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Regularity in residuated lattices

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Abstract

In this paper, we study residuated lattices in order to give new characterizations for dense, regular and Boolean elements in residuated lattices and investigate special residuated lattices in order to obtain new characterizations for the directly indecomposable subvariety of Stonean residuated lattices. Free algebra in varieties of Stonean residuated lattices is constructed. We introduce in residuated lattice a new type of filter called special filter and investigate its properties. Finally, regular filter property in residuated lattices is introduced and is studied in details.

Keywords: (semi) divisible residuated lattice, Boolean element, directly indecomposable algebra, free algebras.

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1 Introduction

Residuation is a fundamental concept of ordered structures and categories. The origin of residuated lattices is in mathematical logic without contraction. They have been investigated by Krull (1924), Dilworth (1939), Ward and Dilworth (1939), Ward (1940), Balbes and Dwinger (1974) and Pavelka (1979). In Idziak (1984) proves that the class of residuated lattices is equational. These lattices have been known under many names: *BCK-lattices* in Idziak (1984), *full BCK-algebras* in Okada and Terui (1999), *FL_{ew}-algebras* in Ono and Komori (1985), and *integral, residuated, commutative monoids* in Höhle (1995).

Ward (1940), Ward and Dilworth (1939) were the first who introduced the concept of a *residuated lattice* as a generalization of ideal lattices of a ring. In their original definition, a residuated lattice is what we would call it an *integral commutative* one. The general definition of a residuated lattice was given by Galatos et al.(2007). They first developed the structural theory of this kind of algebra about residuated lattices. Over the last ten years, with the computers and information in science developing rapidly, the residuated lattice theory made great progress. Many experts and scholars had carried out thorough systematical research into it, they studied it from different points of view. For example, Blount and Tsinakis (2003) took the residuated lattice theory as an expansion of the theory of groups; J. S. Olson (2008) investigated it from the view of variety; Galatos et al. (2007) investigated it from the view of semiring. The theory of residuated lattices was used to develop algebraic counterparts of fuzzy logics in Turunen (1999) and substructural logics in H. Ono (2003).

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Hájek (1998) investigated the notion of BL-algebras and the concepts of filters and prime filters in BL-algebras in order to provide an algebraic proof of the completeness theorem of *Basic Logic* (*BL*, for short), arising from the continuous triangular norms, familiar in the fuzzy logic frame-work. Using prime filters in BL-algebras, he proved the completeness of Basic Logic. Soon after, Turunen (1999) published a study on BL-algebras and their deductive systems.

A weaker logic than *BL* called *Monoidal T-norm Based Logic* (*MTL*, for short) was defined by Esteva and Godo (2001) and proved by Jenei and Montagna (2002) to be the logic of left continuous T-norms and their residua. The algebraic counterpart of this logic is *MTL-algebra*, also introduced by Esteva and Godo (2001). In Esteva and Godo (2001) a residuated lattice L is called *MTL-algebra* if the prelinearity property holds in L.

Recently, Turunen and Mertanen (2008) and D. Buşneag et al.(2013) defined the notion of semi-divisible residuated lattice and investigated their properties. Also, D. Buşneag et al.(2015) investigated the notion of Stonean residuated lattices and they discussed it from the view of ideal theory.

Cignoli (2008) investigated the structure of free algebras in the subvarieties of Stonean residuated lattices and proved that each algebra in a variety $\mathbf{V}$ of bounded residuated lattices can be represented as a weak Boolean product of directly indecomposable algebras in $\mathbf{V}$ over the Stone space of its Boolean skeleton (Theorem 1.3). In fact, free algebras are weak Boolean products of directly indecomposable algebras. In order to obtain his characterization the steps were to consider the sets of regular and dense elements ($Reg(L) = \{x \in L : x^{**} = x\}$, $D(L) = \{x \in L : x^{*} = 0\}$) and characterize the Boolean skeleton of residuated lattices. Based on the importance of regular, dense and Boolean elements in constructing free algebras, we realize that it will be interesting to investigate different notions of regular substructures of residuated lattices ($B(L)$ in Kowalski and Ono (2002), $Reg(L)$ in Cignoli (2008), $R(L)$ and $MV(L)$ in D. Buşneag et al.(2013), $M(L)$ introduced by us) in order to obtain new characterizations for these elements. We notice that the sets $MV(L)$ and $Reg(L)$ coincide. We study the relationship between different notions of regular elements in residuated lattices, with their sets denoted by $B(L), R(L), M(L)$ and $MV(L)$. Moreover, we offer a new characterization for dense (see Theorem 3), regular (see Proposition 8) and Boolean elements in a residuated lattice L (see Theorem 6). Taking as a guide line the case of *special BL-algebras* from Mohtashamnia and Borumand Saeid (2012), we define and study the class of *special residuated lattices*. We show (see Example 7) that the class of special residuated lattices is an independent class than *divisible residuated lattices*, *BL-algebras*, *MTL-algebras* and *G-algebras* and as an application to our study we prove that the class of special residuated lattices is directly indecomposable. Moreover, we prove that L *is a special residuated lattice* iff $B(L) = R(L) = M(L) = MV(L) = \{0, 1\}$, and L *is a Stonean residuated lattice* iff $B(L) = R(L) = M(L) = MV(L)$.

Cignoli (2008) in Theorem 2.4 obtain on a given residuated lattice L and an element $0 \notin L$, a structure $S(L) = L \cup \{0\}$ of a Stonean residuated lattice such that the set of dense elements $D(S(L)) = L$. Based on this result in Lemma 2.5 proved that for a Stonean residuated lattice L, $\{0\} \cup D(L)$ becomes the universe of a subalgebra of L witch is isomorphic with $S(D(L))$. Moreover, L is isomorphic with $S(D(L))$ iff L is directly indecomposable. Following our study on regular substructures of a residuated lattice L we obtain in Corollary 5 an equivalent result with Lemma 2.5, that is, L is a directly indecomposable Stonean residuated lattice iff L is a special residuated lattice.

We discuss briefly the applications of our results on classes of residuated lattices such as *Boolean algebras*, *Stonean residuated lattices* and *hyperarchimedean residuated lattices* and give some characterizations of them.

The paper is organized as follows:

In Section 2, we recall the basic definitions and we put in evidence rules of calculus in a residuated lattice which we need in the rest of the paper.

In Section 3, we define the set $M(L)$ and we investigate the relationship between different notions of regular elements in residuated lattices (see Cignoli (2008), D. Buşneag et al.(2013), D. Buşneag et al.(2015), Turunen and Mertanen (2008)) denoted by $B(L), R(L), M(L)$ and $MV(L)$. In D. Buşneag et al.(2013) proved that $B(L) \subseteq R(L) \subseteq MV(L)$, but the converse does not always hold. Also, they presented some characterizations for $R(L)$ and $MV(L)$. By Theorem 1, we conclude that $B(L) \subseteq R(L) \subseteq M(L) \subseteq MV(L)$, but the converse does not always hold (for that we offer relavent examples of residuated lattices). Based on the works D. Buşneag et al.(2015) and Mohtashamnia and Borumand Saeid (2012) we define the notion of *special residuated lattice*, we give some examples in order to establish the relationship between them and others algebraic structures. We introduce the notion of special filter in residuated lattices and we study its properties in detail.

In Section 4, we give a new characterization for Boolean elements in residuated lattices, we study some classes of directly indecomposable residuated lattices. We propose new characterizations for Stonean and Boolean residuated lattices.

In Section 5, we introduce the regular filter property in residuated lattices and we characterize residuated lattices endowed with this property. Moreover, we study the properties of hyperarchimedean residuated lattices and present some characterizations.

2 Preliminaries

In this section, we recall some basic notions relevant to residuated lattices which will need in the sequel.

Definition 1. [17] A *bounded residuated lattice* is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

(Lr_1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice;
(Lr_2) $(L, \odot, 1)$ is a commutative monoid;
(Lr_3) $\odot$ and $\rightarrow$ form an adjoint pair, i.e., $a \odot x \leq b$ iff $x \leq a \rightarrow b$.

We call them simply *residuated lattices.* Usually, (Lr_3) is called the residuation property. For examples of residuated lattices see Galatos et al.(2007), Hájek (1998), D. Buşneag et al.(2013), Freytes (2004), Kowalski and Ono (2002), Iorgulescu (2008), Piciu (2007) and Turunen (1999). If L is totally ordered, then L is called a *chain.*

For $x \in L$ and $n \geq 1$ we define $x^* = x \rightarrow 0$, $x^{**} = (x^*)^*$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$.

We refer to Hájek (1998), D. Buşneag et al.(2013), Piciu (2007)-Ward and Dilworth (1939) for detailed proofs of these well-known results:

Proposition 1. *For every $x, y, z \in L$, we have:*

(r_1) $x \rightarrow x = 1,\ x \rightarrow 1 = 1,\ 1 \rightarrow x = x$;
(r_2) $x \leq y$ *iff* $x \rightarrow y = 1$;
(r_3) *If* $x \leq y$, *then* $z \odot x \leq z \odot y$, $z \rightarrow x \leq z \rightarrow y$, $y \rightarrow z \leq x \rightarrow z$;
(r_4) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$;
(r_5) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$, $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;
(r_6) $(x \vee y)^* = x^* \wedge y^*$, $(x \wedge y)^* \geq x^* \vee y^*$, $(x \odot y)^{**} \geq x^{**} \odot y^{**}$;
(r_7) $x^{**} \rightarrow y^{**} = y^* \rightarrow x^* = x \rightarrow y^{**} = (x \rightarrow y^{**})^{**}$;
(r_8) $x \odot x^* = 0,\ 1^* = 0,\ 0^* = 1,\ x^{***} = x^*$;
(r_9) $x \leq x^{**},\ x^{**} \leq x^* \rightarrow x,\ x \rightarrow y \leq y^* \rightarrow x^*$;
(r_{10}) $x \rightarrow y \leq (x \rightarrow y)^{**} \leq x^{**} \rightarrow y^{**}$;

Following the above mentioned literature, if we consider the identities:

(i_1) $x \wedge y = x \odot (x \rightarrow y)$ $\quad$ $(divisibility)$;
(i_2) $(x^* \wedge y^*)^* = [x^* \odot (x^* \rightarrow y^*)]^*$ $\quad$ $(semi-divisibility)$;
(i_3) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ $\quad$ $(prelinearity)$;
(i_4) $x^* \vee x^{**} = 1$;
(i_5) $x^2 = x$;

Then the residuated lattice L is called:

(i) *Divisible* if L verifies (i_1);
(ii) *Semi-divisible* if L verifies (i_2);
(iii) *MTL-algebra* if L verifies (i_3);
(iv) *BL-algebra* if L verifies (i_1) and (i_3);
(v) *Stonean* if L verifies (i_4);
(vi) *G-algebra* if L verifies (i_5).

The class of G-algebras defined below represent a subclass of divisible residuated lattices, but different from MTL-algebras (see Buşneag et al. (2013), Buşneag et al. (2015) and Piciu (2007)).

An *MV-algebra* is an algebra $\mathcal{L} = (L, \oplus, *, 0)$ of type $(2, 1, 0)$ satisfying the following equations:

(mv_1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(mv_2) $x \oplus y = y \oplus x$;
(mv_3) $x \oplus 0 = x$;
(mv_4) $x^{**} = x$;
(mv_5) $x \oplus 0^* = 0^*$;
(mv_6) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$, for all $x, y, z \in L$.
Note that axioms $(mv_1) - (mv_3)$ state that $(L, \oplus, *, 0)$ is a commutative monoid.
Every BL-*algebra* L with $x^{**} = x$ for all $x \in L$, is an MV-*algebra*.

On any residuated lattice L (D. Buşneag et al.(2015), Turunen and Mertanen (2008)) we may define an operator $\oplus$ by setting for all $x, y \in L$,

$$x \oplus y = (x^* \odot y^*)^*. \tag{1}$$

By (r_4), the equation (1) is equivalent with

$$x \oplus y = x^* \to y^{**} = y^* \to x^{**}, \text{ for all } x, y \in L. \tag{2}$$

Proposition 2. [8] *For every $x, y, z \in L$, we have:*
(r_{11}) $x \oplus 0 = x^{**},\ x \oplus 1 = 1, x \oplus x^* = 1$;
(r_{12}) $x \oplus y = y \oplus x,\ x, y \leq x \oplus y$;
(r_{13}) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(r_{14}) *if* $x \leq y$, *then* $x \oplus z \leq y \oplus z$.

In what follows we will establish other necessary properties of operator $\oplus$ in a residuated lattice L.

Proposition 3. *Let $x, y, z \in L$. Then:*
(r_{15}) $(x \oplus y)^{**} = x \oplus y = x^{**} \oplus y^{**}$;
(r_{16}) *if* $x \vee y = 1$, *then* $x \oplus y = 1$;
(r_{17}) $x \oplus (y \vee z)^* = (x \oplus y^*) \wedge (x \oplus z^*)$;
(r_{18}) $x \oplus (y^* \vee z^*)^* = (x \oplus y) \wedge (x \oplus z)$;
(r_{19}) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$ *iff* $(x^{**} \to y^{**})^{**} \to y^{**} = (y^{**} \to x^{**})^{**} \to x^{**}$.

Proof. (r_{15}). We obtain successively $(x \oplus y)^{**} = (x^* \odot y^*)^{***} \overset{(r_8)}{=} (x^* \odot y^*)^* = x \oplus y \overset{(r_8)}{=} [(x^{**})^* \odot (y^{**})^*]^* = x^{**} \oplus y^{**}$.

(r_{16}). Since $x \leq x^{**}, y \leq y^{**}$ and $x^{**}, y^{**} \leq x \oplus y$ imply $x \vee y \leq x \oplus y$. Since $1 = x \vee y \leq x \oplus y$, then $x \oplus y = 1$.

(r_{17}). By $(r_5), (r_6), (r_8)$ and (2) we obtain successively $x \oplus (y \vee z)^* \overset{(2)}{=} x^* \to (y \vee z)^{***} \overset{(r_8)}{=} x^* \to (y \vee z)^* \overset{(r_6)}{=} x^* \to (y^* \wedge z^*) \overset{(r_5)}{=} (x^* \to y^*) \wedge (x^* \to z^*) \overset{(r_7)}{=} (y^{**} \to x^{**}) \wedge (z^{**} \to x^{**}) \overset{(2)}{=} (x \oplus y^*) \wedge (x \oplus z^*)$.

(r_{18}). By $(r_5), (r_6), (r_8)$ and (2) we obtain successively $x \oplus (y^* \vee z^*)^* \overset{(2)}{=} x^* \to (y^* \vee z^*)^{***} \overset{(r_8)}{=} x^* \to (y^* \vee z^*)^* \overset{(r_6)}{=} x^* \to (y^{**} \wedge z^{**}) \overset{(r_5)}{=} (x^* \to y^{**}) \wedge (x^* \to z^{**}) \overset{(2)}{=} (x \oplus y) \wedge (x \oplus z)$.

(r_{19}). Since $(x^* \oplus y)^* \oplus y \overset{(2)}{=} (x^{**} \to y^{**})^* \oplus y \overset{(2)}{=} (x^{**} \to y^{**})^{**} \to y^{**}$ and $(y^* \oplus x)^* \oplus x \overset{(2)}{=} (y^{**} \to x^{**})^* \oplus x \overset{(2)}{=} (y^{**} \to x^{**})^{**} \to x^{**}$. Thus, our claim holds. □

In literature (D. Buşneag et al.(2013),Kowalski and Ono (2002)) of residuated lattices the Boolean elements are known under the characterization: if L is a residuated lattice, then $e \in B(L)$ iff $e \vee e^* = 1$.

Proposition 4. [32] *If $e \in B(L)$, then $e^2 = e$, $e = e^{**}$ and $e^* \to e = e$.*

Deductive systems correspond to subsets closed with respect to Modus Ponens, their are also called filters, too. Apart from their logical interest, BL-algebras have important algebraic properties and they have been intensively studied from an algebraic point of view. Filters theory plays an important role in studying logical algebras.

Definition 2. [15] *A filter is a non-empty subset F of L such that*
(F_1) *if* $x \leq y$ *and* $x \in F$, *then* $y \in F$;
(F_2) *if* $x, y \in F$, *then* $x \odot y \in F$.
We denote by $\mathcal{F}(L)$ the set of all filters of L.

Also, $(\mathcal{F}(L), \subseteq)$ is the lattice of filters with respect to the inclusion of sets.

We recall (Piciu (2007),Turunen (1999)) that for a non-empty subset S of L we denote by $\langle S\rangle$ the filter generated by S, and $\langle S\rangle = \{x \in L : s_1 \odot ... \odot s_n \leq x, \text{ for some } s_1, ..., s_n \in S\}$. If $a \in L$, the filter generated by $\{a\}$ will be denoted by $\langle a\rangle$, and $\langle a\rangle = \{x \in L : a^n \leq x \text{ for some } n \geq 1\}$. If $F \in \mathcal{F}(L)$ and $a \in L \setminus F$, then $\langle F \cup \{a\}\rangle$ will be denoted by $F(a)$, and $F(a) = \{x \in L : a^n \to x \in F \text{ for some } n \geq 1\}$. If $F, G \in \mathcal{F}(L)$, then $F \vee G = F \vee_{\mathcal{F}(L)} G = \langle F \cup G\rangle = \{x \in L : b \odot c \leq x \text{ for some } b \in F, c \in G\}$.

A filter F is called *proper* filter if $F \neq L$. We say that a proper filter P is a *prime* filter (Piciu (2007)) if for $x, y \in L$ and $x \vee y \in P$, then $x \in P$ or $y \in P$. We denote by $Spec(L)$ the set of all prime filters of L.

We say that a proper filter P is an *inf-irreductible* filter (Piciu (2007)) if for any two proper filters M, N with $P = M \cap N$ imply $P = M = N$. It is known that prime and inf-irreductible filters are equivalent.

We recall that a proper filter M of L is called *maximal* if M is not strictly contained in any proper filter of L.

Every *maximal filter* M of L is obvious *prime* because, if there exist two proper filters $N, P \in \mathcal{F}(L)$ such that $M = N \cap P$, then $M \subseteq N$ and $M \subseteq P$, by the maximality of M we deduce that $M = N = P$, that is, M is an *inf-irreducible*, so *prime* element in the lattice of filters $(\mathcal{F}(L), \subseteq)$ of L (by the distributivity of the lattice of filters $(\mathcal{F}(L), \subseteq)$ of L).

So, if we denote by $Max(L)$ the set of all maximal filters of L, then $Max(L) \subseteq Spec(L)$.

Proposition 5. [17],[34] *For $M \in \mathcal{F}(L)$, $M \neq L$, the following statements are equivalent:*
(i) *M is maximal;*
(ii) *if $x \notin M$, then there exists $n \geq 1$ such that $(x^n)^* \in M$.*

Definition 3. [13],[25] *The intersection of the maximal filters of a residuated lattice L is called the radical of L and will be denoted by $Rad(L)$.*

Proposition 6. [13] $Rad(L) = \{x \in L : \text{ for every } n \geq 1 \text{ there is } k_n \geq 1 \text{ such that } [(x^n)^*]^{k_n} = 0\}$.

For any non-empty subset X of L, the *co-annihilator* of X is the set

$$^{\perp}X = \{a \in L : \; a \vee x = 1 \text{ for any } x \in X\}.$$

Clearly, $^{\perp}L = \{1\}$ and $^{\perp}\{1\} = L$. Similarly, if $F \in \mathcal{F}(L)$ is a proper filter of L, then by $^{\perp}F = \{a \in L | \; a \vee x = 1 \text{ for any } x \in F\}$ is denoted the *co-annihilator* of F. Clearly, $F \cap^{\perp} F = \{1\}$.

Proposition 7. [33] *$e \in B(L)$ iff $^{\perp}\langle e\rangle = \langle e^*\rangle$ iff $^{\perp\perp}\langle e\rangle = \langle e\rangle$.*

Lemma 1. [33, Remark 3.6] *If $F \in \mathcal{F}(L)$ is a proper filter of L, then $F =^{\perp\perp} F$ iff $L = F \vee^{\perp} F$.*

For $F \in \mathcal{F}(L)$ we define a relation $\equiv_F$ on L by $x \equiv_F y$ iff $x \to y, y \to x \in F$, for all $x, y \in L$ iff $(x \to y) \odot (y \to x) \in F$.

Then $\equiv_F$ is a congruence relation on L. For $x \in L$ we denote by $[x] = x/F$ the class of congruence of x modulo F and $L/F = \{x/F : x \in L\}$.

Define the binary operations $\vee, \wedge, \odot$ and $\to$ on L/F by $(x/F) \vee (y/F) = (x \vee y)/F$, $(x/F) \wedge (y/F) = (x \wedge y)/F$, and $(x/F) \odot (y/F) = (x \odot y)/F$, $(x/F) \to (y/F) = (x \to y)/F$ for all $x, y \in L$.

Then $(L/F, \vee, \wedge, \odot, \to, \mathbf{0}, \mathbf{1})$ is a residuated lattice, which is called the *quotient residuated lattice of L with respect to F*, where $\mathbf{0} = 0/F$ and $\mathbf{1} = 1/F$.

The relation of order on L/F is defined by $(x/F) \leq (y/F)$ iff $x \to y \in F$.

For a non-empty subset S of L we denote by $S/F = \{x/F : x \in S\}$. Clearly, for $x \in L$, $x/F = \mathbf{0}$ iff $x^* \in F$ and $x/F = \mathbf{1}$ iff $x \in F$.

3 Regular and dense elements in residuated lattices

In universal algebra's literature we have: if A is an algebra and $\triangle : A \times A \to A$ is a binary function, then an element $x \in A$ is called *regular* if verifies $x \triangle x = x$. By $Reg_{\triangle}(A)$ is denoted the set of regular elements of A with respect to $\triangle$. In residuated lattices we find different definitions for regular elements in Cignoli 2008, D. Buşneag et al.(2013), Kowalski and Ono (2002), Turunen and Mertanen (2008). In this section we study these notions.

In any residuated lattice L, we consider the subset

$$M_1(L) = \{z \in L : \ z = x \oplus y \text{ for some } x, y \in L \setminus \{0,1\}\}$$

and we denote by

$$M(L) := M_1(L) \cup \{0\}.$$

Lemma 2. *The following statements hold, for any $x, y, z \in L$.*
(i) $x^{**} = x$, for all $x \in M(L)$;
(ii) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$, for all $x \in L$ and $y, z \in M(L)$;
(iii) $y \to z \in M(L)$ for all $y \in M(L)$ and $z \in M(L) \setminus \{0\}$.

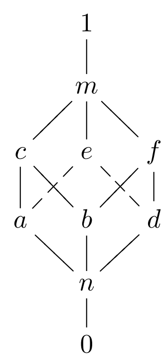
Proof. (i) Let $x \in M(L)$. If $x \in \{0,1\}$, then by (r_8) we get that $x^{**} = x$. If $x \in M(L) \setminus \{0,1\}$, by definition of $M(L)$, there are $a, b \in L \setminus \{0,1\}$ such that $x = a \oplus b$, by (r_8) we have $x^{**} = (a \oplus b)^{**} = [(a^* \odot b^*)^*]^{**} = (a^* \odot b^*)^{***} = (a^* \odot b^*)^* = a \oplus b = x$. Therefore, $x^{**} = x$, for all $x \in M(L)$.

(ii) By (i) and $y, z \in M(L)$, then we obtain successively $x \oplus (y \wedge z) \overset{(2),(i)}{=} x^* \to (y^{**} \wedge z^{**}) \overset{(r_5)}{=} (x^* \to y^{**}) \wedge (x^* \to z^{**}) \overset{(2)}{=} (x \oplus y) \wedge (x \oplus z)$.

(iii) Let $y \in M(L)$, $z \in M(L) \setminus \{0\}$. By (i) and (2) we obtain successively $y \to z = y^{**} \to z^{**} = y^* \oplus z \in M(L)$, then $y \to z \in M(L)$. By definition of $M(L)$, the commutativity and associativity of $\oplus$ we consequence that $M(L)$ is closed under $\oplus$. $\square$

Example 1. By Lemma 2, (iii) we deduce that the set $M(L)$ is not closed with respect to the operation $*$, that is, there are $x \in M(L)$ such that $x^* \notin M(L)$. For that we need an example:

We consider $L = \{0, n, a, b, c, d, e, f, m, 1\}$ with the Hasse diagram:



Then [23] L becomes a residuated lattice relative to the following operations :

$\to$	0	n	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1	1	1
a	f	f	1	f	1	f	1	f	1	1
b	e	e	e	1	1	e	e	1	1	1
c	d	d	e	f	1	d	e	f	1	1
d	c	c	c	c	c	1	1	1	1	1
e	b	b	c	b	c	f	1	f	1	1
f	a	a	a	c	c	e	e	1	1	1
m	n	n	a	b	c	d	e	f	1	1
1	0	n	a	b	c	d	e	f	m	1

$\odot$	0	n	a	b	c	d	e	f	m	1
0	0	0	0	0	0	0	0	0	0	0
n	0	0	0	0	0	0	0	0	0	n
a	0	0	a	0	a	0	a	0	a	a
b	0	0	0	b	b	0	0	b	b	b
c	0	0	a	b	c	0	a	b	c	c
d	0	0	0	0	0	d	d	d	d	d
e	0	0	a	0	a	d	e	d	e	e
f	0	0	0	b	b	d	d	f	f	f
m	0	0	a	b	c	d	e	f	m	m
1	0	n	a	b	c	d	e	f	m	1

By definition $0, 1 \in M(L)$ and we have $n = n \oplus n$, $a = a \oplus a$, $b = b \oplus b$, $c = c \oplus c$, $d = d \oplus d$, $e = e \oplus e$, $f = f \oplus f$. Clearly, $m \neq x \oplus y$ for all $x, y \in L \setminus \{0,1\}$, so $m \notin M(L)$. Therefore, $M(L) = \{0, n, a, b, c, d, e, f, 1\}$.

Since $n \in M(L)$, but $n^* = m \notin M(L)$, then we conclude that there are residuated lattices L such that if $x \in M(L)$, then is not necessary to have $x^* \in M(L)$.

Usually, *a regular element* $x \in L$ is defined as the element such that $x^{**} = x$ and the *set of all regular elements* of L is called the *MV-center* of L and it is denoted by $MV(L)$. It is known that $B(L) \subseteq MV(L)$.

D. Buşneag et al.(2013) introduced the notions of regular and dense elements in any residuated lattice L. They say that an element $x \in L$ is *regular* if for every $y \in L$ we have $(x \to y) \to x = x$. They denote by $R(L)$ the set of all regular elements of L and they showed that $B(L) \subseteq R(L) \subseteq MV(L)$. Also, an element $x \in L$ is *dense* if for every regular element $r \in R(L)$ we have $x \to r = r$. They denote by $D(L)$ the set of all dense elements of L.

In order to avoid misunderstandings we propose to clarify the differences between the subsets $MV(L), M(L)$ and $R(L)$ of any residuated lattice L.

D. Buşneag et al.(2013) present a characterization of regular elements in L. $x \in R(L)$ iff $x^* \to x = x$ iff $x = x^{**}$ and $x^* \odot (x^* \to x) = 0$.

We propose a new characterization of regular elements in L :

Proposition 8. (i) $x \in R(L)$ *iff* $x = x \oplus x$;
(ii) *if* $x, y \in R(L)$, *then* $x \oplus y \in R(L)$;
(iii) $R(L) \subseteq M(L)$;
(iv) $M(L) \subseteq MV(L)$.

Proof. (i) Let $x \in R(L)$. It is known that $x \in R(L)$ iff $x^* \to x = x$ iff $x = x^{**}$ and $x^* \odot (x^* \to x) = 0$ (see D. Buşneag et al.(2013)). We conclude that $x = x^{**}$ and $x = x^* \to x = x^* \to x^{**} = x \oplus x$. Conversely, consider $x \in L$ such that $x = x \oplus x$. By (r_9), $x \leq x^{**}$ and by residuation property we get that $x = x \oplus x = x^* \to x^{**} \geq x^{**}$. We conclude that $x = x^{**}$. It is known that $x \in R(L)$ iff $x^* \to x = x$. So, in order to prove that $x \in R(L)$ it is suffices to show that $x^* \to x = x$. Since $x = x^{**}$ and $x = x \oplus x = x^* \to x^{**}$, it follows that $x = x^* \to x$. Hence $x \in R(L)$.

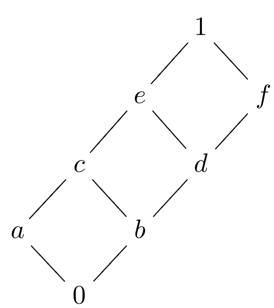
(ii) Consider $x, y \in R(L)$. Then $x = x \oplus x$ and $y = y \oplus y$. By (r_{12}) and (r_{13}) we obtain successively $x \oplus y = (x \oplus x) \oplus (y \oplus y) = (x \oplus y) \oplus (x \oplus y)$. By (i), we obtain $x \oplus y \in R(L)$.

(iii) Let $x \in R(L)$. Then $x = x \oplus x \in M(L)$, thus $R(L) \subseteq M(L)$.

(iv) Let $z \in M(L)$. If $z \in \{0, 1\}$, then clearly $z \in MV(L)$. If $z \in M(L) \setminus \{0, 1\}$, then there exist $x, y \in L \setminus \{0, 1\}$ such that $z = x \oplus y$. By Lemma 2, (i) we have $z = x \oplus y = (x \oplus y)^{**} = z^{**}$. Therefore, $z \in MV(L)$. □

Example 2. The set $R(L)$ is not closed with respect to the operation $*$. For that we consider the example from Example 1. Since $R(L) = \{0, n, a, b, c, d, e, f, 1\}$ and $n \in R(L)$, but $n^* = m \notin R(L)$.

Example 3. By Proposition 8(iii) we have $R(L) \subseteq M(L)$, but the converse does not hold. For the converse, we consider $L = \{0, a, b, c, d, e, f, 1\}$ with the Hasse diagram:



Then [23] L becomes an MTL-algebra with the Hasse diagram: the following operations:

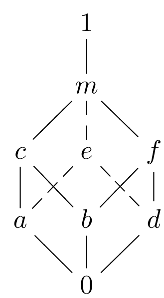
$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	f	1	f	1	f	1	f	1
b	e	e	1	1	1	1	1	1
c	d	e	f	1	f	1	f	1
d	c	c	c	c	1	1	1	1
e	b	c	b	c	f	1	f	1
f	a	a	c	c	e	e	1	1
1	0	a	b	c	d	e	f	1

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	a	0	a	0	a
b	0	0	0	0	0	0	b	b
c	0	a	0	a	0	a	b	c
d	0	0	0	0	d	d	d	d
e	0	a	0	a	d	e	d	e
f	0	0	b	b	d	d	f	f
1	0	a	b	c	d	e	f	1

We can see $R(L) = \{0, a, b, c, f, 1\}$ and $M(L) = \{0, a, b, c, e, f, 1\}$. Therefore, $R(L) \subset M(L)$.

Cignoli (2008) and Turunen and Mertanen (2008) consider for any residuated lattice L the subset $MV(L) := \{x^* \in L : \ x \in L\}$, or equivalently, $MV(L) := \{x \in L : \ x = x^{**}\}$, that is, the MV-center of L.

Example 4. By Proposition 8(iv) we have $M(L) \subseteq MV(L)$, but the converse does not hold. For the converse, we consider $L = \{0, a, b, c, d, e, f, m, 1\}$ with the Hasse diagram:



Then [23] L is a residuated lattice with the following operations:

$\rightarrow$	0	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1
a	m	1	m	1	m	1	m	1	1
b	m	m	1	1	m	m	1	1	1
c	m	m	m	1	m	m	m	1	1
d	m	m	m	m	1	1	1	1	1
e	m	m	m	m	m	1	m	1	1
f	m	m	m	m	m	m	1	1	1
m	m	m	m	m	m	m	m	1	1
1	0	a	b	c	d	e	f	m	1

$\odot$	0	a	b	c	d	e	f	m	1
0	0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	0	0	b
c	0	0	0	0	0	0	0	0	c
d	0	0	0	0	0	0	0	0	d
e	0	0	0	0	0	0	0	0	e
f	0	0	0	0	0	0	0	0	f
m	0	0	0	0	0	0	0	0	m
1	0	a	b	c	d	e	f	m	1

We have $MV(L) = \{0, m, 1\}$ and $M(L) = \{0, 1\}$, so $M(L) \subset MV(L)$ and $M(L) \neq MV(L)$.

Therefore, in the following result we specify the differences between the subsets $R(L)$, $M(L)$ and $MV(L)$ of any residuated lattice L.

Theorem 1. $R(L) \subseteq M(L) \subseteq MV(L)$.

Corollary 1. *The following assertions hold:*
(i) *if* $x \in R(L)$, *then* $x^{**} = x \oplus x = x$, *so* $x^{**} \in R(L)$;
(ii) *if for all* $x \in MV(L)$, $(x^*)^2 = x^*$, *then* $R(L) = MV(L)$.

Proof. (i) Consider $x \in R(L)$, by Proposition 8, we have $x = x \oplus x$. Since $R(L) \subseteq M(L)$ and by (r_{15}), we obtain $x^{**} = (x \oplus x)^{**} = x \oplus x = x$, so $x^{**} \in R(L)$.

(ii) We have $R(L) \subseteq MV(L)$. Now, we prove $MV(L) \subseteq R(L)$. Consider $x \in MV(L)$, by hypothesis we have $(x^*)^2 = x^*$, then we obtain successively $[(x^*)^2]^* = x^{**}$, $x \oplus x = x^{**}$. Since $x \in MV(L)$, then $x = x^{**}$, and so $x \oplus x = x$, that is $x \in R(L)$. Hence $MV(L) \subseteq R(L)$. $\square$

The following result is a consequence of Theorem 1 and Theorem 8 (D. Buşneag et al.(2013)), it represents under which suitable conditions all the sets $R(L)$, $M(L)$ and $MV(L)$ become equals:

Corollary 2. *Let $x \in MV(L)$. Then $R(L) = M(L) = MV(L)$ iff $x^* \odot (x^* \to x) = 0$.*

Proof. We have $R(L) \subseteq M(L) \subseteq MV(L)$. It suffices to prove that $MV(L) \subseteq R(L)$. By hypothesis $x = x^{**}$, (that is, $x \in MV(L)$). By hypothesis and residuation property we obtain $x^* \odot (x^* \to x) = 0$, $x^* \to x \leq x^{**} = x$, so $x^* \to x \leq x$. Since $x \leq x^* \to x$, we deduce $x^* \to x = x$. Let $a \in L$. Then $x^* \leq x \to a$, $(x \to a) \to x \leq x^* \to x = x$, so $(x \to a) \to x \leq x$. Since $x \leq (x \to a) \to x$, we deduce $(x \to a) \to x = x$, for any $a \in L$, that is $x \in R(L)$. Hence $MV(L) \subseteq R(L)$. Therefore, $R(L) = M(L) = MV(L)$.

Conversely, if $x \in MV(L)$, then $x = x^{**}$. Since $x \in R(L)$, by definition for any $a \in L$, $(x \to a) \to x = x$, if we consider $a = 0$, then $x^* \to x = x$. By (r_8), $0 = x \odot x^* = (x^* \to x) \odot x^*$. Hence $x^* \odot (x^* \to x) = 0$. □

A directly consequence of Corollary 1, Corollary 2 and Theorem 8 (D. Buşneag et al.(2013)) is the following:

Corollary 3. *Let $x \in L$. Then the following assertions are equivalent:*
(i) $x \in R(L)$;
(ii) $x \in MV(L)$ and $x^* \odot (x^* \to x) = 0$.

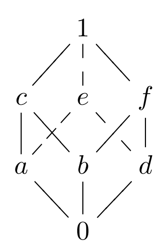
Turunen and Mertanen (2008) proved a necessary and sufficient condition when *the subset $MV(L)$ generates an MV-algebra* (see Theorem 1, Turunen and Mertanen (2008)). We study under which conditions $(M(L), \oplus, *, 0)$ *generates an MV-algebra.* Following Lemma 1 these two sets $M(L)$ and $MV(L)$ are not equivalent!

Theorem 2. *Let $x, y \in L$. If $x^* \to (x^* \to y^*)^* = y^* \to (y^* \to x^*)^*$, then $(M(L), \oplus, *, 0)$ generates an MV-algebra.*

Proof. Let $x, y \in L$ be such that $x^* \to (x^* \to y^*)^* = y^* \to (y^* \to x^*)^*$. By (r_7) for all $x, y \in L$, we obtain successively $x^* \to (x^* \to y^*)^* \overset{(r_7)}{=} (x^* \to y^*)^{**} \to x^{**} \overset{(2)}{=} (x^* \to y^*)^* \oplus x \overset{(r_7)}{=} (y^{**} \to x^{**})^* \oplus x \overset{(2)}{=} (y^* \oplus x)^* \oplus x$. Similarly, $y^* \to (y^* \to x^*)^* = (x^* \oplus y)^* \oplus y$. Hence (mv_6) holds. By Proposition 2 is easy to verify that the conditions $(mv_1) - (mv_5)$ hold in $(M(L), \oplus, *, 0)$. Hence $(M(L), \oplus, *, 0)$ is an $MV-$algebra. □

Now, we give an example of a residuated lattice L which satisfies the condition of Theorem 2.

Example 5. Consider L=$\{0, a, b, c, d, e, f, 1\}$ with the Hasse diagram:



Then [23] L becomes a residuated lattice with the following operations:

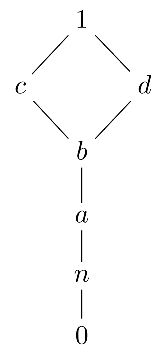
$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	f	1	f	1	f	1	f	1
b	e	e	1	1	e	e	1	1
c	d	e	f	1	d	e	f	1
d	c	c	c	c	1	1	1	1
e	b	c	b	c	f	1	f	1
f	a	a	c	c	e	e	1	1
1	0	a	b	c	d	e	f	1

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	a	0	a	0	a
b	0	0	b	b	0	0	b	b
c	0	a	b	c	0	a	b	c
d	0	0	0	0	d	d	d	d
e	0	a	0	a	d	e	d	e
f	0	0	b	b	d	d	f	f
1	0	a	b	c	d	e	f	1

Then $M(L) = \{0, a, b, c, d, e, f, 1\}$ and $(mv_1) - (mv_6)$ hold in $(M(L), \oplus, *, 0)$. Hence $(M(L), \oplus, *, 0)$ is an $MV-$algebra. Also, the condition from Theorem 2, $x^* \to (x^* \to y^*)^* = y^* \to (y^* \to x^*)^*$, for all $x, y \in L$, holds in L.

The converse of Theorem 2 does not hold, so there exists a residuated lattice L such that $(M(L), \oplus, *, 0)$ generates an MV-algebra and the condition $x^* \to (x^* \to y^*)^* = y^* \to (y^* \to x^*)^*$, for all $x, y \in L$, does not hold.

Example 6. Let $L = \{0, n, a, b, c, d, 1\}$ with the Hasse diagram:



Then [23] L becomes a distributive residuated lattice relative to the following operations:

$\to$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	d	1	1	1	1	1	1
a	n	n	1	1	1	1	1
b	n	n	a	1	1	1	1
c	0	n	a	d	1	d	1
d	n	n	a	c	c	1	1
1	0	n	a	b	c	d	1

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	0	0	0	n	0	n
a	0	0	a	a	a	a	a
b	0	0	a	b	b	b	b
c	0	n	a	b	c	b	c
d	0	0	a	b	b	d	d
1	0	n	a	b	c	d	1

Then $M(L) = \{0, n, 1\}$ is a three-element chain algebra, then $(M(L), \oplus, *, 0)$ generates an MV-algebra, but the condition of Theorem 2 does not hold for the elements a and n. Since $n^{**} = n$ and $d^{**} = d$, then $MV(L) = \{0, n, d, 1\}$, then L is another example of residuated lattice such that $M(L) \subset MV(L)$.

Corollary 4. *Let L be a semi-divisible residuated lattice. Then $(M(L), \oplus, *, 0)$ generates an MV-algebra.*

Proof. Let L be a semi-divisible residuated lattice. Then for all $x, y \in L$, we have $(x^* \wedge y^*)^* \overset{(i_2)}{=} [x^* \odot (x^* \to y^*)]^* \overset{(r_4)}{=} x^* \to (x^* \to y^*)^*$ and $(y^* \wedge x^*)^* \overset{(i_2)}{=} [y^* \odot (y^* \to x^*)]^* \overset{(r_4)}{=} y^* \to (y^* \to x^*)^*$. Since $(x^* \wedge y^*)^* = (y^* \wedge x^*)^*$, we deduce that $x^* \to (x^* \to y^*)^* = y^* \to (y^* \to x^*)^*$, by Theorem 2, the claim holds. $\square$

Remark 1. There exists a residuated lattice L such that $(M(L), \oplus, *, 0)$ generates an MV-algebra, but L is not semi-divisible.

Consider the Example 6, we have $(n^* \wedge d^*)^* = (d \wedge n)^* = n^* = d$ and $[n^* \odot (n^* \to d^*)]^* = [d \odot (d \to n)]^* = (d \odot n)^* = 0^* = 1$, then $(n^* \wedge d^*)^* = d \neq 1 = [n^* \odot (n^* \to d^*)]^*$, so L is not semi-divisible. Since $M(L) = \{0, n, 1\}$, then $(M(L), \oplus, *, 0)$ generates an MV-algebra, but L is not semi-divisible.

We recall that the *set of all dense elements* in a residuated lattice is denoted by $D(L) = \{x \in L : x^* = 0\}$ and if L is distributive then $D(L)$ becomes a filter (see D. Buşneag et al.(2013), Freytes (2004), Mureşan (2010) and Piciu (2007)).

Remark 2. If L is a residuated lattice, then $D(L)$ becomes a proper filter. Indeed, $0 \notin D(L)$. Since $1^* = 0$, then $1 \in D(L)$. If $x \leq y$ and $x \in D(L)$, by (r_3) we get $y^* \leq x^* = 0$, that is, $y \in D(L)$. Let $x, y \in D(L)$. Then $x^* = y^* = 0$. By (r_4), $(x \odot y)^* = (x \odot y) \to 0 = x \to y^* = x \to 0 = x^* = 0$, hence $(x \odot y)^* = 0$, that is, $x \odot y \in D(L)$. Therefore, $D(L)$ is a proper filter of L.

Theorem 3. *In L, the following conditions are equivalent:*
(i) $x^* = 0$, *for any* $x \in L \setminus \{0\}$;
(ii) $(x \to y)^* = (y \to x)^*$, *for any* $x, y \in L \setminus \{0\}$;
(iii) *if* $x \odot y = 0$, *then* $x = 0$ *or* $y = 0$, *for all* $x, y \in L$;
(iv) $B(L) = R(L) = M(L) = MV(L) = \{0, 1\}$.

Proof. (i) $\Rightarrow$ (ii) Let L be a residuated lattice such that $x^* = 0$, for any $x \in L \setminus \{0\}$. Then there exist $0 \neq c, d \in L$ such that $(x \to y)^* = c^* = 0$ and $(y \to x)^* = d^* = 0$, for any $x, y \in L \setminus \{0\}$. If $x \to y = 0$, by (r_2) and (r_4) we obtain successively $x \odot y \leq y \leq x \to y = 0$, $x \odot y = 0$, $(x \odot y) \to 0 = 1$, $x \to (y \to 0) = 1$, $x \to y^* = 1$, then $x \overset{(r_2)}{\leq} y^* = 0$. Hence $x = 0$, which is a contradiction, so $x \to y \neq 0$. We deduce that $(x \to y)^* = (y \to x)^* = 0$, for any $x, y \in L \setminus \{0\}$.

(ii) $\Rightarrow$ (i) Consider L is a residuated lattice such that $(x \to y)^* = (y \to x)^*$, for any $x, y \in L \setminus \{0\}$. Consider $y = 1$, then $(x \to 1)^* = (1 \to x)^*$ for any $x \in L \setminus \{0\}$. Hence $x^* = 0$, for any $x \in L \setminus \{0\}$.

(i) $\Rightarrow$ (iii) Let L be a residuated lattice such that $x^* = 0$, for any $x \in L \setminus \{0\}$. Then the set of all dense elements is $D(L) = L \setminus \{0\}$. Now, we prove that the only element of finite order in L is 0. Suppose that there is $0 < x \in L$ of order $n > 0$, (that is $x^n = 0$). By residuation property we have $x^n = 0$, $x^{n-1} \odot x = 0$, $x^{n-1} \leq x^* = 0$, $x^{n-1} = 0$, hence $ord(x) = n - 1$, which is a contradiction, (because $ord(x) = n$). Therefore, 0 is the only element of finite order.

Now, by Remark 2 we conclude that $D(L) = L \setminus \{0\}$ is a maximal filter of L. Since $D(L) = L \setminus \{0\}$ is a maximal filter of L and $x \odot y = 0$, then $x \notin D(L)$ or $y \notin D(L)$, that is, $x = 0$ or $y = 0$.

(iii) $\Rightarrow$ (i) Let L be a residuated lattice and for all $x, y \in L$, if $x \odot y = 0$, then $x = 0$ or $y = 0$. If $x = y = 0$, then $x^* = y^* = 1$. Now, we consider $x \in L \setminus \{0\}$. Since $x \odot x^* = 0$, we get $x^* = 0$. Therefore, $x^* = 0$, for any $x \in L \setminus \{0\}$.

(i) $\Rightarrow$ (iv) It is clear that $\{0, 1\} \subseteq R(L) \cap M(L) \cap MV(L)$. Suppose by contrary that there exists $x \in L \setminus \{0, 1\}$ such that $x \in R(L)$. By Proposition 8, $x \in R(L)$ iff $x = x \oplus x$, but $x \oplus x = [(x^*)^2]^* = (0^2)^* = 0^* = 1$, so $x = 1$, which is a contradiction. Therefore, $R(L) = \{0, 1\}$. Also if there exists $x \in L \setminus \{0, 1\}$ such that $x \in MV(L)$. By definition $x \in MV(L)$ iff $x^{**} = x$, but $x^{**} = (x^*)^* = 0^* = 1$, hence $x = 1$, which is a contradiction. Therefore, $MV(L) = \{0, 1\}$. Hence $R(L) = M(L) = MV(L) = \{0, 1\}$.

(iv) $\Rightarrow$ (i) Let L be a residuated lattice such that $MV(L) = \{0, 1\}$. In order to prove (i), we suppose by contrary that there is an element $x \in L \setminus \{0\}$ such that $x^* \neq 0$. By (r_8), $x^* = (x^*)^{**}$, then $x^* \in MV(L) = \{0, 1\}$. Because $x^* \neq 0$, we deduce $x^* = 1$, that is $x = 0$, which is a contradiction. Therefore, $x^* = 0$, for all $x \in L \setminus \{0\}$. Clearly, if $MV(L) = \{0, 1\}$, then $B(L) = R(L) = M(L) = MV(L) = \{0, 1\}$.

□

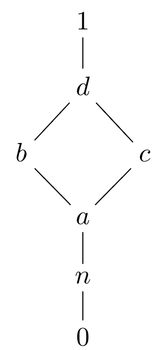
Definition 4. A residuated lattice L which verifies one of the equivalent conditions of Theorem 3 is called *special*. We denote by $SpRL$ the *class of special residuated lattices*.

We notice that every special residuated lattice L is Stonean. Indeed, by Theorem 3, since L is *special*, then $x^* \vee x^{**} = 0 \vee 1 = 1$, for any $x \in L$, hence L is Stonean.

In order to avoid misunderstandings we propose to clarify the differences between the class of *special residuated lattices* and divisible residuated lattices, $MTL-$algebras and $G-$algebras.

In the following example we present a *special residuated lattice* L which is not divisible residuated lattices, $MTL-$algebra or $G-$algebra.

Example 7. Let $L = \{0, n, a, b, c, d, 1\}$ with the Hasse diagram:



Then [23] L becomes a residuated with the operations:

$\rightarrow$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	0	1	1	1	1	1	1
a	0	d	1	1	1	1	1
b	0	c	c	1	c	1	1
c	0	b	b	b	1	1	1
d	0	a	a	b	c	1	1
1	0	n	a	b	c	d	1

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	n	n	n	n	n	n
a	0	n	n	n	n	n	a
b	0	n	n	b	n	b	b
c	0	n	n	n	c	c	c
d	0	n	n	b	c	d	d
1	0	n	a	b	c	d	1

it is easy to see that L is a *special residuated lattice*. Since $b \wedge a = a \neq n = b \odot c = b \odot (b \rightarrow a)$, then the divisibility property (i_1) is not verified, hence it is not a divisible residuated lattice. Since $1 \neq d = c \vee b = (b \rightarrow c) \vee (c \rightarrow b)$, then the prelinearity property (i_3) is not verified, hence it is not an $MTL-$algebra. Since $a^2 = n \neq a$, then the property (i_5) is not verified, we deduce that L is not a $G-$algebra.

We recall (Ciungu (2009), Kowalski and Ono (2002)) that a nontrivial residuated lattice L is *directly indecomposable* iff $B(L) = \{0, 1\}$. Dually, a residuated lattice L which is not directly indecomposable is called *directly decomposable*.

D. Buşneag et al.(2015) introduced the notion of *special Stonean residuated lattices* and they present a characterization for that. Similarly, we define in the general case of residuated lattice the notion of special residuated lattices. In fact, we show that special residuated lattices are the directly indecomposable member of the variety of Stonean residuated lattices, as we can see in the following result:

Corollary 5. *L is a directly indecomposable Stonean residuated lattice iff L is a special residuated lattice.*

Proof. Clearly, L is a special residuated lattice iff $L = \{0\} \cup D(L)$. Cignoli (2008), in Lemma 2.5 proved that L is a directly indecomposable Stonean residuated lattice iff $L = \{0\} \cup D(L)$. Therefore, L is a directly indecomposable Stonean residuated lattice iff L is a special residuated lattice. Also, this result is a consequence of Theorem 3, $(i) \Leftrightarrow (iv)$. □

Remark 3. If L is a special residuated lattice, then it is clear that $D(L) = L \setminus \{0\}$. If L is a nontrivial MV-algebras, then $x^{**} = x$, for all $x \in L$. Hence $D(L) = \{1\}$, then L is not a special residuated lattice. Furthermore, if L is a nontrivial residuated lattice with double negation property, then L is not special.

If L is a special residuated lattice, then $x^{**} = 1$, for all $0 \neq x \in L$. Therefore, L is not an MV-algebra, and double negation property doesn't hold (that is, $x^{**} = x$ for all $x \in L$).

A residuated lattice L is said to be Glivenko iff for all $x \in L$, $(x^{**} \rightarrow x)^{**} = 1$.

Remark 4. (i) Any residuated lattice L with double negation property (that is, $x = x^{**}$, for all $x \in L$) is Glivenko. Indeed, if $x \in L$, then $x = x^{**}$ and $(x^{**} \rightarrow x)^{**} = (x \rightarrow x)^{**} = 1^{**} = 1$.

(ii) If L is a special residuated lattice, then L is Glivenko (the converse does not always hold in L).

Indeed, by (r_9), we have $x \leq x^{**}$, for all $x \in L$, then following Theorem 3 we obtain successively $(x \rightarrow x^{**}) = 1$, $(x \rightarrow x^{**})^* = 0$, $(x^{**} \rightarrow x)^* = 0$, $(x^{**} \rightarrow x)^{**} = 1$. Therefore, L is Glivenko.

We deduce that any special residuated lattice L is Glivenko, but the converse does not always hold. Consider the residuated lattice L from Example 5, then L is a Glivenko residuated lattice, but L is not special (because $a^* = f$).

Corollary 6. *The following assertions are equivalent:*
(i) *L is a special residuated lattice;*
(ii) $x \to y^{**} = y \to x^{**}$, *for all* $x, y \in L \setminus \{0\}$;
(iii) $x^{**} \to y^{**} = y^{**} \to x^{**}$, *for all* $x, y \in L \setminus \{0\}$.

Proof. (i) $\Rightarrow$ (ii). By Theorem 3 and Proposition 1, for all $x, y \in L$, we have $1 = (y^{**} \to y)^{**} \overset{(r_3)}{\leq} [(x \to y^{**}) \to (x \to y)]^{**} \overset{(r_9)}{\leq} [(x \to y^{**}) \to (x \to y)^{**}]^{**} \overset{(r_4)}{=} [(x \to y^{**}) \odot (x \to y)^*]^* \overset{(r_4)}{=} (x \to y^{**}) \to (x \to y)^{**}$. Hence $(x \to y^{**}) \leq (x \to y)^{**}$. By Proposition 1 we have $y \leq y^{**}$, $x \to y \leq x \to y^{**}$, $(x \to y)^{**} \overset{(r_9)}{\leq} (x \to y^{**})^{**} \overset{(r_4)}{=} [(x \odot y^*)^*]^{**} \overset{(r_8)}{=} (x \odot y^*)^* \overset{(r_4)}{=} x \to y^{**}$. Hence $(x \to y)^{**} \leq (x \to y^{**})$. Therefore, $(x \to y^{**}) = (x \to y)^{**}$. Similarly, it follows $(y \to x^{**}) = (y \to x)^{**}$. By Theorem 3 we have $(x \to y)^* = (y \to x)^*$, $(x \to y)^{**} = (y \to x)^{**}$, and so $(x \to y^{**}) = (x \to y)^{**} = (y \to x)^{**} = y \to x^{**}$. Therefore, $x \to y^{**} = y \to x^{**}$.

(ii) $\Rightarrow$ (i). Assume $x \to y^{**} = y \to x^{**}$, for all $0 \neq x, y \in L$. If we consider $x = 1$, then by Proposition 1 we obtain successively $1 \to y^{**} = y \to 1^{**}$, $y^{**} = 1$, $y^{***} = 1^*$, $y^* = 0$. Therefore, by Theorem 3 we deduce L is special.

(ii) $\Leftrightarrow$ (iii). By Proposition 1, (r_7) we have $x \to y^{**} = x^{**} \to y^{**}$ and $y \to x^{**} = y^{**} \to x^{**}$. Since $x \to y^{**} = y \to x^{**}$, then $x^{**} \to y^{**} = x \to y^{**} = y \to x^{**} = y^{**} \to x^{**}$. □

Mohtashamnia and Borumand Saeid (2012) introduce the notion of *special filters* in BL-algebras, in the same manner we consider this notion in residuated lattices and we investigate some properties.

Definition 5. A filter F of a residuated lattice L is said to be *special* iff for all $x, y \in L$, $(x \to y)^* = (y \to x)^*$. We denote by $Fil_{sp}(L)$ the set of all special filters of L.

If we consider the residuated lattice L from Example 7, then $\langle n \rangle = \{n, a, b, c, d, 1\}$ is a special filter of L. Now, if we consider the residuated lattice L from Example 6, then $\langle a \rangle = \{a, b, c, d, 1\}$ is a filter of L, but it is not a special filter because $(c \to d)^* = d^* = n \neq 0 = c^* = (d \to c)^*$.

We recall that an element of finite order of L is called *nilpotent*. We denote by $N(L) = \{x \in L : x^n = 0,\ \text{for some } n \geq 1\}$ the *set of all nilpotent elements* of L. We denote by $D(F) = \{x \in F : x^* = 0\}$ the *set of all dense elements* of a filter F.

Proposition 9. (i) *If F is a proper filter of L, then $D(F)$ is a proper filter of L, too;*
(ii) *If F is a proper filter of L and $x \in F$, then $x \in D(F)$ iff $x^{**} \in D(F)$;*
(iii) *If F is a maximal filter of L, then $x \in D(F)$ iff $x^{**} \in D(F)$.*

Proof. (i) Clearly, $D(F) \subseteq F$. So $0 \notin D(F)$. Since $1^* = 0$, then $1 \in F \cap D(L) = D(F)$. If $x \leq y$ and $x \in D(F)$, then by (r_3) we get $y^* \leq x^* = 0$, that is, $y \in F \cap D(L) = D(F)$. Let $x, y \in D(F)$. So $x \odot y \in F$. Then $x^* = y^* = 0$. By (r_4), $(x \odot y)^* = (x \odot y) \to 0 = x \to y^* = x \to 0 = x^* = 0$, hence $(x \odot y)^* = 0$, that is, $x \odot y \in D(L) \cap F = D(F)$. Therefore, $D(F)$ is a proper filter of L.

(ii) If $x \in D(F)$, since $D(F)$ is a filter of L and $x \leq x^{**}$, then $x^{**} \in D(F)$. Conversely, by hypothesis we have that $x \in F$. Now, if $x^{**} \in D(F)$, then $0 = (x^{**})^* = x^{***} = x^*$, so $x^* = 0$, that is, $x \in D(L)$. Therefore, $x \in D(L) \cap F = D(F)$.

(iii) D. Buşneag et al.(2013) proved that: If F is a maximal filter of L, then $x \in F$ iff $x^{**} \in F$. The rest follows from (ii). □

Lemma 3. *F is a special filter of L iff $D(F) = F$.*

Proof. Let F be a proper filter of L. Now, we prove that F is a special filter of L iff $D(F) = \{x \in F : x^* = 0\} = F$. Clearly, $D(F) \subseteq F$. Let $x \in F$. Then $(1 \to x)^* = (x \to 1)^*$, and so $x^* = 0$. Therefore, $x \in D(F)$, that is, $D(F) = F$.

Conversely, if $D(F) = F$, then $x^* = y^* = 0$, for all $x, y \in F$. Since $x \leq y \to x$, by (r_3) we have $(y \to x)^* \leq x^* = 0$, so $(y \to x)^* = 0$. Since $y \leq x \to y$, by (r_3) we have $(x \to y)^* \leq y^* = 0$, so $(x \to y)^* = 0$. Therefore, $(x \to y)^* = (y \to x)^*$, that is, F is a special filter of L. □

Mohtashamnia and Borumand Saeid (2012) proved that: in a BL-algebra L, any proper filter F of L is a special filter iff L is a special BL-algebra. In residuated lattices we obtain the following generalization:

Theorem 4. *Any proper filter F of L is a special filter iff $L = D(L) \cup N(L)$.*

Proof. By Lemma 3 we have that F is a special filter of L iff $D(F) = F$.

Now, we prove that for all proper filters F of L, $D(F) = F$ iff $L = D(L) \cup N(L)$. Let $D(F) = F$, for all proper filters F of L. In order to prove that $L = D(L) \cup N(L)$, we consider an element $x \in L \setminus \{0\}$ such that $x^* \neq 0$. And we show that $x, x^* \in N(L)$. If there exists a maximal filter M such that $x \in M$, then by hypothesis we deduce that $x^* = 0$, which is a contradiction. Then $x \notin M$. Following Theorem 5, if $x \notin M$, then there is $n \geq 1$ such that $(x^n)^* \in M$ and by hypothesis we obtain $(x^n)^{**} = 0$. Since $x^n \leq (x^n)^{**} = 0$. Then $x^n = 0$, for some $n \geq 1$. Therefore, $x \in N(L)$.

If there exists a maximal filter M such that $x^* \in M$, by hypothesis we deduce that $x^{**} = 0$, and since $x \leq x^{**} = 0$, then $x = 0$, which is a contradiction. Hence $x, x^* \notin M$.

If $x, x^* \notin M$, then $x \vee x^* \notin M$. If by contrary $x \vee x^* \in M$, since M is a prime filter, then $x \in M$ or $x^* \in M$, which is a contradiction. Hence $x \vee x^* \notin M$. Following Theorem 5, there is $t \geq 1$ such that $[(x \vee x^*)^t]^* \in M$, that is, $[(x \vee x^*)^t]^{**} = 0$, and so $(x \vee x^*)^t = 0$. By (r_5) we get $x^t \vee (x^*)^t = 0$. Let $p = max\{n, t\}$, then $x^p \leq x^n = 0$, $x^p \vee (x^*)^p = 0$. Hence $x^p = 0, (x^*)^p = 0$. Therefore, $x, x^* \in N(L)$. We deduce that the set of all elements $x \in L \setminus \{0\}$ such that $x^* \neq 0$ coincide with $N(L)$, hence $L = D(L) \cup N(L)$.

Conversely, we consider $L = D(L) \cup N(L)$. Let F be a proper filter of L. Clearly, $F \subseteq D(L)$. Then $D(F) = F$. □

Example 8. Now, we give an example of a residuated lattice L which satisfies the condition from Theorem 4 and it is not a special residuated lattice. In the residuated lattice L from Example 4 it is easy to see that all the elements of $L \setminus \{1\}$ are nilpotent, the only filter of L is $\{1\}$ and $D(\{1\}) = \{1\}$, so $\{1\}$ is a special filter of L. Therefore, L satisfies the condition from Theorem 4 and it is not a special residuated lattice ($m^* = m \neq 0$). Moreover, since $\{1\}$ is a special filter of L and $L \cong L/\{1\}$, then $L/\{1\}$ is not a special residuated lattice. Therefore, L is a residuated lattice such that $L/\{1\}$ is not a special residuated lattice with $\{1\}$ a special filter of L.

The following result is a consequence of Proposition 9 and Lemma 3:

Corollary 7. *Let F be a maximal filter of L. Then the following assertions are equivalent:*

(i) *F is a special filter of L;*

(ii) $\{1\} = \{[x] \in L/F : [x]^{**} = [1]\}$.

Proof. By Lemma 3 we have that F is a special filter of L iff $F = D(F)$.

$(i) \Rightarrow (ii)$. Let F be a special filter of L. Let $[x] \in L/F$ such that $[x]^{**} = [1]$. Then we have $[x^{**}] = [x]^{**} = [1]$. Hence $x^{**} \in F$. Since F is a maximal filter we get $x \in D(F)$. By hypothesis we get that $x \in F$, therefore $[x] = [1]$. Hence $\{1\} = \{[x] \in L/F : [x]^{**} = [1]\}$.

$(ii) \Rightarrow (i)$. Let $x \in D(F)$. Then $x^{**} \in F$, thus $[x]^{**} = [x^{**}] = [1]$. Hence $[x] \in \{[x] \in L/F : [x]^{**} = [1]\}$. By hypothesis we obtain $[x] = [1]$. Hence $x \in F$. Therefore, $D(F) \subseteq F$. It remains to show that $F \subseteq D(F)$. Let $x \in F$ by (r_9), we have $x \leq x^{**}$. So $x^{**} \in F$. Following Proposition 9, (iii), since F is a maximal filter and $x^{**} \in F$, then $x \in F = D(F)$. Therefore, $F \subseteq Ds(F)$ and we conclude that $F = D(F)$. Therefore, F is a special filter of L. □

We notice an easy consequence of Theorem 4:

Corollary 8. If L is a special residuated lattice, then every filter F of L is special.

4 Boolean elements in residuated lattices

Lemma 4. *If $x \odot y \in B(L)$ such that $x = x \odot x$, $y = y \odot y$ and $x \vee y = 1$, then x and y are Boolean elements, too.*

Proof. By residuation property we obtain that $y \leq (x \to y) \to y$ and $x \leq (x \to y) \to y$, then $x \vee y \leq (x \to y) \to y$. By hypothesis $x \vee y = 1$, then $(x \to y) \to y = 1$, thus $(x \to y) \leq y$. By residuation property we have $y \leq x \to y$, and so $x \to y = y$. In a symmetric manner follows that $y \to x = x$. Since $x \odot y \in B(L)$, then $x \odot y = (x \odot y)^{**} \overset{(r_6)}{\geq} x^{**} \odot y^{**}$. We obtain successively $x^{**} \leq y^{**} \to (x \odot y) \leq y^{**} \to x \leq y \to x = x$, so $x = x^{**}$. In a symmetric manner follows that $y = y^{**}$. Since $x \leq x^* \to x$, then $(x^* \to x)^* \leq x^*$. Since $(x \to x^*)^* \overset{(r_4)}{=} (x \odot x)^{**} = x^{**} = x$, we deduce that $1 = (x^* \to x)^* \vee (x \to x^*)^* \leq x^* \vee x$. Hence $x \vee x^* = 1$. In a symmetric manner follows that $y \vee y^* = 1$. Therefore, x and y are Boolean elements. □

Clearly, the converse of Lemma 4 holds if $y = x^*$. Otherwise, if we consider the residuated lattice L from Example 5 we have $b, f \in B(L), b^* = e \neq f$ such that $b \odot f = b \in B(L), b = b \odot b, f = f \odot f$ and $b \vee f = f \neq 1$.

A directly consequence of Lemma 4 is the following:

Theorem 5. *Let $x, y \in L$ be such that $x \vee y = 1$. Then $x, y \in B(L)$ iff $x \odot y \in B(L)$ and $x = x \odot x$, $y = y \odot y$.*

Remark 5. In Theorem 5 is not necessary to have $y = x^*$. Indeed, if we consider the residuated lattice L from Example 5 we have $b, c, f \in B(L)$ such that $c \odot f = b \in B(L), c = c \odot c, f = f \odot f$ and $c \vee f = 1$, but $f \neq d = c^*$.

D. Buşneag et al.(2013) present a new characterization of Boolean elements in any residuated lattice L. If $x \in L$, then $x \in B(L)$ iff $x \in R(L),\ x \odot x = x$ and $(x \to x^*) \vee (x^* \to x) = 1$.

Now, we offer a new characterization of Boolean elements in a residuated lattice L using the subset $M(L)$.

Lemma 5. *Let $x \in L$. Then the following assertions are equivalent:*

(*i*) $x \in B(L)$;

(*ii*) $x \in M(L)$, $x = x \odot x$ *and* $(x^* \to x)^* \vee (x \to x^*)^* = 1$.

Proof. (*i*) $\Rightarrow$ (*ii*). Let $x \in B(L)$. Then $x^* \in B(L)$. By Proposition 4, $x = x \odot x$, $x^* = x^* \odot x^*$ and $x = x^{**}$, then $x = x^{**} = (x^* \odot x^*)^* = x \oplus x$, so $x \in M(L)$. By Proposition 4 we have $x^* \to x = x$. Since $x \to x^* \overset{(r_4)}{=} (x \odot x)^* = x^*$ we obtain $(x^* \to x)^* \vee (x \to x^*)^* = x^* \vee x^{**} = x^* \vee x = 1$.

(*ii*) $\Rightarrow$ (*i*). Since $x \in M(L)$, by Lemma 2, (*i*), we deduce that $x^{**} = x$. Since $x \leq x^* \to x$, then $(x^* \to x)^* \leq x^*$. Since $(x \to x^*)^* \overset{(r_4)}{=} (x \odot x)^{**} = x^{**} = x$, we deduce that $1 = (x^* \to x)^* \vee (x \to x^*)^* \leq x^* \vee x$. Hence $x \vee x^* = 1$. It is known that $x \in B(L)$ iff $x \vee x^* = 1$, then the claim follows. □

In conclusion, we have the following characterizations of boolean elements in any residuated lattice L.

Theorem 6. *Let $x \in L$. Then the following are equivalent:*
(*i*) $x \in B(L)$;
(*ii*) $x \in MV(L)$ *and* $x \vee x^* = 1$;
(*iii*) $x \in R(L), x = x \odot x$ *and* $(x \to x^*) \vee (x^* \to x) = 1$;
(*iv*) $x \in M(L)$, $x = x \odot x$ *and* $(x^* \to x)^* \vee (x \to x^*)^* = 1$.

By Lemma 1 and Theorem 6, we deduce that $B(L) \subseteq R(L) \subseteq M(L) \subseteq MV(L)$, for any residuated lattice L. Hence, until now, the best way to find Boolean elements in a residuated lattice L is to search for them inside the subset $R(L)$.

Lemma 6. *If L is a Stonean residuated lattice, then $x \in B(L)$ iff $x^{**} = x$.*

Proof. Let L be a Stonean residuated lattice and $x \in L$ be such that $x^{**} = x$. Then $x^* \vee x^{**} = x^* \vee x = 1$, hence $x^* \vee x = 1$. By Theorem 6, we deduce that $x \in B(L)$.

Converse is clear. □

Lemma 7. *If $B(L) = MV(L)$, then L is Stonean.*

Proof. If $B(L) = MV(L) = \{0, 1\}$, by Theorem 3, (iv) we deduce that L is special, so L is Stonean.

If $B(L) = MV(L) \neq \{0, 1\}$, then for an element $x \in L \setminus \{0, 1\}$ there are two possibilities: $x \in B(L)$ or not. In order to prove L is Stonean we suppose by contrary that there is an element $x \in L \setminus \{0, 1\}$ such that $x^* \vee x^{**} \neq 1$. If we consider $x \in B(L) = MV(L)$, then $x = x^{**}$ and $1 = x \vee x^* = x^{**} \vee x^*$, so $x^* \vee x^{**} = 1$. Now, if $x \notin B(L) = MV(L)$, then $x \neq x^{**}$. By (r_8) we have $x^* = x^{***} = (x^*)^{**}$, that is $x^* \in MV(L) = B(L)$, and so $x^* \vee x^{**} = 1$. Therefore, we deduce $x^* \vee x^{**} = 1$, for all $x \in L$, which is a contradiction. Hence L is Stonean. □

The following result is a consequence of Lemma 6 and Lemma 7.

Theorem 7. *L is Stonean iff $B(L) = MV(L)$.*

Remark 6. By Theorem 6, if L is a residuated lattice, then L is Stonean iff $B(L) = R(L) = M(L) = MV(L)$.

Remark 7. There exists residuated lattice L such that $B(L) = R(L) = M(L) = MV(L)$, but L are not Boolean algebras.

Indeed, if we consider the special residuated lattice L from Example 7 we have $B(L) = R(L) = M(L) = MV(L) = \{0, 1\}$. Since $n^* = 0$, then $n \vee n^* = n \vee 0 = n \neq 1$, so L is not Boolean algebra.

We recall [23] that a residuated lattice L with double-negation property is called *involutive*.

Theorem 8. *L is a Boolean algebra iff L is an involutive Stonean residuated lattice.*

Proof. If L is a Boolean algebra, then it is clear that L is an involutive Stonean residuated lattice.

Conversely, if L is an involutive Stonean residuated lattice, then for all $x \in L$, $x^{**} = x$ and $1 = x^* \vee x^{**} = x \vee x^*$, so $x \in B(L)$. Therefore, L is a Boolean algebra. □

5 Filter's regularity

A residuated lattice L which has an unique maximal filter M is called *local residuated lattice*. It is known that L is a directly indecomposable residuated lattice iff 0 and 1 are the only Boolean elements.

The proof that *local residuated lattices* are directly indecomposable was done in Ciungu (2009), we offer an alternative proof using Theorem 6.

Proposition 10. *Any local residuated lattice L is directly indecomposable.*

Proof. Let L be a local residuated lattice. We suppose on the contrary that there exists a Boolean element $x \in B(L) \setminus \{0, 1\}$. Clearly, $x^* \in B(L)$. Since $x, x^* \in B(L)$ and following Theorem 6, (iii) we have $x^2 = x \odot x = x$ and $(x^*)^2 = x^* \odot x^* = x^*$. Since L is a local residuated lattice, let $M \in Max(L)$ be the unique maximal filter of L. Then $\langle x \rangle \subseteq M$ and $\langle x^* \rangle \subseteq M$. Hence, it is necessary to have $x, x^* \subseteq M$. Since M is a filter we deduce that $x \odot x^* = 0 \in M$, hence $M = L$, which is a contradiction. □

Remark 8. The result from Proposition 10 follows from general universal algebraic considerations. More precisely, let A be an algebra in any similarity type. We say that A is *local* if it has a unique maximal congruence ϕ. Now, suppose towards a contradiction that A is not directly indecomposable. Then there is a pair of factor congruences θ_1 and θ_2 which are neither the identity nor the total congruence (see Burris and Sankappanavar's book if necessary). Hence the unique maximal congruence ϕ of A is larger than θ_1 and θ_2. In particular, this implies that the relational product of θ_1 and θ_2 is included in ϕ. Thus the relational product of θ_1 and θ_2 is not the total relation. But this contradicts the assumption that θ_1 and θ_2 are a factor pair. As a consequence, we conclude that A is directly indecomposable.

Corollary 9. *Any special residuated lattice is local.*

Proof. By Corollary 7 (D. Buşneag et al.(2015)), we notice that L is a special residuated lattice iff $L \setminus \{0\}$ is the unique maximal filter. We deduce that any special residuated lattice L is a local residuated lattice. □

Remark 9. There are local residuated lattice L which are not special.

Indeed, if we consider the residuated lattice L from Example 6 we have $\langle a \rangle = \{a, b, c, d, 1\}$ is the unique maximal filter of L, so L is local. Since $n^* = d \neq 0$, then L is not special.

Corollary 10. *Let L be a special residuated lattice. Then $L/D(L)$ is not a special residuated lattice.*

Proof. Since $D(L) = L \setminus \{0\}$ is the unique maximal filter of L, then $L/D(L)$ becomes a residuated lattice. We prove that $L/D(L)$ is a residuated lattice with double negation property and by Remark 3, it follows that $L/D(L)$ is not a special residuated lattice. Suppose by contrary that $L/D(L)$ is not a residuated lattice with double negation property, then there is $x \in L$ such that $(x/D(L))^{**} \neq x/D(L)$. We have $x/D(L) \leq (x/D(L))^{**}$.

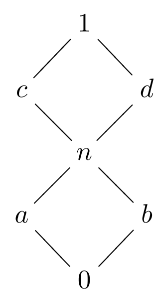
Hence $(x/D(L))^{**} \nleq x/D(L)$, implies that $(x/D(L))^{**} \to x/D(L) \neq 1/D(L)$, implies $x^{**} \to x \notin D(L)$. We deduce that $(x^{**} \to x)^* \neq 0$, which is a contradiction. Hence $L/D(L)$ is a residuated lattice with double negation property. Then $L/D(L)$ is not a special residuated lattice. □

Theorem 9. *$L/D(L)$ is an MV-algebra.*

Proof. In Proposition 3.6, (Mureşan (2010)), if L is a Glivenko residuated lattice, then $L/D(L)$ is an MV-algebra iff L satisfies the equation: $(x^* \to y^*) \to y^* = (y^* \to x^*) \to x^*$, for all $x, y \in L$. By Theorem 3 we have L is a Glivenko residuated lattice such that $(x^* \to y^*) \to y^* = (y^* \to x^*) \to x^* = 0$, for all $x, y \in L$. Therefore, $L/D(L)$ is an MV-algebra. □

Remark 10. If we consider the special residuated lattice L from Example 7, then $D(L) = L \setminus \{0\} = \{n, a, b, c, d, 1\}$ is the unique maximal filter of L and $L/D(L) = \{[x] : \ x \in L\}$, but $[n] = [a] = [b] = [c] = [d] = [1]$, so $L/D(L)$ is a single element algebra, hence $L/D(L)$ is an MV-algebra.

Example 9. [7] *If M is a maximal filter of L, then $x \in M$ iff $x^{**} \in M$. But the converse does not hold in any residuated lattice. Indeed, for the converse we consider $L = \{0, a, b, n, c, d, 1\}$ with the Hasse diagram:*



Then [23] *L becomes a residuated lattice relative to the operations:*

$\to$	0	a	b	n	c	d	1
0	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1
b	a	a	1	1	1	1	1
n	0	a	b	1	1	1	1
c	0	a	b	d	1	d	1
d	0	a	b	c	c	1	1
1	0	a	b	n	c	d	1

$\odot$	0	a	b	n	c	d	1
0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a
b	0	0	b	b	b	b	b
n	0	a	b	n	n	n	n
c	0	a	b	n	c	n	c
d	0	a	b	n	n	d	d
1	0	a	b	n	c	d	1

Clearly, $\langle a\rangle = \{a, n, d, c, 1\}$, $\langle b\rangle = \{b, n, c, d, 1\}$ are maximal filters of L. Since $\langle n\rangle = \{n, c, d, 1\}$ and $\langle n\rangle \subset \langle a\rangle$, we deduce that $\langle n\rangle$ is not a maximal filter of L, but $n^{**} = c^{**} = d^{**} = 1^{**} = 1 \in \langle n\rangle$.

Corollary 11. *If L is a Stonean residuated lattice and $x \in L$. Then the following assertions are equivalent:*
(i) $M \in Max(L)$;
(ii) $M \in Spec(L)$, $x \in M$ iff $x^{**} \in M$.

Proof. $(i) \Rightarrow (ii)$. Clearly, by Remark 9.

$(ii) \Rightarrow (i)$. Let $x \notin M$. By hypothesis, $x^{**} \notin M$. Since L is a Stonean residuated lattice we have $x^* \vee x^{**} = 1$. Since $M \in Spec(L)$ and $x^* \vee x^{**} = 1 \in M$, $x^{**} \notin M$, then $x^* \in M$.

Hence for every $x \in L$, $x \notin M$, there is $n = 1$ such that $(x^1)^* = x^* \in M$. By Proposition 5, (ii) we deduce that $M \in Max(L)$. □

A filter $F \in \mathcal{F}(L)$ of L is called *regular* if $F = ^{\perp\perp} F$. The *set of all regular* filters is denoted by $^{\perp}R(\mathcal{F}(L))$. And $F \in ^{\perp} R(\mathcal{F}(L))$ iff there is $e \in B(L)$ such that $F = \langle e \rangle$ (see Piciu et al.(2008)).

We say that a residuated lattice L has *regular filter property* (or simply, RF-property) if every proper filter $F \in \mathcal{F}(L)$ is regular.

Examples of residuated lattices with RF-property are Boolean algebras (see Proposition 7). By Theorem 3.2, Piciu et al.(2008), we notice that:

Theorem 10. *A residuated lattice L has RF-property iff for every $F \in \mathcal{F}(L)$ a proper filter, there is $e \in B(L)$ such that $F = \langle e \rangle$.*

Clearly, a non chain residuated lattice L with RF-property is a directly decomposable residuated lattice.

Georgescu et al. (2015) called *Gelfand residuated lattices* those residuated lattices in which any prime filter is included in a unique maximal filter. Examples of Gelfand residuated lattices are Stonean residuated lattices (see D. Buşneag et al.(2015)).

Corollary 12. *Every residuated lattice L with RF-property is Gelfand.*

Proof. Let P be a prime filter of L. Then P is contained in a maximal filter. Suppose that there are two distinct maximal filters M_1 and M_2 such that $P \subseteq M_1$ and $P \subseteq M_2$. Since $M_1 \neq M_2$, there is $a \in M_1$ such that $a \notin M_2$. By Proposition 5, there is $n \geq 1$ such that $(a^n)^* \in M_2$. Clearly, $(a^n)^* \notin M_1$. Since $a \notin M_2$ and $(a^n)^* \notin M_1$, then $a \notin P$ and $(a^n)^* \notin P$. By Theorem 10, since L is regular and $M_1 \in \mathcal{F}(L)$, then there is $e \in B(L)$ such that $M_1 = \langle e \rangle$. Clearly, $e^* \notin M_1$. Since $a \in M_1$ and $M_1 = \langle e \rangle$, by (r_4) we obtain successively $e \leq a^n$, $(a^n)^* \leq e^* \in M_2$, hence $e^* \in M_2$.

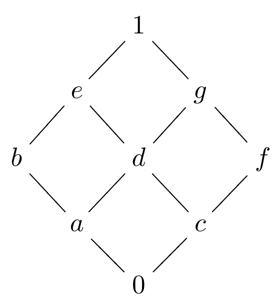
If $e \in P$, since $e \leq a$, we deduce that $a \in P$, which is a contradiction. Hence $e \notin P$.

If $e^* \in P$, since $P \subseteq M_1 \cap M_2$, we deduce that $e^* \in M_1$, which is a contradiction. Hence $e^* \notin P$.

Since P is a prime filter and $e \vee e^* = 1 \in P$ with $e \notin P$ and $e^* \notin P$, which is a contradiction. Hence $M_1 = M_2$. ☐

Example 10. We must to specify that the class of Stonean residuated lattices and residuated lattices with RF-property are different. Also, there are Gelfand residuated lattices which do not have the RF-property.

Consider $L = \{0, a, b, c, d, e, f, g, 1\}$ with the Hasse diagram:



Then [23] L becomes a residuated lattice relative to the following operations:

$\rightarrow$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	g	1	1	g	1	1	g	1	1
b	f	g	1	f	g	1	f	g	1
c	e	e	e	1	1	1	1	1	1
d	d	e	e	g	1	1	g	1	1
e	c	d	e	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	a	b	b	d	e	e	g	1	1
1	0	a	b	c	d	e	f	g	1

$\odot$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	0	a	0	0	a	0	0	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	0	0	0	c	c	c
d	0	0	a	0	0	a	c	c	d
e	0	a	b	0	a	b	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	0	a	c	c	d	f	f	g
1	0	a	b	c	d	e	f	g	1

Clearly, $B(L) = \{0, b, f, 1\}$. Then L is a residuated lattice with RF-property. Since $a^* \vee a^{**} = g \vee a = g \neq 1$, we deduce that L is not Stonean residuated lattice. Hence L is a residuated lattice with RF-property, but L isn't Stonean.

For the converse we consider $L = \{0, a, b, c, d, e, f, g, 1\}$ with the same Hasse diagram as above. Then [21] L becomes a residuated lattice with the following operations:

$\to$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	f	1	1	f	1	1	f	1	1
b	f	g	1	f	g	1	f	g	1
c	b	b	b	1	1	1	1	1	1
d	0	b	b	f	1	1	f	1	1
e	0	a	b	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	0	b	b	c	e	e	f	1	1
1	0	a	b	c	d	e	f	g	1

$\odot$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	a	a	0	a	a	0	a	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	c	c	c	c	c	c
d	0	a	a	c	d	d	c	d	d
e	0	a	b	c	d	e	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	a	a	c	d	d	f	g	g
1	0	a	b	c	d	e	f	g	1

Since $0^* \vee 0^{**} = 1 \vee 0 = 1$, $a^* \vee a^{**} = f \vee b = 1$, $b^* \vee b^{**} = f \vee b = 1$, $c^* \vee c^{**} = b \vee f = 1$, $d^* \vee d^{**} = 0 \vee 1 = 1$, $e^* \vee e^{**} = 0 \vee 1 = 1$, $f^* \vee f^{**} = b \vee f = 1$, $g^* \vee g^{**} = 0 \vee 1 = 1$ and $1^* \vee 1^{**} = 0 \vee 1 = 1$, we deduce that L is a Stonean residuated lattice.

Since $a^2 = a$, we deduce that $\langle a \rangle = \{a, b, d, e, g, 1\}$ and ${}^{\perp\perp}\langle a \rangle = \langle 1 \rangle \neq \langle a \rangle$.

Therefore, L is a Stonean residuated lattice, but L does not have the *RF*-property.

By Theorem 6 (D. Buşneag et al.(2015)), we notice that every Stonean residuated lattice is Gelfand. Therefore, L is a Gelfand residuated lattice, but L does not have the *RF*-property.

Lemma 8. *Let L be a residuated lattice with RF-property. Then $Spec(L) = Max(L)$.*

Proof. Since $Max(L) \subseteq Spec(L)$, then it suffices to prove that any prime filter of L is maximal. We consider $M \in Spec(L)$. In order to prove $M \in Max(L)$, let $x \notin M$.

If there is $F \in \mathcal{F}(L)$ such that $x \in F$. Since L has *RF*-property, then there exists $e \in B(L)$ such that $\langle x \rangle = \langle e \rangle$, $\langle e \rangle \subseteq F$, that is, $x^n = e$, for some $n \geq 1$. Since $1 = e \vee e^* = x^n \vee (x^n)^* \leq x \vee (x^n)^*$, we deduce that $x \vee (x^n)^* = 1 \in M$. Since M is a prime filter and $x \notin M$, we get that $(x^n)^* \in M$.

If there is no $F \in \mathcal{F}(L)$ such that $x \in F$, we deduce that $\langle x \rangle = \langle 0 \rangle$, that is, x is of finite order. Hence $x^n = 0$, for some $n \geq 1$. Therefore, $1 = (x^n)^* \in M$.

In all cases we deduce that for any $x \notin M$, there is $n \geq 1$ such that $(x^n)^* \in M$. Therefore, by Proposition 5, M is maximal. $\square$

D. Buşneag et al. (2010) investigated the properties of hyperarchimedean residuated lattices. An element $a \in L$ is called *archimedean* if there is $n \geq 1$ such that $a^n \in B(L)$; and the lattice L is called *hyperarchimedean* if all its elements are archimedean.

Theorem 11. *L is hyperarchimedean iff L has RF-property.*

Proof. Let L be a hyperarchimedean residuated lattice. Then for any $x \in L$, there exist $n \geq 1$ and $e \in B(L)$ such that $x^n = e$, thus $\langle x \rangle = \langle e \rangle$ and ${}^{\perp}\langle x \rangle ={}^{\perp} \langle e \rangle$. By Proposition 7 we deduce that ${}^{\perp\perp}\langle x \rangle ={}^{\perp\perp} \langle e \rangle = {}^{\perp}\langle e^* \rangle = \langle e^{**} \rangle = \langle e \rangle = \langle x \rangle$, then ${}^{\perp\perp}\langle x \rangle = \langle x \rangle$. Since ${}^{\perp\perp}\langle x \rangle = \langle x \rangle$, for every $x \in L$, we deduce that L has *RF*-property.

Conversely, let L be a residuated lattice with *RF*-property. By Theorem 50 (D. Buşneag et al. (2010)) we deduce that L is hyperarchimedean iff $Spec(L) = Max(L)$. By Lemma 8 we deduce that $Spec(L) = Max(L)$, then L is hyperarchimedean. $\square$

Corollary 13. *L has RF-property iff L is hyperarchimedean iff $Spec(L) = Max(L)$.*

Proof. By Theorem 11 we deduce L is hyperarchimedean iff L has *RF*-property. By Theorem 50 (D. Buşneag et al. (2010))) we deduce L is hyperarchimedean iff $Spec(L) = Max(L)$. Therefore, the claim follows. $\square$

Proposition 11. *L is hyperarhimedean iff for all $x \in L$ there exists $m \geq 1$ such that $x \vee (x^m)^* = 1$.*

Proof. Let $x \in L$. Assume there is $n \geq 1$ such that $x^n \in B(L)$. Then $1 = x^n \vee (x^n)^* \leq x \vee (x^n)^*$, that is $x \vee (x^n)^* = 1$.

Conversely, assume $x \in L$ and for some $n \geq 1$, $x \vee (x^n)^* = 1$. In order to prove that $x^n \in B(L)$, we show by induction that $x^n \vee (x^n)^* = 1$. We obtain successively $1 = 1 \odot 1 = [x \vee (x^n)^*] \odot [x \vee (x^n)^*] \overset{(r_5)}{=} \{x \odot [x \vee (x^n)^*]\} \vee \{(x^n)^* \odot [x \vee (x^n)^*]\} \overset{(r_5)}{=} x^2 \vee [x \odot (x^n)^*] \vee [[x \odot (x^n)^*]] \vee [(x^n)^* \odot (x^n)^*] \leq x^2 \vee (x^n)^*$. Hence $x^2 \vee (x^n)^* = 1$. By (r_5), in similar manner we realize that $1 = 1 \odot 1 = [x^2 \vee (x^n)^*] \odot [x^2 \vee (x^n)^*] \leq x^3 \vee (x^n)^*$. Hence $x^3 \vee (x^n)^* = 1$. More generally, we assume that $x^{n-1} \vee (x^n)^* = 1$ and we realize that $1 = 1 \odot 1 = [x^{n-1} \vee (x^n)^*] \odot [x^{n-1} \vee (x^n)^*] \leq x^n \vee (x^n)^*$. Hence $x^n \vee (x^n)^* = 1$, that is $x^n \in B(L)$. Thus, L is hyperarhimedean. ☐

We recall that if $F, G \in \mathcal{F}(L)$, then $F \vee G = F \vee_{\mathcal{F}(L)} G = \langle F \cup G \rangle = \{x \in L : b \odot c \leq x$ for some $b \in F, c \in G\}$. And $F \wedge G = F \wedge_{\mathcal{F}(L)} G = F \cap G$.

Lemma 9. *L is a hyperarhimedean residuated lattice iff $F \vee^{\perp} F = L$, for any $F \in \mathcal{F}(L)$.*

Lemma 10. *The lattice $\mathcal{F}(L)$ is pseudo-complemented and for any filter F, its pseudo-complement is $^{\perp}F$.*

Proof. By definition $F \wedge^{\perp} F = F \cap^{\perp} F = \{1\}$. Let G be a filter of L such that $F \wedge G = F \wedge_{\mathcal{F}(L)} G = F \cap G = \{1\}$. We shall prove that $G \subseteq^{\perp} F$. Let $a \in G$. For any $x \in F$, we have that $a \vee x \in F \cap G = \{1\}$, since $a \vee x \geq x \in F$ and $a \vee x \geq a \in G$. Hence, $a \vee x = 1$ for any $x \in F$, so $a \in^{\perp} F$. That is, $^{\perp}F$ is the pseudo-complement of F. ☐

The following result is a direct consequence of Lemma 9 and Lemma 10.

Theorem 12. *L is a hyperarhimedean residuated lattice iff $(\mathcal{F}(L), \vee_{\mathcal{F}(L)}, \wedge_{\mathcal{F}(L)}, \{1\}, L)$ is a Boolean algebra.*

6 Conclusions

Different definition of regular elements in residuated lattices, with their sets denoted by $B(L), R(L), M(L)$, $Reg(L)$ and $MV(L)$, are studied (D. Buşneag et al.(2013), D. Buşneag et al.(2015), Cignoli (2008), Turunen and Mertanen (2008), Ciungu (2009)). We study their properties in order to establish the relationship between them and we give a new characterizations for dense, regular and Boolean elements in residuated lattices.

Based on the works Mohtashamnia and Borumand Saeid (2012), Borzooei et al. (2013) and D. Buşneag et al.(2015), we study a new class of directly indecomposable residuated lattices named *special residuated lattices*, we prove that classes of special BL-algebras and integral BL-algebras coincide, we present some characterizations. Moreover, we gave some examples of special residuated lattices and showed the relationship between them and other algebraic structures. In addition for a special residuated lattice L, we proved that the unique maximal filter of L is $L \setminus \{0\}$ and $L/D(L)$ is an MV-algebra. A new characterizations for Stonean residuated lattices and Boolean algebras is given in Theorems 7 and 8.

Also, we consider the notion of *residuated lattices with RF-property*. Finally, we prove that residuated lattices with RF-property are equivalent with hyperarchimedean residuated lattices.

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