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Super- and sub-additive transformations of aggregation functions from the point of view of approximation

F. Kouchakinejad[1] and A. Sipošová[2]

[1] Department of Statistics, Faculty of Mathematics and Computer Science, Shahid Bahonar University of Kerman, Kerman, Iran

[2] Slovak University of Technology in Bratislava, Faculty of Civil Engineering, Department of Mathematics and Descriptive Geometry, Radlinského 11, 810 05 Bratislava, Slovakia

kouchakinezhad@gmail.com, alexandra.siposova@stuba.sk

Abstract

The way super- and sub-additive transformations of aggregation functions are introduced involve suprema and infima taken over simplexes whose dimensions may grow arbitrarily. Exact values of such transformations may thus be hard to determine in general. In this note we discuss methods of algorithmic approximation of such transformations.

Keywords: aggregation function, sub-additive and super-additive transformation, approximation.

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1 Introduction

Representation of inhomogeneous numerical data of different origin by a single value appears to be a necessity in numerous applications in natural and social sciences, and a framework developed for this purpose in mathematics is known as aggregation. Methods of merging inputs into a single value by means of various types of aggregation functions have been extensively studied and subsequently applied in computer science, economics and industry. Surveys on aggregation function can be found e.g. in [1, 2] and in [6] that also outlines the future of research into aggregation.

In this contribution, by an *aggregation function* we will mean any mapping $A:\ [0,\infty[^n \to [0,\infty[$, increasing in every coordinate and such that $A(\mathbf{0}) = A(0,\ldots,0) = 0$. From the viewpoint of applications, important role is played by super- and sub-additivity. An aggregation function A as defined above is *super-additive* if $A(\mathbf{u}+\mathbf{v}) \ge A(\mathbf{u}) + A(\mathbf{v})$ for every $\mathbf{u},\ \mathbf{v} \in [0,\infty[^n$. Similarly, A is *sub-additive* if $A(\mathbf{u}+\mathbf{v}) \le A(\mathbf{u}) + A(\mathbf{v})$ for all $\mathbf{u},\ \mathbf{v} \in [0,\infty[^n$.

Since an aggregation function need not be super- or sub-additive in general, it turned out to be important to consider 'envelopes' of an aggregation function A that would, in some sense, be the super- and sub-additive functions 'closest' to A. This approach was initiated in [4] and has led to a number of follow-ups. By [4], the *super-additive* and *sub-additive* transformations, A^* and A_*, of an aggregation function A, are functions $[0,\infty[^n \to [0,\infty]$ given by

$$A^*(\mathbf{x}) = \sup\ \{\sum_{j=1}^{k} A(\mathbf{x}^{(j)})\ ;\ \ \sum_{j=1}^{k} \mathbf{x}^{(j)} \le \mathbf{x}\}\ ,\quad \text{and} \tag{1}$$

$$A_*(\mathbf{x}) = \inf\ \{\sum_{j=1}^{k} A(\mathbf{x}^{(j)})\ ;\ \ \sum_{j=1}^{k} \mathbf{x}^{(j)} \ge \mathbf{x}\}\ . \tag{2}$$

The fact that the functions A^* and A_* are super-additive and sub-additive, respectively, gave the two transformations their name. For illustration we give two examples of less trivially looking aggregation functions $[0,\infty[^n \to\ [0,\infty[$ in

Corresponding Author: A. Šipošová

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both categories: In the usual notation $\mathbf{x} = (x_1, \ldots, x_n)$, the function $f(\mathbf{x}) = (\sum_{i=1}^n x_i^2)/(1+\sum_{i=1}^n x_i)$ is super-additive, while the function $g(\mathbf{x}) = (\sum_{i=1}^n x_i)/(1 + \sum_{i=1}^n x_i)$ is sub-additive. In dimension one, that is, if $n = 1$, convexity implies super-additivity and concavity implies sub-additivity. This, however, does not extend to $n \geq 2$; the relationship between convexity/concavity and super-/sub-additivity in the multi-dimensional case is quite subtle and we refer he reader to [8] for a detailed discussion on this issue.

We also briefly indicate how the two concepts arose from applications in economics by quoting [4]. For instance, if an aggregation function A is such that $A(\mathbf{x})$ represent production output for a vector $\mathbf{x}$ of manufacturing factors, the optimal output for every vector $\mathbf{x}$ of available resources would, by [4], be modeled by the value of $A^*(\mathbf{x})$; dually, if $A(\mathbf{x})$ is the cost of a set of items represented by $\mathbf{x}$, then an optimal price of items represented by $\mathbf{x}$ for every such vector would be equal to $A_*(\mathbf{x})$. Connections of the transformations (1) and (2) with both theory and applications also come from the so-called sub-decomposition and super-decomposition integrals of aggregation functions (see [3] for details).

We will focus on computation-theoretic aspects of exact determination of values of super- and sub-additive transformations. Namely definitions (1) and (2) in general require calculations ranging over simplices with potentially unbounded dimension. This motivates a study of approximation of such transformations of a given aggregation function. Our aim is to present a twofold way of considering this problem. In Section 2 we propose bounds on the transformations introduced above within a given error term. A different approach is contained in Section 3 by considering transformations of piecewise linear approximations of aggregation functions, restricted due to complexity to a one-dimensional case. A discussion of the results is deferred to the concluding Section 4.

This paper is based on a presentation of both authors at the 6th Iranian Joint Congress on Fuzzy and Intelligent Systems, and its extended abstract appeared in [5]. The original shorter version [5] has been considerably extended here. Moreover, by coupling two approximation approaches of Sections 2 and 3 as proposed in Section 4 we hope to give a new impetus to a further study of approximations of super- and sub-additive transformations.

2 Approximation within a given error term

Let L be a non-empty set of linearly independent unit vectors from $[0,\infty[^n$, so that $1 \leq |L| \leq n$. For each vector $\mathbf{u} \in L$ we let

$$s_{\mathbf{u}} = \limsup_{t\to 0^+} \frac{A(t\mathbf{u})}{t} \quad \text{and} \quad i_{\mathbf{u}} = \liminf_{t\to 0^+} \frac{A(t\mathbf{u})}{t} \ ; \tag{3}$$

being aware that both $s_{\mathbf{u}}$ and $i_{\mathbf{u}}$ may attain the value $+\infty$ in general. Obviously (3) is motivated by directional derivatives in the calculus of several variables, which explains our restriction to unit vectors $\mathbf{u} \in L$. Further, let $\mathbf{s}$ and $\mathbf{i}$ stand for the vectors $(s_{\mathbf{u}})_{\mathbf{u}\in L}$ and $(i_{\mathbf{u}})_{\mathbf{u}\in L}$.

The first estimates we present are motivated by an application of their simplified versions in [7]. Given a vector $\mathbf{x} \in \mathrm{Span}(L)$, that is, $\mathbf{x} = \sum_{\mathbf{u}\in L} c_{\mathbf{u}}\mathbf{u}$, we let $[\mathbf{x}]_L$ denote the coordinate vector $(c_{\mathbf{u}})_{\mathbf{u}\in L}$. Throughout, the notation $\mathbf{s}\cdot[\mathbf{x}]_L$ means the usual dot product of the two vectors, that is, $\mathbf{s}\cdot[\mathbf{x}]_L = \sum_{\mathbf{u}\in L} s_{\mathbf{u}}c_{\mathbf{u}}$; the same is assumed for symbolic notation of the form $\mathbf{s}[\mathbf{x}]_L$ with the dot omitted.

Proposition 1 *For every vector* $\mathbf{x} \in \mathrm{Span}(L)$ *one has* $A^*(\mathbf{x}) \geq \mathbf{s}\cdot[\mathbf{x}]_L$ *and* $A_*(\mathbf{x}) \leq \mathbf{i}\cdot[\mathbf{x}]_L$.

Proof. Given $\mathbf{u} \in L$, let a function $f_{\mathbf{u}} : [0,\infty[\to [0,\infty[$ be defined by $f_{\mathbf{u}}(z) = A(z\mathbf{u})$. For its super-additive transformation $f^*_{\mathbf{u}}$ we have, by [7],

$$f^*_{\mathbf{u}}(z) \geq \left(\limsup_{t\to 0^+} \frac{f_{\mathbf{u}}(t)}{t}\right)\cdot z = s_{\mathbf{u}} z \ .$$

From $A^*(z\mathbf{u}) \geq f^*_{\mathbf{u}}(z)$, super-additivity of A^* and the above inequalities one has

$$A^*(\mathbf{x}) = A^*\left(\sum_{\mathbf{u}\in L} c_{\mathbf{u}}\mathbf{u}\right) \geq \sum_{\mathbf{u}\in L} A^*(c_{\mathbf{u}}\mathbf{u}) \geq \sum_{\mathbf{u}\in L} f^*_{\mathbf{u}}(c_{\mathbf{u}}) \geq \sum_{\mathbf{u}\in L} s_{\mathbf{u}}c_{\mathbf{u}} \ , \tag{4}$$

which proves the first inequality. The proof of the second one, relying on the corresponding lower bound of [7], is analogous. □

The bounds in Proposition 1 are based on the behavior of an aggregation function A near the origin in directions given by the unit vectors $\mathbf{u} \in L$. The problem is that the fractions $A(t\mathbf{u})/t$ appearing in (3) may have drastically different values from $s_{\mathbf{u}}$ and $i_{\mathbf{u}}$ for t sufficiently away from zero. We propose here a possible way out, continuing to use the values of $A(t\mathbf{u})/t$ but taken at arbitrarily preassigned values of t that also depend on the set L. Let $t_\mathbf{u}$ ($\mathbf{u} \in L$) be arbitrarily chosen positive real numbers and let

$$\alpha_\mathbf{u} = \frac{A(t_\mathbf{u}\mathbf{u})}{t_\mathbf{u}} \ . \tag{5}$$

Further, let $\boldsymbol{\alpha} = (\alpha_\mathbf{u})_{\mathbf{u}\in L}$ be the vector with coordinates $\alpha_\mathbf{u}$ for $\mathbf{u} \in L$. We also use $\mathbf{1}$ to denote the all-one $|L|$-dimensional vector.

Theorem 1 *Let $A: \ [0,\infty[^n \to [0,\infty[$ be a continuous aggregation function, with $\alpha_\mathbf{u}$ for $\mathbf{u} \in L$ as introduced in (5). Then, for every $\varepsilon > 0$ there exist positive real numbers $\gamma_\mathbf{u}$ for $\mathbf{u} \in L$ such that for every $\mathbf{x} = \sum_{\mathbf{u}\in L} c_\mathbf{u}\mathbf{u}$ with $c_\mathbf{u} \geq \gamma_\mathbf{u}$ we have*

$$A^*(\mathbf{x}) \geq (\boldsymbol{\alpha} - \varepsilon\mathbf{1})\cdot[\mathbf{x}]_L \quad \text{and} \quad A_*(\mathbf{x}) \leq (\boldsymbol{\alpha} + \varepsilon\mathbf{1})\cdot[\mathbf{x}]_L \ . \tag{6}$$

Proof. We begin with attempting the left-hand side of (6). By the assumption that A is continuous, for any given $\varepsilon > 0$ there is a $\delta > 0$ (and we may also assume that $0 < \delta < t_\mathbf{u}$ for all $\mathbf{u} \in L$) such that for all $\tau \in [0, \delta]$ and all $\mathbf{u} \in L$ it holds that

$$\frac{A((t_\mathbf{u} - \tau)\mathbf{u})}{t_\mathbf{u} - \tau} > \alpha_\mathbf{u} - \varepsilon \ . \tag{7}$$

Let $\mathbf{x} \in [0,\infty[^n$ be any vector such that $\mathbf{x} = \sum_{\mathbf{u}\in I} c_\mathbf{u}\mathbf{u}$ where $c_\mathbf{u} \geq t_\mathbf{u}^2/\delta$ for each $\mathbf{u} \in I$. Also, let $d_\mathbf{u} = \lceil t_\mathbf{u}/\delta \rceil$. Our next claim is:

For every integer m and each $z_\mathbf{u}$ such that $(m-1)t_\mathbf{u} \leq z_\mathbf{u} \leq m t_\mathbf{u}$ there exists a non-negative real $\delta_\mathbf{u} \leq \delta$ such that $z_\mathbf{u} = m t_\mathbf{u} - \delta_\mathbf{u} d_\mathbf{u}$.

To see this, note that the bounds involving m reduce to $(m-1) \leq m - d_\mathbf{u}\delta_\mathbf{u}/t_\mathbf{u} \leq m$. Our claim then amounts to being able to find $\delta_\mathbf{u} \leq \delta$ such that $d_\mathbf{u}\delta_\mathbf{u}/t_\mathbf{u}$ can attain arbitrary values in $[0, 1]$, which is surely possible as $d_\mathbf{u}\delta/t_\mathbf{u} \geq 1$ by the choice of $d_\mathbf{u} = \lceil t_\mathbf{u}/\delta \rceil$.

Take $n_\mathbf{u} = \lceil c_\mathbf{u}/t_\mathbf{u} \rceil$, so that $n_\mathbf{u} - 1 < c_\mathbf{u}/t_\mathbf{u} \leq n_\mathbf{u}$ or, equivalently, $(n_\mathbf{u} - 1)t_\mathbf{u} < c_\mathbf{u} \leq n_\mathbf{u} t_\mathbf{u}$. By our claim applied to $z_\mathbf{u} = c_\mathbf{u}$ there is a $\delta_\mathbf{u} \in [0, \delta]$ such that $c_\mathbf{u} = n_\mathbf{u} t_\mathbf{u} - \delta_\mathbf{u} d_\mathbf{u}$. As this holds for every $\mathbf{u} \in L$, for our vector $\mathbf{x} = \sum_{\mathbf{u}\in L} c_\mathbf{u}\mathbf{u}$ with $c_\mathbf{u} \geq d_\mathbf{u} = \lceil t_\mathbf{u}/\delta \rceil$ for every $\mathbf{u} \in I$, we obtain

$$\mathbf{x} = \sum_{\mathbf{u}\in L} c_\mathbf{u}\mathbf{u} = \sum_{\mathbf{u}\in L} (n_\mathbf{u} t_\mathbf{u} - \delta_\mathbf{u} d_\mathbf{u})\mathbf{u} = \sum_{\mathbf{u}\in L} (n_\mathbf{u} - d_\mathbf{u})(t_\mathbf{u}\mathbf{u}) + \sum_{\mathbf{u}\in L} d_\mathbf{u}(t_\mathbf{u} - \delta_\mathbf{u})\mathbf{u} \ . \tag{8}$$

Note that by our earlier assumption $c_\mathbf{u} \geq t_\mathbf{u}^2/\delta$ we have $\lceil c_\mathbf{u}/t_\mathbf{u} \rceil \geq \lceil t_\mathbf{u}/\delta \rceil$, which means that $n_\mathbf{u} \geq d_\mathbf{u}$. Thus, in both sums that appear in (8), all the coefficient at the unit vectors $\mathbf{u} \in L$ are *positive*. This also means that the right-hand side of (8) is a positive-integral linear combination of the vectors $t_\mathbf{u}\mathbf{u}$ and 'slightly smaller' vectors $(t_\mathbf{u} - \delta_\mathbf{u})\mathbf{u}$; observe that $t_\mathbf{u} - \delta_\mathbf{u} > t_\mathbf{u} - \delta$.

Combining super-additivity of A^*, dominance of A by A^*, definition of $t_\mathbf{u}$ and the inequality (7) we subsequently obtain the following inequalities for our vector $\mathbf{x}$ as above:

$$\begin{aligned}
A^*(\mathbf{x}) &= A^*\left(\sum_{\mathbf{u}\in L}(n_\mathbf{u} - d_\mathbf{u})t_\mathbf{u}\mathbf{u} + \sum_{\mathbf{u}\in L} d_\mathbf{u}(t_\mathbf{u} - \delta_\mathbf{u})\mathbf{u}\right) \\
&\geq \sum_{\mathbf{u}\in L}(n_\mathbf{u} - d_\mathbf{u})A^*(t_\mathbf{u}\mathbf{u}) + \sum_{\mathbf{u}\in L} d_\mathbf{u} A^*((t_\mathbf{u} - \delta_\mathbf{u})\mathbf{u}) \\
&\geq \sum_{\mathbf{u}\in L}(n_\mathbf{u} - d_\mathbf{u})t_\mathbf{u}\alpha_\mathbf{u} + \sum_{\mathbf{u}\in L} d_\mathbf{u}(\alpha_\mathbf{u} - \varepsilon)(t_\mathbf{u} - \delta_\mathbf{u}) \\
&= \sum_{\mathbf{u}\in L}(n_\mathbf{u} t_\mathbf{u} - \delta_\mathbf{u} d_\mathbf{u})(\alpha_\mathbf{u} - \varepsilon) + \varepsilon\sum_{\mathbf{u}\in L}(n_\mathbf{u} - d_\mathbf{u})t_\mathbf{u} \\
&= \sum_{\mathbf{u}\in L} c_\mathbf{u}(\alpha_\mathbf{u} - \varepsilon) + \varepsilon\sum_{\mathbf{u}\in L}(n_\mathbf{u} - d_\mathbf{u})t_\mathbf{u} \ .
\end{aligned}$$

Letting $\mathbf{t} = (t_\mathbf{u})_{\mathbf{u}\in L}$, $\mathbf{n} = (n_\mathbf{u})_{\mathbf{u}\in L}$, $\mathbf{d} = (d_\mathbf{u})_{\mathbf{u}\in L}$, and using the vector $\boldsymbol{\alpha} = (\alpha_\mathbf{u})_{\mathbf{u}\in L}$ introduced earlier, the above conclusion admits the following useful restatement:

If $\mathbf{x} \in \mathrm{Span}(L)$ *is a vector such that the* $\mathbf{u}$*-th coordinate of* $[\mathbf{x}]_L$ *satisfies* $c_{\mathbf{u}} \geq t_{\mathbf{u}}^2/\delta$*, then*

$$A^*(\mathbf{x}) \geq (\boldsymbol{\alpha} - \varepsilon\mathbf{1})\cdot[\mathbf{x}]_L + \varepsilon(\mathbf{n} - \mathbf{d})\cdot\mathbf{t} \ . \tag{9}$$

Moreover, as $\mathbf{n} - \mathbf{d}$ is a vector with no negative coordinate, under the same assumptions we obtain $A^*(\mathbf{x}) \geq (\boldsymbol{\alpha} - \varepsilon\mathbf{1})\cdot[\mathbf{x}]_L$ which is the left-hand side of (6).

A proof of the right-hand side inequality in (6) may be obtained by a minor modification of the approach presented above. The assumptions stated before the inequality (7) imply:

$$\frac{A((t_{\mathbf{u}} - \tau)\mathbf{u})}{t_{\mathbf{u}} - \tau} < \alpha_{\mathbf{u}} + \varepsilon \ . \tag{10}$$

From this point on, using verbatim the same arguments as above until the end of the paragraph following (8), for any vector $\mathbf{x}$ one just needs to combine sub-additivity of A_*, dominance of A_* by A, definition of $t_{\mathbf{u}}$ and the inequality (10) to compile a corresponding chain of estimates:

$$\begin{aligned} A_*(\mathbf{x}) &= A_*\left(\sum_{\mathbf{u}\in L}(n_{\mathbf{u}} - d_{\mathbf{u}})t_{\mathbf{u}}\mathbf{u} + \sum_{\mathbf{u}\in L} d_{\mathbf{u}}(t_{\mathbf{u}} - \delta_{\mathbf{u}})\mathbf{u}\right) \\ &\leq \sum_{\mathbf{u}\in L}(n_{\mathbf{u}} - d_{\mathbf{u}})A_*(t_{\mathbf{u}}\mathbf{u}) + \sum_{\mathbf{u}\in L} d_{\mathbf{u}}A_*((t_{\mathbf{u}} - \delta_{\mathbf{u}})\mathbf{u}) \\ &\leq \sum_{\mathbf{u}\in L}(n_{\mathbf{u}} - d_{\mathbf{u}})t_{\mathbf{u}}\alpha_{\mathbf{u}} + \sum_{\mathbf{u}\in L} d_{\mathbf{u}}(\alpha_{\mathbf{u}} + \varepsilon)(t_{\mathbf{u}} - \delta_{\mathbf{u}}) \\ &= \sum_{\mathbf{u}\in L}(n_{\mathbf{u}}t_{\mathbf{u}} - \delta_{\mathbf{u}}d_{\mathbf{u}})(\alpha_{\mathbf{u}} + \varepsilon) - \varepsilon\sum_{\mathbf{u}\in L}(n_{\mathbf{u}} - d_{\mathbf{u}})t_{\mathbf{u}} \\ &= \sum_{\mathbf{u}\in L} c_{\mathbf{u}}(\alpha_{\mathbf{u}} + \varepsilon) - \varepsilon\sum_{\mathbf{u}\in L}(n_{\mathbf{u}} - d_{\mathbf{u}})t_{\mathbf{u}} \ . \end{aligned}$$

Using the symbols introduced before (9) the result of the above chain can be stated in the form

$$A_*(\mathbf{x}) \leq (\boldsymbol{\alpha} + \varepsilon\mathbf{1})\cdot[\mathbf{x}]_L - \varepsilon(\mathbf{n} - \mathbf{d})\cdot\mathbf{t} \ , \tag{11}$$

implying the right-hand side of (6) and completing the proof. □

It may be of interest to comment on what happens in Theorem 1 if the aggregation function A is already super- or sub-additive. Addressing the former case, suppose for instance that, given a super-additive function $A = A^* : \ [0,\infty[\to [0,\infty[$, one takes for L the set $\{\mathbf{e}_i;\ 1 \leq i \leq n\}$ of the standard unit vectors, so that $[\mathbf{x}]_L = (x_1, \ldots, x_n)$. Let us also make an arbitrary choice of $t_i > 0$, $1 \leq i \leq n$, so that $\alpha = (\alpha_1, \ldots, \alpha_n)$ for $\alpha_i = A(t_i\mathbf{e}_i)/t_i$. Then, the inequality $A(\mathbf{x}) = A^*(\mathbf{x}) \geq (\alpha - \varepsilon\mathbf{1})[\mathbf{x}]_L$ holds for every $\varepsilon > 0$ and (universally) for $\gamma_{\mathbf{e}_i} = t_i$.

3 Approximation by piecewise linear functions

In this section we will develop our ideas only for one-dimensional functions and comment on multi-dimensional cases later.

Let $A = \{a_0, a_1, \ldots, a_n, \ldots\}$ and $B = \{b_0, b_1, \ldots, b_n, \ldots\}$ be two infinite, strictly increasing sequences of real numbers such that $a_0 = b_0 = 0$, and let $\sigma_n = (b_{n+1} - b_n)/(a_{n+1} - a_n)$ for every $n \geq 0$. We will focus on a particular continuous piecewise linear function $f = f_{A,B}$ that depends on the two sequences. The function (of one variable) is defined by

$$f(x) = f_{A,B}(x) = b_n + \sigma_n(x - a_n) \quad \text{for} \quad x \in [a_n, a_{n+1}], \quad n \geq 0; \tag{12}$$

a part of f is depicted in Figure 1.

In what follows we will show that the determination of $f^*(z)$ for an arbitrary $z > 0$ is a finite problem, of (at most) exponential computational complexity. To this end we need to make a number of preliminary estimates.

From now on we fix an arbitrary $z > 0$ and we let n be the unique subscript for which $z \in [a_{n-1}, a_n[$. Further, let X be a finite collection of (not necessarily distinct) elements from the interval $]0, z]$ such that $\sum_{x\in X} = z$. We will begin by observations concerning the sum $f_X(z) = \sum_{x\in X} f(x)$.

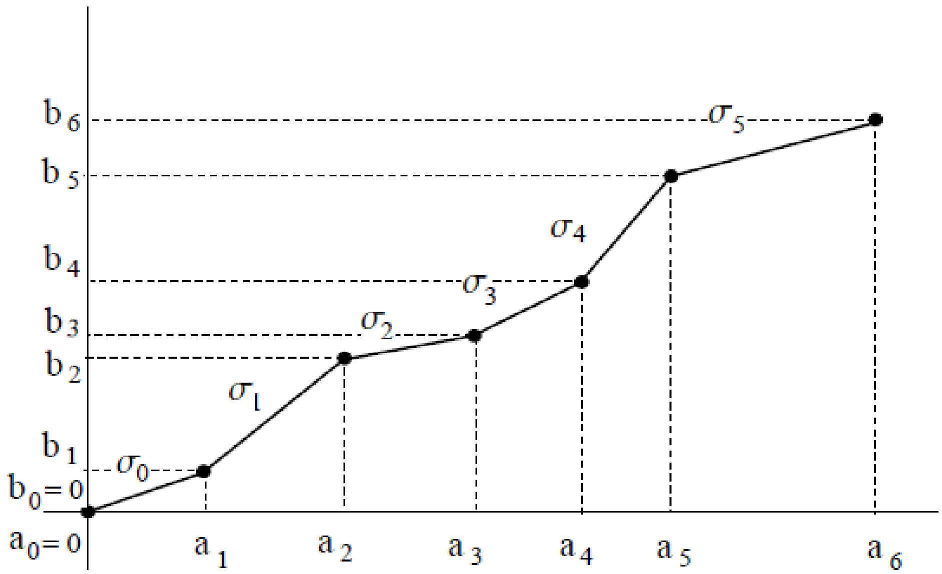

Figure 1: A piecewise linear continuous function $f = f_{A,B}$.

For $i \in \{0, 1, \ldots, n-1\}$ let X_i be the sub-collection of those $x \in X$ that belong to the interval $[a_i, a_{i+1}[$. Consider an i for which X_i is non-empty and consists of $k = k_i$ not necessarily distinct entries $x_{i1}, x_{i2}, \ldots, x_{ik}$. Letting $\overline{x}_i = (x_{i1}+x_{i2}+\ldots+x_{ik})/k \in [a_i, a_{i+1}[$ be the average of the entries in X_i, the part of $f_X(z)$ corresponding to the summation of entries in the interval $[a_i, a_{i+1}[$ becomes

$$\sum_{x\in X_i} f(x) = \sum_{j\le k}(b_i + \sigma_i(x_{ij} - a_i)) = k(b_i + \sigma_i(\overline{x}_i - a_i)) = kf(\overline{x}_i)\ . \tag{13}$$

Thus, instead of the $k = k_i > 0$ entries $x_{i1}, x_{i2}, \ldots, x_{ik} \in [a_i, a_{i+1}[$ we can take k_i times their average $\overline{x}_i \in [a_i, a_{i+1}[$ instead, leaving the value of $f_X(z)$ unchanged. To give a more detailed description, for i such that $|X_i| = k_i > 0$ we can without loss of generality assume that the collection X_i consists of $k_i > 0$ mutually equal entries $z_i = \overline{x}_i \in [a_i, a_{i+1}[$. Then, taking arbitrary $z_i \in [a_i, a_{i+1}[$ for those i for which $|X_i| = k_i = 0$, the entire collection X may be characterized by a choice of two n-tuples $(k_i)_{i<n}$ and $(z_i)_{i<n}$ as above and satisfying $\sum_{i<n} k_i z_i = z$; moreover, by (13) we then have $f_X(z) = \sum_{i<n} k_i f(z_i)$.

Our task of determining the super-additive transformation f^* of our piecewise linear function f given by (12) at an arbitrary $z > 0$ as above therefore reduces to determining

$$f^*(z) = \sup\{\sum_{i<n} k_i(b_i + \sigma_i(z_i - a_i))\} \tag{14}$$

taken over all n-tuples $(k_i)_{i<n}$ of non-negative integers and all n-tuples $(z_i)_{i<n}$ of *positive* real entries with $z_i \in [a_i, a_{i+1}[$ such that $\sum_{i<n} k_i z_i = z$. Our next goal is to show that this is a finite problem, despite of a potential infinitude of choices for the n-tuples $(k_i)_{i,n}$.

Indeed, for any $j \in \{1, \ldots, n-1\}$ one obviously has $z = \sum_{i<n} k_i z_i \ge k_j z_j \ge k_j a_j$ and so $k_j \le z/a_j$, $1 \le j \le n-1$. We will prove a similar estimate for k_0 next. Suppose that $k_0 \ge 2$ and $z_0 < a_1/2$. We may then replace the k_0-tuple of entries equal to z_0 by the (k_0-1)-tuple taking $2z_0 < a_1$ one time and z_0 the remaining (k_0-2)-times. The average of this (k_0-1)-tuple is easily seen to be equal to $z_0 \cdot k_0/(k_o - 1)$, which is larger than z_0 but still smaller than a_1. Repeating this process, mutatis mutandis, we either arrive at a situation when the latest average is at least as large as $a_1/2$, or when k_0 is reduced to 1. The conclusion is that if $k_0 \ge 2$, the k_0 entries z_0 may be replaced with either a single entry from $]0, a_1[$, or by some non-zero number k'_0 of mutually equal entries z'_0 lying in the interval $[a_1/2, a_1[$. With a slight abuse of notation by letting, without loss of generality, $k'_0 = k_0$ and $z'_0 = z_0$ in the second case, we obtain $z = \sum_{i<n} k_i z_i \ge k_0 z_0 \ge k_0 a_1/2$ and so $k_0 \le \max\{1, 2z/a_1\}$.

It follows that in determining $f^*(z)$ for $z > 0$ by (14) it is sufficient to consider only those n-tuples $(k_0, k_1, \ldots, k_{n-1})$ of non-negative integers for which $k_0 \le \max\{1, 2z/a_1\}$ and $k_j \le z/a_j$ for $j \in \{1, \ldots, n-1\}$. But there is just a finite number of such n-tuples, and for each of them the supremum in (14) becomes a linear programming problem in n variables $(z_i)_{i<n}$ of determining

$$\max \sum_{i<n} k_i(b_i + \sigma_i(z_i - a_i)) \quad \text{subject to} \quad \sum_{i<n} k_i z_i = z \ \text{ and } \ a_i \le z_i \le a_{i+1} \ (i < n) \ ; \tag{15}$$

note the slight relaxation of the problem by allowing equality in the constraints (which, of course, does not affect the result). The computational complexity of such an approach for our piecewise linear function $f = f_{A,B}$ at a given $z \in [a_{n-1}, a_n[$ for $n \to \infty$ is clearly bounded by $c_n p(n)$ where $p(n)$ is a polynomial representing the complexity of the above linear program in n variables and c_n is the number of n-tuples of positive integers $(k_i)_{i<n}$ with entries bounded as above, so that $c_n = \max\{1, 2z/a_1\} \prod_{1\le i<n}(z/a_i)$, and hence also $c_n \le z^n/(a_1^2 a_2 \dots a_{n-1})$ for sufficiently large z. We sum up our findings formally, including the result of a completely analogous analysis for the sub-additive transformation.

Theorem 2 *Let $f = f_{A,B}$ be a continuous piecewise linear function given by* (12)*. Then, for any $n \ge 0$ and any $z \in [a_{n-1}, a_n[$ the values of the super-additive transformation $f^*(z)$ and the sub-additive transformations $f_*(z)$ can be determined by a finite set of linear programs of the form* (15)*, with an appropriate minimization modification for f_*. Moreover, the total complexity of such a determination does not exceed $p(n)z^n/(a_1^2 a_2 \dots a_{n-1})$ for sufficiently large z, where $p(n)$ is a polynomial representing the complexity of the linear program* (15) *in n variables (in a maximization or minimization version).* □

Another way to look at (15) is to let the non-negative integers k_i bounded above by z/a_i for $1 \le i \le n-1$ and by $2z/a_1$ for $i = 0$ be additional n variables. In the above notation (and letting Z_0 be the set of non-negative integers) this leads to a quadratic semi-integer program of the form

$$f^*(z) = \max \sum_{i<n} k_i(b_i + \sigma_i(z_i - a_i)) \tag{16}$$

subject to the constraints listed in (15) and additional constraints

$$k_i \in Z_0 \ \ (0 \le i \le n-1), \quad k_0 \le 2z/a_1, \quad k_i \le z/a_i \ \ (1 \le i \le n-1) \ . \tag{17}$$

This semi-integer quadratic program may be approached by available methods in the theory of quadratic programming. We note that, in actual fact, our setting allows for derivation of more restrictions on the integers k_i but it is not our intention to develop more details of this kind here.

4 Conclusion

In this contribution we have attempted to make steps towards understanding the problem of approximating the values of super- and sub-additive transformations of aggregation function, because these transformations are defined by an intrinsically infinite process. We approached this task in two ways.

In section 2 we presented bounds on the values of $A^*(\mathbf{x})$ and $A_*(\mathbf{x})$ for an aggregation function A in terms of a function that is linear in Proposition 1, and linear for sufficiently large arguments in Theorem 1. The estimates in Theorem 1 look at a superficial glance slightly worse because of the built-in error term. However, on the contrary, they may actually be much better than those of Proposition 1 in a number of cases because of much more suitable 'starting values' $t_{\mathbf{u}}$, when compared with the suprema and infima in (3). Since it is rather difficult to illustrate this on a multi-dimensional example within a reasonably small space, we do so in one dimension.

Example 1. *Let $A(x) = \ln(1 + x)$ for $x \in [0, \infty[$ and let $\mathbf{u} = 1$ be the unit one-dimensional vector, that is, just the real number 1 in this case. Further, referring to the notation introduced before the statement of Theorem 1, let $t = t_{\mathbf{u}} > 0$ be arbitrary and let $\alpha = \alpha_{\mathbf{u}} = A(t)/t = \ln(1 + t)/t$. By Theorem 1 one obtains the lower bound $A_*(x) \le (\alpha + \varepsilon)x$ for every given $\varepsilon > 0$ and every sufficiently large x (depending on ε). This is definitely a much better bound on A for large x than $A(x) \le x$ which is implied by Proposition 1. Note that since A is concave, we have $A_* = A$ and with $\varepsilon \to 0$ and $t \to \infty$ the coefficient at x in our bound tends to zero, which is consistent with the derivative of A tending to zero as $x \to \infty$.*

Section 3 is, on the other hand, based on a different philosophy: namely, constructing first a piecewise linear approximation of an aggregation function and *afterwards* developing a method of an *exact* evaluation of the super- and sub-additive transformations of the piecewise linear approximation. In a more precise setting, this scenario could be applied as follows. First, for a given aggregation function A, construct a piecewise linear strictly increasing function f with $f(0) = 0$ and with no accumulation of 'corner points' such that, say, $|A(z) - f(z)| \leq \varepsilon$ for some preassigned $\varepsilon > 0$ and for all z. By the methods of section 3 we then may exactly determine the values of $f^*(z)$ by an algorithm of at most exponential complexity in z. This, of course, poses the interesting question of how good an approximation f for A should be chosen so that f^* would also be a good approximation for A^*. We leave this as an open question but point out that the answer may be non-trivial, as illustrated by the following simple one-dimensional example on the same function as before.

Example 2. *Take again $A(x) = \ln(1 + x)$ for $x \in [0, \infty[$. Let f be a piecewise linear approximation of A such that $|A(z) - f(z)| \leq \varepsilon$ as above* and *such that the 'corner points' of f all lie on the graph of A and have no accumulation point. Then, letting $z \to \infty$, the difference $|A^*(z) - f^*(z)|$ becomes arbitrarily large. The reason for this is as follows.*

On the one hand, it is an easy exercise in calculus to show that $A^*(x) = x$ for every $x \in [0, \infty[$. On the other hand, however, borrowing the notation introduced in Section 3 associated with our approximation function f, it is obvious that f^* is given by the steepest slope $(b_{n+1} - b_n)/(a_{n+1} - a_n)$ which, due to concavity of A, is the first slope, that is, $f^*(z) = ((b_1 - b_0)/(a_1 - a_0))z = (b_1/a_1)z$ where $b_1 = f(a_1) = \ln(1 + a_1)$. Moreover, again due to concavity of A one has $b_1/a_1 < 1$, and this strict inequality finally demonstrates that the difference $|A^*(z) - f^*(z)|$ grows beyond any limit.

In principle, the method of exact determination of f^* and f_* presented in section 3 generalizes to arbitrarily dimensional piecewise linear functions. In order to give a hint for a d-dimensional case with $d \geq 2$, one would have to consider a simplicial decomposition of $[0, \infty[^d$ into d-simplexes (that is, triangles for $d = 2$, tetrahedrons for $d = 3$, and so on) and define f to be linear 'above' each simplex, making sure in addition that f is continuous to get a more appealing situation (although the method of section 3 clearly does not depend on continuity).

We conclude with a hope that more methods of estimating values of the super- and sub-additive transformations of aggregation functions will be found. A possible way to proceed would be to merge the two methods of Sections 2 and 3 with the help of more powerful optimization methods.

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