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A new view on fuzzy automata normed linear structure spaces

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Abstract

In this paper, the concept of fuzzy automata normed linear structure spaces is introduced and suitable examples are provided. The concepts of fuzzy automata α-open sphere, fuzzy automata $\mathscr{N}$-locally compact spaces, fuzzy automata $\mathscr{N}$-Hausdorff spaces are also discussed. Some properties related with fuzzy automata normed linear structure spaces and fuzzy automata $\mathscr{N}$-Hausdorff spaces are discussed.

Keywords: Fuzzy automata normed linear structure spaces, Fuzzy automata $\mathscr{N}$-Hausdorff spaces and Fuzzy automata $\mathscr{N}$-locally compact spaces.

1 Introduction

Zadeh [25] innovated the concept of a fuzzy set in 1965 and then it has invaded almost all branches of mathematics. The notion of an automaton was first fuzzified by Wee [24]. Later, the concepts of fuzzy subsystems and strong fuzzy subsystems of a fuzzy finite state machine (ffsm) were introduced and studied by Malik and Mordeson [16]. In [2], [19], [20], it is shown that certain topological and fuzzy topological concepts can be used in fuzzy automata theory to throw light on the structure of such fuzzy automata, particularly, to obtain certain results pertaining to their connectivity and separation properties. Z. H. Li, P. Li and Y. M. Li, [15] discussed the relationships among several types of fuzzy automata. In [4], [6], [12], [13], [14], [21], [22] the researchers began to work on fuzzy automata with membership values in complete residuated lattice, lattice ordered monoid and some kind of lattices. Ignjatovic, Ciric and Simovic [7] studied the concepts of subsystems, reverse subsystems and double subsystems of a fuzzy automaton in terms of fuzzy relation inequalities and equations. Tiwari, Singh, Sharan and Yadav [22] introduced and studied the concept of bifuzzy core inducing a bifuzzy topology on the state-set of fuzzy automaton. Katsaras [9] introduced the idea of fuzzy norm on a linear space. In 1992, Felbin [3] introduced an idea of a fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space so that the corresponding fuzzy metric associated to this fuzzy norm is of Kaleva and Seikkala type [8]. In 1994, Cheng and Mordeson [1] introduced another idea of a fuzzy norm on a linear space in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [10]. In motivation of the paper Cheng and Mordeson, we have introduced a new definition of a fuzzy norm which is associated fuzzy automata. The novelty of this definition is the validity of this type of fuzzy norm into a family of non-empty states in fuzzy automata. Thus in this paper, the concept of fuzzy automata normed linear structure spaces is introduced and suitable examples are provided. The concepts of fuzzy automata α-open sphere, fuzzy automata $\mathscr{N}$-locally compact spaces, fuzzy automata $\mathscr{N}$-Hausdorff spaces are also discussed. Some properties related with fuzzy automata normed linear structure spaces and fuzzy automata $\mathscr{N}$-Hausdorff spaces are discussed.

Motivated by the work done by some fuzzy topologist [20], [21],[23] on general fuzzy automata, along with fuzzy automata structure space introduced by [11], the notion of fuzzy automata normed linear structure space is introduced. Also the concepts of fuzzy automata α-open sphere and fuzzy automata $\mathscr{N}$-locally compact spaces are also discussed. Some properties related with fuzzy automata normed linear structure spaces and fuzzy automata $\mathscr{N}$-Hausdorff spaces are discussed.

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2 Preliminaries

In this section, some basic concepts of fuzzy automata have been recalled. Also some related results and propositions are studied from various research articles. Some definitions and preliminary results are presented in this section in our form.

Definition 2.1. [19] A fuzzy automaton is a triple $M = (Q, X, \delta)$, where Q is a set (of states of M), X is a monoid (the input monoid of M), whose identity shall be denoted as e and δ is a fuzzy subset of $Q \times X \times Q$, i.e., a map $\delta : Q \times X \times Q \to [0, 1]$, such that $\forall q, p \in Q$, $\forall x, y \in X$,

(i) $\delta(q, e, p) = 1$ or 0, according as $q = p$ or $q \neq p$,

(ii) $\delta(q, xy, p) = \bigvee\{\delta(q, x, r) \wedge \delta(r, x, p) : r \in Q\}$.

Definition 2.2. [17] $\lambda \in I^Q$ is called a fuzzy subsystem of (Q, X, δ) if

$$\lambda(q) \geq \lambda(p) \wedge \delta(p, x, q), \forall p, q \in Q, x \in X$$

Definition 2.3. [19] A Kuratowski fuzzy closure operator on X is a function $c : I^X \to I^X$ satisfying $\forall \alpha \in I, \lambda, \mu \in I^X$,

(i) $c(\alpha) = \alpha$,

(ii) $\lambda \leq c(\lambda)$,

(iii) $c(\lambda \vee \mu) = c(\lambda) \vee (\mu)$,

(iv) $c(c(\lambda)) = c(\lambda)$.

A Kuratowski fuzzy closure operator k on X is called saturated if $\forall \lambda_j \in I^X$, $j \in J$,

$$c(\vee\{\lambda_j : j \in J\}) = \vee\{c(\lambda_j) : j \in J\}.$$

Remark 2.4. Every Kuratowski fuzzy closure operator k on X gives rise to a fuzzy topology on X in which a fuzzy set μ is closed iff $k(\mu) = \mu$.

Proposition 2.5. [2] The function $c : I^Q \to I^Q$ defined as

$$c(\lambda)(q) = \vee\{\vee\{\lambda(p) \wedge \delta(p, x, q) : x \in X\} : p \in Q\}, \forall \lambda \in I^Q, \forall q \in Q$$

is a kuratowski saturated fuzzy closure operator on Q.

This proposition shows that c is a fuzzy closure operator on I^Q. Then c induces a fuzzy topology τ on Q. The fuzzy topology τ is called the fuzzy topology associated with the fuzzy automaton M.

Proposition 2.6. [2] $\lambda \in I^Q$ is a fuzzy subsystem of (Q, X, δ) iff $c(\lambda) = \lambda$ (*i.e.*, iff λ is closed with respect to fuzzy topology induced by c on Q).

Definition 2.7. [2] A fuzzy subset λ of Q is said to be a generating fuzzy set of M if $c(\lambda) = 1$.

Definition 2.8. [18] Suppose that T_1 and T_2 be two topologies on X. T_2 is said to be coarser than T_1 if $T_1 \supset T_2$.

Definition 2.9. [3] A norm on a linear space X is a function $\| \cdot \| : X \to R$ with the following properties:

(i) $\|x\| \geq 0$ for all $x \in X$,

(ii) $\|\lambda x\| = |\lambda| \|x\|$, for all $x \in X$ and $\lambda \in R$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$,

(vi) $\|x\| = 0$ implies that $x = 0$.

A normed linear space $(X, \| \cdot \|)$ is a linear space X equipped with a norm $\| \cdot \|$.

Example 2.10. $C[a, b]$ is a normed linear space with the maximum (or L^∞) norm

$$||f|| = max_{[a,b]}|f(x)|.$$

Definition 2.11. [5] A triangular norm (*t-norm*) T on $[0, 1]$ is defined as an increasing, commutative and associative function $T : [0, 1] \times [0, 1] \to [0, 1]$ satisfying $T(1, x) = x$, for all $x \in [0, 1]$.

3 Fuzzy Automata Normed Linear Structure Spaces

Motivated by the work done by some fuzzy topologist [22], [23],[25] on general fuzzy automata, the notion of fuzzy automata normed linear structure space along with the concepts of fuzzy automata α-open sphere and fuzzy automata $\mathscr{N}$-locally compact spaces are introduced. Some properties related with fuzzy automata normed linear structure spaces and fuzzy automata $\mathscr{N}$-Hausdorff spaces are discussed. Throughout this paper, Q is a non-empty set of states of M. A fuzzy subset λ of Q is characterized by a membership function $\lambda : Q \to I$, where $I = [0, 1]$. The set of all fuzzy subsets of Q will be denoted by I^Q.

Definition 3.1. [11] Let M $= (Q, X, \delta)$ be a fuzzy automaton where Q is a set of states of M. For all $\lambda \in I^Q$ and $q \in Q$, $c(\lambda)(q) = \bigvee_{p\in Q}\{\bigvee_{x\in X}\{\lambda(p) \wedge \delta(p, x, q)\}\}$ is a fuzzy closure operator on Q. Let $\tau = \{\lambda \in I^Q : c(1_Q - \lambda) = 1_Q - \lambda$, where $1_Q - \lambda$ is the fuzzy automata complement of λ $\}$ be the collection of fuzzy sets which satisfies the following conditions:

(i) $0_Q, 1_Q \in \tau$;

(ii) If $\lambda_1, \lambda_2 \in \tau$, then $\lambda_1 \wedge \lambda_2 \in \tau$;

(iii) If $\lambda_i \in \tau$ for each $i \in J$, where J is an indexed set then $\vee\lambda_i \in \tau$.

Then the ordered pair (Q, τ) is said to be a fuzzy automata structure space iff there exists a fuzzy automaton (Q, X, δ) such that τ is a fuzzy topology associated with (Q, X, δ). Moreover, the members of τ are said to be the fuzzy automata open subsystems and their complements are said to be the fuzzy automata closed subsystems.

Notation Throughout this paper, 0_Q takes the membership value $\mu_{0_Q}(q) = 0$, for all $q \in Q$ and 1_Q takes the membership value $\mu_{1_Q}(q) = 1$, for all $q \in Q$.

Example 3.2. Let M $= (Q, X, \delta)$ be a fuzzy automaton, where the set of all states Q is given by $Q = X = \{0, 1, 2, ...\}$ and $\delta : Q \times X \times Q \to [0, 1]$ is given by

$$\delta(q, 0, p) = \begin{cases} 1 & \text{if} \quad q = p \\ 0 & \text{if} \quad q \neq p \end{cases}$$

with $\delta(q, x_0, p) = 0.8, \delta(q, x_0, q) = 0.7, \delta(p, x_0, p) = 0.6$ and $\delta(p, x_0, q) = 0.75$ for a fixed $x_0 \in X$ such that $x_0 \neq 0$ and for fixed $p, q \in Q$. For other $p, q \in Q$ and $x \in X$. Let $\delta(p, x, q) = 0$, $\lambda, \eta \in I^Q$ be defined as follows: $\lambda(p) = 0.62, \lambda(q) = 0.5$, $\eta(p) = 0.34, \eta(q) = 0.25$ and for other $r \in Q, \lambda(r) = 0$ and $\eta(r) = 0$. The Kuratowski saturated fuzzy closure operator $c : I^Q \to I^Q$ on Q is defined as follows,

$$c(\lambda)(q) = \bigvee_{p\in Q} \left\{\bigvee_{x\in X}\{\lambda(p) \wedge \delta(p, x, q) : x \in X\} : p \in Q\right\}$$

for all $\lambda \in I^Q$ and $q \in Q$. It is seen that $c(\lambda) = \lambda, c(\eta) = \eta, c(0_Q) = 0_Q$ and $c(1_Q) = 1_Q$. Let $\tau = \{0_Q, 1_Q, 1_Q - \lambda, 1_Q - \eta\}$. Then clearly τ is a fuzzy automata structure on Q and hence the ordered pair (Q, τ) is a fuzzy automata structure space.

Definition 3.3. Let $M = (Q, X, \delta)$ be a fuzzy automaton. A fuzzy automata normed linear space is a 3-tuple (Q, N, T) where Q is non-empty set of states of M and also it is a linear space over the field $\mathbb{F}$, T is a t-norm and N is a fuzzy set on $Q \times (0, \infty)$, such that for all $p, q \in Q$ and all $s, t > 0$, the following conditions holds:

(i) $N(p, t) > 0$,

(ii) $N(p, t) = 1$, for all $t > 0$ if and only if $p = 0$,

(iii) If $\alpha \neq 0$, then $N(\alpha p, t) = N(p, \frac{t}{|\alpha|})$, $\forall t, \alpha \in \mathbb{F}$,

(iv) $T(N(p, t), N(q, s)) \leq N(p + q, t + s)$, $\forall t, s \in \mathbb{F}$,

(v) $N(p, \cdot)$ is a non-decreasing function of $\mathbb{F}$ and $lim_{t\to\infty}N(p, t) = 1$.

(vi) Assume that for all $p \neq 0, N(p, \cdot)$ is a continuous function on $\mathbb{F}$ and strictly increasing on the subset $\{$ $t : 0 < N(p, t) < 1$ $\}$ of $\mathbb{F}$.

Example 3.4. Let $M = (Q, X, \delta)$ be a fuzzy automaton where $Q = \mathbb{R}^2$ is a vector space over the field $\mathbb{R}$. Let $p = (p_1, p_2) \in \mathbb{R}^2$ and $N : \mathbb{R}^2 \times (0, \infty) \to [0, 1]$ be defined by

$$N(p,t) = \begin{cases} \frac{t^2}{(t\,+\,|p_1|)\,\,(t\,+\,|p_2|)}, & \text{for } t > 0 \\ \\ 0, & \text{for } t \leq 0 \end{cases}$$

and also the t-norm is defined as $T(a, b) = ab$. Then $(\mathbb{R}^2, N, T)$ is a fuzzy automata normed linear space.

Example 3.5. Let $M = (Q, X, \delta)$ be a fuzzy automaton where $Q = \mathbb{C}$ is a vector space over the field $\mathbb{R}$. Let $N : \mathbb{C} \times (0, \infty) \to [0, 1]$ be defined by

$$N(p,t) = \begin{cases} 1, & \text{for} \quad |p| < t \\ 0, & \text{for} \quad |p| \geq t \end{cases}$$

and also the t-norm is defined as $T(a, b) = min\{\ a, b\ \}$. Then $(\mathbb{C}, N, T)$ is a fuzzy automata normed linear space.

Definition 3.6. Let (Q, N, T) be a fuzzy automata normed linear space and let $p \in Q$, $\alpha \in (0, 1)$ and $\epsilon > 0$. The fuzzy set $\mu_\alpha(p, \epsilon)$ where $\mu_\alpha : Q \times (0, \infty) \to I$, be defined over Q by

$$\mu_\alpha(p,\epsilon)(q) = \begin{cases} 1-\alpha, & N(p-q,\epsilon) > \alpha \\ 0, & \text{otherwise} \end{cases}$$

is said to be a fuzzy automata α-open sphere in Q if

$$c(1_Q - \mu_\alpha(p, \epsilon)) = (1_Q - \mu_\alpha(p, \epsilon)).$$

Definition 3.7. Any fuzzy set $\mu \in I^Q$ is called fuzzy automata open subsystem if for $\mu(p) > 0$ and $c(1_Q - \mu) = (1_Q - \mu)$, there exists $\epsilon > 0$ such that $\mu_\alpha(p, \epsilon) \leq \mu$, for some $\alpha \in (0, 1)$ and $\forall p \in Q$.

Definition 3.8. Let Q be non-empty set of states of M. A fuzzy point q_β over Q has the membership function

$$q_\beta(p) = \begin{cases} \beta, & \text{for} \quad q = p \\ 0, & \text{for} \quad q \neq p \end{cases}$$

is also a fuzzy set, for all $p \in Q$, where $0 \leq \beta \leq 1$. q_β is said to have a support q and value β. If the fuzzy point q_β satisfies the conditions of fuzzy automata norm linear on Q, then it is said to be a fuzzy automata $\mathscr{N}$-point in Q. The collection of all fuzzy automata $\mathscr{N}$-points in Q is denoted by $\mathcal{F}\mathscr{N}\mathcal{P}(Q)$.

Proposition 3.9. Let $M = (Q, X, \delta)$ be a fuzzy automaton and let Q be a non-empty set of states of M. Let (Q, N, T) be a fuzzy automata normed linear space. Then the family

$$\tau_{\mathscr{N}} = \{\ \mu \in I^Q : \mu \text{ is fuzzy automata open subsystem }\}$$

is a fuzzy automata normed linear structure on Q.

Proof. (i) Clearly $1_Q(p) = 1$, for all $p \in Q$. Since $\mu_\alpha(p, \epsilon) \leq 1$, for all $\epsilon > 0$, $\alpha \in (0, 1)$ and $c(1_Q - 1_Q) = c(0_Q) = 0_Q = 1_Q - 1_Q$. Hence $1_Q \in \tau_N$. Since $0_Q(p) = 0$, for all $p \in Q$ and $c(1_Q - 0_Q) = c(1_Q) = 1_Q = 1_Q - 0_Q$. Thus $0_Q \in \tau_{\mathscr{N}}$.

(ii) Let $\mu_1, \mu_2 \in \tau_{\mathscr{N}}$ and $(\mu_1 \wedge \mu_2)(p) \geq 0$. We have $\mu_1(p) \geq 0, c(1_Q - \mu_1) = 1_Q - \mu_1$ and $\mu_2(p) \geq 0, c(1_Q - \mu_2) = 1_Q - \mu_2$. So there exist $\alpha_1, \alpha_2 \in (0, 1)$ and $\epsilon_1, \epsilon_2 > 0$ such that $\mu_{\alpha_1}(p, \epsilon_1) \leq \mu_1$ and $\mu_{\alpha_2}(p, \epsilon_2) \leq \mu_2$. Assume that $\alpha = max\ \{\ \alpha_1, \alpha_2\ \}$ and $\epsilon = min\ \{\ \epsilon_1, \epsilon_2\ \}$. If $N(p - q, \epsilon) > \alpha$, then

$$N(p-q,\epsilon_1) \geq N(p-q,\epsilon) > \alpha \geq \alpha_1$$
$$N(p-q,\epsilon_2) \geq N(p-q,\epsilon) > \alpha \geq \alpha_2.$$

Since $N(p - q, \epsilon) > \alpha$, by Definition 3.6 $\mu_\alpha(p, \epsilon)(q) = 1 - \alpha$. Since $\alpha \geq \alpha_1$, $\mu_\alpha(p, \epsilon)(q) = 1 - \alpha \leq 1 - \alpha_1 = \mu_{\alpha_1}(p, \epsilon_1)(q)$ and since $\alpha \geq \alpha_2$, $\mu_\alpha(p, \epsilon)(q) = 1 - \alpha \leq 1 - \alpha_2 = \mu_{\alpha_2}(p, \epsilon_2)(q)$ which implies $\mu_\alpha(p, \epsilon) \leq \mu_{\alpha_1}(p, \epsilon_1)$ and $\mu_\alpha(p, \epsilon) \leq \mu_{\alpha_2}(p, \epsilon_2)$. Therefore

$$\mu_\alpha(p,\epsilon) \leq \mu_{\alpha_1}(p,\epsilon_1) \wedge \mu_{\alpha_2}(p,\epsilon_2) \leq \mu_1 \wedge \mu_2.$$

Also $c(1_Q - (\mu_1 \wedge \mu_2)) = (1_Q - (\mu_1 \wedge \mu_2))$. Hence $(\mu_1 \wedge \mu_2) \in \tau_{\mathscr{N}}$.

(iii) Let $\{\mu_i\} \in \tau_{\mathscr{N}}$, for $i \in J$ and $\bigvee_i \mu_i(p) \geq 0, i \in J$. Then there exists i_0 such that $\mu_{i_0}(p) \geq 0$ and $c(1_Q - \mu_{i_0}) = 1_Q - \mu_{i_0}$. So there exist $\epsilon > 0$ and $\alpha \in (0,1)$ such that $\mu_\alpha(p,\epsilon) \leq \mu_{i_0}$. Therefore $\mu_\alpha(p,\epsilon) \leq \bigvee_i \mu_i$ and $c(1_Q - \bigvee_i \mu_i) = (1_Q - \bigvee_i \mu_i)$, $i \in J$. Hence $\bigvee_i \mu_i \in \tau_{\mathscr{N}}$ where $i \in J$.

Hence $(Q, \tau_{\mathscr{N}})$ is a fuzzy automata normed linear structure space. The members of $\tau_{\mathscr{N}}$ are called the fuzzy automata $\mathscr{N}$-open subsystem and the complement of fuzzy automata $\mathscr{N}$-open subsystem is called fuzzy automata $\mathscr{N}$-closed. □

Definition 3.10. Let $(Q, \tau_{\mathscr{N}})$ be a fuzzy automata normed linear structure space. Let $\lambda \in I^Q$ be any fuzzy automata subsystem. Then the fuzzy automata $\mathscr{N}$-interior of λ is denoted by $\mathcal{FAI}nt_{\mathscr{N}}(\lambda)$ and it is defined as

$$\mathcal{FAI}nt_{\mathscr{N}}(\lambda) = \bigvee \{ \beta \in I^Q : \beta \leq \lambda \text{ and } \beta \text{ is a fuzzy automata } \mathscr{N}\text{-open subsystem} \}.$$

Definition 3.11. Let $(Q, \tau_{\mathscr{N}})$ be a fuzzy automata normed linear structure space. Let $\lambda \in I^Q$ be any fuzzy automata subsystem. Then the fuzzy automata $\mathscr{N}$-closure of λ is denoted by $\mathcal{FAC}l_{\mathscr{N}}(\lambda)$ and it is defined as

$$\mathcal{FAC}l_{\mathscr{N}}(\lambda) = \bigwedge \{ \beta \in I^Q : \lambda \leq \beta \text{ and } \beta \text{ is a fuzzy automata } \mathscr{N}\text{-closed subsystem} \}.$$

Note Throughout this paper, t-norm T is defined as

$$T(a,b) = min \{ a, b \}. \tag{1}$$

Proposition 3.12. Let (Q, N, T) be a fuzzy automata normed linear space such that N satisfying (vi) of Definition 3.3. Then every fuzzy automata α-open subsystem is a fuzzy automata $\mathscr{N}$-open subsystem, for all $\alpha \in (0,1)$.

Proof. Let $p, q, r \in Q, \epsilon > 0$ and $\alpha \in (0,1)$. Suppose that $\mu_\alpha(p,\epsilon)(q) > 0$ and $c(1_Q - \mu_\alpha(p,\epsilon)) = 1_Q - \mu_\alpha(p,\epsilon)$. Thus by Definition 3.6, $N(p-q,\epsilon) > \alpha$. Assume that $0 < \alpha < \alpha_1 < N(p-q,\epsilon)$ for $\alpha_1 \in (0,1)$. By (vi) of Definition 3.3, there exist t where $0 < t < \epsilon$ such that

$$N(p-q,t) > \alpha_1 \tag{2}$$

Let $\delta = \epsilon - t > 0$ and $\mu_{\alpha_1}(q,\delta)(r) > 0$. Hence

$$N(q-r,\delta) > \alpha_1 \tag{3}$$

Thus

$$\begin{aligned} N(p-r,\epsilon) &= N((p-q)+(q-r), t+\delta) \\ &\geq T(N(p-q,t), N(q-r,\delta)) \text{ [By Definition 3.3 (iv)]} \\ &\geq min \{ N(p-q,t), N(q-r,\delta) \} \\ &\geq min \{ \alpha_1, \alpha_1 \} \text{ [By Equation (2) and (3)]} \\ N(p-r,\epsilon) &\geq \alpha_1 \\ \therefore N(p-r,\epsilon) &\geq \alpha \end{aligned}$$

Further, by Definition 3.6, $\mu_\alpha(p,\epsilon)(r) = 1-\alpha$. So $\mu_{\alpha_1}(q,\delta)(r) = 1-\alpha_1 \leq 1-\alpha = \mu_\alpha(p,\epsilon)(r)$. Therefore $\mu_{\alpha_1}(q,\delta) \leq \mu_\alpha(p,\epsilon)$. Hence $\mu_\alpha(p,\epsilon)$ is a fuzzy automata $\mathscr{N}$-open subsystem in $(Q, \tau_{\mathscr{N}})$. □

Proposition 3.13. Let (Q, N, T) be a fuzzy automata normed linear space. Then fuzzy automata α-open sphere is a fuzzy convex set, for all $\alpha \in (0,1)$.

Proof. Let $p \in Q$ and $\alpha \in (0,1)$. Suppose that $q, r \in Q$ and $\xi \in [0,1]$. Thus

$$\begin{aligned} N(p-(\xi q + (1-\xi)r), \epsilon) &\geq min\{N(\xi(p-q), \xi\epsilon), N((1-\xi)(p-r), (1-\xi)\epsilon)\} \\ &= min\{N(p-q,\epsilon), N((p-r),\epsilon)\}. \end{aligned}$$

Hence $\mu_\alpha(p,\epsilon)(\xi q + (1-\xi)r) \geq min\{\mu_\alpha(p,\epsilon)(q), \mu_\alpha(p,\epsilon)(r)\}$.

Thus $\mu_\alpha(p,\epsilon)$ is a fuzzy convex set. □

Definition 3.14. A fuzzy automata normed linear structure space $(Q, \tau_{\mathscr{N}})$ is said to be fuzzy automata $\mathscr{N}$-Hausdorff space if for any two fuzzy automata $\mathscr{N}$-points p_λ, $q_\beta \in \mathcal{FNP}(Q)$ with $p_\lambda \not q q_\beta$, there exist fuzzy automata $\mathscr{N}$-open subsystems $\delta, \rho \in I^Q$ in $(Q, \tau_{\mathscr{N}})$ such that $p_\lambda \leq \delta$, $q_\beta \leq \rho$ and $\delta \not q \rho$.

Proposition 3.15. Let (Q, N, T) be a fuzzy automata normed linear space such that N satisfying (vi) of Definition 3.3. Then the fuzzy automata normed linear structure space $(Q, \tau_{\mathscr{N}})$ is fuzzy automata $\mathscr{N}$-Hausdorff.

Proof. Let p_λ, $q_\beta \in \mathcal{F}\mathscr{N}\mathcal{P}(Q)$ with $p \neq q$. Then $p - q \neq 0$. Thus by (ii) of Definition 3.3, there exists a $t_0 > 0$ such that $N(p - q, t_0) < 1$. So, for $\alpha \in (0, 1)$, $N(p - q, t_0) < \alpha$. For $r \in Q$, let the fuzzy sets $\delta, \rho \in I^Q$ be defined as follows:

$$\delta(r) = \begin{cases} 1, & r = p \\ \mu_\alpha(p, \frac{t_0}{2})(r), & \text{otherwise} \end{cases}$$

$$\rho(r) = \begin{cases} 1, & r = q \\ \mu_\alpha(q, \frac{t_0}{2})(r), & \text{otherwise} \end{cases}$$

Clearly δ and ρ are fuzzy automata $\mathscr{N}$-open and $p_\lambda \leq \delta, q_\beta \leq \rho$. Now let us assume that δ q ρ. Then there exist $p_0 \in Q$ such that $(\delta + \rho)(p_0) > 1 > 0$. Hence $[\mu_\alpha(p, t_0/2) + \mu_\alpha(q, t_0/2)](p_0) > 0$.

This implies that $\mu_\alpha(p, t_0/2)(p_0) > 0$ which implies $N(p - p_0, t_0/2) > \alpha$ or $\mu_\alpha(q, t_0/2)(p_0) > 0$ which implies $N(q - p_0, t_0/2) > \alpha$. Therefore

$$\begin{aligned} N(p - q, t_0) &= N(p - p_0 + p_0 - q, \frac{t_0}{2} + \frac{t_0}{2}) \\ &\geq T(N(p - p_0, \frac{t_0}{2}), N(q - p_0, \frac{t_0}{2})) \text{ [By Definition 3.3 (iv)]} \\ &\geq min \; \{N(p - p_0, \frac{t_0}{2}), N(q - p_0, \frac{t_0}{2})\} \text{ [By Equation (1)]} \\ &\geq min \; \{\alpha, \alpha\} \\ &\geq \alpha \end{aligned}$$

Hence $N(p - q, t_0) \geq \alpha$ which is a contradiction to the fact that $N(p - q, t_0) < \alpha$. Hence $\delta \not{q} \rho$. Therefore $(Q, \tau_{\mathscr{N}})$ is a fuzzy automata $\mathscr{N}$-Hausdorff space. □

Definition 3.16. Let $(Q, \tau_{\mathscr{N}})$ and $(R, \vartheta_{\mathscr{N}})$ be two fuzzy automata normed linear structure spaces. A function $\varphi : (Q, \tau_{\mathscr{N}}) \to (R, \vartheta_{\mathscr{N}})$ is said to be fuzzy automata $\mathscr{N}$-continuous if for each fuzzy automata $\mathscr{N}$-open subsystem $\mu \in I^R$ in $(R, \vartheta_{\mathscr{N}})$, $\varphi^{-1}(\mu) \in I^Q$ is fuzzy automata $\mathscr{N}$-open in $(Q, \tau_{\mathscr{N}})$.

Proposition 3.17. Let $(Q, \tau_{\mathscr{N}})$ be a fuzzy automata normed linear structure space. Let $\delta \in I^Q$ be a fuzzy set and $p_\lambda \in \mathcal{F}\mathscr{N}\mathcal{P}(Q)$ be a fuzzy automata $\mathscr{N}$-point such that $p_\lambda \leq \delta$. Then $\vee\{\ p_\lambda : p_\lambda \leq \delta\ \} = \delta$.

Proof. The proof is obvious. □

Proposition 3.18. Let $(Q, \tau_{\mathscr{N}})$ be a fuzzy automata normed linear structure space. A fuzzy set $\mu \in I^Q$ is fuzzy automata $\mathscr{N}$-open subsystem if and only if $\forall p_\lambda \in \mathcal{F}\mathscr{N}\mathcal{P}(Q)$ and $p_\lambda \leq \mu$, there exists a fuzzy automata $\mathscr{N}$-open subsystem $\gamma \in I^Q$ such that $p_\lambda \leq \gamma \leq \mu$.

Proof. Let $\mu \in I^Q$ is fuzzy automata $\mathscr{N}$-open subsystem. Let $\mathcal{B}$ denotes a base for $\tau_{\mathscr{N}}$. Then $\mu = \vee_{i \in J} \gamma_i$ where $\gamma_i \in \mathcal{B}, \forall i \in J$. Let $p_\lambda \leq \mu$. Then $p_\lambda \leq \vee_{i \in J} \gamma_i$ which implies $p_\lambda \leq \gamma_i \leq \mu$, for some $i \in J$.

Conversely, let $p_\lambda \leq \mu$ and $p_\lambda \in \mathcal{F}\mathscr{N}\mathcal{P}(Q)$. Then there exists a fuzzy automata $\mathscr{N}$-open subsystem $\gamma \in I^Q$ such that $p_\lambda \leq \gamma \leq \mu$. Then taking arbitrary union, we get

$$\begin{aligned} \vee_i \{p_\lambda : p_\lambda \leq \mu\} \leq \vee_i \gamma \leq \mu, \; i \; \in \; J \\ \mu \leq \vee_i \gamma \leq \mu, i \in J, \quad \{ \text{ by Proposition 3.17 } \} \\ \therefore \mu = \vee_i \gamma, i \in J. \end{aligned}$$

Since arbitrary union of fuzzy automata $\mathscr{N}$-open subsystem is fuzzy automata $\mathscr{N}$-open, $\mu = \vee_i \gamma, i \in J$ is also fuzzy automata $\mathscr{N}$-open. □

Definition 3.19. Let $(Q, \tau_{\mathscr{N}})$ be a fuzzy automata normed linear structure space. Let $P \subseteq Q$ and χ_P is the characteristic function of P, then the collection $\tau_{\mathscr{N}_P} = \{\eta|_P = \eta \wedge \chi_P : \eta \in \tau_{\mathscr{N}}\}$ is a fuzzy automata structure on P, called the fuzzy automata normed linear substructure and the pair $(P, \tau_{\mathscr{N}_P})$ is called a fuzzy automata normed linear substructure space of $(Q, \tau_{\mathscr{N}})$.

Proposition 3.20. Let $(Q, \tau_{\mathscr{N}})$ be a fuzzy automata normed linear structure space. Let $(R, \tau_{\mathscr{N}_R})$ be a fuzzy automata normed linear substructure space of $(Q, \tau_{\mathscr{N}})$. If $(Q, \tau_{\mathscr{N}})$ is a fuzzy automata $\mathscr{N}$-Hausdorff space, then fuzzy automata normed linear substructure space $(R, \tau_{\mathscr{N}_R})$ is also a fuzzy automata $\mathscr{N}$-Hausdorff space.

Proof. Let $(Q, \tau_{\mathscr{N}})$ be a fuzzy automata $\mathscr{N}$-Hausdorff space and $(R, \tau_{\mathscr{N}_R})$ be a fuzzy automata normed linear substructure space of $(Q, \tau_{\mathscr{N}})$. Let $p_\lambda, q_\beta \in \mathcal{FNP}(R)$ be any two fuzzy automata $\mathscr{N}$-points in R such that $p = q$ and $p_\lambda \not q q_\beta$. Since $(Q, \tau_{\mathscr{N}})$ be a fuzzy automata $\mathscr{N}$-Hausdorff space and $p_\lambda, q_\beta \in \mathcal{FNP}(Q)$ is also fuzzy automata $\mathscr{N}$-points in Q, there exists fuzzy automata $\mathscr{N}$-open subsystems $\delta, \rho \in I^Q$ in $(Q, \tau_{\mathscr{N}})$ such that $p_\lambda \leq \delta, q_\beta \leq \rho$ and $\delta \not q \rho$. Further, $p_\lambda \leq (\delta \wedge \chi_R)$, $q_\beta \leq (\rho \wedge \chi_R)$ and $(\delta \wedge \chi_R) \not q (\rho \wedge \chi_R)$. By Definition 3.19, $(\delta \wedge \chi_R)$ and $(\rho \wedge \chi_R)$ are fuzzy automata $\mathscr{N}$-open subsystems in $(R, \tau_{\mathscr{N}_R})$. Hence $(R, \tau_{\mathscr{N}_R})$ is fuzzy automata $\mathscr{N}$-Hausdorff space. □

Definition 3.21. Let $(Q, \tau_{\mathscr{N}})$ ba a fuzzy automata normed linear structure space. A family $\{ \lambda_i \in I^Q : i \in J$ where J is an indexed set $\}$ of fuzzy automata $\mathscr{N}$-open subsystems in $(Q, \tau_{\mathscr{N}})$ is called a fuzzy automata $\mathscr{N}$-open cover of $(Q, \tau_{\mathscr{N}})$ if $\bigvee_{i \in J} \lambda_i = 1_Q$.

Definition 3.22. A fuzzy automata normed linear structure space $(Q, \tau_{\mathscr{N}})$ is said to be a fuzzy automata $\mathscr{N}$-compact space iff for every fuzzy automata $\mathscr{N}$-open cover of $(Q, \tau_{\mathscr{N}})$, there exists a finite subset J_0 of J such that $\bigvee_{i \in J_0} \lambda_i = 1_Q$.

Definition 3.23. Let $(Q, \tau_{\mathscr{N}})$ be a fuzzy automata normed linear structure space. A family $\mathfrak{F} = \{ \lambda_i \in I^Q : i \in J$ where J is an indexed set$\}$ of fuzzy automata subsystems is a fuzzy automata cover of a fuzzy automata subsystem $\mu \in I^Q$ iff $\mu \leq \bigvee \{\lambda_i : \lambda_i \in \mathfrak{F} \}$. It is a fuzzy automata $\mathscr{N}$-open cover iff each member of $\mathfrak{F}$ is a fuzzy automata $\mathscr{N}$-open subsystem.

Definition 3.24. Let $(Q, \tau_{\mathscr{N}})$ be a fuzzy automata normed linear structure space. A fuzzy automata subsystem $\lambda \in I^Q$ is said to be fuzzy automata $\mathscr{N}$-compact in $(Q, \tau_{\mathscr{N}})$ provided for every collection $\{ \beta_i \in I^Q : i \in J$ where J is an indexed set $\}$ of fuzzy automata $\mathscr{N}$-open subsystems in $(Q, \tau_{\mathscr{N}})$ such that $\lambda \leq \vee\{\beta_i\}_{i \in J}$, there exists a finite subset J_0 of J such that $\lambda \leq \vee\{\beta_i\}_{i \in J_0}$.

Proposition 3.25. Let $(Q, \tau_{\mathscr{N}})$ be a fuzzy automata normed linear structure space and let $\lambda \in I^Q$ be a fuzzy automata $\mathscr{N}$-closed subsystem of $(Q, \tau_{\mathscr{N}})$. If $(Q, \tau_{\mathscr{N}})$ is fuzzy automata $\mathscr{N}$-compact, then λ is also fuzzy automata $\mathscr{N}$-compact.

Proof. Let $(Q, \tau_{\mathscr{N}})$ be a fuzzy automata $\mathscr{N}$-compact space. Let $\lambda \in I^Q$ be a fuzzy automata $\mathscr{N}$-closed subsystem. Let $\mathcal{B} = \{\mu_i \in I^Q : i \in J\}$ be a collection of fuzzy automata $\mathscr{N}$-open subsystems of $(Q, \tau_{\mathscr{N}})$ such that $\lambda \leq \bigvee_{i \in J}\{\mu_i\} = 1_Q$. Let $\mathcal{B}' = \{\mu_i\}_{i \in J} \vee \{(1_Q - \lambda)\}$. Then $\lambda \leq \{\vee_{i \in J} \mu_i\} \bigvee (1_Q - \lambda) \leq 1_Q$. Since $(Q, \tau_{\mathscr{N}})$ is a fuzzy automata $\mathscr{N}$-compact space, there exists a finite subset J_0 of J such that $\lambda \leq \{\vee_{i \in J_0} \mu_i\} \bigvee (1_Q - \lambda)\} \leq \vee_{i \in J_0} \mu_i \leq 1_Q$. Hence λ is fuzzy automata $\mathscr{N}$-compact. □

Remark 3.26. Let $(Q, \tau_{\mathscr{N}})$ be fuzzy automata $\mathscr{N}$-Hausdorff space. Then every fuzzy automata $\mathscr{N}$-compact subsystem $\sigma \in I^Q$ is fuzzy automata $\mathscr{N}$-closed subsystem.

Definition 3.27. A fuzzy automata normed linear structure space $(Q, \tau_{\mathscr{N}})$ is said to be fuzzy automata $\mathscr{N}$-locally compact if and only if for every fuzzy automata point $p_\lambda \in \mathcal{FNP}(Q)$, there exists a fuzzy automata $\mathscr{N}$-open subsystem $\mu \in I^Q$ in $(Q, \tau_{\mathscr{N}})$ such that $p_\lambda \leq \mu$ and μ is fuzzy automata $\mathscr{N}$-compact.

Proposition 3.28. Let $(Q, \tau_{\mathscr{N}})$ be a fuzzy automata $\mathscr{N}$-Hausdorff space. Then the following statements are equivalent:

(i) $(Q, \tau_{\mathscr{N}})$ is fuzzy automata $\mathscr{N}$-locally compact;

(ii) For each fuzzy automata point $p_\lambda \in \mathcal{FNP}(Q)$, there exists a fuzzy automata $\mathscr{N}$-open subsystem $\beta \in I^Q$ in $(Q, \tau_{\mathscr{N}})$ such that $p_\lambda \leq \beta$ and $\mathcal{FACl}_{\mathscr{N}}(\beta)$ is fuzzy automata $\mathscr{N}$-compact.

Proof. **(i)⇒(ii)**

Since $(Q, \tau_{\mathscr{N}})$ is fuzzy automata $\mathscr{N}$-locally compact, for each fuzzy automata point $p_\lambda \in \mathcal{FNP}(Q)$, there exists a fuzzy automata $\mathscr{N}$-open subsystem $\beta \in I^Q$ in $(Q, \tau_{\mathscr{N}})$ such that $p_\lambda \leq \beta$ and β is fuzzy automata $\mathscr{N}$-compact. Since $(Q, \tau_{\mathscr{N}})$ is fuzzy automata $\mathscr{N}$-Hausdorff, by Remark 3.26, β is fuzzy automata $\mathscr{N}$-closed. Thus $\beta = \mathcal{FACl}_{\mathscr{N}}(\beta)$. Hence $p_\lambda \leq \beta = \mathcal{FACl}_{\mathscr{N}}(\beta)$ and $\mathcal{FACl}_{\mathscr{N}}(\beta)$ is fuzzy automata $\mathscr{N}$-compact.

(ii)⇒(i)

Suppose that $p_\lambda \in \mathcal{FNP}(Q)$ and there exists a fuzzy automata $\mathscr{N}$-open subsystem $\beta \in I^Q$ in $(Q, \tau_{\mathscr{N}})$ such that $p_\lambda \leq \beta$. Given that $\mathcal{FACl}_{\mathscr{N}}(\beta)$ is fuzzy automata $\mathscr{N}$-compact. Since $\beta \leq \mathcal{FACl}_{\mathscr{N}}(\beta)$, β is fuzzy automata $\mathscr{N}$-compact. Hence $p_\lambda \leq \beta$ and β is fuzzy automata $\mathscr{N}$-compact. Therefore $(Q, \tau_{\mathscr{N}})$ is fuzzy automata $\mathscr{N}$-locally compact. □

Proposition 3.29. Let $(Q,\tau_{\mathscr{N}})$ be a fuzzy automata $\mathscr{N}$-Hausdorff space. Then $(Q,\tau_{\mathscr{N}})$ is fuzzy automata $\mathscr{N}$-locally compact at $p_\lambda \in \mathcal{FNP}(Q)$ if and only if for every fuzzy automata $\mathscr{N}$-open subsystem $\beta \in I^Q$ such that $p_\lambda \leq \beta$, there exists a fuzzy automata $\mathscr{N}$-open subsystem $\rho \in I^Q$ such that $p_\lambda \leq \rho$, $\mathcal{FACl}_{\mathscr{N}}(\rho)$ is fuzzy automata $\mathscr{N}$-compact and $\mathcal{FACl}_{\mathscr{N}}(\rho) \leq \beta$.

Proof. Let us assume that $(Q,\tau_{\mathscr{N}})$ is fuzzy automata $\mathscr{N}$-locally compact at $p_\lambda \in \mathcal{FNP}(Q)$. By Definition 3.27, there exists a fuzzy automata $\mathscr{N}$-open subsystem $\beta \in I^Q$ such that $p_\lambda \leq \beta$ and β is fuzzy automata $\mathscr{N}$-compact. Since $(Q,\tau_{\mathscr{N}})$ is a fuzzy automata $\mathscr{N}$-Hausdorff space, by Remark 3.26, β is fuzzy automata $\mathscr{N}$-closed. Thus $\beta = \mathcal{FACl}_{\mathscr{N}}(\beta)$. Consider a fuzzy automata point $q_\delta \in \mathcal{FNP}(Q)$ such that $q_\delta \leq (1_Q - \beta)$. Since $(Q,\tau_{\mathscr{N}})$ is fuzzy automata $\mathscr{N}$-Hausdorff space, by Definition 3.14, there exist fuzzy automata $\mathscr{N}$-open subsystems $\rho \in I^Q$ and $\sigma \in I^Q$ such that $p_\lambda \leq \rho$ and $q_\delta \leq \sigma$ and $\rho \not q\, \sigma$. Let $\delta = \rho \wedge \beta$. Hence $\delta \leq \beta$ implies $\mathcal{FACl}_{\mathscr{N}}(\delta) \leq \mathcal{FACl}_{\mathscr{N}}(\beta) = \beta$. Since $\mathcal{FACl}_{\mathscr{N}}(\delta)$ is fuzzy automata $\mathscr{N}$-closed and β is fuzzy automata $\mathscr{N}$-compact, by Proposition 3.25, it follows that $\mathcal{FACl}_{\mathscr{N}}(\delta)$ is fuzzy automata $\mathscr{N}$-compact. Thus $p_\lambda \leq \mathcal{FACl}_{\mathscr{N}}(\delta) \leq \beta$ and $\mathcal{FACl}_{\mathscr{N}}(\beta)$ is fuzzy automata $\mathscr{N}$-compact.

The converse follows from Proposition 3.28 (ii). □

Proposition 3.30. Let $(Q,\tau_{\mathscr{N}})$ and (R,φ_N) be any two fuzzy automata normed linear structure spaces. Let $\psi,\omega : (R,\varphi_N) \to (Q,\tau_{\mathscr{N}})$ be any two fuzzy automata $\mathscr{N}$-continuous functions and also let $(\psi,\omega) : (R,\varphi_N) \to (Q\times Q,\tau_{\mathscr{N}} \times \tau_{\mathscr{N}})$ be a function. If $\alpha_\Delta \in I^{Q\times Q}$ is fuzzy automata $\mathscr{N}$-closed in $(Q\times Q,\tau_{\mathscr{N}} \times \tau_{\mathscr{N}})$ where α_Δ is given by

$$\alpha_\Delta = \begin{cases} \chi_\Delta, & \text{if } p = q \\ \\ 0_{Q\times Q}, & \text{if } p \neq q \end{cases}$$

and $\Delta = \{(x,x) \in Q\times Q\}$, then $(\psi,\omega)^{-1}(\alpha_\Delta)$ is also fuzzy automata $\mathscr{N}$-closed in (R,φ_N).

Proof. Let $\psi,\omega : (R,\varphi_{\mathscr{N}}) \to (Q,\tau_{\mathscr{N}})$ be any two fuzzy automata $\mathscr{N}$-continuous functions. Consider the function $(\psi,\omega) : (R,\varphi_{\mathscr{N}}) \to (Q\times Q,\tau_{\mathscr{N}} \times \tau_{\mathscr{N}})$, defined by

$$(\psi,\omega)(p) = (\psi(p),\omega(p)) \text{ (for every } p \in R).$$

To prove this function is fuzzy automata $\mathscr{N}$-continuous. It is sufficient to show that the inverse image of each fuzzy automata $\mathscr{N}$-open subsystem in $(Q\times Q,\tau_{\mathscr{N}} \times \tau_{\mathscr{N}})$ is fuzzy automata $\mathscr{N}$-open in $(R,\varphi_{\mathscr{N}})$. Let us take fuzzy automata $\mathscr{N}$-open subsystem $\delta \times \rho \in I^{Q\times Q}$ in $(Q\times Q,\tau_N \times \tau_{\mathscr{N}})$ where $\delta,\rho \in I^Q$ are fuzzy automata $\mathscr{N}$-open subsystems in $(Q,\tau_{\mathscr{N}})$. Then for every $p \in R$

$$\begin{aligned} ((\psi,\omega)^{-1}(\delta\times\rho))(p) &= (\delta\times\rho)(\psi,\omega)(p) \\ &= (\delta\times\rho)(\psi(p),\omega(p)) \\ &= min\{\delta(\psi(p)),\rho(\omega(p))\} \\ &= min\{\psi^{-1}(\delta(p)),\omega^{-1}(\rho(p))\} \\ &= (\psi^{-1}(\delta)\wedge\omega^{-1}(\rho))(p). \end{aligned}$$

Therefore, $(\psi,\omega)^{-1}(\delta\times\rho) = \psi^{-1}(\delta)\wedge\omega^{-1}(\rho)$.

Using fuzzy automata $\mathscr{N}$-continuity of ψ and ω, we find that $\psi^{-1}(\delta)$ and $\omega^{-1}(\rho)$ are fuzzy automata $\mathscr{N}$-open in $(R,\varphi_{\mathscr{N}})$. Therefore, $\psi^{-1}(\delta)\wedge\omega^{-1}(\rho)$ is fuzzy automata $\mathscr{N}$-open in $(R,\varphi_{\mathscr{N}})$. Hence (ψ,ω) is fuzzy automata $\mathscr{N}$-continuous.

Given that α_Δ is a fuzzy automata $\mathscr{N}$-closed subsystem in $(Q\times Q,\tau_{\mathscr{N}} \times \tau_{\mathscr{N}})$. Since (ψ,ω) is fuzzy automata $\mathscr{N}$-continuous, $\mu = (\psi,\omega)^{-1}\alpha_\Delta$ is fuzzy automata $\mathscr{N}$-closed in $(R,\varphi_{\mathscr{N}})$. □

Remark 3.31. Let $(Q,\tau_{\mathscr{N}})$ be a fuzzy automata normed linear structure space. If $(Q,\tau_{\mathscr{N}})$ is fuzzy automata $\mathscr{N}$-Hausdorff, then $\alpha_\Delta \in I^{X\times X}$ is fuzzy automata $\mathscr{N}$-closed in $(Q\times Q,\tau_{\mathscr{N}} \times \tau_{\mathscr{N}})$.

Proposition 3.32. Let $(Q,\tau_{\mathscr{N}})$ and $(R,\nu_{\mathscr{N}})$ be any two fuzzy automata normed linear structure spaces and let $\psi,\varphi : (Q,\tau_{\mathscr{N}}) \to (R,\nu_{\mathscr{N}})$ be any two fuzzy automata $\mathscr{N}$-continuous functions. If $(R,\nu_{\mathscr{N}})$ is fuzzy automata $\mathscr{N}$-Hausdorff, then the fuzzy automata subsystem $\delta \in I^Q$ defined as:

$$\delta(p) = \begin{cases} 1_Q, & \text{if } p \in \mathfrak{B} \\ \\ 0_Q, & \text{otherwise} \end{cases}$$

where $\mathfrak{B} = \{\ p \in Q : \psi(p) = \varphi(p)\ \}$, is fuzzy automata $\mathscr{N}$-closed.

Proof. Let $\psi : (Q, \tau_{\mathscr{N}}) \to (R, \nu_{\mathscr{N}})$ and $\varphi : (Q, \tau_{\mathscr{N}}) \to (R, \nu_{\mathscr{N}})$ be any two fuzzy automata $\mathscr{N}$-continuous functions. Let us define a function

$$(\psi, \varphi) : (Q, \tau_{\mathscr{N}}) \to (R \times R, \nu_{\mathscr{N}} \times \nu_{\mathscr{N}})$$

which is given by $(\psi, \varphi)(p) = (\psi(p), \varphi(p)), \forall p \in Q$. Now to prove (ψ, φ) is fuzzy automata $\mathscr{N}$-continuous function. For this, let us consider a fuzzy automata $\mathscr{N}$-open subsystem $\alpha \times \beta \in I^{R \times R}$ in $(R \times R, \nu_{\mathscr{N}} \times \nu_{\mathscr{N}})$. Then,

$$\begin{aligned}(\psi, \varphi)^{-1}(\alpha \times \beta)(p) &= (\alpha \times \beta)((\psi, \varphi)(p)), \ \forall \ p \ \in \ Q \\ &= (\alpha \times \beta)(\psi(p), \varphi(p)) \ \forall \ p \ \in \ Q \\ &= \min\{ \ \alpha(\psi(p)), \beta(\varphi(p)) \ \} \ \forall \ p \ \in \ Q \\ &= (\psi^{-1}(\alpha) \wedge \varphi^{-1}(\beta))(p), \ \forall \ p \ \in \ Q\end{aligned}$$

Since ψ and φ are two fuzzy automata $\mathscr{N}$-continuous functions from $(Q, \tau_{\mathscr{N}})$ to $(R, \nu_{\mathscr{N}})$, $\psi^{-1}(\alpha) \in \tau_{\mathscr{N}}$ and $\varphi^{-1}(\beta) \in \tau_{\mathscr{N}}$ are fuzzy automata $\mathscr{N}$-open subsystems in $(Q, \tau_{\mathscr{N}})$. Thus $[\psi^{-1}(\alpha) \wedge \varphi^{-1}(\beta)] \in \tau_{\mathscr{N}}$. Therefore (ψ, φ) is a fuzzy automata $\mathscr{N}$-continuous function.

Since $(R, \nu_{\mathscr{N}})$ is fuzzy automata $\mathscr{N}$-Hausdorff, by Remark 3.31, α_Δ is fuzzy automata $\mathscr{N}$-closed in $(R \times R, \nu_{\mathscr{N}} \times \nu_{\mathscr{N}})$. Thus $(\psi, \varphi)^{-1}\alpha_\Delta$ is fuzzy automata $\mathscr{N}$-closed in $(Q, \tau_{\mathscr{N}})$ since (ψ, φ) is fuzzy automata $\mathscr{N}$-continuous. Hence it is enough to prove that $(\psi, \varphi)^{-1}\alpha_\Delta = \delta$. Let

$$(\psi, \varphi)^{-1}\alpha_\Delta(p) = \alpha_\Delta((\psi, \varphi)(p)), \ \forall \ p \ \in \ Q$$

$$= \begin{cases} 1, & \text{if } \psi(p) = \varphi(p) \\ 0, & \text{otherwise} \end{cases}$$

$$= \delta(p), \forall p \in Q$$

Therefore, δ is a fuzzy automata $\mathscr{N}$-closed subsystem in $(Q, \tau_{\mathscr{N}})$. □

Proposition 3.33. Let $(Q, \tau_{\mathscr{N}})$ and (R, φ_N) be any two fuzzy automata normed linear structure spaces. Let $\phi : (R, \varphi_N) \to (Q, \tau_{\mathscr{N}})$ be any fuzzy automata $\mathscr{N}$-continuous function and let $\nu : (R, \varphi_N) \to (Q \times Q, \tau_{\mathscr{N}} \times \tau_{\mathscr{N}})$ be a function. Then the graph ν of ϕ defined by $\nu(q) = (q, \phi(q))$, for each $q \in R$, is a fuzzy set in $(R \times Q, \varphi_N \times \tau_{\mathscr{N}})$.

Proof. Let $\phi : (R, \varphi_{\mathscr{N}}) \to (Q, \tau_{\mathscr{N}})$ be a fuzzy automata $\mathscr{N}$-continuous function. Let $\psi_Q : (R \times Q, \varphi_{\mathscr{N}} \times \tau_{\mathscr{N}}) \to (Q, \tau_{\mathscr{N}})$ defined by $\psi_Q(q, p) = p$ where $p \in Q$ and $q \in R$ and let $\psi_R : (R \times Q, \varphi_{\mathscr{N}} \times \tau_{\mathscr{N}}) \to (R, \varphi_{\mathscr{N}})$ defined by $\psi_R(q, p) = q$ where $p \in Q$ and $q \in R$. Let ψ_Q and ψ_R be two fuzzy automata $\mathscr{N}$-continuous functions. Let $\psi_Q = \vartheta$ and $\phi \circ \psi_R = \varpi$. Since ψ_Q, ψ_R, ϕ are fuzzy automata $\mathscr{N}$-continuous functions, ϖ is also fuzzy automata $\mathscr{N}$-continuous function. Thus

$$\begin{aligned}\{(q, p) \in R \times Q : \vartheta(q, p) = \varpi(q, p), q \in R, p \in Q\} \\ &= \{(q, p) \in R \times Q : \psi_Q(q, p) = \phi \circ \psi_R(q, p)\} \\ &= \{(q, p) \in R \times Q : \psi_Q(q, p) = \phi(\psi_R(q, p)\} \\ &= \{(q, p) \in R \times Q : p = \phi(q)\} \\ &= \{(q, \phi(q) \in R{\times}Q : q \in R\}.\end{aligned}$$

Hence

$$\Lambda = \{(q, \phi(q) \in R \times Q : q \in R\}.$$

Also

$$\beta_\Lambda = \begin{cases} \chi_\Lambda, & \text{if } p = q \\ \\ 0_{R \times Q}, & \text{if } p \neq q. \end{cases}$$

Therefore $\beta_\Lambda \in I^{R \times Q}$ is a fuzzy set in $(R \times Q, \varphi_{\mathscr{N}} \times \tau_{\mathscr{N}})$. □

4 Conclusions

The concepts of fuzzy automata α-open sphere and fuzzy automata $\mathscr{N}$-open subsystem in fuzzy automata normed linear structure spaces are introduced and studied. Also in this connection the concepts of fuzzy automata $\mathscr{N}$-Hausdorff spaces and fuzzy automata $\mathscr{N}$-locally compact spaces are established and their properties are discussed. Also several basic results have been proved.

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