that there exists $x \in \overline{\mu}_t$ such that x is not an element in $a \circ \overline{\mu}_t$. The reproduction axiom of $(H, \circ)$ asserts that there exists $b \in H$ such that $x \in a \circ b$. We consider the following two cases for b:

- Case $b \in \overline{\mu}_t$. We get that $x \in a \circ b \subseteq a \circ \overline{\mu}_t$ which is a contradiction.
- Case b is not an element in $\overline{\mu}_t$. We get that $b \in H - \overline{\mu}_t$. And having $x \in a \circ b$ implies that $x \in a \circ (H - \overline{\mu}_t)$. Since $x \in \overline{\mu}_t$, it follows that x is not in $H - \overline{\mu}_t$. Thus, $x \in a \circ (H - \overline{\mu}_t) - (H - \overline{\mu}_t) \subseteq a \circ \overline{\mu}_t$ which is a contradiction.

We can prove that $\overline{\mu}_t \circ a = \overline{\mu}_t$, by applying condition 3, in a similar manner. □

Proposition 3.42. *Let $(H, \circ)$ be a hypergroup (or H_v-group). Then every subhypergroup (or H_v-subgroup) of H is a lower level H_v-subgroup of an anti-fuzzy subhypergroup (H_v-subgroup) of H.*

Proof. Let M be a subhypergroup (or H_v-subgroup) of H. For a fixed complex number $t_0 = se^{i\theta}$, $s \in [0, 1[, \theta \in [0, 2\pi[$, the fuzzy subset μ is defined as follows:

$$\mu(x) = \begin{cases} t_0, & \text{if } x \in M; \\ 1e^{2\pi i}, & \text{otherwise.} \end{cases}$$

We have $M = \overline{\mu}_{t_0}$ and $\overline{\mu}_t = \begin{cases} \emptyset, & \text{if } t < t_0; \\ M, & \text{if } t_0 \leq t < 1e^{2\pi i}; \\ H, & \text{if } t = 1e^{2\pi i}. \end{cases}$ is either the empty set a subhypergroup (or H_v-subgroup) of H. Then, by Theorem 3.36, we get that μ is a anti-fuzzy subhypergroup (H_v-subgroup) of H. □

Proposition 3.43. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex anti-fuzzy subhypergroup (H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Define $\overline{\overline{\mu}}$ as follows:*

$$\overline{\overline{\mu}} = \{x \in H : \mu_A(x) = 0e^{0i}\}.$$

Then $\overline{\overline{\mu}}$ is empty or subhypergroup (H_v-subgroup) of H.

Proof. Let $x, y \in \overline{\overline{\mu}} \neq \emptyset$. We want to show that $a \circ \overline{\overline{\mu}} = \overline{\overline{\mu}} = \overline{\overline{\mu}} \circ a$ for all $a \in \overline{\overline{\mu}}$. Let $x \in \overline{\overline{\mu}}$ and $z \in a \circ x$. Having $\mu_A(z) \leq \max\{\mu_A(a), \mu_A(x)\} = 0e^{0i}$ implies that $\mu_A(z) = 0e^{0i}$ and thus $z \in a \circ x \subseteq \overline{\overline{\mu}}$. For all $a, x \in \overline{\overline{\mu}}$, there exists $y \in H$ such that $x \in a \circ y$ and $\mu_A(y) \leq \max\{\mu_A(a), \mu_A(x)\} = 0e^{0i}$. The latter implies that $\mu_A(y) = 0e^{0i}$ and thus $y \in \overline{\overline{\mu}}$. □

Proposition 3.44. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subhypergroup (H_v-subgroup) of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Define the set $\overline{supp}$ as follows:*

$$\overline{supp} = \{x \in H : \mu_A(x) < 1e^{2\pi i}\}.$$

Then supp(μ) is empty or subhypergroup (H_v-subgroup) of H.

Proof. Let $x, y \in \overline{supp} \neq \emptyset$. We want to show that $a \circ \overline{supp} = \overline{supp} = \overline{supp} \circ a$ for all $a \in \overline{supp}$. Let $x \in \overline{supp}$ and $z \in a \circ x$. Having $\mu_A(z) \leq \max\{\mu_A(a), \mu_A(x)\} > 0e^{0i}$ implies that $\mu_A(z) < 1e^{2\pi i}$ and thus $z \in a \circ x \subseteq \overline{supp}$. For all $a, x \in \overline{supp}$, there exists $y \in H$ such that $x \in a \circ y$ and $\mu_A(y) \leq \max\{\mu_A(a), \mu_A(x)\} < 1e^{2\pi i}$. The latter implies that $\mu_A(y) < 1e^{2\pi i}$ and thus $y \in \overline{supp}$. □

Theorem 3.45. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex fuzzy subhypergroup (or H_v-subgroup) of H if and only if A^c is a complex anti-fuzzy subhypergroup (or H_v-subgroup) of H.*

Proof. The statement A is a complex fuzzy subhypergroup (or H_v-subgroup) of H is equivalent, by Theorem 3.10, to having r_A a fuzzy subhypergroup (or H_v-subgroup) of H and w_A a π-fuzzy subhypergroup (or H_v-subgroup) of H. The latter is equivalent, by Theorem 2.9, to having r_A^c an anti-fuzzy subhypergroup (or H_v-subgroup) of H and w_A^c a π-anti-fuzzy subhypergroup (or H_v-subgroup) of H. Theorem 3.32 completes the proof. □

Corollary 3.46. *Let $(H, \circ)$ be a hypergroup (or H_v-group) and A be a (homogeneous) complex fuzzy subset of H with membership function $\mu_A(x) = r_A(x)e^{iw_A(x)}$. Then A is a complex fuzzy and anti-fuzzy subhypergroup (or H_v-subgroup) of H if and only if A^c is a complex fuzzy and anti-fuzzy subhypergroup (or H_v-subgroup) of H.*

Proof. The proof results from Theorem 3.46. □

Example 3.47. Let $(H, \circ)$ be the biset hypergroup, i.e., $x \circ y = \{x, y\}$ for all $x, y \in H$ and let μ be any homogeneous complex fuzzy subset of H. Then, by Propositions 3.17 and 3.39, μ and μ^c are complex fuzzy and anti-fuzzy subhypergroup of H.

Example 3.48. Let $(H, \circ)$ be any hypergroup (H_v-group) with the complex fuzzy subset μ of H as: $\mu(x) = re^{i\theta}$ where $r \in [0, 1], \theta \in [0, 2\pi]$ are fixed real numbers. Then μ and μ^c are both: homogeneous complex fuzzy and anti-fuzzy subhypergroups of H.

4. Conclusion

This paper contributed to the study of fuzzy subhyperstructures by introducing the concepts of complex fuzzy (anti-fuzzy) subhyperstructures and investigating their properties.

For future work, we may define the generalized complex fuzzy subhyperstructures and investigate their properties.

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