

## Negations and aggregation operators based on a new hesitant fuzzy partial ordering

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### Abstract

Based on a new hesitant fuzzy partial ordering proposed by Garmendia et al. [7], in this paper a fuzzy disjunction  $\mathbf{D}$  on the set  $\mathbf{H}$  of finite and nonempty subsets of the unit interval and a t-conorm  $\mathbf{S}$  on the set  $\bar{\mathbf{B}}$  of equivalence class on the set of finite bags of unit interval based on this partial ordering are introduced respectively. Then, hesitant fuzzy negations  $N_n$  on  $\mathbf{H}$  and  $\mu_n$  on  $\bar{\mathbf{B}}$  are proposed. Particularly, their De Morgan's laws are investigated with respect to binary operations  $\mathbf{C}$  and  $\mathbf{D}$  on  $\mathbf{H}$ , as well as  $\mathbf{T}$  and  $\mathbf{S}$  on  $\bar{\mathbf{B}}$  respectively, where  $\mathbf{C}$  is a commutative fuzzy conjunction on  $(\mathbf{H}, \leq_H)$  and  $\mathbf{T}$  is a t-norm on  $(\bar{\mathbf{B}}, \leq_B)$ . Finally, the new hesitant fuzzy aggregation operators are presented on  $\mathbf{H}$  and  $\bar{\mathbf{B}}$  and their more general forms are given. Moreover, the validity of the aggregation operations is illustrated by a numerical example on decision making.

**Keywords:** Hesitant fuzzy sets, Finite subsets of the unit interval, Partial ordering, t-conorm, Negation, Aggregation operation.

## 1 Introduction

As the generalization of fuzzy sets proposed by Zadeh [27], hesitant fuzzy sets allow us the possibility of assigning more than a value of the unit interval to an object of a universe of discourse. The theory of hesitant fuzzy sets has been found to be useful to deal with uncertainty of informations when there is doubt or hesitation. It is well known that fuzzy negations and aggregation operators are significant mathematical tools in approximate reasoning [9, 22] and decision making [10, 11, 13, 23, 24]. The flourishing achievements on hesitant fuzzy negations and hesitant aggregation operators (see [1, 3, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 21, 22, 23, 24]) have been obtained.

Partial order relation is the necessity for defining negations and aggregation operators on the set of hesitant fuzzy elements. Different hesitant fuzzy negations and aggregation operators have been generated by different hesitant partial orderings (see [1, 3, 15, 16]). It is very vital that how to compare the hesitant fuzzy elements with different cardinalities on the set  $\mathbf{H}$  of nonempty and finite subsets of  $[0, 1]$ . Xu and Xia [25] have proposed a partial ordering on  $\mathbf{H}$  and investigated distance and similarity measures. Based on Xu-Xia-partial ordering, Santos et al. [15] have explored typical hesitant fuzzy negations on  $\mathbf{H}$ . Thereafter, a constructive method of typical hesitant triangular norms and the notion of aggregation functions for typical hesitant fuzzy elements regarding Xu-Xia-partial order have been proposed in [16]. Bedregal et al. [1] have proposed a partial ordering based on  $\alpha$ -normalization and then studied the typical hesitant triangular norms and aggregation functions. In addition, typical hesitant fuzzy negations based on this partial order have been presented in [3].

In [7], on the one hand, Garmendia et al. have pointed out that these partial orderings given by above seem acceptable in some occasions but can be too radical in others. In order to compare two hesitant fuzzy elements with different cardinalities, L.Garmendia et al. proposed a more balanced partial ordering on  $\mathbf{H}$  and presented that  $\mathbf{H}$  with this partial ordering is a bounded partially ordered set instead of lattice. However,  $\bar{\mathbf{B}}$  (i.e., the set of equivalence class

on the set of finite bags of unit interval) and some subsets of  $\mathbf{H}$  have the lattice structure. On the other hand, they investigated t-norms on  $\mathbf{H}$  and on  $\bar{\mathbf{B}}$ . Nevertheless, hesitant fuzzy negations and aggregation operations based on this more balanced partial ordering on the  $\mathbf{H}$  and on  $\bar{\mathbf{B}}$  have not been studied. Hence, in this paper we will utilize this partial ordering to construct hesitant fuzzy negations and aggregation operations on  $\mathbf{H}$  and on  $\bar{\mathbf{B}}$ , which would be beneficial to research for hesitant fuzzy approximate reasoning and decision making.

The rest of the paper is arranged as follows. Section 2 briefly reviews some related definitions and properties. Based on the partial ordering proposed by Garmendia et al. [7], Section 3 investigates t-conorms on  $(\mathbf{H}, \leq_H)$  and  $(\bar{\mathbf{B}}, \leq_B)$ . Section 4 proposes negations on  $(\mathbf{H}, \leq_H)$  and  $(\bar{\mathbf{B}}, \leq_B)$  and investigates their algebraic properties. Section 5 defines new aggregation operators and their more general forms on  $(\mathbf{H}, \leq_H)$  and  $(\bar{\mathbf{B}}, \leq_B)$ . Section 6 utilizes a special aggregation operator on  $(\mathbf{H}, \leq_H)$  to deal with decision making. Section 7 summarizes the conclusion.

## 2 Preliminaries

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief. In order to simplify notations, throughout this paper, suppose that any hesitant fuzzy element  $A = \{a_1, a_2, \dots, a_n\} \in \mathbf{H}$  always satisfies increasing order, i.e.,  $a_1 < a_2 < \dots < a_n$ .

**Definition 2.1.** [7] Let  $A = \{a_1, a_2, \dots, a_n\}$  be a finite subset of the unit interval and  $r \in N$ .  $A_r \in [0, 1]^{rn}$  is the vector of  $rn$  coordinates defined as  $A_r = (\overbrace{a_1, \dots, a_1}^{r \text{ times}}, \overbrace{a_2, \dots, a_2}^{r \text{ times}}, \dots, \overbrace{a_n, \dots, a_n}^{r \text{ times}})$ .

**Definition 2.2.** [7] Let  $A = \{a_1, a_2, \dots, a_n\}$  and  $B = \{b_1, b_2, \dots, b_m\}$  be two finite subsets of the unit interval.  $lcm(n, m)$  represents the least common multiple of  $n$  and  $m$ . Rewriting  $A_{\frac{lcm(n, m)}{n}} = (c_1, c_2, \dots, c_{lcm(n, m)})$  and  $B_{\frac{lcm(n, m)}{m}} = (d_1, d_2, \dots, d_{lcm(n, m)})$ ,  $A \leq_H B$  if and only if  $c_i \leq d_i$  for all  $i = 1, 2, \dots, lcm(n, m)$ .

**Lemma 2.3.** [7] Let  $A = \{a_1, a_2, \dots, a_n\}$  and  $B = \{b_1, b_2, \dots, b_m\}$  be two finite subsets of the unit interval and  $r \in N$ .  $A \leq_H B$  if and only if every coordinate of  $A_{r \frac{lcm(n, m)}{n}}$  is smaller than or equal to the corresponding coordinate of  $B_{r \frac{lcm(n, m)}{m}}$ .

In [7], it is proved that  $(\mathbf{H}, \leq_H)$  is a bounded partially ordered set instead of a lattice structure (the smallest element and biggest element are  $\{0\}$  and  $\{1\}$  respectively) and the subset  $(\mathbf{H}_R, \leq_H)$  of the partially ordered set  $(\mathbf{H}, \leq_H)$  is defined in following way.

**Definition 2.4.** [7] Let  $R = r_1, r_2, \dots, r_n, \dots$  be a sequence of natural numbers with  $r_{i+1}$  a multiple of  $r_i$  for every  $i \geq 0$ ,  $H_{r_i}$  is the set of finite subsets of unit interval of cardinality  $r_i$ .  $H_R = \bigcup_{i \geq 1} H_{r_i}$ .

It is notable that every element of set  $A = \{a_1, a_2, \dots, a_n\}$  in  $(\mathbf{H}, \leq_H)$  is different. Now, it is allowed that some elements of set  $A$  in  $(\mathbf{H}, \leq_H)$  are repeated, the so called bags or multisets [26]. A finite bag of the unit interval can be represented as a vector  $\vec{v} = (a_1, a_2, \dots, a_n)$  of  $[0, 1]^n$  and we will always assume that  $a_1 \leq a_2 \leq \dots \leq a_n$ . The set of finite bags of  $[0, 1]$  will be denoted by  $\mathbf{B}$ .

**Definition 2.5.** [7] Let  $\vec{v} = (a_1, a_2, \dots, a_n) \in \mathbf{B}$  and  $r \in N$ ,  $\vec{v}_r$  is defined by  $\vec{v}_r = (\overbrace{a_1, \dots, a_1}^{r \text{ times}}, \overbrace{a_2, \dots, a_2}^{r \text{ times}}, \dots, \overbrace{a_n, \dots, a_n}^{r \text{ times}})$ .

**Definition 2.6.** [7] Let  $\vec{u} = (a_1, a_2, \dots, a_n)$  and  $\vec{v} = (b_1, b_2, \dots, b_m)$  be two finite bags of the unit interval and  $lcm(n, m)$  is the least common multiple of  $n$  and  $m$ . Rewriting  $\vec{u}_{\frac{lcm(n, m)}{n}} = (c_1, c_2, \dots, c_{lcm(n, m)})$  and  $\vec{v}_{\frac{lcm(n, m)}{m}} = (d_1, d_2, \dots, d_{lcm(n, m)})$ ,  $\vec{u} \leq_B \vec{v}$  if and only if  $c_i \leq d_i$  for all  $i = 1, 2, \dots, lcm(n, m)$ .

**Proposition 2.7.** [7] The relation  $\leq_B$  on  $\mathbf{B}$  is a preorder (i.e., it is reflexive and transitive).

**Lemma 2.8.** [7] Given two bags  $\vec{u}$  and  $\vec{v}$  of  $\mathbf{B}$ ,  $\vec{u} \leq_B \vec{v}$  and  $\vec{v} \leq_B \vec{u}$  if and only if there exists  $\vec{w} \in \mathbf{B}$  and  $r, s \in N$  such that  $\vec{u} = \vec{w}_r$  and  $\vec{v} = \vec{w}_s$ .

**Definition 2.9.** [7] On  $\mathbf{B}$  consider the equivalence relation  $\sim$  defined for all  $\vec{u}, \vec{v} \in \mathbf{B}$  by  $\vec{u} \sim \vec{v}$  if and only if there exists  $\vec{w} \in \mathbf{B}$  and  $r, s \in N$  such that  $\vec{u} = \vec{w}_r$  and  $\vec{v} = \vec{w}_s$  and denote the quotient  $\mathbf{B}/\sim$  by  $\bar{\mathbf{B}}$ .

The vector of a class with the smallest number of coordinates will be called its canonical representative. In [7],  $\leq_B$  is compatible with  $\sim$  and  $(\bar{\mathbf{B}}, \leq_B)$  is a lattice.

In [2] t-norms are defined on bounded partially ordered sets. The concept of general conjunction is provided by [6].

**Definition 2.10.** [6] An operation  $C: [0, 1]^2 \rightarrow [0, 1]$  is a fuzzy conjunction if

- (i) It is increasing with respect to each variable.
- (ii)  $C(1, 1) = 1, C(0, 0) = C(0, 1) = C(1, 0) = 0$ .

**Definition 2.11.** [7] Let  $\mathbf{P} = (P, \leq_P, 0, 1)$  be a bounded partially ordered set. An operation  $\mathbf{C}: P^2 \rightarrow P$  is a fuzzy conjunction if

- (i) It is increasing with respect to each variable.
- (ii)  $\mathbf{C}(1, 1) = 1, \mathbf{C}(0, 0) = \mathbf{C}(0, 1) = \mathbf{C}(1, 0) = 0$ .

**Definition 2.12.** [2] A  $t$ -norm  $T$  on a bounded partially ordered set  $\mathbf{P} = (P, \leq_P, 0, 1)$  is a binary operation on  $P$  that for all  $x, y, z \in P$  satisfies

- (i)  $T(x, 1) = x$  (neutral element)
- (ii) If  $x \leq_P y$ , then  $T(x, z) \leq_P T(y, z)$  (monotonicity)
- (iii)  $T(x, y) = T(y, x)$  (commutativity)
- (iv)  $T(x, T(y, z)) = T(T(x, y), z)$ .

**Definition 2.13.** [7] Let  $T$  be a  $t$ -norm on  $[0, 1]$ . On  $\mathbf{H}$  the binary operation  $\mathbf{C}$  is defined in the following way. For two elements  $A = \{a_1, a_2, \dots, a_n\}$  and  $B = \{b_1, b_2, \dots, b_m\}$  of  $\mathbf{H}$ , rewriting  $A_{\frac{lcm(n,m)}{n}} = (c_1, c_2, \dots, c_{lcm(n,m)})$  and  $B_{\frac{lcm(n,m)}{m}} = (d_1, d_2, \dots, d_{lcm(n,m)})$ ,

$$\mathbf{C}(A, B) = \{T(c_1, d_1), T(c_2, d_2), \dots, T(c_{lcm(n,m)}, d_{lcm(n,m)})\}. \quad (1)$$

**Definition 2.14.** [7] Let  $T$  be a  $t$ -norm on  $[0, 1]$ . On  $\bar{\mathbf{B}}$  the binary operation  $\mathbf{T}$  is defined in the following way. For two elements  $[\vec{u}] = [(a_1, a_2, \dots, a_n)]$  and  $[\vec{v}] = [(b_1, b_2, \dots, b_m)]$  of  $\bar{\mathbf{B}}$ , rewriting  $\vec{u}_{\frac{lcm(n,m)}{n}} = (c_1, c_2, \dots, c_{lcm(n,m)})$  and  $\vec{v}_{\frac{lcm(n,m)}{m}} = (d_1, d_2, \dots, d_{lcm(n,m)})$ ,

$$\mathbf{T}([\vec{u}], [\vec{v}]) = [(T(c_1, d_1), T(c_2, d_2), \dots, T(c_{lcm(n,m)}, d_{lcm(n,m)}))]. \quad (2)$$

In [7], it is shown that the  $\mathbf{C}$  on  $(\mathbf{H}, \leq_H)$  is a commutative fuzzy conjunction instead of a  $t$ -norm and  $\mathbf{T}$  on  $(\bar{\mathbf{B}}, \leq_B)$  is a  $t$ -norm. Moreover,  $\mathbf{C}$  is proved that it is a  $t$ -norm on the subset  $\mathbf{H}_R$  of  $\mathbf{H}$ .

### 3 T-conorms on $(\mathbf{H}, \leq_H)$ and $(\bar{\mathbf{B}}, \leq_B)$

In this section, the binary operations  $\mathbf{D}$  on  $(\mathbf{H}, \leq_H)$  and  $\mathbf{S}$  on  $(\bar{\mathbf{B}}, \leq_B)$  associated with a  $t$ -conorm  $S$  on  $[0, 1]$  are introduced respectively. The following properties are investigated:  $\mathbf{D}$  on  $\mathbf{H}$  is a commutative fuzzy disjunction instead of a  $t$ -conorm. However,  $\mathbf{D}$  on  $\mathbf{H}_R$  is a  $t$ -conorm. Moreover,  $\mathbf{S}$  on  $\bar{\mathbf{B}}$  is a  $t$ -conorm.

**Definition 3.1.** Let  $\mathbf{P} = (P, \leq_P, 0, 1)$  be a bounded partially ordered set. An operation  $\mathbf{D}: P^2 \rightarrow P$  is called a fuzzy disjunction if

- (i) it is increasing with respect to each variable.
- (ii)  $\mathbf{D}(0, 0) = 0, \mathbf{D}(0, 1) = \mathbf{D}(1, 0) = \mathbf{D}(1, 1) = 1$ .

Definition 3.1 presents the notion of a general fuzzy disjunction on a bounded partially ordered set. The fuzzy disjunction on a bounded partially ordered set is precisely a classical fuzzy disjunction introduced by Batyrshin et al. [4] when the bounded partially ordered set is the interval  $[0, 1]$ .

Let us present the definition of a  $t$ -conorm on a bounded partially ordered set.

**Definition 3.2.** A  $t$ -conorm  $S$  on a bounded partially ordered set  $\mathbf{P} = (P, \leq_P, 0, 1)$  is a binary operation on  $P$  that for all  $x, y, z \in P$  satisfies

- (i)  $S(x, 0) = x$  (neutral element)
- (ii) If  $x \leq_P y$ , then  $S(x, z) \leq_P S(y, z)$  (monotonicity)
- (iii)  $S(x, y) = S(y, x)$  (commutativity)
- (iv)  $S(x, S(y, z)) = S(S(x, y), z)$ .

From a  $t$ -conorm  $S$  on  $[0, 1]$ , an operation  $\mathbf{D}$  can be defined on  $\mathbf{H}$ .

**Definition 3.3.** Let  $S$  be a  $t$ -conorm on  $[0, 1]$ . On  $\mathbf{H}$  the binary operation  $\mathbf{D}$  is defined in the following way. For two elements  $A = \{a_1, a_2, \dots, a_n\}$  and  $B = \{b_1, b_2, \dots, b_m\}$  of  $\mathbf{H}$ , rewriting  $A_{\frac{lcm(n,m)}{n}} = (c_1, c_2, \dots, c_{lcm(n,m)})$  and  $B_{\frac{lcm(n,m)}{m}} = (d_1, d_2, \dots, d_{lcm(n,m)})$ ,

$$\mathbf{D}(A, B) = \{S(c_1, d_1), S(c_2, d_2), \dots, S(c_{lcm(n,m)}, d_{lcm(n,m)})\}.$$

(3)

**Example 3.4.** If  $A = \{0.3, 0.6\}$  and  $B = \{0.2, 0.5, 0.6\}$  and  $S$  is the Product  $t$ -conorm ( $a \oplus b = a + b - ab$ ), then  $A_3 = (0.3, 0.3, 0.3, 0.6, 0.6, 0.6)$ ,  $B_2 = (0.2, 0.2, 0.5, 0.5, 0.6, 0.6)$  and

$$\mathbf{D}(A, B) = \{0.44, 0.44, 0.65, 0.8, 0.84, 0.84\} = \{0.44, 0.65, 0.8, 0.84\}.$$

Notably the last equality is obtained by transforming the multiset into a set.

**Lemma 3.5.** Let  $A = \{a_1, a_2, \dots, a_n\}$  and  $B = \{b_1, b_2, \dots, b_m\}$  be two finite subsets of the unit interval and  $r \in \mathbb{N}$ . Rewriting  $A_{r \frac{lcm(n,m)}{n}} = (c_1, c_2, \dots, c_{rlcm(n,m)})$  and  $B_{r \frac{lcm(n,m)}{m}} = (d_1, d_2, \dots, d_{rlcm(n,m)})$ ,

$$\mathbf{D}(A, B) = \{S(c_1, d_1), S(c_2, d_2), \dots, S(c_{rlcm(n,m)}, d_{rlcm(n,m)})\}.$$

(4)

**Proposition 3.6.** Let  $S$  be a  $t$ -conorm on  $[0, 1]$ . Then  $\mathbf{D}$  is a commutative fuzzy disjunction on  $(\mathbf{H}, \leq_H)$ .

*Proof.* (i) Obviously, it follows that  $\mathbf{D}$  satisfied commutativity from Definition of  $\mathbf{D}$ .

(ii) Let  $A = \{a_1, a_2, \dots, a_n\}$ ,  $B = \{b_1, b_2, \dots, b_m\}$  and  $C = \{c_1, c_2, \dots, c_p\}$  be finite subsets of the unit interval. Rewriting  $A_{\frac{lcm(n,m,p)}{n}} = (d_1, d_2, \dots, d_{lcm(n,m,p)})$ ,  $B_{\frac{lcm(n,m,p)}{m}} = (e_1, e_2, \dots, e_{lcm(n,m,p)})$  and  $C_{\frac{lcm(n,m,p)}{p}} = (f_1, f_2, \dots, f_{lcm(n,m,p)})$ . If  $A \leq_H B$ , then, from Lemma 2.3,  $d_i \leq e_i$  for all  $i = 1, 2, \dots, lcm(n, m, p)$ , so  $S(d_i, f_i) \leq S(e_i, f_i)$ . From Lemma 3.5, it follows that  $\mathbf{D}(A, C) \leq \mathbf{D}(B, C)$ .

(iii) The conditions  $\mathbf{D}(\{0\}, \{0\}) = \{0\}$ ,  $\mathbf{D}(\{0\}, \{1\}) = \mathbf{D}(\{1\}, \{0\}) = \mathbf{D}(\{1\}, \{1\}) = \{1\}$  are easily proved.  $\square$

Since it does not satisfy associativity, the fuzzy disjunction  $\mathbf{D}$  is not a  $t$ -conorm on  $(\mathbf{H}, \leq_H)$ . In fact, we could show the following counter-example:

Let  $A = \{0.5, 0.6\}$ ,  $B = \{0.4, 0.5, 0.7\}$ ,  $C = \{0.2, 0.3, 0.5\}$  and  $S$  is a Product  $t$ -conorm ( $a \oplus b = a + b - ab$ ). Then

$$\mathbf{D}(\mathbf{D}(A, B), C) = \{0.76, 0.8, 0.825, 0.86, 0.9, 0.94\},$$

$$\mathbf{D}(A, \mathbf{D}(B, C)) = \{0.76, 0.825, 0.86, 0.94\}.$$

Clearly,  $\mathbf{D}(\mathbf{D}(A, B), C) \neq \mathbf{D}(A, \mathbf{D}(B, C))$ . Therefore,  $\mathbf{D}$  is not a  $t$ -conorm on  $(\mathbf{H}, \leq_H)$ .

**Proposition 3.7.** Let  $R = r_1, r_2, \dots, r_n, \dots$  be a sequence of natural numbers with  $r_{i+1}$  a multiple of  $r_i$  for every  $i \geq 0$ ,  $H_{r_i}$  is the set of finite subsets of unit interval of cardinality  $r_i$  and  $H_R = \bigcup_{i \geq 1} H_{r_i}$ . Then  $\mathbf{D}: \mathbf{H}_R^2 \rightarrow \mathbf{H}_R$  is a  $t$ -conorm on  $\mathbf{H}_R$ .

On  $(\bar{\mathbf{B}}, \leq_B)$  we can also derive a binary operation  $\mathbf{S}$  from a  $t$ -conorm on  $[0, 1]$  in a similar way as Definition 3.3. In this case,  $\mathbf{S}$  is a  $t$ -conorm.

**Definition 3.8.** Let  $S$  be a  $t$ -conorm on  $[0, 1]$ . On  $\bar{\mathbf{B}}$  the binary operation  $\mathbf{S}$  is defined in the following way. For two elements  $[\vec{u}] = [(a_1, a_2, \dots, a_n)]$  and  $[\vec{v}] = [(b_1, b_2, \dots, b_m)]$  of  $\bar{\mathbf{B}}$ , rewriting  $\vec{u}_{\frac{lcm(n,m)}{n}} = (c_1, c_2, \dots, c_{lcm(n,m)})$  and  $\vec{v}_{\frac{lcm(n,m)}{m}} = (d_1, d_2, \dots, d_{lcm(n,m)})$ ,

$$\mathbf{S}([\vec{u}], [\vec{v}]) = [(S(c_1, d_1), S(c_2, d_2), \dots, S(c_{lcm(n,m)}, d_{lcm(n,m)}))].$$

(5)

**Example 3.9.** If  $\vec{u} = (0.3, 0.6)$  and  $\vec{v} = (0.2, 0.5, 0.6)$  and  $S$  is the Product  $t$ -conorm ( $a \oplus b = a + b - ab$ ), then  $\vec{u}_3 = (0.3, 0.3, 0.3, 0.6, 0.6, 0.6)$ ,  $\vec{v}_2 = (0.2, 0.2, 0.5, 0.5, 0.6, 0.6)$  and  $\mathbf{S}([\vec{u}], [\vec{v}]) = [(0.44, 0.44, 0.65, 0.8, 0.84, 0.84)]$ .

**Proposition 3.10.** Let  $S$  be a  $t$ -conorm on  $[0, 1]$ . Then  $\mathbf{S}$  is a  $t$ -conorm on  $(\bar{\mathbf{B}}, \leq_B)$ .

*Proof.* Let  $\vec{u} = \{a_1, a_2, \dots, a_n\}$ ,  $\vec{v} = \{b_1, b_2, \dots, b_m\}$  and  $\vec{w} = \{c_1, c_2, \dots, c_p\}$  be finite bags of the unit interval. Rewriting  $\vec{u}_{\frac{lcm(n,m,p)}{n}} = (d_1, d_2, \dots, d_{lcm(n,m,p)})$ ,  $\vec{v}_{\frac{lcm(n,m,p)}{m}} = (e_1, e_2, \dots, e_{lcm(n,m,p)})$  and  $\vec{w}_{\frac{lcm(n,m,p)}{p}} = (f_1, f_2, \dots, f_{lcm(n,m,p)})$ .

(i) It is trivial to prove that  $\mathbf{S}([\vec{u}], [\vec{0}]) = [\vec{u}]$ .

(ii) Monotonicity: If  $\vec{u} \leq_B \vec{v}$ , then  $d_i \leq e_i$  for all  $i = 1, 2, \dots, lcm(n, m, p)$  and from this  $S(d_i, f_i) \leq S(e_i, f_i)$ . Hence,  $\mathbf{S}(\vec{u}, \vec{w}) \leq \mathbf{S}(\vec{v}, \vec{w})$ .

(iii) Commutativity is easily proved from the commutativity of  $S$ .

(iv) Associativity:

$$\begin{aligned} & \mathbf{S}([\vec{u}], \mathbf{S}([\vec{v}], [\vec{w}])) \\ &= \mathbf{S}([(d_1, d_2, \dots, d_{lcm(n,m,p)}], \mathbf{S}([(e_1, e_2, \dots, e_{lcm(n,m,p)}], [(f_1, f_2, \dots, f_{lcm(n,m,p)}])])) \\ &= \mathbf{S}([(d_1, d_2, \dots, d_{lcm(n,m,p)}], [(S(e_1, f_1), S(e_2, f_2), \dots, S(e_{lcm(n,m,p)}, f_{lcm(n,m,p)}))])) \\ &= [(S(d_1, S(e_1, f_1)), S(d_2, S(e_2, f_2)), \dots, S(d_{lcm(n,m,p)}, S(e_{lcm(n,m,p)}, f_{lcm(n,m,p)})))] \\ &= [(S(S(d_1, e_1), f_1), S(S(d_2, e_2), f_2), \dots, S(S(d_{lcm(n,m,p)}, e_{lcm(n,m,p)}), f_{lcm(n,m,p)})))] \\ &= \mathbf{S}([(S(d_1, e_1), S(d_2, e_2), \dots, S(d_{lcm(n,m,p)}, e_{lcm(n,m,p)}))], [(f_1, f_2, \dots, f_{lcm(n,m,p)})]) \\ &= \mathbf{S}(\mathbf{S}([\vec{u}], [\vec{v}]), [\vec{w}]). \end{aligned}$$

□

## 4 Negations on $(\mathbf{H}, \leq_H)$ and $(\bar{\mathbf{B}}, \leq_B)$

In this section, we will define negation operations  $N_n$  on  $(\mathbf{H}, \leq_H)$  and  $\mu_n$  on  $(\bar{\mathbf{B}}, \leq_B)$  associated with a negation  $n$  on  $[0, 1]$  and investigate whether binary operations  $\mathbf{C}$  and  $\mathbf{D}$  ( $\mathbf{T}$  and  $\mathbf{S}$ ) with respect to negation  $N_n$  ( $\mu_n$ ) satisfy De Morgan's law or not. In the following, we will recall definition of negation on  $[0, 1]$ .

**Definition 4.1.** [20] A decreasing function  $n : [0, 1] \rightarrow [0, 1]$  such that  $n(0) = 1$  and  $n(1) = 0$  is said to be negation. If, additionally,  $n(n(x)) = x$  holds for all  $x \in [0, 1]$ , it is said to be a strong negation.

This definition can be extended to any bounded partially ordered set.

**Definition 4.2.** [8] Let  $A$  be a set and  $\leq_A$  be a partial order in  $A$  such that  $(A, \leq_A)$  has a minimum element  $\text{Min}_{\leq_A}$  and a maximum element  $\text{Max}_{\leq_A}$ . A negation in  $(A, \leq_A)$  is a function  $N : A \rightarrow A$  such that  $N$  is decreasing,  $N(\text{Min}_{\leq_A}) = \text{Max}_{\leq_A}$  and  $N(\text{Max}_{\leq_A}) = \text{Min}_{\leq_A}$ . If, additionally,  $N(N(x)) = x$  holds for all  $x \in A$ , it is said to be a strong negation.

**Definition 4.3.** Let  $A = \{a_1, a_2, \dots, a_n\} \in \mathbf{H}$ , and  $n$  is a negation in  $[0, 1]$ . The operation  $N_n$  associated with  $n$  is defined as

$$N_n(A) = \{n(a_n), n(a_{n-1}), \dots, n(a_1)\}. \quad (6)$$

**Lemma 4.4.** Let  $A = \{a_1, a_2, \dots, a_n\} \in \mathbf{H}$ , and  $n$  is a negation in  $[0, 1]$ . Rewriting  $A_r = (\overbrace{a_1, \dots, a_1}^{r \text{ times}}, \overbrace{a_2, \dots, a_2}^{r \text{ times}}, \dots, \overbrace{a_n, \dots, a_n}^{r \text{ times}}) = (c_1, c_2, \dots, c_{rn})$ ,  $r \in \mathbf{N}$ ,

$$N_n(A) = \{n(c_{rn}), \dots, n(c_2), n(c_1)\}. \quad (7)$$

**Proposition 4.5.** Let  $N_n : \mathbf{H} \rightarrow \mathbf{H}$  be the operation associated with a negation  $n$  in  $[0, 1]$ . Then  $N_n$  is a negation in  $(\mathbf{H}, \leq_H)$ .

*Proof.* **Proof.** (i)  $N_n(\{0\}) = \{1\}$  and  $N_n(\{1\}) = \{0\}$  are easily proved by definition of  $N_n$ .

(ii) Let  $A = \{a_1, a_2, \dots, a_n\}$  and  $B = \{b_1, b_2, \dots, b_m\}$  be any two elements of  $\mathbf{H}$ . Rewriting  $A_{\frac{lcm(n,m)}{n}} = (c_1, c_2, \dots, c_{lcm(n,m)})$  and  $B_{\frac{lcm(n,m)}{m}} = (d_1, d_2, \dots, d_{lcm(n,m)})$ , applying Lemma 4.4, so  $A \leq_H B \Rightarrow c_i \leq d_i \Rightarrow n(c_i) \geq n(d_i) \Rightarrow N_n(B) \leq_H N_n(A)$ . Hence,  $N_n$  is a negation on  $(\mathbf{H}, \leq_H)$ . □

**Proposition 4.6.** Let  $N_n : \mathbf{H} \rightarrow \mathbf{H}$  be the operation associated with a negation  $n$  in  $[0, 1]$ . Then  $N_n$  is involutive, that is,  $N_n(N_n(A)) = A$ ,  $\forall A \in \mathbf{H}$  if and only if  $n$  is strong.

*Proof.* “ $\Leftarrow$ ” if  $n$  is strong, then  $n(n(x)) = x$ ,  $\forall x \in [0, 1]$ . Hence,  $\forall A = \{a_1, a_2, \dots, a_n\} \in \mathbf{H}$ ,

$$\begin{aligned} N_n(N_n(A)) &= N_n(\{n(a_n), n(a_{n-1}), \dots, n(a_1)\}) \\ &= \{n(n(a_1)), \dots, n(n(a_{n-1})), n(n(a_n))\} \\ &= \{a_1, a_2, \dots, a_n\} = A. \end{aligned}$$

“ $\Rightarrow$ ” if negation  $n$  is not strong, there exists  $u \in [0, 1]$  such that  $n(n(u)) \neq u$ . Without loss of generality, assuming  $u$  is unique, let  $B = \{u, b_2, \dots, b_m\} \in \mathbf{H}$  and  $n(n(b_i)) = b_i$  for all  $i = 2, 3, \dots, m$ . Then  $N_n(N_n(B)) = \{n(n(u)), b_2, \dots, b_m\} \neq B$ . It is contradictory with  $N_n$  which is involutive.  $\square$

**Proposition 4.7.** *Let  $N_n : \mathbf{H} \rightarrow \mathbf{H}$  be the operation associated with a negation  $n$  in  $[0, 1]$ . Then,  $N_n$  is a strong negation in  $\mathbf{H}$  if and only if  $n$  is strong.*

*Proof.* It is directly proved from Propositions 4.5 and 4.6.  $\square$

**Proposition 4.8.** *Let  $N_n : \mathbf{H} \rightarrow \mathbf{H}$  be the operation associated with an injective negation  $n$  in  $[0, 1]$ . Then  $N_n$  is a negation in  $\mathbf{H}_R$ .*

*Proof.* On the one hand  $N_n$  is closed in  $\mathbf{H}_R$  because  $n$  is an injective negation, on the other hand  $\mathbf{H}_R$  is the subset of the  $\mathbf{H}$ , therefore from Proposition 4.5, the  $N_n$  is a negation in  $\mathbf{H}_R$ .  $\square$

**Remark 4.9.** *In Proposition 4.8, the  $n$  must be injective, otherwise  $N_n$  may be not closed in  $\mathbf{H}_R$ . Indeed, we can show the following counter-example:*

*Assuming*

$$n(x) = \begin{cases} 1 - 2x, & 0 \leq x \leq 0.5; \\ 0, & 0.5 \leq x \leq 1 \end{cases}$$

*Obviously,  $n$  is a negation in  $[0, 1]$  but is not injective. Let  $A = \{0.3, 0.6, 0.7\} \in \mathbf{H}_R$ . Then*

$$N_n(A) = \{n(0.7), n(0.6), n(0.3)\} = \{0, 0, 0.4\} = \{0, 0.4\},$$

*it follows from the definition of  $\mathbf{H}_R$  that  $N_n(A) = \{0, 0.4\} \notin \mathbf{H}_R$ .*

**Proposition 4.10.** *Let  $N_n : \mathbf{H} \rightarrow \mathbf{H}$  be the operation associated with an injective negation  $n$  in  $[0, 1]$ . Then  $N_n$  is strong in  $\mathbf{H}_R$  if and only if  $n$  is strong in  $[0, 1]$ .*

*Proof.* Since  $\mathbf{H}_R$  is the subset of the  $\mathbf{H}$ , from Proposition 4.6,  $N_n$  is involutive in  $\mathbf{H}_R$  if and only if  $n$  is strong. Additionally, from Proposition 4.8, then  $N_n$  is strong in  $\mathbf{H}_R$  if and only if  $n$  is strong in  $[0, 1]$ .  $\square$

On  $(\bar{\mathbf{B}}, \leq_B)$  we can also derive a negation operation  $\mu_n$  from a negation operation  $n$  on  $[0, 1]$  in a similar way as in Definition 4.3.

**Definition 4.11.** *Let  $n$  be a negation operation in  $[0, 1]$ . The operation  $\mu_n$  on  $\bar{\mathbf{B}}$  is defined in the following way: for any element  $[\vec{u}] = [(a_1, a_2, \dots, a_n)] \in \bar{\mathbf{B}}$ ,*

$$\mu_n([\vec{u}]) = [(n(a_n), n(a_{n-1}), \dots, n(a_1))]. \quad (8)$$

**Lemma 4.12.** *Let any  $[\vec{u}] = [(a_1, a_2, \dots, a_n)] \in \bar{\mathbf{B}}$ , and  $n$  is a negation in  $[0, 1]$ . Rewriting  $\vec{u}_r = (\overbrace{a_1, \dots, a_1}^{r \text{ times}}, \overbrace{a_2, \dots, a_2}^{r \text{ times}}, \dots, \overbrace{a_n, \dots, a_n}^{r \text{ times}}) = (c_1, c_2, \dots, c_{rn})$ ,  $r \in N$ ,*

$$\mu_n([\vec{u}]) = [(n(c_{rn}), \dots, n(c_2), n(c_1))]. \quad (9)$$

**Proposition 4.13.** *Let  $\mu_n : \bar{\mathbf{B}} \rightarrow \bar{\mathbf{B}}$  be the operation associated with a negation  $n$  in  $[0, 1]$ . Then  $\mu_n$  is a negation in  $(\bar{\mathbf{B}}, \leq_B)$ .*

*Proof.* (1)  $\mu_n([\vec{0}]) = [\vec{1}]$  and  $\mu_n([\vec{1}]) = [\vec{0}]$  are easily proved by definition of  $\mu_n$ .

(2) Let  $[\vec{u}] = [(a_1, a_2, \dots, a_n)]$  and  $[\vec{v}] = [(b_1, b_2, \dots, b_m)]$  be any two elements of  $\bar{\mathbf{B}}$ . Rewriting  $\vec{u}_{lcm(n,m)} = (c_1, c_2, \dots, c_{lcm(n,m)})$  and  $\vec{v}_{lcm(n,m)} = (d_1, d_2, \dots, d_{lcm(n,m)})$ , applying Lemma 4.12, so  $\vec{u} \leq_B \vec{v} \Rightarrow c_i \leq d_i \Rightarrow n(c_i) \geq n(d_i) \Rightarrow \mu_n([\vec{v}]) \leq_B \mu_n([\vec{u}])$ . Hence,  $\mu_n$  is a negation on  $(\bar{\mathbf{B}}, \leq_B)$ .  $\square$

**Proposition 4.14.** *Let  $\mu_n : \bar{\mathbf{B}} \rightarrow \bar{\mathbf{B}}$  be the operation associated with a negation  $n$  in  $[0, 1]$ . Then  $\mu_n$  is involutive, that is,  $\mu_n(\mu_n([\vec{u}])) = [\vec{u}]$ ,  $\forall [\vec{u}] \in \bar{\mathbf{B}}$  if and only if  $n$  is strong.*

*Proof.* “ $\Leftarrow$ ” if  $n$  is strong, then  $n(n(x)) = x, \forall x \in [0, 1]$ . For all  $[\vec{u}] = [(a_1, a_2, \dots, a_n)] \in \bar{\mathbf{B}}$ ,

$$\begin{aligned}\mu_n(\mu_n([\vec{u}])) &= \mu_n([(n(a_n), n(a_{n-1}), \dots, n(a_1))]) \\ &= [(n(n(a_1)), \dots, n(n(a_{n-1})), n(n(a_n)))] \\ &= [(a_1, a_2, \dots, a_n)] = [\vec{u}].\end{aligned}$$

“ $\Rightarrow$ ” if negation  $n$  is not strong, there exist  $u \in [0, 1]$  such that  $n(n(u)) \neq u$ . Without loss of generality, assuming  $u$  is unique, let  $[\vec{v}] = [(u, b_2, \dots, b_m)] \in \bar{\mathbf{B}}$  and  $n(n(b_i)) = b_i$  for all  $i = 2, 3, \dots, m$ . Then  $\mu_n(\mu_n([\vec{v}])) = [(n(n(u)), b_2, \dots, b_m)] \neq [\vec{v}]$ , it is contradictory with  $\mu_n$  which is involutive.  $\square$

**Proposition 4.15.** *Let  $\mu_n : \bar{\mathbf{B}} \rightarrow \bar{\mathbf{B}}$  be the operation associated with a negation  $n$  in  $[0, 1]$ . Then  $\mu_n$  is strong in  $(\bar{\mathbf{B}}, \leq_B)$  if and only if  $n$  is strong.*

*Proof.* It is directly proved from Propositions 4.13 and 4.14.  $\square$

In [7], binary operations  $\mathbf{C}$  on  $\mathbf{H}$  and  $\mathbf{T}$  on  $\bar{\mathbf{B}}$  associated with t-norm  $T$  in  $[0, 1]$  are defined respectively. In the following we will analyze whether  $\mathbf{C}$  and  $\mathbf{D}$  satisfies De Morgan’s law or not with respect to  $N_n$ .

**Proposition 4.16.** *Let  $N_n$  be an operation associated with the strong negation  $n$ ,  $T$  and  $S$ , t-norm and dual t-conorm, respectively, with respect to  $n$ . For any two elements  $A = \{a_1, a_2, \dots, a_n\}$  and  $B = \{b_1, b_2, \dots, b_m\}$  of  $\mathbf{H}$ , rewriting  $A_{\frac{1cm(n,m)}{n}} = (c_1, c_2, \dots, c_{1cm(n,m)})$  and  $B_{\frac{1cm(n,m)}{m}} = (d_1, d_2, \dots, d_{1cm(n,m)})$ ,  $\mathbf{C}$  and  $\mathbf{D}$  are defined as*

$$\begin{aligned}\mathbf{C}(A, B) &= \{T(c_1, d_1), T(c_2, d_2), \dots, T(c_{1cm(n,m)}, d_{1cm(n,m)})\}, \\ \mathbf{D}(A, B) &= \{S(c_1, d_1), S(c_2, d_2), \dots, S(c_{1cm(n,m)}, d_{1cm(n,m)})\},\end{aligned}$$

it holds that

$$N_n(\mathbf{C}(A, B)) = \mathbf{D}(N_n(A), N_n(B)) \text{ and } N_n(\mathbf{D}(A, B)) = \mathbf{C}(N_n(A), N_n(B)).$$

(10)

*Proof.*  $N_n(\mathbf{C}(A, B)) = \{n(T(c_{1cm(n,m)}, d_{1cm(n,m)})), \dots, n(T(c_2, d_2)), n(T(c_1, d_1))\}$   
 $= \{S(n(c_{1cm(n,m)}), n(d_{1cm(n,m)})), \dots, S(n(c_2), n(d_2)), S(n(c_1), n(d_1))\}$   
 $= \mathbf{D}(N_n(A), N_n(B)).$

The proof of  $N_n(\mathbf{D}(A, B)) = \mathbf{C}(N_n(A), N_n(B))$  is similar.  $\square$

**Example 4.17.** Assuming  $n(x) = 1 - x$  and  $T$  and  $S$  are Product t-norm ( $a \otimes b = ab$ ) and Product t-conorm ( $a \oplus b = a + b - ab$ ) respectively. Obviously,  $T$  and  $S$  are dual with respect to the strong negation  $n$ . For two elements  $A = \{0.1, 0.3\}$  and  $B = \{0.2, 0.3, 0.4\}$  of  $\mathbf{H}$ , rewriting  $A_3 = \{0.1, 0.1, 0.1, 0.3, 0.3, 0.3\}$  and  $B_2 = \{0.2, 0.2, 0.3, 0.3, 0.4, 0.4\}$ . Then

$$\begin{aligned}N_n(\mathbf{C}(A, B)) &= N_n(\{0.02, 0.02, 0.03, 0.09, 0.12, 0.12\}) \\ &= N_n(\{0.02, 0.03, 0.09, 0.12\}) = \{0.88, 0.91, 0.97, 0.98\}\end{aligned}$$

and

$$\begin{aligned}\mathbf{D}(N_n(A), N_n(B)) &= \mathbf{D}(\{0.7, 0.9\}, \{0.6, 0.7, 0.8\}) \\ &= \{0.88, 0.88, 0.91, 0.97, 0.98, 0.98\} = \{0.88, 0.91, 0.97, 0.98\}.\end{aligned}$$

Clearly,

$$N_n(\mathbf{C}(A, B)) = \mathbf{D}(N_n(A), N_n(B)).$$

Similarly,  $N_n(\mathbf{D}(A, B)) = \mathbf{C}(N_n(A), N_n(B)).$

**Remark 4.18.** If  $n$  is not strong, the Proposition 4.16 may be not true. Indeed, we can show the following counter-example:

Assuming

$$n(x) = \begin{cases} 1 - 2x, & 0 \leq x \leq 0.4; \\ 0, & 0.4 < x \leq 1 \end{cases}$$

and although the other conditions of Example 4.17. Then

$$N_n(\mathbf{C}(A, B)) = N_n(\{0.02, 0.02, 0.03, 0.09, 0.12, 0.12\})$$

$$= N_n(\{0.02, 0.03, 0.09, 0.12\}) = \{0.76, 0.82, 0.94, 0.96\}$$

and

$$\begin{aligned} \mathbf{D}(N_n(A), N_n(B)) &= \mathbf{D}(\{0.4, 0.8\}, \{0.2, 0.4, 0.6\}) \\ &= \{0.52, 0.52, 0.64, 0.88, 0.92, 0.92\} = \{0.52, 0.64, 0.88, 0.92\}. \end{aligned}$$

Clearly,  $N_n(\mathbf{C}(A, B)) \neq \mathbf{D}(N_n(A), N_n(B))$ .

**Proposition 4.19.** *Let  $\mu_n$  be an operation associated with the strong negation  $n$ ,  $T$  and  $S$ ,  $t$ -norm and dual  $t$ -conorm, respectively, with respect to  $n$ . For two any elements  $[\vec{u}] = [(a_1, a_2, \dots, a_n)]$  and  $[\vec{v}] = [(b_1, b_2, \dots, b_m)]$  of  $\bar{\mathbf{B}}$ , rewriting  $\vec{u}_{lcm(n,m)} = (c_1, c_2, \dots, c_{lcm(n,m)})$  and  $\vec{v}_{lcm(n,m)} = (d_1, d_2, \dots, d_{lcm(n,m)})$ ,  $\mathbf{T}$  and  $\mathbf{S}$  are defined as*

$$\begin{aligned} \mathbf{T}([\vec{u}], [\vec{v}]) &= [(T(c_1, d_1), T(c_2, d_2), \dots, T(c_{lcm(n,m)}, d_{lcm(n,m)}))], \\ \mathbf{S}([\vec{u}], [\vec{v}]) &= [(S(c_1, d_1), S(c_2, d_2), \dots, S(c_{lcm(n,m)}, d_{lcm(n,m)}))], \end{aligned}$$

it holds that

$$\mu_n(\mathbf{T}([\vec{u}], [\vec{v}])) = \mathbf{S}(\mu_n([\vec{u}]), \mu_n([\vec{v}])) \text{ and } \mu_n(\mathbf{S}([\vec{u}], [\vec{v}])) = \mathbf{T}(\mu_n([\vec{u}]), \mu_n([\vec{v}])).$$

(11)

*Proof.*  $\mu_n(\mathbf{T}([\vec{u}], [\vec{v}])) = [(n(T(c_{lcm(n,m)}, d_{lcm(n,m)})), \dots, n(T(c_2, d_2)), n(T(c_1, d_1)))] = [(S(n(c_{lcm(n,m)}), n(d_{lcm(n,m)})), \dots, S(n(c_2), n(d_2)), S(n(c_1), n(d_1)))] = \mathbf{S}(\mu_n([\vec{u}]), \mu_n([\vec{v}]))$ .

The proof of  $\mu_n(\mathbf{S}([\vec{u}], [\vec{v}])) = \mathbf{T}(\mu_n([\vec{u}]), \mu_n([\vec{v}]))$  is similar.  $\square$

**Example 4.20.** Assuming  $n(x) = 1 - x$  and  $T$  and  $S$  are Łukasiewicz  $t$ -norm ( $a \otimes_{Lu} b = (a + b - 1) \vee 0$ ) and Łukasiewicz  $t$ -conorm ( $a \oplus_{Lu} b = (a + b) \wedge 1$ ) respectively. Obviously,  $T$  and  $S$  are dual with respect to the strong negation  $n$ . For two elements  $[\vec{u}] = [(0.5, 0.7)]$  and  $[\vec{v}] = [(0.5, 0.6, 0.8)]$  of  $\bar{\mathbf{B}}$ , rewriting  $[\vec{u}_3] = [(0.5, 0.5, 0.5, 0.7, 0.7, 0.7)]$  and  $[\vec{v}_2] = [(0.5, 0.5, 0.6, 0.6, 0.8, 0.8)]$ . Then

$$\mu_n(\mathbf{T}([\vec{u}], [\vec{v}])) = \mu_n([(0, 0, 0.1, 0.3, 0.5, 0.5)]) = [(0.5, 0.5, 0.7, 0.9, 1, 1)]$$

and

$$\mathbf{S}(\mu_n([\vec{u}]), \mu_n([\vec{v}])) = \mathbf{S}([(0.3, 0.5)], [(0.2, 0.4, 0.5)]) = [(0.5, 0.5, 0.7, 0.9, 1, 1)].$$

Clearly,

$$\mu_n(\mathbf{T}([\vec{u}], [\vec{v}])) = \mathbf{S}(\mu_n([\vec{u}]), \mu_n([\vec{v}])).$$

Similarly,  $\mu_n(\mathbf{S}([\vec{u}], [\vec{v}])) = \mathbf{T}(\mu_n([\vec{u}]), \mu_n([\vec{v}]))$ .

**Remark 4.21.** If  $n$  is not strong, the Proposition 4.19 may be not true. Indeed, we can show the following counterexample:

Assuming

$$n(x) = \begin{cases} 1 - 2x, & 0 \leq x \leq 0.4; \\ 0, & 0.4 < x \leq 1 \end{cases}$$

and although the other conditions of Example 4.20. Then

$$\mu_n(\mathbf{T}([\vec{u}], [\vec{v}])) = \mu_n([(0, 0, 0.1, 0.3, 0.5, 0.5)]) = [(0, 0, 0.4, 0.8, 1, 1)]$$

and

$$\mathbf{S}(\mu_n([\vec{u}]), \mu_n([\vec{v}])) = \mathbf{S}([(0, 0)], [(0, 0, 0)]) = [(0, 0, 0, 0, 0, 0)].$$

Clearly,  $\mu_n(\mathbf{T}([\vec{u}], [\vec{v}])) \neq \mathbf{S}(\mu_n([\vec{u}]), \mu_n([\vec{v}]))$ .



## 5 Aggregation operations on $(\mathbf{H}, \leq_H)$ and $(\bar{\mathbf{B}}, \leq_B)$

In this section, new aggregation operations will be proposed on  $(\mathbf{H}, \leq_H)$  and  $(\bar{\mathbf{B}}, \leq_B)$  and they will be extended to more general form. Next, let us recall some conceptions about aggregation operation.

**Definition 5.1.** [5] *An  $n$ -ary aggregation function is an increasing function  $M : [0, 1]^n \rightarrow [0, 1]$  such that  $M(0, \dots, 0) = 0$  and  $M(1, \dots, 1) = 1$ .*

This definition can be extended to the bounded partially ordered sets.

**Definition 5.2.** [17] *Let  $A$  be a set and  $\leq_A$  be a partial order in  $A$  such that  $(A, \leq_A)$  has a minimum element  $\text{Min}_{\leq_A}$  and a maximum element  $\text{Max}_{\leq_A}$ . An aggregation operation in  $(A, \leq_A)$  is a function  $\mathbf{M} : A^n \rightarrow A$  such that  $\mathbf{M}$  is increasing,  $\mathbf{M}(\text{Min}_{\leq_A}, \dots, \text{Min}_{\leq_A}) = \text{Min}_{\leq_A}$  and  $\mathbf{M}(\text{Max}_{\leq_A}, \dots, \text{Max}_{\leq_A}) = \text{Max}_{\leq_A}$ .*

**Definition 5.3.** Let  $M$  be an  $n$ -ary aggregation function in  $[0, 1]$ . For  $A_i = \{a_1^i, a_2^i, \dots, a_{m_i}^i\} \in \mathbf{H}$  and  $e = \text{lcm}(m_1, m_2, \dots, m_n)$ , rewriting  $(A_i)_{\frac{e}{m_i}} = (c_1^i, c_2^i, \dots, c_e^i)$ ,  $i = 1, 2, \dots, n$ , an  $n$ -ary function  $\mathbf{M} : \mathbf{H}^n \rightarrow \mathbf{H}$  is defined as

$$\mathbf{M}(A_1, A_2, \dots, A_n) = \{M(c_1^1, \dots, c_1^n), M(c_2^1, \dots, c_2^n), \dots, M(c_e^1, \dots, c_e^n)\}. \quad (12)$$

**Lemma 5.4.** *Let  $M$  be an  $n$ -ary aggregation function in  $[0, 1]$ . For  $A_i = \{a_1^i, a_2^i, \dots, a_{m_i}^i\} \in \mathbf{H}$  and  $e = \text{lcm}(m_1, m_2, \dots, m_n)$ , rewriting  $(A_i)_{r \frac{e}{m_i}} = (c_1^i, c_2^i, \dots, c_{r e}^i)$ ,  $i = 1, 2, \dots, n$ ,  $r \in N$ ,*

$$\mathbf{M}(A_1, A_2, \dots, A_n) = \{M(c_1^1, \dots, c_1^n), M(c_2^1, \dots, c_2^n), \dots, M(c_{r e}^1, \dots, c_{r e}^n)\}. \quad (13)$$

**Proposition 5.5.** *Let  $\mathbf{M} : \mathbf{H}^n \rightarrow \mathbf{H}$  be the  $n$ -ary operation associated with an aggregation function  $M$  in  $[0, 1]$ . Then,  $\mathbf{M}$  is an aggregation operation in  $\mathbf{H}$ .*

*Proof.* (i)  $\mathbf{M}(\{0\}, \{0\}, \dots, \{0\}) = \{0\}$  and  $\mathbf{M}(\{1\}, \{1\}, \dots, \{1\}) = \{1\}$ .

(ii) Assuming  $A_i = \{a_1^i, a_2^i, \dots, a_{m_i}^i\}$  and  $B_i = \{b_1^i, b_2^i, \dots, b_{p_i}^i\}$  are the elements of  $\mathbf{H}$  for all  $i = 1, 2, \dots, n$ . Let  $\delta = \text{lcm}(m_1, m_2, \dots, m_n)$ ,  $\varepsilon = \text{lcm}(p_1, p_2, \dots, p_n)$  and  $\eta = \text{lcm}(\delta, \varepsilon)$ , rewriting  $(A_i)_{\frac{\eta}{m_i}} = (c_1^i, c_2^i, \dots, c_\eta^i)$  and  $(B_i)_{\frac{\eta}{p_i}} = (d_1^i, d_2^i, \dots, d_\eta^i)$ , then from Lemma 5.4

$$\mathbf{M}(A_1, A_2, \dots, A_n) = \{M(c_1^1, \dots, c_1^n), M(c_2^1, \dots, c_2^n), \dots, M(c_\eta^1, \dots, c_\eta^n)\}$$

and

$$\mathbf{M}(B_1, B_2, \dots, B_n) = \{M(d_1^1, \dots, d_1^n), M(d_2^1, \dots, d_2^n), \dots, M(d_\eta^1, \dots, d_\eta^n)\}.$$

For all  $i = 1, 2, \dots, n$ ,  $A_i \leq_H B_i \Rightarrow c_j^i \leq d_j^i \Rightarrow M(c_j^1, \dots, c_j^n) \leq M(d_j^1, \dots, d_j^n)$  for all  $j = 1, 2, \dots, \eta$ . Hence,  $\mathbf{M}(A_1, A_2, \dots, A_n) \leq_H \mathbf{M}(B_1, B_2, \dots, B_n)$ . So  $\mathbf{M}$  is an aggregation function in  $\mathbf{H}$ .  $\square$

**Example 5.6.** Let  $A_1 = \{0.2, 0.3\}$ ,  $A_2 = \{0.4, 0.5\}$  and  $A_3 = \{0.1, 0.3, 0.4\}$  be the elements of  $\mathbf{H}$ , and  $M(x_1, x_2, x_3) = \frac{x_1 + x_2 + x_3}{3}$ . Rewriting

$$(A_1)_3 = \{0.2, 0.2, 0.2, 0.3, 0.3, 0.3\},$$

$$(A_2)_3 = \{0.4, 0.4, 0.4, 0.5, 0.5, 0.5\},$$

$$(A_3)_2 = \{0.1, 0.1, 0.3, 0.3, 0.4, 0.4\}.$$

Then

$$\mathbf{M}(A_1, A_2, A_3) = \{M(0.2, 0.4, 0.1), M(0.2, 0.4, 0.1), M(0.2, 0.4, 0.3), M(0.3, 0.5, 0.3), M(0.3, 0.5, 0.4), M(0.3, 0.5, 0.4)\} = \{\frac{7}{30}, \frac{7}{30}, 0.3, \frac{11}{30}, 0.4, 0.4\} = \{\frac{7}{30}, 0.3, \frac{11}{30}, 0.4\}.$$

**Corollary 5.7.** *Let  $M_1 \leq M_2 \leq \dots \leq M_n \leq \dots$  be a sequence of  $n$ -ary aggregation functions in  $[0, 1]$ . For  $A_i = \{a_1^i, a_2^i, \dots, a_{m_i}^i\} \in \mathbf{H}$  and  $e = \text{lcm}(m_1, m_2, \dots, m_n)$ , rewriting  $(A_i)_{\frac{e}{m_i}} = (c_1^i, c_2^i, \dots, c_e^i)$ ,  $i = 1, 2, \dots, n$ , an  $n$ -ary function  $\mathbf{M} : \mathbf{H}^n \rightarrow \mathbf{H}$  is an aggregation operation in  $\mathbf{H}$  if  $\mathbf{M}$  is defined as*

$$\mathbf{M}(A_1, A_2, \dots, A_n) = \{M_1(c_1^1, \dots, c_1^n), M_2(c_2^1, \dots, c_2^n), \dots, M_e(c_e^1, \dots, c_e^n)\}.$$

(14)

*Proof.* The proof is similar to that of Proposition 5.5.  $\square$

**Proposition 5.8.** *Let  $M : \mathbf{H}^n \rightarrow \mathbf{H}$  be the  $n$ -ary operation associated with an injective aggregation function  $M$  in  $[0, 1]$ . Then  $\mathbf{M}$  is an aggregation operation in  $\mathbf{H}_R$ .*

*Proof.* Firstly,  $\mathbf{M}$  is closed in  $\mathbf{H}_R$  because  $M$  is an injective aggregation function  $M$  in  $[0, 1]$ . Secondly  $\mathbf{H}_R$  is the subset of  $\mathbf{H}$ . Therefore, it follows from Proposition 5.5 that  $\mathbf{M}$  is an aggregation operation in  $\mathbf{H}_R$ .  $\square$

**Remark 5.9.** In Proposition 5.8, the  $M$  must be injective, otherwise it may be not closed in  $\mathbf{H}_R$ . In fact, we can show the following counter-example:

Assuming

$$f(u, v) = \begin{cases} u, & v = 1 \\ v, & u = 1 \\ 0, & \text{otherwise} \end{cases},$$

it is a t-norm in  $[0, 1]$ , so it is an aggregation function, which is not injective. Let  $A = \{0.1, 1\}$  and  $B = \{0.1, 0.3, 0.4, 1\}$  be two elements of  $\mathbf{H}_R$ , rewriting  $A_2 = \{0.1, 0.1, 1, 1\}$ . Then

$$\mathbf{M}(A, B) = \{f(0.1, 0.1), f(0.1, 0.3), f(1, 0.4), f(1, 1)\} = \{0, 0, 0.4, 1\} = \{0, 0.4, 1\},$$

but  $\{0, 0.4, 1\} \notin \mathbf{H}_R$  here.

On  $(\bar{\mathbf{B}}, \leq_B)$  we can also derive an  $n$ -ary operation  $\mathbf{G}$  from an aggregation function  $M$  on  $[0, 1]$  in similar way as Definition 5.3.

**Definition 5.10.** Let  $M$  be an  $n$ -ary aggregation function in  $[0, 1]$ . For  $[\vec{u}_i] = [(a_1^i, a_2^i, \dots, a_{m_i}^i)] \in \bar{\mathbf{B}}$  and  $e = \text{lcm}(m_1, m_2, \dots, m_n)$ , rewriting  $(\vec{u}_i)_{\frac{e}{m_i}} = (c_1^i, c_2^i, \dots, c_e^i)$ ,  $i = 1, 2, \dots, n$ . An  $n$ -ary function  $\mathbf{G} : \bar{\mathbf{B}}^n \rightarrow \bar{\mathbf{B}}$  is defined as

$$\mathbf{G}([\vec{u}_1], [\vec{u}_2], \dots, [\vec{u}_n]) = [(M(c_1^1, \dots, c_1^n), M(c_2^1, \dots, c_2^n), \dots, M(c_e^1, \dots, c_e^n))].$$

(15)

**Proposition 5.11.** *Let  $\mathbf{G} : \bar{\mathbf{B}}^n \rightarrow \bar{\mathbf{B}}$  be the  $n$ -ary operation associated with an aggregation function  $M$  on  $[0, 1]$ . Then  $\mathbf{G}$  is an aggregation operation in  $\bar{\mathbf{B}}$ .*

*Proof.* (i)  $\mathbf{G}([\vec{0}], [\vec{0}], \dots, [\vec{0}]) = [\vec{0}]$  and  $\mathbf{G}([\vec{1}], [\vec{1}], \dots, [\vec{1}]) = [\vec{1}]$ .

(ii) Assuming that  $[\vec{u}_i] = [(a_1^i, a_2^i, \dots, a_{m_i}^i)]$  and  $[\vec{v}_i] = [(b_1^i, b_2^i, \dots, b_{p_i}^i)]$  are the elements of  $\bar{\mathbf{B}}$  for all  $i = 1, 2, \dots, n$ . Let  $\delta = \text{lcm}(m_1, m_2, \dots, m_n)$ ,  $\varepsilon = \text{lcm}(p_1, p_2, \dots, p_n)$  and  $\eta = \text{lcm}(\delta, \varepsilon)$ , rewriting  $(\vec{u}_i)_{\frac{\eta}{m_i}} = (c_1^i, c_2^i, \dots, c_\eta^i)$  and  $(\vec{v}_i)_{\frac{\eta}{p_i}} = (d_1^i, d_2^i, \dots, d_\eta^i)$ , then

$$\mathbf{G}([\vec{u}_1], [\vec{u}_2], \dots, [\vec{u}_n]) = [(M(c_1^1, \dots, c_1^n), M(c_2^1, \dots, c_2^n), \dots, M(c_\eta^1, \dots, c_\eta^n))]$$

and

$$\mathbf{G}([\vec{v}_1], [\vec{v}_2], \dots, [\vec{v}_n]) = [(M(d_1^1, \dots, d_1^n), M(d_2^1, \dots, d_2^n), \dots, M(d_\eta^1, \dots, d_\eta^n))].$$

For all  $i = 1, 2, \dots, n$ ,  $\vec{u}_i \leq_B \vec{v}_i \Rightarrow c_j^i \leq d_j^i \Rightarrow M(c_j^1, \dots, c_j^n) \leq M(d_j^1, \dots, d_j^n)$  for all  $j = 1, 2, \dots, \eta$ . Hence,  $\mathbf{G}([\vec{u}_1], [\vec{u}_2], \dots, [\vec{u}_n]) \leq_B \mathbf{G}([\vec{v}_1], [\vec{v}_2], \dots, [\vec{v}_n])$ . So  $\mathbf{G}$  is an aggregation function in  $\bar{\mathbf{B}}$ .  $\square$

**Example 5.12.** Let  $[\vec{u}_1] = [(0.1, 0.2)]$ ,  $[\vec{u}_2] = [(0.1, 0.3)]$  and  $[\vec{u}_3] = [(0.1, 0.3, 0.3)]$  be the elements of  $\bar{\mathbf{B}}$ , and  $M(x_1, x_2, x_3) = \frac{x_1 + x_2 + x_3}{3}$ . Rewriting

$$(\vec{u}_1)_3 = (0.1, 0.1, 0.1, 0.2, 0.2, 0.2),$$

$$(\vec{u}_2)_3 = (0.1, 0.1, 0.1, 0.3, 0.3, 0.3),$$

$$(\vec{u}_3)_2 = (0.1, 0.1, 0.3, 0.3, 0.3, 0.3).$$

Then

$$\mathbf{G}([\vec{u}_1], [\vec{u}_2], [\vec{u}_3]) = [(M(0.1, 0.1, 0.1), M(0.1, 0.1, 0.1), M(0.1, 0.1, 0.3), M(0.2, 0.3, 0.3), M(0.2, 0.3, 0.3), M(0.3, 0.3, 0.3))] = [(0.1, 0.1, \frac{1}{6}, \frac{8}{30}, \frac{8}{30}, 0.3)].$$

**Corollary 5.13.** Let  $M_1 \leq M_2 \leq \dots \leq M_n \leq \dots$  be a sequence of  $n$ -ary aggregation functions in  $[0, 1]$ . For  $[\vec{u}_i] = [(a_1^i, a_2^i, \dots, a_{m_i}^i)] \in \bar{\mathbf{B}}$  and  $e = \text{lcm}(m_1, m_2, \dots, m_n)$ , rewriting  $(\vec{u}_i)_{\frac{e}{m_i}} = (c_1^i, c_2^i, \dots, c_e^i)$ ,  $i = 1, 2, \dots, n$ . An  $n$ -ary function  $\mathbf{G} : \bar{\mathbf{B}}^n \rightarrow \bar{\mathbf{B}}$  is an aggregation operation in  $\bar{\mathbf{B}}$  if  $\mathbf{G}$  is defined as

$$\mathbf{G}([\vec{u}_1], [\vec{u}_2], \dots, [\vec{u}_n]) = [(M_1(c_1^1, \dots, c_1^n), M_2(c_2^1, \dots, c_2^n), \dots, M_n(c_e^1, \dots, c_e^n))]. \quad (16)$$

*Proof.* The proof is similar to that of Proposition 5.11. □

## 6 Decision making based on hesitant fuzzy information

In this section, we will utilize a special aggregation operation  $\mathbf{M}$  and the score function value of hesitant fuzzy element [24](i.e. score function value of hesitant fuzzy element  $h$  is  $s(h) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma$ , where  $\#h$  is the number of elements in  $h$ , and for two hesitant fuzzy elements  $h_1, h_2 \in \mathbf{H}$ , if  $s(h_1) < s(h_2)$ , then  $h_1 < h_2$ ; if  $s(h_1) = s(h_2)$ , then  $h_1 = h_2$ .) to deal with decision making based on hesitant fuzzy information.

**Example 6.1.** [10, 24] *The enterprise's board of directors, which includes five members, is to plan the development of large projects (strategy initiatives) for the following five years. Suppose there are four possible projects  $Y_i$  ( $i = 1, 2, 3, 4$ ) to be evaluated. It is necessary to compare these projects to select the most important of them as well as order them from the point of view of their importance, taking into account four attributes suggested by the Balanced Scorecard methodology (it should be noted that all of them are of the maximization type):  $G_1$ : financial perspective,  $G_2$ : the customer satisfaction,  $G_3$ : internal business process perspective, and  $G_4$ : learning and growth perspective. And suppose that the weight vector of the attributes is  $w = (0.2, 0.3, 0.15, 0.35)^T$ .*

In the following, we use the proposed method to determine the optimal project.

**Step 1.** The decision matrix  $H = (h_{ij})_{4 \times 4}$  is given in **Table 1**, where  $h_{ij}(i, j = 1, 2, 3, 4)$  is in the form of hesitant fuzzy elements.

**Step 2.** Let  $M(x_1, x_2, x_3, x_4) = 0.2 \times x_1 + 0.3 \times x_2 + 0.15 \times x_3 + 0.35 \times x_4$ . Then by utilizing aggregation operation  $\mathbf{M}$  of Definition 23, the hesitant fuzzy elements  $h_i(i = 1, 2, 3, 4)$  for the projects  $Y_i(i = 1, 2, 3, 4)$  could be obtained. That is,

$$\begin{aligned} h_1 &= \mathbf{M}(h_{11}, h_{12}, h_{13}, h_{14}) = \mathbf{M}((0.2, 0.4, 0.7), (0.2, 0.6, 0.8), (0.2, 0.3, 0.6, 0.7, 0.9), \\ &(0.3, 0.4, 0.5, 0.7, 0.8)) = (0.235, 0.235, 0.235, 0.285, 0.285, 0.445, 0.525, 0.525, 0.525, \\ &0.61, 0.73, 0.73, 0.795, 0.795, 0.795) = (0.235, 0.285, 0.445, 0.525, 0.61, 0.73, 0.795), \\ h_2 &= \mathbf{M}(h_{21}, h_{22}, h_{23}, h_{24}) = \mathbf{M}((0.2, 0.4, 0.7, 0.9), (0.1, 0.2, 0.4, 0.5), (0.3, 0.4, 0.6, \\ &0.9), (0.5, 0.6, 0.8, 0.9)) = (0.29, 0.41, 0.63, 0.78), \\ h_3 &= \mathbf{M}(h_{31}, h_{32}, h_{33}, h_{34}) = \mathbf{M}((0.3, 0.5, 0.6, 0.7), (0.2, 0.4, 0.5, 0.6), (0.3, 0.5, 0.7, \\ &0.8), (0.2, 0.5, 0.6, 0.7)) = (0.235, 0.47, 0.585, 0.685), \\ h_4 &= \mathbf{M}(h_{41}, h_{42}, h_{43}, h_{44}) = \mathbf{M}((0.3, 0.5, 0.6), (0.2, 0.4), (0.5, 0.6, 0.7), (0.8, 0.9)) \\ &= (0.475, 0.475, 0.53, 0.625, 0.66, 0.66) = (0.475, 0.53, 0.625, 0.66). \end{aligned}$$

**Step 3.** Calculate the score values  $s(h_i)(i = 1, 2, 3, 4)$  of  $h_i$ :

$$s(h_1) = 0.5179, s(h_2) = 0.5275, s(h_3) = 0.4938, s(h_4) = 0.5725.$$

**Step 4.** By ranking  $s(h_i)(i = 1, 2, 3, 4)$  of  $h_i$ ,  $h_4 > h_2 > h_1 > h_3$ , which means that  $Y_4$  is the optimal project. This result is consistent with the conclusion in [24], so the validity of this method is verified.

**Table 1** Hesitant fuzzy decision matrix

	$G_1$	$G_2$	$G_3$	$G_4$
$Y_1$	(0.2, 0.4, 0.7)	(0.2, 0.6, 0.8)	(0.2, 0.3, 0.6, 0.7, 0.9)	(0.3, 0.4, 0.5, 0.7, 0.8)
$Y_2$	(0.2, 0.4, 0.7, 0.9)	(0.1, 0.2, 0.4, 0.5)	(0.3, 0.4, 0.6, 0.9)	(0.5, 0.6, 0.8, 0.9)
$Y_3$	(0.3, 0.5, 0.6, 0.7)	(0.2, 0.4, 0.5, 0.6)	(0.3, 0.5, 0.7, 0.8)	(0.2, 0.5, 0.6, 0.7)
$Y_4$	(0.3, 0.5, 0.6)	(0.2, 0.4)	(0.5, 0.6, 0.7)	(0.8, 0.9)

## 7 Conclusions

The paper has introduced two binary operations  $\mathbf{D}$  on  $(\mathbf{H}, \leq_H)$  and  $\mathbf{S}$  on  $(\bar{\mathbf{B}}, \leq_B)$  respectively. It is shown that  $\mathbf{D}$  is not a t-conorm on  $(\mathbf{H}, \leq_H)$ . However,  $\mathbf{D}$  is a t-conorm on  $(\mathbf{H}_R, \leq_H)$  and  $\mathbf{S}$  is a t-conorm on  $(\bar{\mathbf{B}}, \leq_B)$ . In addition, we have proposed two negations  $N_n$  on  $(\mathbf{H}, \leq_H)$  and  $\mu_n$  on  $(\bar{\mathbf{B}}, \leq_B)$  respectively and investigated their De Morgan's laws with respect to  $\mathbf{C}$  and  $\mathbf{D}$  on  $(\mathbf{H}, \leq_H)$  as well as  $\mathbf{T}$  and  $\mathbf{S}$  on  $(\bar{\mathbf{B}}, \leq_B)$ . Two aggregation operations and their general forms have been provided respectively on  $(\mathbf{H}, \leq_H)$  and on  $(\bar{\mathbf{B}}, \leq_B)$ . Moreover, the validity of the aggregation operation on  $(\mathbf{H}, \leq_H)$  has been illustrated by a numerical example on decision making.

The next research directions are to find reasonable ways to construct triangular norms on  $\mathbf{H}$  from triangular norms on  $[0, 1]$  and investigate the application of negations and aggregation operators based on the hesitant fuzzy partial ordering introduced by [7].

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