

Convex structures via convex L -subgroups of an L -ordered group

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Abstract

In this paper, we first characterize the convex L -subgroup of an L -ordered group by means of four kinds of cut sets of an L -subset. Then we consider the homomorphic preimages and the product of convex L -subgroups. After that, we introduce an L -convex structure constructed by convex L -subgroups. Furthermore, the notion of the degree to which an L -subset of an L -ordered group is a convex L -subgroup is proposed and characterized. An L -fuzzy convex structure results from convex L -subgroup degree is imported naturally, and its L -fuzzy convexity preserving mappings is investigated.

Keywords: L -ordered group, convex L -subgroup, L -convex structure, convex L -subgroup degree, L -fuzzy convex structure.

1 Introduction

The initial concept of convex structures is mainly defined and studied in \mathbb{R}^n in the pioneering works of Newton, Minkowski and others [4, 8], and it has been accepted to be of increasing importance in the study of extremum problems in areas of applied mathematics. Actually, convex structures exists in so many mathematical research areas, such as vector spaces, metric spaces, graphs, matroids, median algebras, lattices and so on.

By axiomatizing the properties of convex sets in \mathbb{R}^n , the notion of convex structures (also called abstract convexity) on a non-empty set X is introduced in [26], where a convex structure on X is defined to be a subfamily of 2^X which contains both the empty set \emptyset and X itself and which is closed under arbitrary intersections and nested unions. With the development of fuzzy set theory which was introduced by Zadeh [39], the notion of convex structures has been extended to fuzzy setting. In 1994, Rosa [18, 19] proposed the concept of a fuzzy convex structure as a subfamily of $[0, 1]^X$. In 2009, Y. Maruyama developed the definition of a fuzzy convex structure by generalizing the lattice $[0, 1]$ to a completely distributive lattice L in [15], which is called an L -convex structure. For this kind of fuzzy convex structure, there are some other studies [12, 16, 17, 20] in the last two years. In 2014, Shi and Xiu [23] introduced a new approach to the fuzzification of convex structures, which is called an M -fuzzifying convex structure. There is also some other research work on M -fuzzifying convexity, such as [24, 28, 29, 30, 32, 31]. Recently, Shi and Xiu [25] present a more general approach to the the fuzzification of convex structures, which is called an (L, M) -fuzzy convex structure. It is also a generalization of an L -convex structure and an M -fuzzifying convex structure.

Ordered convexity is first studied on partially ordered sets [26], it can also be imported to semilattices and lattices. Based on that, Zhong and Shi lately introduced a kind of L -convex structure on partially ordered sets [43]. Almost at the same time, Li and Shi [14] presented an L -fuzzy convex structure on a lattice by studying the L -convex fuzzy sublattice degrees. Follows their idea, L -fuzzy convex structures can also be constructed in many other mathematical structures such as vector spaces, metric spaces, and order groups. With the development of fuzzy mathematics and fuzzy logic, fuzzy order has attracted more and more attentions. There are so many literatures focus on it [10, 13, 33, 34, 35, 36, 37, 38, 40, 41, 42]. In this paper, we focus on the mathematical structure of L -ordered group, which is a group with a fuzzy order. In an L -ordered group, convex L -subgroup is an important and basic concept which is

introduced in literature [5]. Some characterizations of convex L -subgroups are given in this literature and it is shown that quotient L -ordered groups can be constructed by means of convex L -subgroups. In literature [7], it is discussed that how to construct a convex L -subgroup by an L -subset in an L -ordered group. In the present paper, we first characterize convex L -subgroups by means of four kinds of cut sets of L -subsets and L -ordered group homomorphisms, and we also investigate the product of convex L -subgroups. After that, an L -convex structure constructed by convex L -subgroups is introduced and its L -convexity preserving mappings are discussed. What's more, the degree to which an L -subset of an L -ordered group is a convex L -subgroup is considered and the convexity degree of the product of finite L -subsets are analyzed. Then an L -fuzzy convex structure is proposed naturally and its L -fuzzy convexity preserving mappings are discussed.

The paper is organized as follows. In Section 2, we recall some necessary definitions and results which are needed later on. In Section 3, we characterize convex L -subgroups of an L -ordered group by means of four kinds of cut sets of L -subsets and L -ordered group homomorphisms, then we introduce an L -convex structure constructed by convex L -subgroups and study its L -convexity preserving mappings. In Section 4, the notion of the degree to which an L -subset of an L -ordered group is a convex L -subgroup is proposed and its characterizations are given. In Section 5, An L -fuzzy convex structure results from the convex L -subgroup degree is introduced naturally, and L -fuzzy convexity preserving mappings are analyzed. In the final section, we summarize the results and draw a conclusion.

2 Preliminaries

Throughout this paper, unless otherwise stated, L always denotes a complete Heyting algebra or a frame. In other words, L is a complete lattice satisfying the infinite distributive law of finite meets over arbitrary joins i.e., $a \wedge (\bigvee_{b \in B} b) = \bigvee_{b \in B} (a \wedge b)$ for any $a \in L$ and $B \subseteq L$. The smallest element and the largest element in L are denoted by 0 and 1, respectively. An element $a \in L$ is called a co-prime element if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$, and the set of non-zero co-prime elements in L is denoted by $J(L)$. An element $a \in L$ is called a prime element if $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$, and the set of non-unit prime elements in L is denoted by $P(L)$. For all $a, b \in L$, we say that a is wedge below b , in symbols $a \prec b$, if for every subset $D \subseteq L$ with $b \leq \bigvee D$, there exists $d \in D$ such that $a \leq d$. Similarly, we define $b \prec^{op} a$ if and only if for every subset $D \subseteq L$ with $\bigwedge D \leq b$, there exists $d \in D$ such that $d \leq a$.

Let $\beta(b) = \{a \in L \mid a \prec b\}$, $\beta^*(b) = \beta(b) \cap J(L)$; $\alpha(b) = \{a \in L \mid b \prec^{op} a\}$, $\alpha^*(b) = \alpha(b) \cap P(L)$.

In particular, L is a completely distributive lattice [11, 9] if and only if $b = \bigvee \beta(b)$ for each $b \in L$. In this case, $b = \bigvee \beta(b) = \bigvee \beta^*(b) = \bigwedge \alpha(b) = \bigwedge \alpha^*(b)$, and $\beta(\bigvee_{i \in I} b_i) = \bigcup_{i \in I} \beta(b_i)$, $\alpha(\bigwedge_{i \in I} b_i) = \bigcup_{i \in I} \alpha(b_i)$ (refer to [27]).

Lemma 2.1. *Let L be a completely distributive lattice. Then the following statements are equivalent:*

- (1) $x \leq y$;
- (2) for each $a \in J(L)$, $a \leq x \implies a \leq y$;
- (3) for each $a \in P(L)$, $x \not\leq a \implies y \not\leq a$;
- (4) for each $a \in P(L)$, $a \notin \alpha(x) \implies a \notin \alpha(y)$;
- (5) for each $a \in J(L)$, $a \in \beta(x) \implies a \in \beta(y)$.

Proof. The proof is similar to that of Lemma 2.1 in [14]. □

For any non-empty set X , let 2^X be the family of all subsets of X , and L^X the set of all L -subsets of X . Then L^X is also a complete lattice by defining \leq on L^X pointwisely. We always do not discriminate an element $a \in L$ with the constant function $\bar{a} : X \rightarrow L$ such that $\bar{a}(x) = a$ for all $x \in X$.

For each $A \in L^X$ and $a \in L$, we can define

$$A_{[a]} = \{x \in X \mid A(x) \geq a\},$$

$$A^{(a)} = \{x \in X \mid A(x) \not\leq a\}.$$

If L is a completely distributive lattice, we can define

$$A^{[a]} = \{x \in X \mid a \notin \alpha(A(x))\},$$

$$A_{(a)} = \{x \in X \mid a \in \beta(A(x))\}.$$

Definition 2.2. [21] *Let $A \in L^X$ and $B \in L^X$. We define the product of A and B , in symbols $A \times B$, to be an L -subset of $X \times Y$:*

$$(A \times B)(x, y) = A(x) \wedge B(y) \quad (\forall (x, y) \in X \times Y).$$

Proposition 2.3. [43] *Let L be a complete Heyting algebra. If $\{A_i \mid i \in I\} \subseteq L^X$ is non-empty and totally ordered, then $(\bigvee_{i \in I} A_i(x)) \wedge (\bigvee_{j \in I} A_j(y)) = \bigvee_{i \in I} (A_i(x) \wedge A_i(y))$ for all $x, y \in X$.*

For each map $f : X \rightarrow Y$, there is a well-known L -forward powerset operator $f_L^\rightarrow : L^X \rightarrow L^Y$ defined by $f_L^\rightarrow(A)(y) = \bigvee_{f(x)=y} A(x)$ for all $y \in Y, A \in L^X$, and an L -backward powerset operator $f_L^\leftarrow : L^Y \rightarrow L^X$ defined by $f_L^\leftarrow(B) = B \circ f$ for each $B \in L^Y$.

Next, we recall the definitions of L -convexity and (L, M) -fuzzy convexity, respectively.

Definition 2.4. [15] A subset $\mathcal{C} \subseteq L^X$ is called an L -convex structure, if it satisfies the following conditions:

(LC1) $\bar{0}, \bar{1} \in \mathcal{C}$;

(LC2) $\bigwedge_{i \in I} A_i \in \mathcal{C}$ for all non-empty family $\{A_i \mid i \in I\} \subseteq \mathcal{C}$;

(LC3) $\bigvee_{i \in I} A_i \in \mathcal{C}$ for all family $\{A_i \mid i \in I\} \subseteq \mathcal{C}$ which is non-empty and totally ordered.

The pair (X, \mathcal{C}) is called an L -convex space. The members of \mathcal{C} are called L -convex sets. For each $A \in L^X$, $co(A) = \bigwedge \{C \in L^X \mid A \leq C \in \mathcal{C}\}$ is called L -convex hull of A , which is the smallest L -convex set including A . It is easily seen that $A \in \mathcal{C} \iff co(A) = A$.

Definition 2.5. [15] Let (X, \mathcal{C}_X) and (Y, \mathcal{C}_Y) be L -convex spaces. A mapping $f : X \rightarrow Y$ is said to be L -convexity preserving if $f_L^\leftarrow(D) \in \mathcal{C}_X$ for each $D \in \mathcal{C}_Y$. f is said to be L -convex-to-convex if $f_L^\rightarrow(C) \in \mathcal{C}_Y$ for each $C \in \mathcal{C}_X$.

Definition 2.6. [25] A mapping $\mathcal{C} : L^X \rightarrow M$ is called an (L, M) -fuzzy convex structure on X if it satisfies:

(LMC1) $\mathcal{C}(\bar{0}) = \mathcal{C}(\bar{1}) = 1$;

(LMC2) $\mathcal{C}(\bigwedge_{i \in I} A_i) \geq \bigwedge_{i \in I} \mathcal{C}(A_i)$ for all non-empty family $\{A_i \mid i \in I\} \subseteq \mathcal{C}$;

(LMC3) $\mathcal{C}(\bigvee_{i \in I} A_i) \geq \bigwedge_{i \in I} \mathcal{C}(A_i)$ for all family $\{A_i \mid i \in I\} \subseteq \mathcal{C}$ which is non-empty and totally ordered.

The pair (X, \mathcal{C}) is called an (L, M) -fuzzy convex space.

Definition 2.7. [25] Let (X, \mathcal{C}_X) and (Y, \mathcal{C}_Y) be (L, M) -fuzzy convex spaces. A mapping $f : X \rightarrow Y$ is said to be (L, M) -fuzzy convexity-preserving if $\mathcal{C}_X(f_L^\leftarrow(D)) \geq \mathcal{C}_Y(D)$ for all $D \in L^Y$. f is said to be (L, M) -fuzzy convex-to-convex if $\mathcal{C}_X(D) \leq \mathcal{C}_Y(f_L^\rightarrow(D))$ for all $D \in L^X$.

In particular, if $M = 2$, then an $(L, 2)$ -fuzzy convex structure is the L -convex structure in Definition 2.4. If $L = 2$, then an $(2, M)$ -fuzzy convex structure is the M -fuzzifying convex structure in [23]. If $M = L = 2$, then an $(2, 2)$ -fuzzy convex structure is exactly the classical convex structure in [26].

In the sequel of this section, let us recall some concepts and results about fuzzy partial order which are needed later on.

Definition 2.8. [10, 40, 37] A fuzzy preorder e (also called an L -preorder) on X is an L -relation satisfying:

(1) $\forall x \in X, e(x, x) = 1$;

(2) $\forall x, y, z \in X, e(x, y) \wedge e(y, z) \leq e(x, z)$.

A fuzzy preorder e is called a fuzzy partial order (also called an L -order) if it satisfies

(3) $\forall x, y \in X, e(x, y) = e(y, x) = 1 \implies x = y$.

Then (X, e) is called a fuzzy partially ordered set or an L -poset for simplicity.

Remark 2.9. The original definition of a fuzzy partial order proposed by Fan and Zhang in [10, 40] is based on complete Heyting algebras, and it is generalized onto complete residuated lattices in [37]. In order to study fuzzy relational systems, Bělohlávek also defines and studies an \mathbf{L} -order on a set [2, 3], and in [33, 34], Yao verifies that a fuzzy partial order in the sense of Fan and Zhang is equivalent to an \mathbf{L} -order in the sense of Bělohlávek.

Let (X, e) be an L -poset. Then it is easy to check that $(X, e_{[a]})$ and $(X, e^{(a)})$ are preordered sets for each $a \in L$. In particular, if L is a completely distributive lattice. Then $(X, e^{[a]})$ is also an preordered set for each $a \in L$. In addition, if L satisfies $\beta(b \wedge c) = \beta(b) \cap \beta(c)$ for all $b, c \in L$. Then $(X, e_{(a)})$ is also a preordered set for each $a \in L$.

Proposition 2.10. [34, 13] Let $\{(X_i, e_i) \mid i \in I\}$ be a non-empty family of L -posets. Put $X = \prod_{i \in I} X_i$ and define $e = \prod_{i \in I} e_i : X \times X \rightarrow L$ by $\forall x = (x_i)_{i \in I}, y = (y_i)_{i \in I} \in X$:

$$e(x, y) = \bigwedge_{i \in I} e_i(x_i, y_i).$$

Then (X, e) is an L -poset, called the product L -poset of $\{(X_i, e_i) \mid i \in I\}$.

Definition 2.11. [40, 13] Let $(X, e_X), (Y, e_Y)$ be L -posets. A mapping $f : X \rightarrow Y$ is said to be L -order-preserving (resp., L -order-reversing) if $e_X(x, y) \leq e_Y(f(x), f(y))$ (resp., $e_X(x, y) \leq e_Y(f(y), f(x))$) for all $x, y \in X$.

Example 2.12. By the definition of product of L -posets, it is easy to see that each projection $p_i : \prod_{i \in I} X_i \rightarrow X_i$ is L -order-preserving.

3 Convex L -subgroups and the L -convex structure constructed by them

In this section, we first give characterizations of a convex L -subgroup in terms of four kinds of cut sets of an L -subset. Then we consider the homomorphic preimage and the product of convex L -subgroups. After that, we introduce an L -convex structure induced by convex L -subgroups in an L -ordered group.

Definition 3.1. [1] An ordered (A preordered) group (G, \leq) is a group $(G, \cdot, 1)$ together with an order (a preordered) \leq on G such that $x \leq y \implies axb \leq ayb$ for all $x, y, a, b \in G$.

Definition 3.2. [5, 7] An L -(pre)ordered group (G, e) is a group $(G, \cdot, 1)$ together with an L -(pre)order e on G such that $e(x, y) \leq e(axb, ayb)$ for all $x, y, a, b \in G$.

Proposition 3.3. Let (G, e) be an L -(pre)ordered group and $a \in L$. Then

- (1) $(G, e_{[a]})$ is a preordered group,
- (2) $(G, e^{(a)})$ is a preordered group,
- (3) if L is a completely distributive lattice, then $(G, e^{[a]})$ is a preordered group,
- (4) if L is a completely distributive lattice with $\beta(b \wedge c) = \beta(b) \cap \beta(c)$ for all $b, c \in L$, then $(G, e_{(a)})$ is a preordered group.

Proof. The proofs are trivial and omitted here. □

Proposition 3.4. [6] If (G_1, e_1) and (G_2, e_2) are L -ordered groups, then $(G_1 \times G_2, e_1 \times e_2)$ is an L -ordered group.

Definition 3.5. [5, 6, 7] Let (G, e) be an L -ordered group and $A \in L^G$. Then A is said to be an L -subgroup if $A(x) \wedge A(y) \leq A(xy^{-1})$ for all $x, y \in G$.

Proposition 3.6. [6] Let (G, e) be an L -ordered group, and $\{A_i\}_{i \in I}$ be a family of L -subgroups. Then $\bigwedge_{i \in I} A_i$ is an L -subgroup of (G, e) .

Definition 3.7. [26] Let (G, \leq) be an ordered group (or a preordered group) and $A \subseteq G$. If $x \in A, y \in A \implies z \in A$ for all $x \leq z \leq y$, then A is said to be convex. If A is not only convex but also a subgroup of $(G, \cdot, 1)$, then A is called a convex subgroup of (G, \leq) .

Definition 3.8. [5, 7] Let (G, e) be an L -(pre)ordered group and $A \in L^G$. Then A is said to be convex or an L -ordered convex set if $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ for all $x, y, z \in G$, and A is called a convex L -subgroup if A is not only an L -ordered convex set but also an L -subgroup.

Now, we are going to give some equivalent descriptions for convex L -subgroups of an L -ordered group.

Proposition 3.9. Let (G, e) be an L -ordered group. If $A \in L^G$ is a convex L -subgroup, then

- (1) $\forall a \in L, A_{[a]}$ is a convex subgroup of $(G, e_{[a]})$,
- (2) $\forall a \in P(L), A^{(a)}$ is a convex subgroup of $(G, e^{(a)})$.

Proof. (1). We all known that $A_{[a]}$ is a subgroup of $(G, \cdot, 1)$ by Theorem 3.7 in [21], so it just needs to prove $A_{[a]}$ is convex in $(G, e_{[a]})$. For each $a \in L$, suppose $x, y \in A_{[a]}$ and $(x, z), (z, y) \in e_{[a]}$. Then $A(x) \geq a, A(y) \geq a$ and $e(x, z) \geq a, e(z, y) \geq a$. Since A is an L -ordered convex set, that is, $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$. Therefore $A(z) \geq a$, in other words, $z \in A_{[a]}$. Hence $A_{[a]}$ is convex in $(G, e_{[a]})$ by definition.

(2). According to Theorem 3.7 in [21], to prove (2) it suffices to prove $A^{(a)}$ is convex in $(G, e^{(a)})$. For each $a \in P(L)$, suppose $x, y \in A^{(a)}$ and $(x, z), (z, y) \in e^{(a)}$. Then $A(x) \not\leq a, A(y) \not\leq a$ and $e(x, z) \not\leq a, e(z, y) \not\leq a$. It results $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \not\leq a$ because of $a \in P(L)$. Note that A is an L -ordered convex set, that is, $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$. Then $A(z) \not\leq a$, in other words, $z \in A^{(a)}$. Thus $A^{(a)}$ is convex in $(G, e^{(a)})$. □

Theorem 3.10. Let (G, e) be an L -ordered group. Then $A \in L^G$ is a convex L -subgroup of (G, e) if and only if $A_{[a]}$ is a convex subgroup of $(G, e_{[a]})$ for all $a \in L$.

Proof. Necessity is proved in Proposition 3.9.

Sufficiency. Obviously A is an L -subgroup of $(G, \cdot, 1)$ by Theorem 3.7 in [21], so it suffices to prove $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ for all $x, y, z \in X$. Set $a = e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y)$. Then $x, y \in A_{[a]}$ and $(x, z), (z, y) \in e_{[a]}$. Since $A_{[a]}$ is convex in $(G, e_{[a]})$, so $z \in A_{[a]}$. That is, $A(z) \geq a$, and so $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$, as desired. □

Theorem 3.11. *Let (G, e) be an L -ordered group with L be a completely distributive lattice, and $A \in L^G$. Then the following conditions are equivalent:*

- (1) A is a convex L -subgroup of (G, e) ;
- (2) $A_{[a]}$ is a convex subgroup of $(G, e_{[a]})$ for all $a \in L$,
- (3) $A_{[a]}$ is a convex subgroup of $(G, e_{[a]})$ for all $a \in P(L)$,
- (4) $A^{(a)}$ is a convex subgroup of $(G, e^{(a)})$ for all $a \in P(L)$,
- (5) $A^{[a]}$ is a convex subgroup of $(G, e^{[a]})$ for all $a \in L$,
- (6) $A^{[a]}$ is a convex subgroup of $(G, e^{[a]})$ for all $a \in P(L)$.

Proof. By Proposition 3.9, we have (1) \implies (2), (4), and (2) \implies (3), (5) \implies (6) is obvious. So we just need to prove (3) \implies (1), (4) \implies (1), (1) \implies (5) and (6) \implies (1).

(3) \implies (1). By Theorem 3.7 in [21], it suffices to prove $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ for all $x, y, z \in X$. For each $a \in P(L)$, if $a \leq e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y)$, then $x, y \in A_{[a]}$ and $(x, z), (z, y) \in e_{[a]}$. Since $A_{[a]}$ is convex in $(G, e_{[a]})$, so $z \in A_{[a]}$. That is, $A(z) \geq a$, then $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ according to Lemma 2.1.

(4) \implies (1). By Theorem 3.7 in [21], it just need to prove that A is an L -ordered convex set, that is to prove $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ for all $x, y, z \in X$. According to Lemma 2.1, it suffices to prove for each $a \in P(L)$, $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \not\leq a$ implies $A(z) \not\leq a$. In fact, $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \not\leq a$ implies $x, y \in A^{(a)}$ and $(x, z), (z, y) \in e^{(a)}$. So $z \in A^{(a)}$ since $A^{(a)}$ is convex in $(G, e^{(a)})$, that is, $A(z) \not\leq a$, as needed.

(1) \implies (5). For each $a \in L$, suppose $x, y \in A^{[a]}$ and $(x, z), (z, y) \in e^{[a]}$. Then $a \notin \alpha(e(x, z)), a \notin \alpha(e(z, y))$ and $a \notin \alpha(A(x)), a \notin \alpha(A(y))$. Thus $a \notin \alpha(e(x, z)) \cup \alpha(e(z, y)) \cup \alpha(A(x)) \cup \alpha(A(y)) = \alpha(e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y))$. Since A is an L -ordered convex set, that is, $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$, then we have $a \notin \alpha(A(z))$ by Lemma 2.1. It implies $z \in A^{[a]}$, and so $A^{[a]}$ is convex in $(G, e^{[a]})$. By Theorem 3.7 in [21], we can conclude that $A^{[a]}$ is a convex subgroup of $(G, e^{[a]})$.

(6) \implies (1). By Theorem 3.7 in [21], it suffices to prove that $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ for all $x, y, z \in X$. According to Lemma 2.1, it suffices to prove that for each $a \in P(L)$, $a \notin \alpha(e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y))$ implies $a \notin \alpha(A(z))$. In fact, $a \notin \alpha(e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y)) = \alpha(e(x, z)) \cap \alpha(e(z, y)) \cap \alpha(A(x)) \cap \alpha(A(y))$ implies $a \notin \alpha(e(x, z)), a \notin \alpha(e(z, y))$ and $a \notin \alpha(A(x)), a \notin \alpha(A(y))$. In other words, $x, y \in A^{[a]}$ and $(x, z), (z, y) \in e^{[a]}$. So $z \in A^{[a]}$ since $A^{[a]}$ is convex in $(G, e^{[a]})$. That is, $a \notin \alpha(A(z))$, as desired. \square

Theorem 3.12. *Let (G, e) be an L -ordered group with L be a completely distributive lattice, and $A \in L^G$. If $\beta(b \wedge c) = \beta(b) \cap \beta(c)$ for all $b, c \in L$, then the following conditions are equivalent:*

- (1) A is a convex L -subgroup of (G, e) ;
- (2) $\forall a \in L, A_{(a)}$ is a convex subgroup of $(G, e_{(a)})$;
- (3) $\forall a \in J(L), A_{(a)}$ is a convex subgroup of $(G, e_{(a)})$.

Proof. First, we can easily check that A is an L -subgroup of $(G, \cdot, 1)$ if and only if $A_{(a)}$ is a subgroup of $(G, \cdot, 1)$ for all $a \in L$ or $a \in J(L)$.

So to prove (1) \implies (2), it just needs to show $A_{(a)}$ is convex in $(G, e_{(a)})$ for each $a \in L$. In other words, for $x, y \in A_{(a)}$ and $(x, z), (z, y) \in e_{(a)}$, it suffices to prove $z \in A_{(a)}$. In fact, $x, y \in A_{(a)}$ and $(x, z), (z, y) \in e_{(a)}$ implies $a \in \beta(A(x)), a \in \beta(A(y))$ and $a \in \beta(e(x, z)), a \in \beta(e(z, y))$. Since $\beta(x \wedge y) = \beta(x) \cap \beta(y)$ for all $x, y \in L$, so we have $a \in \beta(A(x)) \cap \beta(A(y)) \cap \beta(e(x, z)) \cap \beta(e(z, y)) = \beta(e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y))$. Because A is an L -ordered convex set, that is, $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$, then we have $a \in \beta(A(z))$ by Lemma 2.1. It implies $z \in A_{(a)}$, as required.

(2) \implies (3) is obvious.

To prove (3) \implies (1), it suffices to prove $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ for all $x, y, z \in X$. According to Lemma 2.1, it suffices to prove $a \in \beta(e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y))$ implies $a \in \beta(A(z))$ for each $a \in J(L)$. In fact, $a \in \beta(e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y))$ implies $a \in \beta(e(x, z)), a \in \beta(e(z, y)), a \in \beta(A(x)), a \in \beta(A(y))$. In other words, $x, y \in A_{(a)}$ and $(x, z), (z, y) \in e_{(a)}$. So $z \in A_{(a)}$ since $A_{(a)}$ is convex in $(G, e_{(a)})$. That is, $a \in \beta(A(z))$, as needed. \square

Definition 3.13. [5] *Let (G_1, e_1) and (G_2, e_2) be L -ordered groups. A mapping $f : G_1 \rightarrow G_2$ is called an L -ordered group homomorphism if it is a group homomorphism and L -order preserving.*

Proposition 3.14. *Let (G_1, e_1) and (G_2, e_2) be L -ordered groups, and $f : G_1 \rightarrow G_2$ be an L -ordered group homomorphism. If $B \in L^{G_2}$ is a convex L -subgroup of (G_2, e_2) , then $f_L^-(B)$ is a convex L -subgroup of (G_1, e_1) .*

Proof. $\forall x, y \in G_1$, we have

$$\begin{aligned} f_L^{\leftarrow}(B)(x) \wedge f_L^{\leftarrow}(B)(y) &= B(f(x)) \wedge B(f(y)) \\ &\leq B(f(x)f(y)^{-1}) \\ &= B(f(xy^{-1})) \\ &= f_L^{\leftarrow}(B)(xy^{-1}). \end{aligned}$$

It implies $f_L^{\leftarrow}(B)$ is an L -subgroup of $(G_1, \cdot, 1)$.

Moreover, for all $x, y, z \in G_1$, we have $e_1(x, z) \wedge e_1(z, y) \wedge f_L^{\leftarrow}(B)(x) \wedge f_L^{\leftarrow}(B)(y) \leq e_2(f(x), f(z)) \wedge e_2(f(z), f(y)) \wedge B(f(x)) \wedge B(f(y)) \leq B(f(z)) = f_L^{\leftarrow}(B)(z)$. It implies that $f_L^{\leftarrow}(B)$ is an L -ordered convex set. Therefore, we can conclude that $f_L^{\leftarrow}(B)$ is a convex L -subgroup of (G_1, e_1) . \square

In what follows, we are about to discuss the product of convex L -subgroups in L -ordered groups.

Proposition 3.15. *Let (G_1, e_1) and (G_2, e_2) be L -ordered groups. If $A \in L^{G_1}$ and $B \in L^{G_2}$ are convex L -subgroups, then $A \times B$ is a convex L -subgroup in the product L -ordered group $(G_1 \times G_2, e)$ where $e = e_1 \times e_2$.*

Proof. The proof is straightforward. \square

In classical case, $A \times B$ is convex in $(G_1 \times G_2, \leq)$ if and only if A is convex in (G_1, \leq) and B is convex in (G_2, \leq) . However, the converse of Proposition 3.15 does not hold, please see the following example.

Example 3.16. *Let $L = \{0, a, b, 1\}$ with \leq defined as $0 \leq a, b \leq 1$, $a \vee b = 1$ and $a \wedge b = 0$. (G_1, e_1) and (G_2, e_2) be L -ordered group, where $G_1 = \{x_1, x_2, x_3\}$ and $G_2 = \{y_1, y_2, y_3\}$, $e_1 : G_1 \times G_1 \rightarrow L$ and $e_2 : G_2 \times G_2 \rightarrow L$ are presented in Tabel 1 and Tabel 2:*

Let $A \in L^{G_1}$ defined by $A(x_1) = A(x_3) = a$, $A(x_2) = 0$ and $B \in L^{G_2}$ defined by $B(y_1) = B(y_3) = b$, $B(y_2) = 0$. Then

G_1	x_1	x_2	x_3
x_1	1	a	a
x_2	0	1	a
x_3	0	0	1

Table 1: $e_1 : G_1 \times G_1 \rightarrow L$

G_2	y_1	y_2	y_3
y_1	1	b	b
y_2	0	1	b
y_3	0	0	1

Table 2: $e_2 : G_2 \times G_2 \rightarrow L$

for each $(x, y) \in G_1 \times G_2$, we have $(A \times B)(x, y) = 0$, it results that $A \times B$ is a convex L -subgroup of $(G_1 \times G_2, e_1 \times e_2)$. But $e_1(x_1, x_2) \wedge e_1(x_2, x_3) \wedge A(x_1) \wedge A(x_3) = a \not\leq A(x_2)$, which implies that A is not a convex L -subgroup of (G_1, e_1) .

The product L -ordered group of (G_1, e_1) and (G_2, e_2) can easily generalized to the finite case, so we have the following theorem.

Theorem 3.17. *Let $\{(G_i, e_i)\}_{i=1}^n$ be a non-empty family of L -ordered groups, and for each $1 \leq i \leq n$, $A_i \in L^{G_i}$ be a convex L -subgroup of (G_i, e_i) . Then $\prod_{i=1}^n A_i$ is a convex L -subgroup of $(\prod_{i=1}^n G_i, e)$ where $e = \prod_{i=1}^n e_i$.*

In the rest part of this section, we are about to introduce an L -convex structure in an L -ordered group.

From now on, we use \mathcal{C}_G to denote the set of all convex L -subgroups of an L -ordered group (G, e) .

Theorem 3.18. *Let (G, e) be an L -ordered group. Then \mathcal{C}_G is an L -convex structure, and (G, \mathcal{C}_G) is an L -convex space.*

Proof. **(LC1)** is obvious.

(LC2). Refer to Proposition 3.6 in [7].

To prove **(LC3)**. Suppose $\{A_i \mid i \in I\} \subseteq \mathcal{C}_G$ be non-empty and totally ordered. Then for all $x, y \in G$, we have

$$\begin{aligned} \left(\bigvee_{i \in I} A_i\right)(x) \wedge \left(\bigvee_{i \in I} A_i\right)(y) &= \bigvee_{i \in I} \left(A_i(x) \wedge A_i(y)\right) \\ &\leq \bigvee_{i \in I} A_i(xy^{-1}) \\ &= \left(\bigvee_{i \in I} A_i\right)(xy^{-1}). \end{aligned}$$

This implies $\bigvee_{i \in I} A_i$ is an L -subgroup. Next, we show that $\bigvee_{i \in I} A_i$ is an L -ordered convex set in (G, e) . Take $\forall x, y, z \in G$, we have

$$\begin{aligned} e(x, z) \wedge e(z, y) \wedge \left(\bigvee_{i \in I} A_i\right)(x) \wedge \left(\bigvee_{i \in I} A_i\right)(y) \\ &= e(x, z) \wedge e(z, y) \wedge \bigvee_{i \in I} \left(A_i(x) \wedge A_i(y)\right) \\ &= \bigvee_{i \in I} \left(e(x, z) \wedge e(z, y) \wedge A_i(x) \wedge A_i(y)\right) \\ &\leq \bigvee_{i \in I} A_i(z) = \left(\bigvee_{i \in I} A_i\right)(z). \end{aligned}$$

That is to say, $\bigvee_{i \in I} A_i$ is an L -ordered convex set. Thus $\bigvee_{i \in I} A_i \in \mathcal{C}_G$. \square

Theorem 3.19. *Let (G_1, e_1) and (G_2, e_2) be L -ordered groups and $f : G_1 \rightarrow G_2$ be an L -ordered group homomorphism. Then f is L -convexity preserving from (G_1, \mathcal{C}_{G_1}) to (G_2, \mathcal{C}_{G_2}) .*

Proof. It follows from Theorem 3.14. \square

Employing this result, it accesses immediately the following corollary.

Corollary 3.20. *If (G, e) is the product L -ordered group of $\{(G_i, e_i) \mid i \in I\}$. Then each projection $p_i : (G, e) \rightarrow (G_i, e_i)$ is L -convexity preserving.*

Proof. It follows from Theorem 3.19. \square

4 Convex L -subgroup degree

From section 3, we can see that in an L -ordered group (G, e) , an L -subset A is either a convex L -subgroup when both A is an L -subgroup and the inequality $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ holds for all $x, y, z \in G$, or not a convex L -subgroup when A is not an L -subgroup or the inequality does not hold. There is no other case. But from the view of fuzzy logic, the L -subset A may be an L -subgroup to some extent and the inequality also can be established to some extent. In other words, the L -subset A may be a convex L -subgroup to some extent. Based on this consideration, we are going to introduce the notion of the degree to which an L -subset is a convex L -subgroup.

First of all, on a complete Heyting algebras L , there exists a well-known implication operator $\rightarrow : L \times L \rightarrow L$ as the right adjoint for the operator \wedge defined by

$$a \rightarrow b = \bigvee \{x \in L \mid a \wedge x \leq b\}.$$

As we all known, in a group, an L -subset may be an L -subgroup to some extent. Let us review the definition of L -subgroup degree.

Definition 4.1. [22] *Let G be a group and $A \in L^G$. The L -subgroup degree $m(A)$ of A is defined as*

$$m(A) = \bigwedge_{x, y \in X} \left((A(x) \wedge A(y)) \rightarrow A(xy^{-1}) \right).$$

Similar to the way of considering L -subgroup degree, we can consider the convex degree to describe the extent of an L -subset be a convex set with respect to an L -order. Integrated L -subgroup degree and convex degree, we define convex L -subgroup degree as following.

Definition 4.2. Let (G, e) be an L -ordered group and $A \in L^G$. The convex L -subgroup degree $CLD(A)$ of A is defined as:

$$CLD(A) = m(A) \wedge \bigwedge_{x, y, z \in X} \left((e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y)) \rightarrow A(z) \right).$$

Remark 4.3. $CLD(A)$ can be regarded as the degree to which A is a convex L -subgroup. Obviously A is a convex L -subgroup if and only if $CLD(A) = 1$.

Analogous to Lemma 3.4 in [22], we can immediately obtain the following lemma.

Lemma 4.4. Let (G, e) be an L -ordered group and $A \in L^G$, $a \in L$. Then $CLD(A) \geq a$ if and only if $m(A) \geq a$ and $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \wedge a \leq A(z)$ for all $x, y, z \in G$.

By Lemma 4.4, we can easily obtain the following lemma.

Lemma 4.5. Let (G, e) be an L -ordered group, and $A \in L^G$. Then $CLD(A) = \bigvee \{a \in L \mid a \wedge A(x) \wedge A(y) \leq A(xy^{-1}), a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z), \forall x, y, z \in G\}$.

Proof. On one hand, $CLD(A) \geq \bigvee \{a \in L \mid a \wedge A(x) \wedge A(y) \leq A(xy^{-1}), a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z), \forall x, y, z \in G\}$ just follows from Lemma 4.4.

On the other hand, suppose $CLD(A) = b$, then $b \leq (A(x) \wedge A(y)) \rightarrow A(xy^{-1})$ and $b \leq (e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y)) \rightarrow A(z)$ for all $x, y, z \in G$. It implies $b \wedge A(x) \wedge A(y) \leq A(xy^{-1})$, $b \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$. Hence, $CLD(A) \leq \bigvee \{a \in L \mid a \wedge A(x) \wedge A(y) \leq A(xy^{-1}), a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z), \forall x, y, z \in G\}$. \square

In the next, we are about to give characterizations of convex L -subgroup degree of an L -subset by means of four kinds of levels.

Theorem 4.6. Let (G, e) be an L -ordered group, and $A \in L^G$. Then

$$CLD(A) = \bigvee \{a \in L \mid \forall b \leq a, A_{[b]} \text{ is a convex subgroup of } (G, e_{[b]})\}.$$

Proof. Suppose $a \in L$ with $a \wedge A(x) \wedge A(y) \leq A(xy^{-1})$ and $a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ for all $x, y, z \in G$. For each $b \leq a$, if $x, y \in A_{[b]}$, then we have $A(xy^{-1}) \geq a \wedge b \geq b$. It results $xy^{-1} \in A_{[b]}$, and so $A_{[b]}$ is a subgroup of G . Moreover, if $(x, z), (z, y) \in e_{[b]}$, then $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \geq b$. It results $A(z) \geq (b \wedge a) = b$. So $z \in A_{[b]}$, and it implies $A_{[b]}$ is convex in $(G, e_{[b]})$. Therefore, $CLD(A) \leq \bigvee \{a \in L \mid \forall b \leq a, A_{[b]} \text{ is a convex subgroup of } (G, e_{[b]})\}$ from Lemma 4.5.

Conversely, for every $a \in L$ with $A_{[b]}$ is convex subgroup of $(G, e_{[b]})$ for all $b \leq a$, we need to prove that $a \wedge A(x) \wedge A(y) \leq A(xy^{-1})$ and $a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ for all $x, y, z \in X$. Suppose $a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \geq b$, then we have $b \leq a$, $x, y \in A_{[b]}$ and $(x, z), (z, y) \in e_{[b]}$. Therefore, $z \in A_{[b]}$ since $A_{[b]}$ is convex. That is $A(z) \geq b$, so $a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$. Analogously, we can get $a \wedge A(x) \wedge A(y) \leq A(xy^{-1})$. \square

Theorem 4.7. If L is a completely distributive lattice, and (G, e) is an L -ordered group, then for all $A \in L^G$, we have

$$CLD(A) = \bigvee \{a \in L \mid \forall b \in P(L), a \not\leq b, A^{(b)} \text{ is a convex subgroup of } (X, e^{(b)})\}.$$

Proof. Assume $a \in L$ with $a \wedge A(x) \wedge A(y) \leq A(xy^{-1})$ and $a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ for all $x, y, z \in G$. For each $b \in P(L)$ such that $a \not\leq b$, if $x, y \in A^{(b)}$, then $A(xy^{-1}) \not\leq b$. That is to say, $xy^{-1} \in A^{(b)}$, so $A^{(b)}$ is a subgroup of G . Besides, if $(x, z), (z, y) \in e^{(b)}$, then $e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \not\leq b$. It results $A(z) \not\leq b$. So $z \in A^{(b)}$, and it implies $A^{(b)}$ is convex in $(G, e^{(b)})$. Thus $CLD(A) \leq \bigvee \{a \in L \mid \forall b \in P(L), a \not\leq b, A^{(b)} \text{ is a convex subgroup of } (G, e^{(b)})\}$.

Conversely, for every $a \in L$ with $A^{(b)}$ is a convex subgroup of $(G, e^{(b)})$ for all $b \in P(L)$ such that $a \not\leq b$, we need to prove that $a \wedge A(x) \wedge A(y) \leq A(xy^{-1})$ and $a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ for all $x, y, z \in G$. Suppose $b \in P(L)$ and $a \wedge A(x) \wedge A(y) \not\leq b$, then $a \not\leq b$, $x, y \in A^{(b)}$. Since $A^{(b)}$ is a subgroup, so $A(xy^{-1}) \not\leq b$. It implies $a \wedge A(x) \wedge A(y) \leq A(xy^{-1})$. Suppose $a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \not\leq b$, then we have $a \not\leq b$, $x, y \in A^{(b)}$ and $(x, z), (z, y) \in e^{(b)}$. Therefore, $A(z) \not\leq b$ since $A^{(b)}$ is convex. It results $a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ by Lemma 2.1. \square

Theorem 4.8. *If L is a completely distributive lattice, and (G, e) is an L -ordered group, then for all $A \in L^G$, we have*

$$CLD(A) = \bigvee \{a \in L \mid \forall b \notin \alpha(a), A^{[b]} \text{ is a convex subgroup of } (G, e^{[b]})\}.$$

Proof. Suppose $a \in L$ with $a \wedge A(x) \wedge A(y) \leq A(xy^{-1})$ and $a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ for all $x, y, z \in G$. For each $b \notin \alpha(a)$, if $x, y \in A^{[b]}$, then $b \notin \alpha(A(x)), b \notin \alpha(A(y))$. By Lemma 2.1, we have $b \notin \alpha(A(xy^{-1}))$. That is, $xy^{-1} \in A^{[b]}$, and so $A^{[b]}$ is a subgroup of G . Besides, if $(x, z), (z, y) \in e^{[b]}$, then $b \notin \alpha(e(x, z)), b \notin \alpha(e(z, y))$. It implies $b \notin \alpha(e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y))$, and so $b \notin \alpha(A(z))$. That is, $z \in A^{[b]}$, and so $A^{[b]}$ is convex in $(X, e^{[b]})$. According to Lemma 4.5, we have $CLD(A) \leq \bigvee \{a \in L \mid \forall b \leq a, A^{[b]} \text{ is convex in } (X, e^{[b]})\}$.

Conversely, for every $a \in L$ with $A^{[b]}$ is a convex subgroup of $(G, e^{[b]})$ for all $b \notin \alpha(a)$, we need to prove that $a \wedge A(x) \wedge A(y) \leq A(xy^{-1})$ and $a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ for all $x, y, z \in G$. Assume $b \notin \alpha(a \wedge A(x) \wedge A(y))$, then we have $b \notin \alpha(a), b \notin \alpha(A(x)), b \notin \alpha(A(y))$. Since $A^{[b]}$ is a subgroup, then we have $b \notin \alpha(A(xy^{-1}))$. It results $a \wedge A(x) \wedge A(y) \leq A(xy^{-1})$ by Lemma 2.1.

Suppose $b \notin \alpha(a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y))$, then we have $b \notin \alpha(a), b \notin \alpha(A(x)), b \notin \alpha(A(y)), b \notin \alpha(e(x, z)), b \notin \alpha(e(z, y))$. In other words, $x, y \in A^{[b]}$ and $(x, z), (z, y) \in e^{[b]}$. Therefore, $z \in A^{[b]}$ since $A^{[b]}$ is convex. That is $b \notin \alpha(A(z))$, so $a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$. The proof is completed. \square

Theorem 4.9. *Let L be a completely distributive lattice with $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ for all $a, b \in L$, and (G, e) be an L -ordered group. Then for all $A \in L^G$, we have $CLD(A) = \bigvee \{a \in L \mid \forall b \in \beta(a), A_{(b)} \text{ is a convex subgroup in } (G, e_{(b)})\}$.*

Proof. Assume $a \in L$ with $a \wedge A(x) \wedge A(y) \leq A(xy^{-1})$ and $a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ for all $x, y, z \in G$. For each $b \in \beta(a)$, if $x, y \in A_{(b)}$, then $b \in \beta(A(x)), b \in \beta(A(y))$, so $b \in \beta(a \wedge A(x) \wedge A(y))$. It results $b \in \beta(A(xy^{-1}))$, which implies $xy^{-1} \in A_{(b)}$. Thus $A_{(b)}$ is a subgroup. Besides, if $(x, z), (y, z) \in e_{(b)}$, then we have $b \in \beta(e(x, z)), b \in \beta(e(z, y))$. It implies $b \in \beta(a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y))$. So $b \in \beta(A(z))$, and it indicates $z \in A_{(b)}$. Hence $A_{(b)}$ is convex in $(G, e_{(b)})$. According to Lemma 4.5, we have $CLD(A) \leq \bigvee \{a \in L \mid \forall b \leq a, A_{(b)} \text{ is a convex subgroup of } (G, e_{(b)})\}$.

Conversely, for every $a \in L$ such that $A_{(b)}$ is a convex subgroup of $(G, e_{(b)})$ for all $b \in \beta(a)$, we need to prove that $a \wedge A(x) \wedge A(y) \leq A(xy^{-1})$ and $a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ for all $x, y, z \in G$. Suppose $b \in \beta(a \wedge A(x) \wedge A(y))$, then we have $b \in \beta(a), b \in \beta(A(x)), b \in \beta(A(y))$. It indicates $x, y \in A_{(b)}$, and so $b \in \beta(A(xy^{-1}))$ since $A_{(b)}$ is a subgroup. Therefore, $a \wedge A(x) \wedge A(y) \leq A(xy^{-1})$.

Suppose $b \in \beta(a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y))$, then we have $b \in \beta(a), b \in \beta(A(x)), b \in \beta(A(y)), b \in \beta(e(x, z)), b \in \beta(e(z, y))$. It results $x, y \in A_{(b)}$ and $(x, z), (z, y) \in e_{(b)}$. Therefore, $z \in A_{(b)}$ since $A_{(b)}$ is convex. That is $b \in \beta(A(z))$, so $a \wedge e(x, z) \wedge e(z, y) \wedge A(x) \wedge A(y) \leq A(z)$ by Lemma 2.1. As required. \square

Theorem 4.10. *Let $(G_1, e_1), (G_2, e_2)$ be L -ordered groups, and $(G_1 \times G_2, e)$ be their product. Then $CLD(A) \wedge CLD(B) \leq CLD(A \times B)$ for all $A \in L^{G_1}$ and $B \in L^{G_2}$,*

Proof. For any $x \in G_1 \times G_2$, suppose $x = (x_1, x_2)$. Then we have

$$CLD(A \times B) = m(A \times B) \wedge \bigwedge_{x, y, z \in G_1 \times G_2} \left((e(x, z) \wedge e(z, y) \wedge (A \times B)(x) \wedge (A \times B)(y)) \rightarrow (A \times B)(z) \right).$$

That is,

$$CLD(A \times B) = m(A \times B) \wedge \bigwedge_{x, y, z \in G_1 \times G_2} \left((e_1(x_1, z_1) \wedge e_2(x_2, z_2) \wedge e_1(z_1, y_1) \wedge e_2(z_2, y_2) \wedge A(x_1) \wedge B(x_2) \wedge A(y_1) \wedge B(y_2)) \rightarrow (A(z_1) \wedge B(z_2)) \right)$$

It results

$$\begin{aligned} CLD(A \times B) &\geq m(A) \wedge \bigwedge_{x_1, y_1, z_1 \in G_1} \left((e_1(x_1, z_1) \wedge e_1(z_1, y_1) \wedge A(x_1) \wedge A(y_1)) \rightarrow A(z_1) \right) \wedge m(B) \wedge \bigwedge_{x_2, y_2, z_2 \in G_2} \left((e_2(x_2, z_2) \wedge e_2(z_2, y_2) \wedge B(x_2) \wedge B(y_2)) \rightarrow B(z_2) \right) \\ &= CLD(A) \wedge CLD(B). \end{aligned}$$

□

Theorem 4.11. Let $(G_1, e_1), (G_2, e_2)$ be L -ordered groups and $B \in L^{G_2}$. If the mapping $f : G_1 \rightarrow G_2$ is an L -ordered group homomorphism, then $CLD(f_L^{\leftarrow}(B)) \geq CLD(B)$.

Proof. For every $B \in L^{G_2}$, we have

$$\begin{aligned}
& \bigwedge_{x_1, z_1, y_1 \in G_1} \left((e_1(x_1, z_1) \wedge e_1(z_1, y_1) \wedge f_L^{\leftarrow}(B)(x_1) \wedge f_L^{\leftarrow}(B)(y_1)) \right. \\
& \quad \left. \rightarrow f_L^{\leftarrow}(B)(z_1) \right) \\
& \geq \bigwedge_{x_1, z_1, y_1 \in G_1} \left((e_2(f(x_1), f(z_1)) \wedge e_2(f(z_1), f(y_1)) \wedge B(f(x_1)) \right. \\
& \quad \left. \wedge B(f(y_1))) \rightarrow B(f(z_1)) \right) \\
& \geq \bigwedge_{x_2, z_2, y_2 \in G_2} \left((e_2(x_2, z_2) \wedge e_2(z_2, y_2) \wedge B(x_2) \wedge B(y_2)) \rightarrow B(z_2) \right).
\end{aligned}$$

Then from Theorem 3.12(3) in [22], we have $CLD(f_L^{\leftarrow}(B)) \geq CLD(B)$. □

5 L -fuzzy convex structure results from convex L -subgroup degree

In this section, by means of convex L -subgroup degree, we obtain an L -fuzzy convex structure on an L -ordered group. Furthermore, we discuss some properties of this kind of L -fuzzy convex structure.

For an L -ordered group (G, e) , let $A \in L^G$. Then $CLD(A)$ can be regarded as the degree to which A is a convex L -subgroup, and CLD can be naturally considered as a mapping from L^G to L by $A \rightarrow CLD(A)$. By the following theorem, we know that CLD is an L -fuzzy convex structure on the L -ordered group (G, e) , which is called the L -fuzzy convex structure induced by convex L -subgroup degree.

Theorem 5.1. Let (G, e) be an L -ordered group. Then the mapping $CLD : L^G \rightarrow L$ defined by $A \rightarrow CLD(A)$ is an L -fuzzy convex structure.

Proof. (LMC1). $CLD(\bar{0}) = CLD(\bar{1}) = 1$ is obvious.

(LMC2). For all $\{A_i \mid i \in I\} \subseteq L^G$, we have

$$\begin{aligned}
CLD\left(\bigwedge_{i \in I} A_i\right) &= m\left(\bigwedge_{i \in I} A_i\right) \wedge \bigwedge_{x, y, z \in G} \left(\left(e(x, z) \wedge e(z, y) \wedge \bigwedge_{i \in I} A_i(x) \right. \right. \\
& \quad \left. \left. \wedge \bigwedge_{i \in I} A_i(y) \right) \rightarrow \bigwedge_{i \in I} A_i(z) \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
CLD\left(\bigwedge_{i \in I} A_i\right) &= m\left(\bigwedge_{i \in I} A_i\right) \wedge \bigwedge_{i \in I} \bigwedge_{x, y, z \in X} \left(\left(e(x, z) \wedge e(z, y) \wedge \bigwedge_{i \in I} A_i(x) \right. \right. \\
& \quad \left. \left. \wedge \bigwedge_{i \in I} A_i(y) \right) \rightarrow A_i(z) \right) \\
&\geq \bigwedge_{i \in I} \left(m(A_i) \wedge \bigwedge_{x, y, z \in X} \left(\left(e(x, z) \wedge e(z, y) \wedge A_i(x) \right. \right. \right. \\
& \quad \left. \left. \wedge A_i(y) \right) \rightarrow A_i(z) \right) \\
&= \bigwedge_{i \in I} CLD(A_i).
\end{aligned}$$

(LMC3). Suppose $\{A_i \mid i \in I\} \subseteq L^G$ is non-empty and totally ordered, then we have

$$\begin{aligned}
m\left(\bigvee_{i \in I} A_i\right) &= \bigwedge_{x, y \in G} \left(\bigvee_{i \in I} A_i(x) \wedge \bigvee_{i \in I} A_i(y) \rightarrow \bigvee_{i \in I} A_i(xy^{-1}) \right) \\
&= \bigwedge_{x, y \in G} \left(\bigvee_{i \in I} A_i(x) \wedge A_i(y) \rightarrow \bigvee_{i \in I} A_i(xy^{-1}) \right) \\
&\geq \bigwedge_{i \in I} \bigwedge_{x, y, z \in G} \left(A_i(x) \wedge A_i(y) \rightarrow A_i(xy^{-1}) \right) \\
&= \bigwedge_{i \in I} m(A_i). \\
\\
CLD\left(\bigvee_{i \in I} A_i\right) &= m\left(\bigvee_{i \in I} A_i\right) \wedge \bigwedge_{x, y, z \in G} \left(\left(e(x, z) \wedge e(z, y) \wedge \bigvee_{i \in I} A_i(x) \right. \right. \\
&\quad \left. \left. \wedge \bigvee_{i \in I} A_i(y) \right) \rightarrow \bigvee_{i \in I} A_i(z) \right) \\
&= m\left(\bigvee_{i \in I} A_i\right) \wedge \bigwedge_{x, y, z \in G} \left(\left(e(x, z) \wedge e(z, y) \wedge \right. \right. \\
&\quad \left. \left. \bigvee_{i \in I} (A_i(x) \wedge A_i(y)) \right) \rightarrow \bigvee_{i \in I} A_i(z) \right) \\
&\geq m\left(\bigvee_{i \in I} A_i\right) \wedge \bigwedge_{i \in I} \bigwedge_{x, y, z \in G} \left(\left(e(x, z) \wedge e(z, y) \right. \right. \\
&\quad \left. \left. \wedge A_i(x) \wedge A_i(y) \right) \rightarrow A_i(z) \right) \\
&\geq \bigwedge_{i \in I} CLD(A_i).
\end{aligned}$$

□

In what follows, some properties of the L -fuzzy convex structure in an L -ordered group are discussed, where L is a completely distributive lattice.

From [25], we know that \mathcal{C} is an L -fuzzy convex structure if and only if $\mathcal{C}_{[a]}$ is an L -convex structure for all $a \in L \setminus \{0\}$ or $\mathcal{C}^{[a]}$ is an L -convex structure for all $a \in \alpha(0)$. Then the following theorem is obvious.

Theorem 5.2. *Let (G, e) be an L -ordered group. Then the following statements are equivalent:*

- (1) CLD is an L -fuzzy convex structure induced by convex L -subgroup degree on G ;
- (2) $\forall a \in J(L)$, $CLD_{[a]}$ is an L -convex structure on G ;
- (3) $\forall a \in \alpha(0)$, $CLD^{[a]}$ is an L -convex structure on G ;

By Theorem 4.11, we have the following theorem.

Theorem 5.3. *Let $(G_1, e_1), (G_2, e_2)$ be L -ordered groups. If the mapping $f : G_1 \rightarrow G_2$ is an L -ordered group homomorphism, then $f : (G_1, CLD_{G_1}) \rightarrow (G_2, CLD_{G_2})$ is an L -fuzzy convexity-preserving mapping.*

The following theorem will present some equivalent characterizations of L -fuzzy convexity-preserving mappings.

Theorem 5.4. *Let $(G_1, e_1), (G_2, e_2)$ be L -ordered groups and $f : G_1 \rightarrow G_2$ be a mapping. Then the following statements are equivalent:*

- (1) $f : (G_1, CLD_{G_1}) \rightarrow (G_2, CLD_{G_2})$ is an L -fuzzy convexity-preserving mapping.
- (2) $f : (G_1, (CLD_{G_1})_{[a]}) \rightarrow (G_2, (CLD_{G_2})_{[a]})$ is an L -convexity-preserving mapping for each $a \in L \setminus \{0\}$.
- (3) $f : (G_1, (CLD_{G_1})_{[a]}) \rightarrow (G_2, (CLD_{G_2})_{[a]})$ is an L -convexity-preserving mapping for each $a \in J(L)$.
- (4) $f : (G_1, (CLD_{G_1})^{[a]}) \rightarrow (G_2, (CLD_{G_2})^{[a]})$ is an L -convexity-preserving mapping for each $a \in \alpha(0)$.
- (5) $f : (G_1, (CLD_{G_1})^{[a]}) \rightarrow (G_2, (CLD_{G_2})^{[a]})$ is an L -convexity-preserving mapping for each $a \in \alpha^*(0)$.

Proof. (1) \implies (2). It suffices to show that $f_L^{\leftarrow}(B) \in (CLD_{G_1})_{[a]}$ for any $B \in (CLD_{G_2})_{[a]}$. Since $B \in (CLD_{G_2})_{[a]}$, we have $CLD_{G_2}(B) \geq a$. Note that f is an L -fuzzy convexity-preserving mapping, we have $CLD_{G_1}(f_L^{\leftarrow}(B)) \geq CLD_{G_2}(B) \geq a$. It implies $f_L^{\leftarrow}(B) \in (CLD_{G_1})_{[a]}$, as needed.

(2) \implies (3) is obvious.

(3) \implies (1). Take any $B \in L^{G_2}$, and suppose $CLD_{G_2}(B) \geq a$ for any $a \in J(L)$. That is to say, $B \in (CLD_{G_2})_{[a]}$. Since f is an L -convexity-preserving mapping, we have $f_L^{\leftarrow}(B) \in (CLD_{G_1})_{[a]}$. It implies $CLD_{G_1}(f_L^{\leftarrow}(B)) \geq a$. By Lemma 2.1, we conclude that $CLD_{G_1}(f_L^{\leftarrow}(B)) \geq CLD_{G_2}(B)$. So f is an L -fuzzy convexity-preserving.

(1) \implies (4). It suffices to show that $f_L^{\leftarrow}(B) \in (CLD_{G_1})^{[a]}$ for any $B \in (CLD_{G_2})^{[a]}$. Since $B \in (CLD_{G_2})^{[a]}$, we have $a \notin \alpha(CLD_{G_2}(B))$. Since f is an L -fuzzy convexity-preserving mapping, we have $CLD_{G_1}(f_L^{\leftarrow}(B)) \geq CLD_{G_2}(B)$. It implies that $a \notin \alpha(CLD_{G_1}(f_L^{\leftarrow}(B)))$, i.e., $f_L^{\leftarrow}(B) \in (CLD_{G_1})^{[a]}$, as required.

(4) \implies (5) is obvious.

(5) \implies (1). Take any $B \in L^{G_2}$, and suppose $a \notin \alpha(CLD_{G_2}(B))$ for any $a \in P(L)$. That is to say, $B \in (CLD_{G_2})^{[a]}$. Since f is an L -convexity-preserving mapping, we have $f_L^{\leftarrow}(B) \in (CLD_{G_1})^{[a]}$. It implies $a \notin \alpha(CLD_{G_1}(f_L^{\leftarrow}(B)))$. By Lemma 2.1, we know that $CLD_{G_1}(f_L^{\leftarrow}(B)) \geq CLD_{G_2}(B)$. So f is an L -fuzzy convexity-preserving mapping. \square

6 Conclusions

In this paper, we first characterize convex L -subgroups of an L -ordered group by means of four kinds of cut sets of L -subsets. Then we introduce the L -convex structure constructed by convex L -subgroups and analyze its properties. Furthermore, we discuss the degree to which an L -subset is a convex L -subgroup. Then an L -fuzzy convex structure results from the convex L -subgroup degree is proposed naturally and its properties are studied.

Acknowledgement

The authors would like to thank the anonymous reviewers and the area editor for their careful reading and valuable comments. This work is supported by Doctoral Scientific Research Foundation of Shandong Jianzhu University (XNBS1344), the National Natural Science Foundation of China (61471409) and Shandong Province Social Science Planning Research Project(14DGLJ06).

References

- [1] M. E. Anderson, T. H. Feil, *Lattice-ordered Groups: An Introduction*, Springer, Dordrecht, 1988.
- [2] R. Bělohlávek, *Fuzzy Relational Systems: Foundation and Principles*, Kluwer Academic/Plenum Press, New York, 2002.
- [3] R. Bělohlávek, *Concept lattices and order in fuzzy logic*, Annals of Pure and Applied Logic, **128** (2004), 277–298.
- [4] M. Berger, *Convexity*, American Mathematical Monthly, **97**(8) (1990), 650–678.
- [5] R. A. Borzooei, A. Dvurecenskij, O. Zahiri, *L-ordered and L-lattice ordered groups*, Information Sciences, **314** (2015), 118–134.
- [6] R. A. Borzooei, F. Hosseini, O. Zahiri, *(Totally) L-ordered groups*, Journal of Intelligent and Fuzzy Systems, **30** (2016), 1489–1498.
- [7] R. A. Borzooei, F. Hosseini, O. Zahiri, *Convex L-lattice subgroups in L-ordered groups*, Categories and General Algebraic Structures with Applications, **9**(1) (2018), 139–161.
- [8] W. A. Coppel, *Axioms for convexity*, Bulletin of the Australian Mathematical Society, **47**(2) (1993), 179–197.
- [9] B. A. Davey, H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 2001.
- [10] L. Fan, *A new approach to quantitative domain theory*, Electronic Notes in Theoretical Computer Science, **45** (2001), 77–87.
- [11] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, Cambridge, 2003.

- [12] Q. Jin, L. -Q. Li, *On the embedding of L -convex spaces in stratified L -convex spaces*, SpringerPlus, **5**(1) (2016), 1610.
- [13] H. -L. Lai, D. -X. Zhang, *Fuzzy preorder and fuzzy topology*, Fuzzy Sets and Systems, **157** (2006), 1865–1885.
- [14] J. Li, F. -G. Shi, *L -fuzzy convexity induced by L -convex fuzzy sublattice degree*, Iranian Journal of Fuzzy Systems, **14**(5) (2017), 83–102.
- [15] Y. Maruyama, *Lattices-valued fuzzy convex geometry*, RIMS Kokyuroku, **1641** (2009), 22–37.
- [16] B. Pang, Y. Zhao, *Characterizations of L -convex spaces*, Iranian Journal of Fuzzy System, **13**(4) (2016), 51–61.
- [17] B. Pang, F. -G. Shi, *Subcategories of the category of L -convex spaces*, Fuzzy Sets and Systems, **313** (2017), 61–74.
- [18] M. V. Rosa, *On fuzzy topology fuzzy convexity spaces and fuzzy local convexity*, Fuzzy Sets and Systems, **62** (1994), 97–100.
- [19] M. V. Rosa, *A study of fuzzy convexity with special reference to separation properties*, PhD. thesis, Cochin University of Science and Technology, 1994.
- [20] C. Shen, F.,-G. Shi, *L -convex systems and the categorical isomorphism to Scott-hull operators*, Iranian Journal of Fuzzy Systems, **15**(2) (2018), 23–40.
- [21] F. -G. Shi, *L -fuzzy relations and L -fuzzy subgroups*, The Journal of Fuzzy Mathematics, **8** (2000), 491–499.
- [22] F. -G. Shi, *L -fuzzy subgroup degrees and L -fuzzy normal subgroup degrees*, Journal of Advanced Research in Pure Mathematics, **3**(4) (2011), 92–108.
- [23] F. -G. Shi, Z. -Y. Xiu, *A new approach to the fuzzification of convex structures*, Journal of Applied Mathematics, **5** (2014), 1–12.
- [24] F. -G. Shi, E. -Q. Li, *The restricted hull operator of M -fuzzifying convex structures*, Iranian Journal of Fuzzy Systems, **30** (2015), 409–421.
- [25] F. -G. Shi, Z. -Y. Xiu, *(L, M) -fuzzy convex structures*, Journal of Nonlinear Science and its Applications, **10**(7) (2017), 3655–3669.
- [26] M. L. J. Van De Vel, *Theory of Convex Structures*, North-Holland, New York, 1993.
- [27] G. -J. Wang, *Theory of topological molecular lattices*, Fuzzy Sets and Systems, **47** (1992), 351–376.
- [28] X. -Y. Wu, S. -Z. Bai, *On M -fuzzifying JHC convex structures and M -fuzzifying Peano interval spaces*, Journal of Intelligent and Fuzzy Systems, **30** (2016), 2447–2458.
- [29] X. -Y. Wu, E. -Q. Li, S. -Z. Bai, *Geometric properties of M -fuzzifying convex structures*, Journal of Intelligent and Fuzzy Systems, **32** (2017), 4273–4284.
- [30] X. -Y. Wu, B. Davvaz, S. -Z. Bai, *On M -fuzzifying convex matroids and M -fuzzifying independent structures*, Journal of Intelligent and Fuzzy Systems, **33** (2017), 269–280.
- [31] Z.-Y. Xiu, B. Pang, *M -fuzzifying cotopological spaces and M -fuzzifying convex spaces as M -fuzzifying closure spaces*, Journal of Intelligent and Fuzzy Systems, **33** (2017), 613–620.
- [32] Z. -Y. Xiu, F. -G. Shi, *M -fuzzifying interval spaces*, Iranian Journal of Fuzzy Systems, **14**(1) (2017), 145–162.
- [33] W. Yao, *L -fuzzy Scott topology and Scott convergence of stratified L -filters on fuzzy dcpos*, Electronic Notes in Theoretical Computer Science, **257** (2009), 135–152.
- [34] W. Yao, *Quantitative domains via fuzzy sets: part I: continuity of fuzzy directed complete posets*, Fuzzy Sets and Systems, **161** (2010), 973–987.
- [35] W. Yao, *An approach to fuzzy frames via fuzzy posets*, Fuzzy Sets and Systems, **166** (2011), 75–89.
- [36] W. Yao, *An approach to the fuzzification of complete lattices*, Journal of Intelligent and Fuzzy Systems, **26**(5) (2014), 2239–2249.

- [37] W. Yao, L. S. Xu, *Fuzzy Galois connections on fuzzy posets*, Mathematical Logic Quarterly, **55** (2009), 84–91.
- [38] W. Yao, B. Zhao, *A duality between Ω -categories and algebraic Ω -categories*, Electronic Notes in Theoretical Computer Science, **301** (2014), 153–168.
- [39] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 238–353.
- [40] Q. -Y. Zhang, L. Fan, *Continuity in quantitative domains*, Fuzzy Sets and Systems, **154** (2005), 118–131.
- [41] Q. -Y. Zhang, L. Fan, *A kind of L-fuzzy complete lattice and adjoint functor theorem for LF-Poset*, Report on the Fourth International Symposium on Domain Theory, Hunan University, Changsha, China, June 2006.
- [42] Q. Zhang, W. Xie, L. Fan, *Fuzzy complete lattices*, Fuzzy Sets and Systems, **160** (2009), 2275–2291.
- [43] Y. Zhong, F. -G. Shi, *Formulations of L-convex hulls on some algebraic structures*, Journal of Intelligent and Fuzzy Systems, **33** (2017), 1385–1395.