EQUALITY PROPOSITIONAL LOGIC AND ITS EXTENSIONS

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Abstract. We introduce a new formal logic, called equality propositional logic. It has two basic connectives, $\wedge$ (conjunction) and $\equiv$ (equivalence). Moreover, the $\Rightarrow$ (implication) connective can be derived as $A \Rightarrow B := (A \wedge B) \equiv A$. We formulate the equality propositional logic and demonstrate that the resulting logic has reasonable properties such as Modus Ponens(MP) rule, Hypothetical Syllogism(HS) rule and completeness, etc. Especially, we provide two ways to prove the completeness of this logic system. We also introduce two extensions of equality propositional logic. The first one is involutive equality propositional logic, which is equality propositional logic with double negation. The second one adds prelinearity which is rich enough to enjoy the strong completeness property. Finally, we introduce additional connective Δ(delta) in equality propositional logic and demonstrate that the resulting logic holds soundness and completeness.

1. Introduction

It is well know that the development of mathematical logic can be taken as two basic directions. The first one directs that the basic connective is implication and the basic inference rule is Modus Ponens (from A and $A \Rightarrow B$ infer B). The second one directs that the basic connective is logical equivalence and the basic inference rules are Equanimity (from A and $A \equiv B$ infer B) and Leibniz (from $A \equiv B$ infer $C[p := A] \equiv C[p := B]$). The connective of equality (equivalence) seems to be more essential connective than implication (see [4]). In [14], the author constructs foundations of mathematics using equality in higher-order logic (type theory), and the formal proofs can be more effectively formed in an equational style. Various arguments in favor of such approach have been given in [13], namely that the equational style makes it possible to present calculations in a rigorous manner, without complexity and detail overwhelming. Hence, proofs in this style are relatively easy to construct.

Equality algebras were initially introduced by Jenei in [15], motivated from EQ-algebras [16]. It has two basic binary operations, meet operation $\wedge$ and an equality operation $\sim$, and one constant 1. It was proved that any equality algebra has a corresponding BCK-meet-semilattice and any BCK(D)-meet-semilattice (with distributivity property) has a corresponding equality algebra in [5, 6, 15]. In [19], the authors study the relations between equality algebras and other logical algebras

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such as residuated lattice, MTL-algebra, BL-algebra, MV-algebra, Hertz-algebra, Heyting-algebra, Boolean-algebra, EQ-algebra and hoop-algebra. Some related properties of equality algebras and EQ-algebras were mentioned in [2, 3, 10, 17, 18]. Moreover, EQ-logics are based on a special algebra (EQ-algebra) of truth values. It were introduced in [7, 8] and can be taken as a special class of fuzzy logics that differ from residuated fuzzy logics [11, 12]. Motivated by [7] we introduce a new formal logic, called equality propositional logic. It is based on equality algebra of truth values. Unlike classical logic, this paper attempt to develop equality propositional logics by equivalence instead of implication. And the inference rules are Equanimity and Lebniz while the rule of modus ponens is derived.

This paper is organized as follows. In Section 2, we review some basic definitions and properties that will be used in the reminder of the paper. In Section 3, we introduce equality propositional logic and show its main properties. In Section 4, we introduce involutive equality propositional logic, which is equality propositional logic with the law of double negation. In Section 5, we introduce prelinear equality propositional logic, which is equality propositional logic with the law of prelinearity, and prove a stronger completeness. In Section 6, we introduce delta equality propositional logic (equality propositional logic with delta connective). It mainly starts with extension of equality algebra by the delta operation.

It should be emphasized that formal proofs proceed on many places in equational style. This means that the proof consists of a sequence of equivalences, each of which being derived from the previous ones using inference rules. Thus, we may start with a formula to be proved and end up with an axiom. This is the reason why equational proofs can be more easily automated.

2. Preliminaries

In this section, we recall some definitions and results which will be used in the following sections and we shall not cite them every time while they are used.

Definition 2.1. [10, 16] An **EQ-algebra** is an algebra $\mathcal{A} = (A, \wedge, \odot, \sim, 1)$ of type (2,2,2,0) such that the following axioms are fulfilled: for all $a, b, c, d \in A$,

(a1) $(A, \wedge, 1)$ is a $\wedge$-semilattice with top element 1,
(a2) $(A, \odot, 1)$ is a monoid and $\odot$ is isotone in both arguments w.r.t $\leq$,
(a3) $a \sim a = 1$, *(reflexivity axiom)*
(a4) $((a \wedge b) \sim c) \odot (d \sim a) \leq c \sim (d \wedge b)$, *(substitution axiom)*
(a5) $(a \sim b) \odot (c \sim d) \leq (a \sim c) \sim (b \sim d)$, *(congruence axiom)*
(a6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$, *(monotonicity axiom)*
(a7) $a \odot b \leq a \sim b$. *(boundedness axiom)*

The operation $\wedge$ is called meet (infimum), $\sim$ is called fuzzy equality and $\odot$ is called multiplication. We write $a \leq b$ (and $b \geq a$) if and only if $a \wedge b = a$, for all $a, b \in A$, as usual. Clearly, $\leq$ is a partial order. For all $a, b \in A$, define the following two derived operations, $\to$ implication and $\leftrightarrow$ equivalence operation by

$$a \to b = a \sim (a \wedge b),\ a \leftrightarrow b = (a \to b) \wedge (b \to a).$$

There are other special types of EQ-algebras, see below.

Definition 2.2. [9, 10, 16] Let A be an EQ-algebra. We say that it is
(1) **separated**, if for all $a, b \in A$, $a \sim b = 1$ implies $a = b$,
(2) **good**, if for all $a \in A$, $a \sim 1 = a$,
(3) **lattice-ordered EQ-algebra**, if the underlying $\wedge$-semilattice is a lattice,
(4) **lattice EQ-algebra** (ℓEQ-algebra), if it is lattice-order in which the following substitution axiom holds: for all $a, b, c, d \in A$,

$$((a \vee b) \sim c) \odot (d \sim a) \leq ((d \vee b) \sim c),$$

(5) **residuated**, if for all $a, b, c \in A$, $(a \odot b) \wedge c = a \odot b$ if and only if $a \wedge ((b \wedge c) \sim b) = a$,
(6) **prelinear**, if for all $a, b \in A$, 1 is the unique upper bound of the set $\{(a \to b), (b \to a)\}$ in A.

It is well know that EQ-algebras open the door to an alternative development of fuzzy (many-valued) logic with the basic connective being a fuzzy equality instead of an implication. But the product operation in EQ-algebras is quite loosely related to the other operations, if it is replaced by another binary operation smaller or equal than the original product we still obtain an EQ-algebras, and this fact might make it difficult to obtain algebraic. For this reason, a new structure equality algebra was introduced, and its mainly aim is to find something similar to EQ-algebras but without a product. See the following definitions and propositions.

Definition 2.3. [15] An **equality algebra** is an algebra $\mathcal{E} = (E, \sim, \wedge, 1)$ of type (2,2,0) such that the following axioms are fulfilled, for all $a, b, c \in E$,
(e1) $(E, \wedge, 1)$ is a commutative idempotent integral monoid, i.e. $\wedge$-semilattice with the top element 1,
(e2) $a \sim b = b \sim a$,
(e3) $a \sim a = 1$,
(e4) $a \sim 1 = a$,
(e5) $a \leq b \leq c$ implies $a \sim c \leq b \sim c$ and $a \sim c \leq a \sim b$,
(e6) $a \sim b \leq (a \wedge c) \sim (b \wedge c)$,
(e7) $a \sim b \leq (a \sim c) \sim (b \sim c)$.

Owing to the definitions of operations in equality algebras are similar to EQ-algebras, so we will not repeat them in the sequel.

In what follows, we agree that $\sim$ and $\to$ have higher priority than $\wedge$.

Proposition 2.4. [15] *Let $\mathcal{E} = (E, \sim, \wedge, 1)$ be an equality algebra. Consider the following statements:*
($e5_a$) $a \sim (a \wedge b \wedge c) \leq a \sim (a \wedge b)$, for all $a, b, c \in E$.
($e5_b$) $a \sim (a \wedge b) \leq (a \wedge c) \sim (a \wedge b \wedge c)$, for all $a, b, c \in E$.
($e5_{a'}$) $a \to (b \wedge c) \leq a \to b$, for all $a, b, c \in E$.
($e5_{b'}$) $a \to b \leq (a \wedge c) \to b$, for all $a, b, c \in E$,.
We have that (e5) is equivalent to $\{(e5_a), (e5_b)\}$ which in turn is equivalent to $\{(e5_{a'}), (e5_{b'})\}$.

Proposition 2.5. [3, 15, 19] *Let $\mathcal{E} = (E, \sim, \wedge, 1)$ be an equality algebra. Then the followings hold: for all $a, b, c \in E$,*

(1) $a \sim b \leq a \leftrightarrow b \leq a \to b$,
(2) $a \leq (a \sim b) \sim b$,
(3) $a \sim b = 1$ *if and only if* $a = b$,
(4) $a \to b = 1$ *if and only if* $a \leq b$,
(5) $a \to b = 1$ *and* $b \to a = 1$ *imply* $a = b$,
(6) $1 \to a = a$, $a \to 1 = 1$, $a \to a = 1$,
(7) $a \leq b \to a$, $a \to (b \to a) = 1$,
(8) $a \leq (a \to b) \to b$, $a \to ((a \to b) \to b) = 1$,
(9) $a \to b \leq (b \to c) \to (a \to c)$, $(a \to b) \to ((b \to c) \to (a \to c)) = 1$,
(10) $a \leq b \to c$ *if and only if* $b \leq a \to c$,
(11) $a \to (b \to c) = b \to (a \to c)$,
(12) $a \leftrightarrow a = 1$, $a \leftrightarrow 1 = a$,
(13) $b \leq a$ *implies* $a \leftrightarrow b = a \to b = a \sim b$,
(14) $a \leq b$ *implies* $b \to c \leq a \to c$, $a \to c \leq c \to b$,
(15) $a \to b = a \to (a \wedge b)$,
(16) $a \sim b \leq (c \to a) \sim (c \to b)$,
(17) $a \sim b \leq (c \to a) \to (c \to b)$,
(18) $a \to b \leq (c \to a) \to (c \to b)$,
(19) $a \to b \leq (a \wedge c) \to (b \wedge c)$,
(20) $a \to b = ((a \to b) \to b) \to b$,
(21) $b \leq a \sim (a \wedge b)$, $a \sim b \leq a \sim (a \wedge b)$,
(22) $b \leq a$ *implies* $((a \sim b) \sim b) \sim b = a \sim b$,
(23) $a \leq (a \sim a \wedge b) \sim b$, $b \leq (a \sim a \wedge b) \sim b$.

Definition 2.6. [19] An equality algebra $\mathcal{E} = (E, \sim, \wedge, 1)$ is **bounded** if there exists an element $0 \in E$ such that $0 \leq a$, for all $a \in E$. In a bounded equality algebra $\mathcal{E}$, we define the negation" $'$ " on $\mathcal{E}$ by $a' = a \to 0 = a \sim 0$, for all $a \in E$. If $(a')' = a$, for all $a \in E$, then the bounded equality algebra E is called **involutive**.

Definition 2.7. [19] Let $\mathcal{E} = (E, \sim, \wedge, 1)$ be an equality algebra. Then,
(1) $\mathcal{E}$ is called **prelinear**, if 1 is the unique upper bound of the set $\{a \to b, b \to a\}$, for all $a, b \in E$.
(2) $\mathcal{E}$ is called **commutative**, if $(a \to b) \to b = (b \to a) \to a$, for all $a, b \in E$.

Definition 2.8. [19] A **lattice equality algebra** is an equality algebra which has lattice structure.

Proposition 2.9. [19] *Any lattice equality algebra* $\mathcal{E} = (E, \sim, \wedge, 1)$ *is prelinear if and only if the following identity holds, for all* $a, b, c \in E$,

$$(a \wedge b) \to c = (a \to c) \vee (b \to c).$$

3. Equality propositional logics

In this section, we will introduce the notion of equality propositional logics and investigate some of their properties.

Definition 3.1. (1) The language of equality propositional logic consists of propositional variables $p, q, ...,$ binary connectives $\wedge, \equiv$ and a truth (logical) constant $\top$.

(2) Each propositional variable is a formula, $\top$ is a formula and if A, B are formulas, then $A \wedge B$ (conjunction), $A \equiv B$(equality or equivalence) are formulas. $\Rightarrow$ (implication) and $\Leftrightarrow$ (bi-implication) are defined as the following shorts:

$$A \Rightarrow B := (A \wedge B) \equiv A,$$
$$A \Leftrightarrow B := (A \Rightarrow B) \wedge (B \Rightarrow A).$$

(3) The set of all formulas for the given language J is denoted by F_J.

Definition 3.2. The following formulas are axioms of the equality propositional logics: for all $A, B, C \in F_J$,

(E1) $(A \equiv \top) \equiv A$,
(E2) $(A \wedge B) \equiv (B \wedge A)$,
(E3) $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$,
(E4) $A \wedge A \equiv A$,
(E5) $A \wedge \top \equiv A$,
(E6) $(A \Rightarrow (B \wedge C)) \Rightarrow (A \Rightarrow B)$,
(E7) $(A \Rightarrow B) \Rightarrow ((A \wedge C) \Rightarrow B)$,
(E8) $(A \equiv B) \Rightarrow ((A \wedge C) \equiv (B \wedge C))$,
(E9) $(A \equiv B) \Rightarrow ((A \equiv C) \equiv (B \equiv C))$,
(E10) $(A \equiv B) \equiv (B \equiv A)$.

The inference rules of equality propositional logics are Equanimity rule(EA) and Leibniz rule(Leib):

$$\text{(EA)} \ \frac{A, A \equiv B}{B},$$
$$\text{(Leib)} \ \frac{A \equiv B}{C[p := A] \equiv C[p := B]},$$

where by $C[p := X]$ we denote a formula resulting from C by replacing all occurrences of the variable p in C by the formula X. In the following, $C[p := X]$ also called as a "C-part".

The formal proofs proceed mainly in an equational style, which are sequences of formulas of the form $A_1 \equiv A_2$,..., $A_{n-1} \equiv A_n$ such that each of the individual theorems $A_i \equiv A_{i+1}$ has an independent proof.

A formula A is called **provable** (notation: $\vdash A$) if it is the last member of a proof in equality propositional logics.

Remark 3.3. Let us comment on these axioms in equality propositional logics, for all $A, B, C \in F_J$. (E1) expresses $\vdash (A \equiv \top) \equiv A$ for each formula A and logical constant $\top$. (E2)-(E4) express commutativity, associativity and idempoteny of $\wedge$. (E5) expresses $\vdash A \wedge \top \equiv A$ for each formula A and constant $\top$. (E6)-(E7) say monotonicity axioms. (E8) says that $(A \wedge C) \equiv (B \wedge C)$ follows from $A \equiv B$. (E9) says that $(A \equiv C) \equiv (B \equiv C)$ follows from $A \equiv B$. (E10) expresses commutativity of $\equiv$.

Proposition 3.4. *In equality propositional logics, the following formulas hold: for all $A, B, C \in F_J$,*
(1) $\vdash A \equiv A$,
(2) $A \equiv B \vdash B \equiv A$,
(3) $A \equiv \top \vdash A$,
(4) $A \vdash A \equiv \top$,
(5) $A \equiv B, B \equiv C \vdash A \equiv C$,
(6) $A \wedge D \equiv C, A \equiv B \vdash B \wedge D \equiv C$,
(7) $A, A \Rightarrow B \vdash B$, *(MP)*
(8) $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow C$, *(HS)*
(9) $A \Rightarrow (B \equiv C), B \equiv D \vdash A \Rightarrow (D \equiv C)$,
(10) $A \Rightarrow (B \equiv C), C \equiv D \vdash A \Rightarrow (B \equiv D)$,
(11) $(A \equiv D) \equiv C, A \equiv B \vdash (B \equiv D) \equiv C$.

Proof. The proof of this proposition is similar to that given in paper [7] and so is omitted. □

Proposition 3.5. *In equality propositional logics, the following formulas hold: for all $A, B, C \in F_J$,*
(1) $\top \Rightarrow A \vdash (\top \Rightarrow A) \equiv A$,
(2) $A, B \vdash A \wedge B$,
(3) $A \Rightarrow (B \equiv C), B \vdash A \Rightarrow C$,
(4) $\vdash (A \equiv B) \Rightarrow (A \Rightarrow B)$,
(5) $\vdash ((A \wedge B) \wedge C) \Rightarrow (B \wedge C)$,
(6) $A \Rightarrow B, C \Rightarrow D \vdash A \wedge C \Rightarrow B \wedge D$,
(7) $A \equiv B, C \equiv D \vdash A \wedge C \equiv B \wedge D$,
(8) $\vdash B \Rightarrow (A \Rightarrow B)$.

Proof.
(1) $\top \wedge A \equiv \top$ (i.e.$\top \Rightarrow A$) (assumption)
$\Leftrightarrow \langle (Leib) + axiom(E2); \text{"}C - part\text{"} : p \equiv \top \rangle$
$A \wedge \top \equiv \top$
$\Leftrightarrow \langle (Leib) + axiom(E5); \text{"}C - part\text{"} : p \equiv \top \rangle$
$A \equiv \top$
$\Leftrightarrow \langle axiom(E1) \rangle$
A

(2) A (assumption)
$\Leftrightarrow \langle (EA) + axiom(E5) + Proposition\ 3.4(2) \rangle$
$A \wedge \top$
$\Leftrightarrow \langle (Leib) + assumption : B + Proposition 3.4(2) + Proposition\ 3.4(4); \text{"}C - part\text{"} : A \wedge p \rangle$
$A \wedge B$

(3) The proof is similar to Lemma 6(h) in paper [7].

(4) $(A \equiv B) \Rightarrow ((A \wedge A) \equiv (B \wedge A))$ (axiom($E8$))
$\Leftrightarrow \langle (Leib) + axiom(E4); \text{"}C - part\text{"} : (A \equiv B) \Rightarrow (p \equiv (B \wedge A)) \rangle$

$(A \equiv B) \Rightarrow (A \equiv (B \wedge A))$
$\Leftrightarrow \langle (Leib) + axiom(E10); \text{“}C - part\text{”} : (A \equiv B) \Rightarrow p \rangle$
$(A \equiv B) \Rightarrow ((A \wedge B) \equiv A)$
$\Leftrightarrow \langle the\ definition\ of\ implication \Rightarrow \rangle$
$(A \equiv B) \Rightarrow (A \Rightarrow B)$

(5) $A \wedge (B \wedge C) \equiv (B \wedge C) \wedge A$ (axiom($E2$))
$\Leftrightarrow \langle (Leib) + axiom(E4); \text{“}C - part\text{”} : A \wedge p \equiv (B \wedge C) \wedge A \rangle$
$A \wedge ((B \wedge C) \wedge (B \wedge C)) \equiv (B \wedge C) \wedge A$
$\Leftrightarrow \langle (Leib) + axiom(E3); \text{“}C - part\text{”} : p \equiv (B \wedge C) \wedge A \rangle$
$(A \wedge (B \wedge C)) \wedge (B \wedge C) \equiv (B \wedge C) \wedge A$
$\Leftrightarrow \langle (Leib) + axiom(E2); \text{“}C - part\text{”} : (A \wedge (B \wedge C)) \wedge (B \wedge C) \equiv p \rangle$
$(A \wedge (B \wedge C)) \wedge (B \wedge C) \equiv A \wedge (B \wedge C)$
$\Leftrightarrow \langle the\ definition\ of\ implication \Rightarrow \rangle$
$((A \wedge B) \wedge C) \Rightarrow (B \wedge C)$

(6) First,
$((A \wedge B) \wedge C) \Rightarrow (B \wedge C)$ (Proposition 3.5(5))
$\Leftrightarrow \langle (Leib) + assumption : (A \wedge B) \equiv A; \text{“}C - part\text{”} : (p \wedge C) \Rightarrow (B \wedge C) \rangle$
$(A \wedge C) \Rightarrow (B \wedge C)$
Then,
$((C \wedge B) \wedge D) \Rightarrow (B \wedge D)$ (Proposition 3.5(5))
$\Leftrightarrow \langle (Leib) + axiom(E2) + axiom(E3); \text{“}C - part\text{”} : p \Rightarrow (B \wedge D) \rangle$
$(B \wedge (C \wedge D)) \Rightarrow (B \wedge D)$
$\Leftrightarrow \langle (Leib) + assumption : (C \wedge D) \equiv C; \text{“}C - part\text{”} : (B \wedge p) \Rightarrow (B \wedge D) \rangle$
$(B \wedge C) \Rightarrow (B \wedge D)$
Finally,
we use these results and Proposition 3.4(8) to get $\vdash A \wedge C \Rightarrow B \wedge D$.

(7) First,
$(A \equiv B) \Rightarrow ((A \wedge C) \equiv (B \wedge C))$ (axiom($E8$))
$\Leftrightarrow \langle (EA) + assumption : A \equiv B \rangle$
$(A \wedge C) \equiv (B \wedge C)$
Then,
$(C \equiv D) \Rightarrow ((C \wedge B) \equiv (D \wedge B))$ (axiom($E8$))
$\Leftrightarrow \langle (EA) + assumption : C \equiv D \rangle$
$(C \wedge B) \equiv (D \wedge B)$
$\Leftrightarrow \langle (Leib) + axiom(E2); \text{“}C - part\text{”} : p \equiv (D \wedge B) \rangle$
$(B \wedge C) \equiv (D \wedge B)$
Finally,
By the above results and Proposition 3.4(5) we get $\vdash A \wedge C \equiv B \wedge D$.

(8) The proof is similar to Lemma 6(r) in paper [7]. □

Definition 3.6. Let E be an equality algebra. A **truth evaluation** $e : F_J \to E$ is defined as follows: if $p \in F_J$ is a propositional variable then $e(p) \in E$. Furthermore,
(1) $e(\top) = 1$,

(2) $e(A \wedge B) = e(A) \wedge e(B)$,
(3) $e(A \equiv B) = e(A) \sim e(B)$,
for all formulas $A, B \in F_J$.

A formula $A \in F_J$ is called **tautology** if $e(A) = 1$ for each truth evaluation $e : F_J \to E$. Thus a formula is a tautology, this means that the formula is absolutely true under any evaluation. Formulas A, B are called **semantically equivalent** if $e(A) = e(B)$ for each truth evaluation e.

Remark 3.7. Let $A, B \in F_J$. Note that A, B are semantically equivalent if and only if $A \Leftrightarrow B$ is a tautology. Also observe that if A and B are semantically equivalent then $A \equiv C$ and $B \equiv C$ are semantically equivalent, also $C \equiv A$ and $C \equiv B$, and similarly for other connective $\wedge$.

Proposition 3.8. *All axioms of the equality propositional logics are tautologies.*

Proof. Let $A, B, C \in F_J$. Suppose $e(A) = a$, $e(B) = b$, $e(C) = c$, for each truth evaluation $e : F_J \to E$. $(E1) - (E5), (E8) - (E10)$ are tautologies. We only need to verify $(E6)$ and $(E7)$. By Proposition 2.4 and Proposition 2.5(4), we have $e((A \Rightarrow (B \wedge C)) \Rightarrow (A \Rightarrow B)) = (e(A) \to (e(B) \wedge e(C))) \to (e(A) \to e(B)) = (a \to (b \wedge c)) \to (a \to b) = 1$, and $e((A \Rightarrow B) \Rightarrow ((A \wedge C) \Rightarrow B)) = (e(A) \to e(B)) \to ((e(A) \wedge e(C)) \to e(B)) = (a \to b) \to ((a \wedge c) \to b) = 1$. □

Proposition 3.9. *Let $e : F_J \to E$ be a truth evaluation, where E is a support of an equality algebra. The inference rules of the equality propositional logics are sound in the following sense:*
(1) if $e(A) = 1$ and $e(A \equiv B) = 1$ then $e(B) = 1$, for all $A, B \in F_J$,
(2) if $e(B \equiv C) = 1$ then $e(A[p := B] \equiv A[p := C]) = 1$, for all $A, B, C \in F_J$.

Proof. Suppose $e(A) = a$, $e(B) = b$, $e(C) = c$,

(1) If $e(A) = a = 1$ and $e(A \equiv B) = a \sim b = 1$, we have $1 \sim b = 1$. By Proposition 2.5(3), we get $b = 1$. Therefore, $e(B) = 1$.

(2) By induction the complexity of the formula A. If A is either $\top$ or q (other than p), then $e(A[p := B] \equiv A[p := C]) = e(A \equiv A) = e(A) \sim (A) = a \sim a = 1$. On the other hand, if A is p then $e(A[p := B] \equiv A[p := C]) = e(B \equiv C) = 1$.

For induction step, we choose an arbitrary nonatomic A and prove $e(A[p := B] \equiv A[p := C]) = 1$, that is $e(A[p := B]) \sim e(A[p := C]) = 1$. By Proposition 2.5(3), we conclude that $e(A[p := B]) = e(A[p := C])$. By induction hypothesis (IH), we get that the claim $e(D[p := B]) = e(D[p := C])$ is true for all formulas less complex than A.

Let A be $E \wedge F$ and the IH applies to E and F. Now, $A[p := B] \equiv A[p := C]$ implies $(E[p := B] \wedge F[p := B]) \equiv (E[p := C] \wedge F[p := C])$ and $e(E[p := B] \wedge F[p := B]) = e(E[p := B]) \wedge e(F[p := B]) = e(E[p := C]) \wedge e(F[p := C]) = e(E[p := C] \wedge F[p := C])$ (by IH), thus we get $e(A[p := B]) = e(A[p := C])$.

Let A be $E \equiv F$. The proof of this case is left as an exercise. □

Proposition 3.10. *(Soundness) Each provable formula in equality propositional logics is a tautology.*

Proof. The proof is straightforward. This is because all axioms are tautologies (Proposition 3.8) and Equanimity rule, Leibniz rule preserve tautologicity (Proposition 3.9) in equality propositional logics. □

The following is the Lindenbaum-Tarski technique.

Definition 3.11. Define $A \approx B$ by $\vdash A \equiv B$, for all $A, B \in F_J$.

It follows from Proposition 3.4(1), Proposition 3.4(2) and Proposition 3.4(5) that $\approx$ is an equivalence relation on F_J. Let us denote $[A]$ by an equivalence class of A and put $\overline{E} = \{[A] | A \in F_J\}$. Finally we define
(1) $1 = [\top]$,
(2) $[A] \wedge [B] = [A \wedge B]$,
(3) $[A] \sim [B] = [A \equiv B]$.

Proposition 3.12. *The algebra* $\overline{\mathcal{E}} = (\overline{E}, \sim, \wedge, 1)$ *is an equality algebra.*

Proof. Let $A, B \in F_J$. We note that $[A] \leq [B]$ iff $[A] \wedge [B] = [A]$ iff $\vdash (A \wedge B) \equiv A$ iff $\vdash A \Rightarrow B$ iff $\vdash (A \Rightarrow B) \equiv \top$ iff $[A] \rightarrow [B] = [\top]$. Particularly, we obtain $[A] \leq [\top]$ by axiom $(E5)$. By the axioms $(E2) - (E5)$, we can get (e1). Moreover, we see that axiom $(e2)$ follows from $(E10)$, $(e3)$ from Proposition 3.4(1), $(e4)$ from $(E1)$, $(e5)$ from $(E6)$ and $(E7)$, $(e6)$ from $(E8)$ and $(e7)$ from $(E9)$. Therefore, the algebra $\overline{\mathcal{E}} = (\overline{E}, \sim, \wedge, 1)$ is an equality algebra.

□

Proposition 3.13. *(Completeness) In equality propositional logics, the following statements are equivalent: for each formula* $A \in F_J$,
(1) $\vdash A$,
(2) $e(A) = 1$ *for each equality algebra* $\mathcal{E}$ *and for each truth evaluation* $e : F_J \rightarrow E$.

Proof. (1) $\Rightarrow$ (2): The proof is straightforward from Soundness.

(2) $\Rightarrow$ (1): By the Proposition 3.12, algebra $\overline{\mathcal{E}}$ of equivalence classes of formulas is an equality algebra. Thus, if (2) holds then it also holds for each $e : F_J \rightarrow \overline{E}$. If $e(A) = 1$, then it means that $[A] = [\top]$, i.e. $\vdash A \equiv \top$. Therefore, by Proposition 3.4(3) we get $\vdash A$. □

In the following, we only give the mainly property of basic EQ-logic, more details about EQ-logic see paper [7].

Proposition 3.14. [7] *(Completeness) In basic EQ-logic, the following statements are equivalent: for each formula* $A \in F_J$,
(1) $\vdash A$,
(2) $e(A) = 1$ *for each EQ-algebra* $\mathcal{E}$ *and for each truth evaluation* $e : F_J \rightarrow E$.

Because equality algebras are derived from EQ-algebras and are their special subclass, the same maybe hold also for the equality propositional logics. For this reason, we consider to construct a bridge from the EQ-logic to equality propositional logic, so that completeness of the latter turns out to be a straightforward consequence of the completeness of the former.

Proposition 3.15. *Each basic EQ-logic is an equality propositional logic.*

Proof. From the definitions of EQ-logic and equality propositional logic, we only need to prove $(E7)-(E10)$. For $(E7)$, this is a consequence of Lemma 5(b) in paper [7]. For $(E8)$, it follows from Theorem 1(d) in paper [7] that $a \sim b \leq (a \wedge c) \sim (b \wedge c)$, for all $a, b, c \in A$, where A is an EQ-algebra. Thus, $(a \sim b) \rightarrow (a \wedge c) \sim (b \wedge c)) = 1$. By Proposition 3.14, we have $\vdash (A \equiv B) \Rightarrow ((A \wedge C) \equiv (B \wedge C))$. For $(E9)$ and $(E10)$, those are similar to Lemma 6(i) and Lemma 6(l), respectively, in paper [7]. □

The above proposition illustrates the completeness of equality propositional logic on the other hand. Through the completeness of equality propositional logic, we can naturally get the following corollary.

Corollary 3.16. *In equality propositional logics, the following formulas hold: for all* $A, B, C \in F_J$,
(1) $\vdash (A \equiv B) \Rightarrow (A \Rightarrow B)$,
(2) $\vdash A \Rightarrow ((A \equiv B) \equiv B)$,
(3) $\vdash (A \Rightarrow B) \equiv (A \Rightarrow (A \wedge B))$,
(4) $\vdash (A \equiv B) \Rightarrow ((C \Rightarrow A) \equiv (C \Rightarrow B))$,
(5) $\vdash (A \equiv B) \Rightarrow ((C \Rightarrow A) \Rightarrow (C \Rightarrow B))$,
(6) $\vdash (A \Rightarrow B) \Rightarrow ((C \Rightarrow A) \Rightarrow (C \Rightarrow B))$,
(7) $\vdash (A \equiv B) \Rightarrow ((C \wedge A) \Rightarrow (C \wedge B))$,
(8) $\vdash (A \equiv B) \equiv (((A \Rightarrow B) \Rightarrow B) \Rightarrow B)$,
(9) $\vdash (A \equiv B) \Rightarrow (A \equiv (A \wedge B))$,
(10) $\vdash A \Rightarrow ((A \equiv (A \wedge B)) \equiv B)$,
(11) $\vdash B \Rightarrow ((A \equiv (A \wedge B)) \equiv B)$.

Proof. It is a straightforward consequence of Proposition 2.5 and Proposition 3.13. □

4. **Involutive equality propositional logics**

In this section, we introduce the notion of involutive equality propositional logics and we get some results in this logic systems.

We modify the language J of the equality propositional logic by replacing the logical constant $\top$ by $\bot$. Furthermore, we introduce the following shorts of formulas, for all $A, B \in F_J$,
(A) $\top := \bot \equiv \bot$,
(B) $\neg A := A \equiv \bot$,
(C) $A \vee B := \neg(\neg A \wedge \neg B)$.

Definition 4.1. Axioms of **involutive equality propositional logic** are those of the equality propositional logic plus the following ones: for all $A \in F_J$,
(E11) $(A \wedge \bot) \equiv \bot$ (we can also write it as $\bot \Rightarrow A$),
(E12) $\neg\neg A \equiv A$ (involution),
Axioms $(E11)$ and $(E12)$ characterize the basic properties of $\bot$.

Semantic of involutive equality propositional logic is formed by involutive equality algebra.

Similarly to the definition of tautology in Definition 3.6, a formula $A \in F_J$ is called **contradiction** if $e(A) = 0$ for each truth evaluation $e : F_J \to E$, where E is involutive equality algebra. In particular, we define $e(\bot) = 0$ for all truth evaluation $e : F_J \to E$.

Proposition 4.2. *All axioms of the involutive equality propositional logics are tautologies.*

Proof. Let $A \in F_J$. By Proposition 3.8, we only need to prove that $(E11)$ and $(E12)$ are tautologies. For axiom $(E11)$, we have $e((A \wedge \bot) \equiv \bot)$=$(a \wedge 0) \sim 0 = 0 \sim 0 = 1$. It is follows from Definition 2.6 that axiom $(E12)$ holds. □

Proposition 4.3. *Let $e : F_J \to E$ be a truth evaluation, where E is a support of an involutive equality algebra. The inference rules of the involutive equality propositional logic are sound in the following sense, for all $A, B, C \in F_J$,*
(1) if $e(A) = 1$ and $e(A \equiv B) = 1$ then $e(B) = 1$,
(2) if $e(B \equiv C) = 1$ then $e(A[p := B] \equiv A[p := C]) = 1$.

Proof. The proof is similar to Proposition 3.9. □

Proposition 4.4. *(Soundness) Each provable formula in involutive equality propositional logics is a tautology.*

Proof. This proof is a straightforward consequence from Proposition 4.2 and Proposition 4.3. □

Similarly to Proposition 3.12 and Proposition 3.13, we get the following propositions.

Proposition 4.5. *The algebra $\overline{\mathcal{E}} = (\overline{E}, \wedge, \sim, 1)$ is an involutive equality algebra.*

Proof. Since the algebra $\overline{\mathcal{E}} = \langle \overline{E}, \wedge, \sim, 1\rangle$ is an algebra of equivalence classes of formulas. It is clear that $\overline{\mathcal{E}}$ is involutive equality algebra.

□

Proposition 4.6. *(Completeness) In involutive equality propositional logics, the following statements are equivalent: for each formula $A \in F_J$:*
(1) $\vdash A$,
(2) $e(A) = 1$ for each involutive equality algebra $\mathcal{E}$ and for each truth evaluation $e : F_J \to E$.

Proof. The proof is similar to Proposition 3.13. □

5. **Prelinear equality propositional logics**

In this section, we introduce the concept of prelinear equality propositional logics.

The language of this logic is the same as that of equality propositional logic extended by a short of formula, for all $A, B \in F_J$,
(D) $A \vee B := ((A \Rightarrow B) \Rightarrow B) \wedge ((B \Rightarrow A) \Rightarrow A)$.

Definition 5.1. Axioms of **prelinear equality propositional logic** are those of the equality propositional logic plus the following one, for all $A, B, C \in F_J$,
(E13) $((A \wedge B) \Rightarrow C) \equiv ((A \Rightarrow C) \vee (B \Rightarrow C))$.

Semantic of prelinear equality propositional logic is formed by prelinear equality algebra.

Proposition 5.2. *All axioms of the prelinear equality propositional logics are tautologies.*

Proof. Let $A, B, C \in F_J$. By Proposition 3.8, we only need to prove $(E13)$ is a tautology. By Proposition 2.9, we have $e(((A \wedge B) \Rightarrow C) \equiv ((A \Rightarrow C) \vee (B \Rightarrow C)))$=$((a \wedge b) \to c) \sim ((a \to c) \vee (b \to c)) = 1$. □

Proposition 5.3. *Let $e : F_J \to E$ be a truth evaluation, where E is a support of a prelinear equality algebra. The inference rules of the prelinear equality propositional logic are sound in the following sense, for all $A, B, C \in F_J$,*
(1) if $e(A) = 1$ and $e(A \equiv B) = 1$ then $e(B) = 1$,
(2) if $e(B \equiv C) = 1$ then $e(A[p := B] \equiv A[p := C]) = 1$.

Proof. The proof is similar to Proposition 3.9. □

Proposition 5.4. *(Soundness) Each provable formula in prelinear equality propositional logics is a tautology.*

Proof. This is a straightforward consequence from Proposition 5.2 and Proposition 5.3. □

Proposition 5.5. *The algebra $\overline{\mathcal{E}} = (\overline{E}, \wedge, \sim, 1)$ is a prelinear equality algebra.*

Proof. The proof is similar to Proposition 3.12.

□

Proposition 5.6. *(Completeness) In prelinear equality propositional logics, the following statements are equivalent: for each formula $A \in F_J$,*
(1) $\vdash A$,
(2) $e(A) = 1$ for each prelinear equality algebra E and for each truth evaluation $e : F_J \to E$.

Proof. The proof is similar to Proposition 3.13. □

Definition 5.7. A **theory** T is a set of formulas, called special axioms of the theory. A **proof** in a theory T is a sequence $A_1, A_2, ..., A_n$ of formulas such that each A_i either is an axiom of equality propositional logic or is special axiom of T or follows from some preceding A_j, A_k by Equanimity rule or Leibniz rule. A is **provable** in T (notation $T \vdash A$) if it is the last member of a proof. An **evaluation** e is a **model of** T **if** $e(A) = 1$ **for each formula**$A \in T$ **(all special axioms are true for each truth evaluation** e**).**

In the following, we show that strong soundness and strong completeness hold in prelinear equality propositional logics.

Proposition 5.8. *(Strong soundness) Let $A \in F_J$. If $T \vdash A$ then A is true in each model of T (whenever e is a model of T then $e(A) = 1$).*

Proof. The proof of this proposition can be completed by the method analogous to that used above. □

Proposition 5.9. *(Strong completeness) In prelinear equality propositional logics, the following statements are equivalent: for every formula $A \in F_J$ and for every theory T,*

(1) $T \vdash A$,

(2) $e(A) = 1$ for each prelinear equality algebra E and for each each model e of T.

Proof. (1) $\Rightarrow$ (2): All axioms of T are true in all models of T.

(2) $\Rightarrow$ (1): If (2) holds, then it also holds for each model e of T, where $\overline{\mathcal{E}} = (\overline{E}, \wedge, \sim, 1)$ is a prelinear equality algebra (the Lindenbaum algebra). If $e(A) = 1$ then it means that $[A] = [\top]$, i.e. $T \vdash A \equiv \top$. Therefore, by Proposition 3.4(3) we get $T \vdash A$.

□

6. Delta equality algebras and delta equality propositional logics

As we all know, delta connective (Δ) was first introduced by Bazz in [1]. In 2015, M. Dyba and V. Novák apply delta connective to EQ$_\Delta$-logic and EQ$_\Delta$-algebra in [8]. In order to further study the role of delta connective in logic algebras, we extend equality algebra by delta connective. In particular, the resulting algebra is a semantic system of delta equality propositional logic (equality propositional logic with delta connective).

Now we will enrich equality algebras with unary operation Δ, which need to satisfy some conditions as in the following definition.

Definition 6.1. A **delta equality algebra** is an algebra $\mathcal{E}_\Delta = (E, \wedge, \sim, \Delta, 0, 1)$, which is an equality algebra $\mathcal{E}$ with bottom element 0 extended by a unary additional operation $\Delta : E \to E$ fulfilling the following axioms: for all $a, b \in E$,

(eΔ1) $\Delta 1 = 1$,

(eΔ2) $\Delta a \leq a$,

(eΔ3) $\Delta a \leq \Delta\Delta a$,

(eΔ4) $\Delta(a \sim b) \leq \Delta a \sim \Delta b$,

(eΔ5) $\Delta(a \wedge b) = \Delta a \wedge \Delta b$.

Axiom (eΔ1) expresses the fundamental property that 1 is preserved. Axioms (eΔ2), (eΔ3) characterize Δ as a certain closure operation. Axioms (eΔ4), (eΔ5) characterize distributivity of Δ over connectives.

In the following, we give an example of delta equality algebra.

Example 6.2. Let $E = \{0, a, b, c, d, 1\}$ with $0 < a, b < c < d < 1$. Define the operation $\sim$ as follows,

$\sim$	0	a	b	c	d	1
0	1	d	d	d	c	0
a	d	1	c	d	c	a
b	d	c	1	d	c	b
c	d	d	d	1	d	c
d	c	c	c	d	1	d
1	0	a	b	c	d	1

By the definition of $\to$ we obtain the implication as follows,

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	1	1
b	d	d	1	1	1	1
c	d	d	d	1	1	1
d	c	c	c	d	1	1
1	0	a	b	c	d	1

The Δ operation is defined by $\Delta 1 = 1$, and $\Delta x = 0$ otherwise.

It is easy to verify that $(E, \wedge, \sim, \Delta, 0, 1)$ is a delta equality algebra.

Now we give some properties of delta equality algebra.

Proposition 6.3. *Let $\mathcal{E}_\Delta$ be a delta equality algebra. Then the following properties hold: for all $a, b, c \in E$,*

(1) $\Delta(a) = \Delta\Delta(a)$,
(2) $a \leq b$ implies $\Delta(a) \leq \Delta(b)$,
(3) $\Delta(a \to b) \leq \Delta(a) \to \Delta(b)$,
(4) $\Delta(a \sim b) \leq \Delta(a) \to \Delta(b)$,
(5) If $\mathcal{E}_\Delta$ is a bounded delta equality algebra, then $\Delta(0) = 0$,
(6) $\Delta(a) = 1$ if and only if $a = 1$,
(7) $\Delta(a) \leq b$ if and only if $\Delta(a) \leq \Delta(b)$,
(8) $\Delta(E) = Fix_\Delta$, where $Fix_\Delta = \{a \in E | \Delta(a) = a\}$,
(9) Fix_Δ is closed under $\wedge$,
(10) If $b \leq a$, then $\Delta(a) \to \Delta(b) = \Delta(a) \sim \Delta(b)$,
(11) $\Delta(a \sim b) \leq (a \wedge c) \sim (b \wedge c)$,
(12) If $\Delta(E) = E$, then $\Delta = id_E$,
(13) $Ker(\Delta) = \{1\}$, where $Ker(\Delta) = \{a \in E | \Delta(a) = 1\}$,
(14) $\Delta(a) = a$ or $\Delta(a)$ and a are not comparable,
(15) If E is linearly order, then $\Delta = id_E$.

Proof. (1) This is a straightforward consequence of $(e\Delta 2)$ and $(e\Delta 3)$.

(2) The proof is similar to Lemma 3.4(b) in paper [8].

(3) The proof is similar to Lemma 3.4(c) in paper [8].

(4) By (2)(3) and Proposition 2.5(1), we have $\Delta(a \sim b) \leq \Delta(a \to b) \leq \Delta a \to \Delta b$.

(5) It follows from $(e\Delta 2)$ that $\Delta 0 \leq 0$. Hence $\Delta 0 = 0$.

(6) If $\Delta(a) = 1$ for some $a \in E$, then by $(e\Delta 2)$ we have $1 = \Delta(a) \leq a$. Thus $a = 1$. The converse follows by $(e\Delta 1)$.

(7) If $\Delta(a) \leq b$, we have $\Delta\Delta(a) \leq \Delta(b)$ by (2). We also get $\Delta\Delta(a) = \Delta(a)$ by (1). Thus $\Delta(a) \leq \Delta(b)$. Conversely, if $\Delta(a) \leq \Delta(b)$, then we have $\Delta(a) \leq \Delta(b) \leq b$.

(8) Let $b \in \Delta(E)$. So there exists $a \in E$ such that $b = \Delta(a)$. By (1), we have $\Delta(b) = \Delta\Delta(a) = \Delta(a) = b$. It follows that $b \in Fix_\Delta$. Conversely, if $b \in Fix_\Delta$, we have $b \in \Delta(E)$. Hence $\Delta(E) = Fix_\Delta$.

(9) By $(e\Delta5)$, we obtain that Fix_Δ is closed under $\wedge$.

(10) Since $b \leq a$, we have $\Delta(b) \leq \Delta(a)$ and $\Delta(a) \to \Delta(b) = \Delta(a) \sim (\Delta(a) \wedge \Delta(b)) = \Delta(a) \sim \Delta(b)$.

(11) This is a straightforward consequence of $(e6)$ and $(e\Delta2)$.

(12) For any $a \in E$, we have $a = \Delta(a_0)$ for some $a_0 \in E$. By (1), we have $\Delta(a) = \Delta(\Delta(a_0)) = \Delta(a_0) = a$. Hence, $\Delta = id_E$.

(13) Assume that $a \in E$ and $a \neq 1$, such that $\Delta(a) = 1$. By $(e\Delta2)$, we have $1 = \Delta(a) \leq a$, and hence $a = 1$, which is a contradiction. Therefore, $Ker(\Delta) = \{1\}$.

(14) Assume $a \in E$ such that $\Delta(a) \neq a$, and $\Delta(a)$ and a are comparable. Then $\Delta(a) < a$ or $a < \Delta(a)$. It follows from (1)(2), we have $\Delta(a) < \Delta(a)$, which is a contradiction.

(15) It is clear from (14). □

From the above Proposition 6.3, one can see that $\Delta(E)$ is closed under the operation $\wedge$. However, the following example shows that $\Delta(E)$ is not a subalgebra of $\mathcal{E}_\Delta$ since it, in general, is not closed under $\sim$.

Example 6.4. Let $E = \{0, a, b, c, 1\}$ with $0 < a < b, c < 1$. Consider the operation $\sim$ and $\to$ defined by the following tables:

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	c	b	a
b	0	c	1	0	b
c	0	b	0	1	c
1	0	a	b	1	b

$\to$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $\mathcal{E} = (E, \wedge, \sim, 0, 1)$ is an equality algebra. Now, we define $\Delta(0) = 0$, $\Delta(a) = a$, $\Delta(b) = a$, $\Delta(c) = c$, $\Delta(1) = 1$. One can easily check that $\mathcal{E}_\Delta = (E, \wedge, \sim, \Delta, 0, 1)$ is a delta equality algebra. However, $\Delta(E)$ is not a subalgebra of $\mathcal{E}_\Delta$ since $a \sim c = b \notin \Delta(E)$.

Proposition 6.5. *Let $\mathcal{E}_\Delta$ be a delta equality algebra. Then the followng conditions are equivalent:*

(1) $\Delta(a \to b) \leq (a \wedge c) \to (b \wedge c)$, for all $a, b, c \in E$,

(2) $\Delta(b) \leq c \to (b \wedge c)$, for all $b, c \in E$,

(3) $\Delta(b) \leq d \to (d \wedge (c \to (b \wedge c)))$, for all $b, c, d \in E$.

Proof. (1) $\Rightarrow$ (2): Taking $a = 1$, we obtain that (2) holds.

(2) $\Rightarrow$ (3): Assume (2) holds. By $(e\Delta3)$, we get $\Delta(b) \leq \Delta\Delta(b) \leq \Delta(c \to (b \wedge c))$. By (2) again, we obtain that $\Delta(b) \leq \Delta(d \to (d \wedge (c \to (b \wedge c)))) \leq d \to (d \wedge (c \to (b \wedge c)))$.

(3) $\Rightarrow$ (2): Taking $d = 1$, we obtain that (2) holds.

(2) $\Rightarrow$ (1): Assume (2) holds. We have $\Delta(a \to b) \leq (a \wedge c) \to ((a \to b) \wedge (a \wedge c))$. Thus, $\Delta(a \to b) \leq (a \wedge c) \to ((a \to b) \wedge a \wedge c)$. Furthermore, according to Proposition 2.5(14), we have $\Delta(a \to b) \leq (a \wedge c) \to ((a \to b) \wedge a \wedge c) \leq (a \wedge c) \to (b \wedge c)$. $\square$

The language of delta equality propositional logic is the language of the equality propositional logic extended by the unary connective Δ.

Definition 6.6. Axioms of **delta equality propositional logic** are those of the equality propositional logic plus the following ones, for all $A, B \in F_J$,
(EΔ1) $\Delta A \Rightarrow A$,
(EΔ2) $\Delta A \Rightarrow \Delta\Delta A$,
(EΔ3) $\Delta(A \equiv B) \Rightarrow \Delta A \equiv \Delta B$,
(EΔ4) $\Delta(A \wedge B) \equiv \Delta A \wedge \Delta B$.

The inference rules of delta equality propositional logics are Equanimity rule(EA) and Leibniz rule(Leib) and the Necessitation rule

$$\text{(N)}\ \frac{A}{\Delta A}.$$

It is similar to Definition 3.6, we also put $e(\Delta A) = \Delta e(A)$ for truth evaluation $e : F_J \to E$.

Proposition 6.7. *All axioms of delta equality propositional logics are tautologies.*

Proof. This is straightforward consequence by using the axioms and properties of delta equality algebra. $\square$

Proposition 6.8. *Let $e : F_J \to E$ be a truth evaluation, where E is a support of a delta equality algebra. The inference rules of delta equality propositional logic are sound in the following sense: for all $A, B, C \in F_J$,*
(1) if $e(A) = 1$ and $e(A \equiv B) = 1$ then $e(B) = 1$,
(2) if $e(B \equiv C) = 1$ then $e(A[p := B] \equiv A[p := C]) = 1$,
(3) if $e(A) = 1$ then $e(\Delta A) = 1$.

Proof. The proof is a straightforward consequence from Proposition 3.9 and $e(\Delta A) = \Delta e(A) = 1$. $\square$

Proposition 6.9. *(Soundness) Each provable formula in delta equality propositional logic is a tautology.*

Proof. The proof is a straightforward consequence from Proposition 6.7 and Proposition 6.8. $\square$

Proposition 6.10. *In delta equality propositional logics, the following formulas hold: for all $A, B \in F_J$,*
(1) $\vdash \Delta\top \equiv \top$,
(2) $\vdash \Delta(A \Rightarrow B) \Rightarrow (\Delta A \Rightarrow \Delta B)$.

Proof. The proof of this proposition is similar to Lemma 3.3(b) and Lemma 3.3(c) in paper [8].

$\square$

It is similar to Definition 3.11, we also put $\Delta[A] = [\Delta A]$.

Proposition 6.11. *The algebra* $\overline{\mathcal{E}}_\Delta = (\overline{E}, \wedge, \sim, \Delta, 1)$ *is a delta equality algebra.*

Proof. We only need to verify all axioms of delta equality algebra. For axioms $(e1)-(e7)$, see the proof of Proposition 3.12. For axioms $(e\Delta1)$-$(e\Delta5)$, see Proposition 6.10(1) and $(e\Delta1)$-$(e\Delta4)$.

□

Proposition 6.12. *(Completeness) In delta equality propositional logics, the following statements are equivalent: for each formula* $A \in F_J$*,*

(1) $\vdash A$*,*

(2) $e(A) = 1$ *for delta equality algebra* $\mathcal{E}_\Delta$ *and a truth evaluation* $e : F_J \to E$*.*

Proof. (1) $\Rightarrow$ (2): The proof is straightforward from Soundness.

(2) $\Rightarrow$ (1): By Proposition 6.11 the algebra $\bar{\mathcal{E}}_\Delta$ of equivalence classes of formulas is a delta equality algebra. Thus, if (2) holds then it also holds for $e : F_J \to \overline{E}$. If $e(A) = 1$ then it means that $[A] = [\top]$, i.e. $\vdash A \equiv \top$. Therefore, by Proposition 3.4(3), we get $\vdash A$. □

Corollary 6.13. *In delta equality propositional logics, the following formulas hold: for all* $A, B, C \in F_J$*,*

(1) $\vdash (\Delta A \equiv B) \Rightarrow (\Delta A \Rightarrow \Delta B)$*,*

(2) $\Delta A \Rightarrow B \vdash \Delta A \Rightarrow \Delta B$*,*

(3) $\Delta A \Rightarrow \Delta B \vdash \Delta A \Rightarrow B$*,*

(4) $B \Rightarrow A \vdash ((\Delta A \Rightarrow \Delta B) \equiv (\Delta A \equiv \Delta B))$*,*

(5) $\vdash \Delta(A \equiv B) \Rightarrow ((C \wedge A) \equiv (C \wedge B))$*.*

Proof. It is straightforward consequence of Proposition 6.3 and Proposition 6.12.

□

7. Conclusions

In this paper, we introduce equality propositional logics based on equality algebra of truth values. Its basic connectives are conjunction and fuzzy equality/ equivalence instead of implication. So, the natural style of formal proofs in equality propositional logics is equational. Moreover, the inference rules are Equanimity and Lebniz while the Modus Ponens rule is derived. We also introduce several kinds of extensions of equality propositional logic such as involutive, prelinear. Finally, we introduce a special class of equality propositional logics that is extended by the additional unary connective of Δ. Note that this connective is very useful in fuzzy logics because it possible to distinguish crisp formulas from the other ones.

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