International Journal Of Industrial Electronics Control and Optimization
Print ISSN: 3517-2645
Online ISSN: 3568-2645
Homepage: https://ieco.usb.ac.ir/

Observer-Based Distributed Consensus Control for Nonlinear Lipschitz and One-Side Lipschitz Fractional-Order Multi-Agent Systems

Farshid AazamManesh [1] | Elham Amini Boroujeni [2] | Fateme Bazarkhak [2] | Mahdi Pourgholi [3]

Technical and Vocational School, Technical and Vocational University, Ilam, Iran. 1
Department of Electrical and Computer Engineering, Faculty of Engineering, Kharazmi University, Tehran, Iran. 2
Faculty of Electrical Engineering, Shahid Beheshti University, Tehran, Iran.3
Corresponding author's email: m_pourgholi@sbu.ac.ir

Article Info

Article type:
Research Article

Article history:
Received: 21-December-2024
Received in revised form: 01-March-2025
Accepted: 12-April-2025
Published online: 22-Dec-2025

Keywords:
Consensus,
Distributed Control,
Fractional-Order Systems,
Nonlinear Systems,
Observer Design.

ABSTRACT

In this paper, an observer-based controller design for fractional-order multi-agent systems is discussed. By introducing a novel algorithm and leveraging appropriate lemmas and theoretical frameworks, we propose a stable observer and a distributed consensus protocol tailored for multi-agent systems within the Lipschitz and one-sided Lipschitz classes of nonlinear systems. Lipschitz systems have a bounded rate of change, ensuring proportional output to input differences, while one-sided Lipschitz systems relax this constraint, allowing differential growth in one direction for efficiency. The stability of the observer and the controller in achieving the consensus problem is demonstrated using the Lyapunov's second method. The proposed approach is rigorously developed, ensuring that the designed observer and controller meet the necessary stability criteria. Extensive simulation results validate the theoretical findings, showcasing the method's effectiveness and robustness in practical scenarios. Specifically, the simulations demonstrate that the proposed method achieves global Mittag-Leffler stability, with the estimated states converging to the actual states with minimal deviation. The method's advantages include its ability to handle a broader class of nonlinear systems, including those with large Lipschitz constants, and its robustness to uncertainties and nonlinearities. These simulations confirm the theoretical predictions and illustrate the practical applicability of our approach in real-world multi-agent systems, such as swarm robotics, power grids, and sensor networks.

I. Introduction

Fractional-order calculus has emerged as a powerful mathematical tool for modeling and control in various engineering domains, including control systems [1-3], nuclear reactor analysis [4], and biological systems such as tumor growth models, reaction-diffusion processes, and bacterial chemotaxis in diffusion gradient chambers [5, 6]. Unlike traditional integer-order models, fractional-order systems leverage fractional derivative and integral operators, enabling more accurate descriptions of complex dynamic behaviors [7]. For instance, fractional-order models have proven superior in characterizing friction phenomena in real-world engineering applications compared to their integer-order counterparts [8]. Consequently, significant research efforts have been directed toward developing fractional-order controllers to enhance the performance of closed-loop systems [9, 10].

A critical challenge in control engineering is that not all system states are directly measurable. This limitation has spurred considerable interest in observer-based controller design and stability analysis. Recent advancements include the development of observer-based controllers for linear systems with unknown inputs using linear matrix inequality (LMI) techniques [11], as well as for nonlinear fractional-order systems by reformulating Lyapunov stability conditions into LMIs [12]. Reduced-order observers have also been proposed for nonlinear fractional-order systems satisfying Lipschitz conditions, with stability guarantees derived using Lyapunov methods [13]. Notably, the one-sided Lipschitz condition, a generalization of the traditional Lipschitz class, offers greater flexibility by allowing

Publisher: University of Sistan and Baluchestan

negative Lipschitz constants, thereby expanding the scope of applicable matrix inequalities [14, 15]. Building on these foundations, researchers have designed full-order and reduced-order observers for one-sided Lipschitz nonlinear fractional-order systems using LMI-based approaches [16]. Additionally, adaptive observer designs have been developed for systems satisfying one-sided Lipschitz and quadratic inner-boundedness conditions [17, 18].

Parallel to these developments, multi-agent systems (MAS) have garnered significant attention due to their broad applicability in areas such as unmanned aerial vehicle coordination [19], spacecraft interaction control [20], mobile robot rendezvous [21], underwater vehicle operations [22], traffic management [23], and data density control [24]. MAS consist of interconnected agents that collaboratively perform complex tasks through local interactions, offering a cost-effective and scalable alternative to centralized systems. A fundamental objective in MAS is achieving consensus, wherein the states or outputs of all agents converge to a common value [25]. While consensus in integer-order MAS has been extensively studied [26, 27], the extension to fractional-order systems remains an active area of research. For example, consensus in second-order MAS with nonlinear dynamics has been investigated under both fixed and switching topologies [28], and distributed control strategies have been proposed for one-sided Lipschitz nonlinear MAS [29]. Fractional-order proportional-integral (FOPI) controllers have also been employed to regulate DC microgrids using consensus-based approaches [30], and consensus in fractional-order MAS has been analyzed using LMI techniques and Razumikhin theory [31]. However, these studies often assume full state measurability, limiting their practical applicability.

Motivated by these advancements and challenges, this paper proposes a novel observer-based consensus control algorithm for one-sided Lipschitz nonlinear fractional-order multi-agent systems. Leveraging Lyapunov's second method, the proposed approach formulates sufficient stability conditions in the form of LMIs, which are solved using the YALMIP toolbox. The contributions of this work are threefold: (1) a systematic observer design for state estimation in fractional-order MAS, (2) a consensus control framework for one-sided Lipschitz nonlinear systems, and (3) numerical validation of the proposed method's efficacy.

The remainder of the paper is organized as follows: Section II presents the mathematical preliminaries and problem formulation. Section III details the observer design for state estimation and consensus control. Section IV provides numerical simulations to demonstrate the proposed method's effectiveness. In section V limitations of the proposed method and suggests directions for future research are provided. Finally, Section VI concludes the paper with key insights and future research directions.

II. Preliminaries

The relationship between agents in a multi-agent factorial system is determined by a weighted graph. A graph $G(v, \mathcal{E})$ is a pair that consists of a set of vertices $v\,(G) = \{v_1, v_2, \dots v_N\}$ and edges $\mathcal{E}\,(G) \subseteq \{(v_i, v_j) \colon v_i, v_j \,\epsilon\, v\,(G)\}$ (i.e., the graph is in general directed and has no self-loops). The graph is said to be undirected if $(v_i, v_j) \in \mathcal{E}\,(G) \Leftrightarrow (v_j, v_i) \in \mathcal{E}\,(G)$. The weighting matrix $W = [w_{ij}] \,\epsilon R^{N\times N}$ for a graph is a matrix with $w_{ij} \geq 0$ elements ($w_{ij} > 0$, if with $w_{ij} \in \mathcal{E}\,(G)$, and $w_{ij} = 0$ otherwise). The in-degree matrix D is defined as $D = diag(d_1, d_2, d_3, \dots, d_N)$ with $d_i = \sum_{j=1}^{N} w_{ij}$. The Laplacian matrix L of directional graph G is defined as $L = D - W$ [32].

A directed tree where all the vertices of the graph are connected by its constituent edges is called a spanning tree of a directed graph [33].

A. Problem Formulation

Definition 1 [34]: The uniform formula of a fractional integral with $\alpha \in (0, 1)$ is defined by

$$ {}_{t0}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \,, \tag{1} $$

where ${}_{t0}D_t^{-\alpha} f(t)$ is the fractional integral of order α of a function $f(t)$, $\Gamma(.)$ is the Gamma function. For an arbitrary real number p. The Caputo derivative operator of fractional order is defined by

$$ {}_{t0}^{C}D_t^{p} f(t) = {}_{t0}D_t^{-q} \left[\frac{d^{[p]+1}}{dt^{[p]+1}} f(t) \right]. \tag{2} $$

where $q = [p] - p + 1$ and $[p]$ stands for the integer part of p.

Definition 2. [35]: Mittag-Leffler function as a complex-valued function of a complex argument z can be presented as:

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \tag{3} $$

where $\alpha, \beta \in C$, $Re\{\alpha\} > 0$, and C is the set of complex numbers, when $\beta = 1$, $E_\alpha(z) = E_{\alpha,1}(z)$.

Lemma 1. [36]: If $x \in R^n$ be a differentiable vector:

$$ \frac{1}{2} {}_{t_0}^{C}D_t^{\alpha}\{x^T(t)x(t)\} \leq x^T(t) {}_{t_0}^{C}D_t^{\alpha} x(t), \quad \forall t \geq t_0 \,, \alpha \in (0,1). \tag{4} $$

Now, suppose the dynamics of each agent can be described as [37]:

$$ {}_{t_0}^{C}D_t^{\alpha} x_i = Ax_i + Bu_i + F_i(x_i, u_i) \,, \quad i\epsilon\{1,2, \dots N\}. \tag{5} $$

$$ y_i = Cx_i $$

where $A\epsilon R^{n\times n}$, $B\epsilon R^{n\times m}$, $C\epsilon R^{m\times n}$, $F_i(x_i,u_i)\epsilon R^{n\times 1}$ is the nonlinear term, and ${}^{C}_{t_0}D^{\alpha}_{t}$ represents the Caputo fractional derivative.

Definition 3. [38]: The vector value function $F_i(x_i,u_i): R^{n}\times R^{m} \rightarrow R^{n}$ is a nonlinear Lipschitz continuous function with a Lipschitz constant r such that:

$$\|F_i(x_{i_1},u_i)\| - \|F_i(x_{i_2},u_i)\| \le r\|x_{i_1}-x_{i_2}\|, \quad (6)$$
$$x_{i_1},x_{i_2}\in R^{n}, u_i\in R^{m}.$$

Definition 4. [38]: The vector value function $F_i(x_i,u_i): R^{n}\times R^{m} \rightarrow R^{n}$ is one-sided Lipschitz, where γ_s is a one-sided Lipschitz constant such that:

$$< sF_i(x_{i_1},u_i) - sF_i(x_{i_2},u_i),\ (x_{i_1}-x_{i_2}) > \le \quad (7)$$
$$\gamma_s\|x_{i_1}-x_{i_2}\|^2, \quad x_{i_1},x_{i_2}\in R^{n}, u_i\in R^{m}.$$

where s is a symmetric positive definite matrix, $\langle\cdot,\cdot\rangle$ denotes the inner product and $\|\cdot\|$ is the Euclidean norm.

Definition 5. [39]: Suppose $x_{eq}=0$ is the equilibrium point of the system (5) with $0<\alpha<1$ and t_0=0. Now, suppose $V(t,x(t)): [0,\infty)\times R^{n}\mapsto R$ is a continuously differentiable function and local Lipschitz with respect to x such that:

$$\alpha_1\|x\|^{a} \le V(t,x(t)) \le \alpha_2\|x\|^{ab} \quad (8)$$

Likewise,

$${}^{C}_{t_0}\mathcal{D}^{\beta}_{t}\ V(t,x(t)) \le -\alpha_3\|x\|^{ab}, \quad t\ge t_0, \quad (9)$$
$$x\in\mathbb{D}, \quad \beta\in(0,1).$$

where α_1, α_2, α_3 and a, b are positive fixed numbers. In this case, the equilibrium point $x_{eq}=0$ is Mittag–Leffler stable. If the assumptions hold globally on R^{n} then $x_{eq}=0$ is globally Mittag–Leffler stable.

Considering the quasi-state vector of the batch system as $x=[x_1^T,\dots x_N^T]^T$, the nonlinear term is $\mathbf{F}(\boldsymbol{x},\boldsymbol{u})=[F_1(x_1,u_1),\dots,F_N(x_N,u_N)]^T$ and u_i is input vector for each i^{th} agent.

Assumption1. [13] The pair (A,C) is observable, $s_i=s_i^T$ is a positive definite matrix, and we can get a positive scalar θ such that:

$$-\theta s_i - A^T s_i - s_i A + C^T C = 0 \quad (10)$$

Assumption 2. For a linear system, if the pair (A,B) is stabilizable, then there exists a matrix k that can be obtained such that $\lambda(A+Bk)<\frac{\alpha\pi}{2}$, and $x_i^T s_i x_i$ can be chosen as a Lyapunov function such that:

$$s_i(A+Bk)+(A+Bk)^T s_i = -Q_i,\ Q_i>0. \quad (11)$$

Now, by considering the control law for distributed systems:

$$u_i = -ck\sum_{j\in N_i} w_{ij}(\hat{x}_j-\hat{x}_i). \quad (12)$$

where $\hat{x}_i$and $\hat{x}_j$ are state estimates in (16). In addition, $c>0\in R$, N_i is the agent i^{th} neighbors, $k\in R^{m\times n}$ is feedback matrix and w_{ij} is considered equal to 1 for any $j\in N_i$ in this model. The general system dynamics (5) can be rewritten as follows:

$${}^{C}_{t_0}D^{\alpha}_{t}x = (I_N\otimes A)x-(cL\otimes Bk)\hat{x} + \mathbf{F}(\boldsymbol{x},\boldsymbol{u}) \quad (13)$$
$$y=(I_N\otimes C)x.$$

where $\otimes$ represents the Kronecker product, $L=L(G)$ is the Laplacian matrix graph G. In addition, if $I_N\otimes A=\bar{A}$, $-(cL\otimes Bk)=\overline{\mathrm{B}}$, and $(I_N\otimes C)=\bar{C}$, then (13) can be rewritten as follows:

$${}^{C}_{t_0}D^{\alpha}_{t}x = \bar{A}\boldsymbol{x}+\overline{\mathrm{B}}\hat{x}+\mathbf{F}(\boldsymbol{x},\boldsymbol{u}) \quad (14)$$
$$y=\bar{C}x$$

Lemma 2. [17] Considering the nonlinear Lipschitz continuous function in (6) and Assumption 2 are met, then r has the following upper bound:

$$r<\frac{\lambda_{min}(Q)}{2\lambda_{max}(s)}. \quad (15)$$

where $Q=diag([Q_1,Q_2,\dots,Q_N])$ and $s=diag([s_1,s_2,\dots,s_N])$. Then, the feedback law $u_i=-ck\sum_{j\in N_i}w_{ij}(\hat{x}_j-\hat{x}_i)$ globally Mittag Leffler stabilizes the system (14).

III. Main Results

A. Design observer

A.1. Observer for Lipschitz class of nonlinear system

In this section, a fractional-order state observer for system (5) is introduced as follows:

$${}^{C}_{t_0}D^{\alpha}_{t}\hat{x}_i = A\hat{x}_i + Bu_i + \mathbf{F}(\hat{\boldsymbol{x}},\boldsymbol{u}) + \boldsymbol{\mathcal{L}_i}(C\hat{x}_i - y_i); t\ge t_0. \quad (16)$$

where $\hat{x}_i\,\epsilon R^{N.d}$ are state estimates, $\boldsymbol{\mathcal{L}_i}$ is the observer of the gain. According to (12), (16) can be rewritten as:

$${}^{C}_{t_0}D^{\alpha}_{t}\hat{x} = (I_N\otimes A)\hat{x}-(cL\otimes Bk)\hat{x} + \boldsymbol{F}(\hat{\boldsymbol{x}},\boldsymbol{u}) + (I_N\otimes\boldsymbol{\mathcal{L}_i})(\bar{C}\hat{x}-y), \quad t\ge t_0. \quad (17)$$

Considering the definitions given for the parameters of (14), then (17) yields that:

$$ {}_{t_0}^{C}D_t^{\alpha}\hat{x} = \bar{A}\hat{x} + \bar{B}\hat{x} + \mathbf{F}(\hat{\boldsymbol{x}},\boldsymbol{u}) + \boldsymbol{\mathcal{L}}(\bar{C}\hat{x}-y), \qquad t \geq t_0. \tag{18} $$

where $(I_N \otimes \boldsymbol{\mathcal{L}}_i) = \boldsymbol{\mathcal{L}}$.

Theorem1. A sufficient condition for the stability of the observer (18) is that the Lipschitz constant is smaller than:

$$ r < \frac{\lambda_{min}(\theta s)}{2\lambda_{max}(s)} \tag{19} $$

Proof:

Note that if observer (18) can provide an accurate estimate of the state vector in (14), then:

$$ \lim_{t\to\infty}(\hat{x}-x) = 0 \tag{20} $$

The difference between the system's state vector and the observer's state vector is defined as follows:

$$ e = \hat{x} - x \tag{21} $$

By considering the Lyapunov function candidate as:

$$ V(e) = e^T s e \tag{22} $$

and substituting (21) in (22), we have:

$$ V(e) = (\hat{x}-x)^T\, s(\hat{x}-x) \tag{23} $$

Taking Caputo derivative from (21) yields

$$ \begin{aligned} {}_{t_0}^{C}D_t^{\alpha}e &= (I_N \otimes A)\hat{x} - (cL \otimes Bk)\hat{x} + \mathbf{F}(\hat{\boldsymbol{x}},\boldsymbol{u}) \\ &\quad + (I_N \otimes \boldsymbol{\mathcal{L}}_i)(\bar{C}\hat{x}-y) \\ &\quad - (I_N \otimes A)x + (cL \otimes Bk)\hat{x} \\ &\quad - \mathbf{F}(\boldsymbol{x},\boldsymbol{u}) \\ &= \big((I_N \otimes A) + \boldsymbol{\mathcal{L}}\bar{C}\big)e + \Delta\mathbf{F} \\ &= (\bar{A} + \boldsymbol{\mathcal{L}}\bar{C})e + \Delta\mathbf{F} \end{aligned} \tag{24} $$

where $(I_N \otimes A) = \bar{A}$ and $\Delta\mathbf{F} = \mathbf{F}(\hat{\mathbf{x}},\mathbf{u}) - \mathbf{F}(\mathbf{x},\mathbf{u})$. Assumption 1 can be defined for N −agent:

$$ \bar{A}^T s + s\bar{A} = -\theta s + \bar{C}^T\bar{C} \tag{25} $$

Using condition (6), (24), (25), and Lemma1, we have:

If $\boldsymbol{\mathcal{L}} = -\frac{1}{2}s^{-1}\,\bar{C}^T$ is considered, then the sufficient condition for ${}_{t_0}^{C}D_t^{\alpha}V(e) < 0$ is $r < \frac{\lambda_{min}(\theta s)}{2\lambda_{max}(s)}$ and the proof is completed.

A.2. Observer for One-Sided Lipschitz

Theorem2. When Lemma 2 is met, the sufficient condition for stability of observer (18) is that the Lipschitz constant is smaller than:

$$ \begin{aligned} {}_{t_0}^{C}D_t^{\alpha}V(e) &\leq 2e^T(t)\, s\, {}_{t_0}^{C}D_t^{\alpha}e(t) \\ &\leq e^T[-\theta s + \bar{C}^T\bar{C} + \bar{C}^T\boldsymbol{\mathcal{L}}^T s \\ &\quad + s\boldsymbol{\mathcal{L}}\bar{C}]e + 2e^T s\Delta F \\ &\leq -(\theta s - \bar{C}^T\bar{C} - \bar{C}^T\boldsymbol{\mathcal{L}}^T s \\ &\quad - s\boldsymbol{\mathcal{L}}\bar{C})\|e\|^2 + 2\|s\|\|e\|\|\Delta F\| \\ &\leq -\lambda_{min}(\theta s - \bar{C}^T\bar{C} \\ &\quad - 2s\boldsymbol{\mathcal{L}}\bar{C})\|e\|^2 + 2r\lambda_{max}(s)\|e\|^2. \end{aligned} \tag{26} $$

$$ \lambda_{min}(\theta s) - 2\gamma_s > 0 \tag{27} $$

Proof:

From (4), (22), and (24) it can be concluded that:

$$ \begin{aligned} {}_{t_0}^{C}D_t^{\alpha}V(e) &\leq e^T[-\theta s + \bar{C}^T\bar{C} + \bar{C}^T\boldsymbol{\mathcal{L}}^T s + s\boldsymbol{\mathcal{L}}\bar{C}]e \\ &\quad + 2e^T s\Delta F \\ &\leq -e^T\theta s e + e^T\bar{C}^T\bar{C}e \\ &\quad + e^T\bar{C}^T\boldsymbol{\mathcal{L}}^T s e + e^T s\boldsymbol{\mathcal{L}}\bar{C}e \\ &\quad + 2e^T s\Delta F \end{aligned} \tag{28} $$

Also, using (7), (28) yields:

$$ \begin{aligned} {}_{t_0}^{C}D_t^{\alpha}V(e) &\leq -\lambda_{min}(\theta s - \bar{C}^T\bar{C} - 2s\boldsymbol{\mathcal{L}}\bar{C})|e|^2 \\ &\quad + 2\gamma_s|e|^2 \end{aligned} \tag{29} $$

If $\boldsymbol{\mathcal{L}} = -\frac{1}{2}s^{-1}\,\bar{C}^T$ then $\lambda_{min}(\theta s) - 2\gamma_s > 0$ and the proof is completed.

The propose observer-based distributed Control for a multi-agent system is illustrated in Figure 1

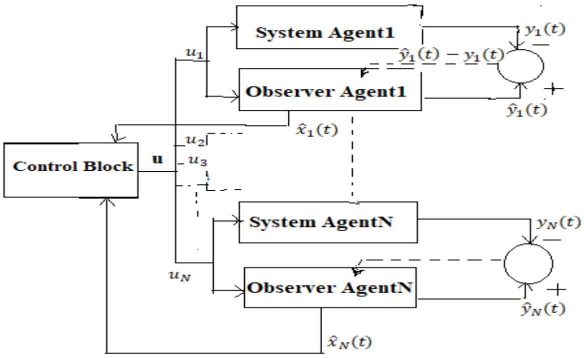

Figure 1. Block diagram portraying the system agents along with the observer and the feedback control unit

B. Achieving consensus

Theorem 3. If Assumptions 1 and 2, and Lipschitz condition (6) are such that:

$$r < \frac{1}{2} inf(\frac{1}{\lambda_{max}(s)}, \frac{\lambda_{min}(\theta s)}{\lambda_{max}(s)}) \tag{30}$$

Then, observer (18) and feedback controller $u_i = -ck\sum_{j\in N_i} w_{ij}(\hat{x}_j - \hat{x}_i)$ will be globally Mittag-Leffler stable for the dynamical system (14).

Proof:

By utilizing (18), (21), and (24), we have:

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha} e(t) = \left(\bar{A} - \frac{1}{2}s^{-1}\ \bar{C}^T\bar{C}\right)e \ + \mathbf{F}(\hat{\mathbf{x}}, \mathbf{u}) \\ - \mathbf{F}(\hat{\mathbf{x}} - \mathbf{e}, \mathbf{u}). \end{aligned} \tag{31}$$

By introducing the following Lyapunov function:

$$V(\hat{x}, e) = \delta V_1(\hat{x}) + V_2(e) \quad , \quad \delta > 0 \tag{32}$$

where $V_1(\hat{x})$ and $V_2(e)$ are:

$$\begin{aligned} V_1(\hat{x}) &= \hat{x}^T s \hat{x} \\ V_2(e) &= e^T s e \end{aligned} \tag{33}$$

Taking derivative from both sides, we have

$${}_{t_0}^{C}D_t^{\alpha} V(\hat{x}, e) = \delta\, {}_{t_0}^{C}D_t^{\alpha} V_1(\hat{x}) + {}_{t_0}^{C}D_t^{\alpha} V_2(e) \tag{34}$$

Using (18), (33), (34) and Theorem 1, it can be concluded that:

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha} V_1(\hat{x}) \le \hat{x}^T[s(\bar{A} + \bar{B}) + (\bar{A} + \bar{B})^T s]\hat{x} \\ + 2\hat{x}^T s\mathbf{F}(\hat{\boldsymbol{x}}, \boldsymbol{u}) - \hat{x}\bar{C}^T\bar{C}e. \end{aligned} \tag{35}$$

Inspired by Assumption 2 for N-agent and the Cauchy-Schwarz inequality:

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha} V_1(\hat{x}) \le -\|\hat{x}\|^2 + 2\hat{x}^T s\mathbf{F}(\hat{x}, \mathbf{u}) \\ + \|\hat{x}\|\|\bar{C}^T\bar{C}\|\|e\| \end{aligned} \tag{36}$$

Using condition Lipschitz and knowing that $\mathbf{F}(\mathbf{0}, \mathbf{u}) = 0$, then

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha} V(\hat{x}, e) \le \delta[-1 + 2r\lambda_{max}(s)]\|\hat{x}\|^2 \\ + \delta\|\bar{C}^T\bar{C}\|\|e\|\|\hat{x}\| \\ + D_{t_0,t}^{\alpha} V_2(e). \end{aligned} \tag{37}$$

If (30) holds, according to theorem 1, when $= \lambda_{min}(\theta s) - 2r\lambda_{max}(s) > 0$, then:

$${}_{t_0}^{C}D_t^{\alpha} V_2(e) \le -l\|e\|^2 \tag{38}$$

Now considering $1 - 2r\lambda_{max}(s) > 0$, and $\|\bar{C}^T\bar{C}\| > 0$. Thus, by selecting δ as (39) the Mittag- Leffler stability ensures the stability of the observer:

$$\delta < \frac{\left(1 - 2r\lambda_{max}(s)\right)l}{\|\bar{C}^T\bar{C}\|^2} \tag{39}$$

Algorithm 1 is proposed for distributed observer-based control for consensus in fractional-order systems *(DOC-FO*) for *N-agent* where θ has a real value.

Algorithm1: DOC − FO Algorithm for N − Agent

```
    $Input: A, B, C, \theta$
1.  $if \ \theta > 0$
2.  Give (10) ← $\theta$
3.  $Compute\ s \leftarrow (10)$
4.  $if \ s > 0 \ and \ s^T = s$
5.  $else$
6.  $back\ to\ step$ 1
7.  $Compute\ k \leftarrow (11)$
8.  $if \ Re(\bar{A} + \bar{B}) < 0$
9.  $u_i = ck\sum_{j\in N_i} w_{ij}(\hat{x}_j - \hat{x}_i)$
10. $else$
11. $back\ to\ step$ 2
12. $end$
13. $end$
```

IV. Nmerical Example

The simulation is based on five agents with the observer and control unit as shown in Figure 2 and the relations (14) and (18) for each agent**.**

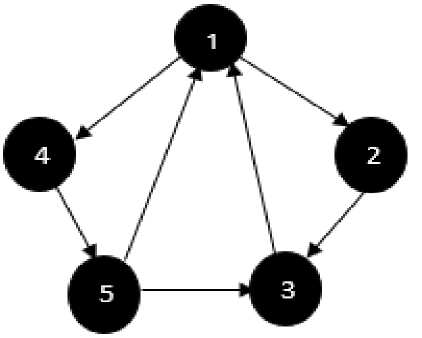

Figure 2. Topology of graph

The matrices of the system (5) considering $x_i \in R^2$ are shown in (40). The order of the fractional derivative is considered as: $\alpha = 0.9$.

$$\begin{aligned} A_i = \begin{bmatrix} 0 & 2 \\ -1 & -2 \end{bmatrix}, \quad B = [0\ \ 1]^T \ , \ C \\ = [\,1\ \ 0], \\ F_i(x_i, u_i) \\ = \frac{1}{6}\begin{bmatrix} \sqrt{{x_{i_1}}^2 + {x_{i_2}}^2} \\ sin(x_{i_1})\, cos\,(u_i) \end{bmatrix} \end{aligned} \tag{40}$$

where $= I$, $s_i = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.5 \end{bmatrix}$ and $r \simeq 0.16 < \frac{\lambda_{min}(Q)}{\lambda_{max}(\theta s)} = 1$, with $\theta = 4$, the pair (A, C) is observable and by solving (10), we have:

$$r < \frac{1}{2}\left(\frac{\lambda_{min}(\theta s)}{\lambda_{max}(s)}\right) = 1 \; and \;\; \gamma_s < \frac{1}{2}(\lambda_{min}(\theta s)) = 0.5 \tag{41}$$

The system (40) is global Mittag-Leffler stable and estimated states have converged towards the true values. Feedback matrix$k = [0 \;\; 0.4804]$, and $c = 2$ for the following initial conditions:

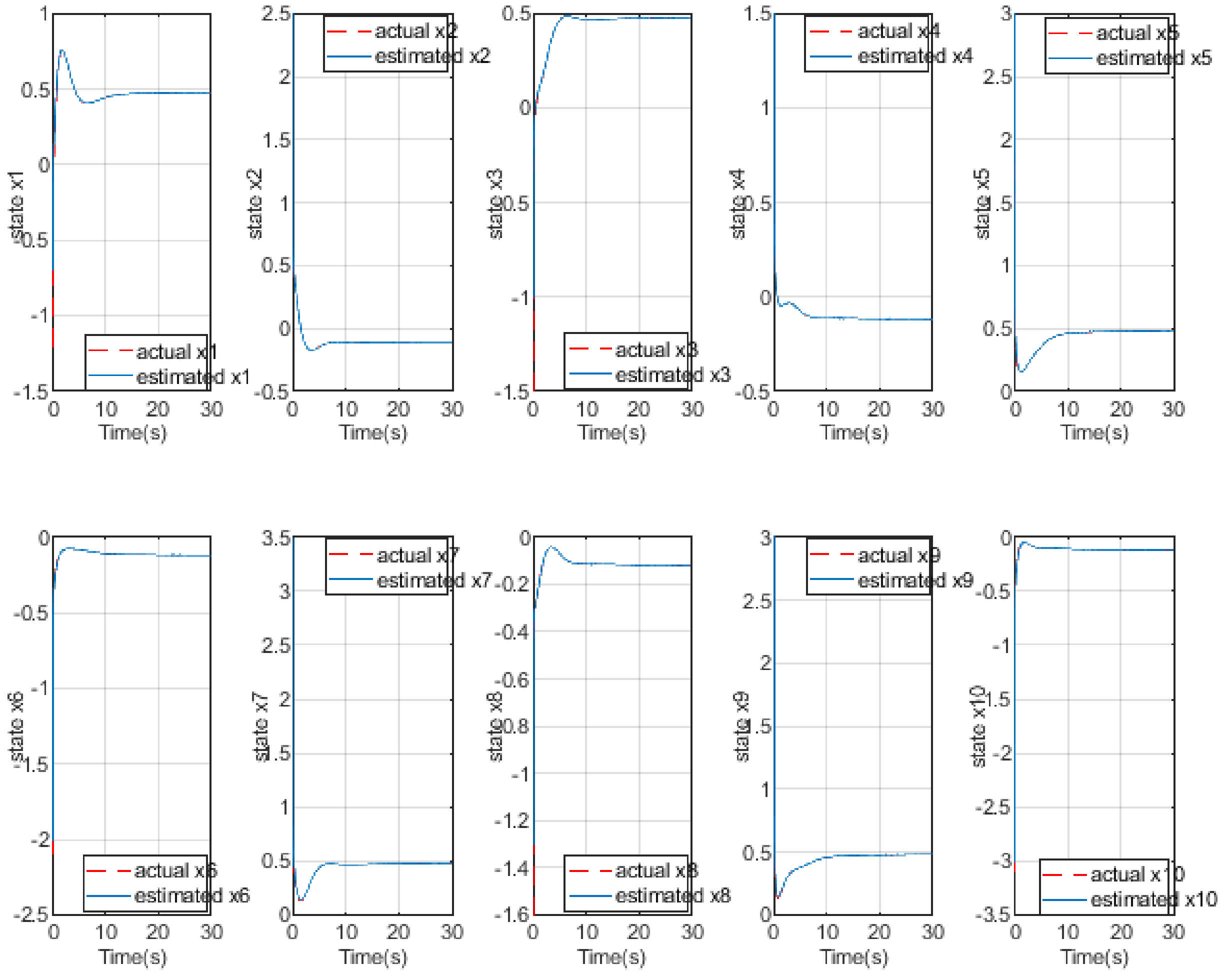

Figure 3 presents the simulation results

$x_1(0) = [-1.5; 2]$, $\hat{x}_1(0) = [-0.8; 2.5]$,

$x_2(0) = [-1.5; 1.2]$, $\hat{x}_2(0) = [-1.1; 1.5]$,

$x_3(0) = [2.5; -2.1]$, $\hat{x}_3(0) = [2.9; -2]$,

$x_4(0) = [3; -1.6]$, $\hat{x}_4(0) = [3.4; -1.3]$,

$x_5(0) = [2.4; -3.1]$, $\hat{x}_5(0) = [2.8; -3]$,

Figure 3 presents the simulation results comparing the actual system states with the estimated states obtained from the proposed observer. The results demonstrate that the estimated states closely track the actual states, with minimal deviation observed over the simulation period. This alignment indicates the effectiveness of the estimation algorithm in accurately reconstructing the system states. Furthermore, the convergence of the estimated states to the actual states reliability of the proposed methodology under the given operating conditions. These findings underscore the potential of the approach for practical applications in state estimation and control systems.

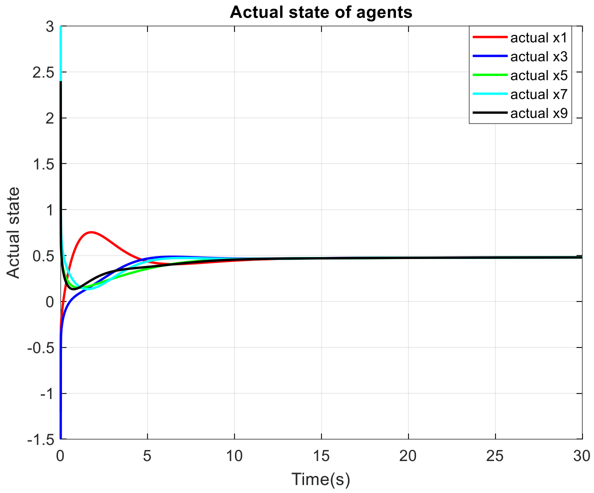

Figure 4. Consensus for the first actual state

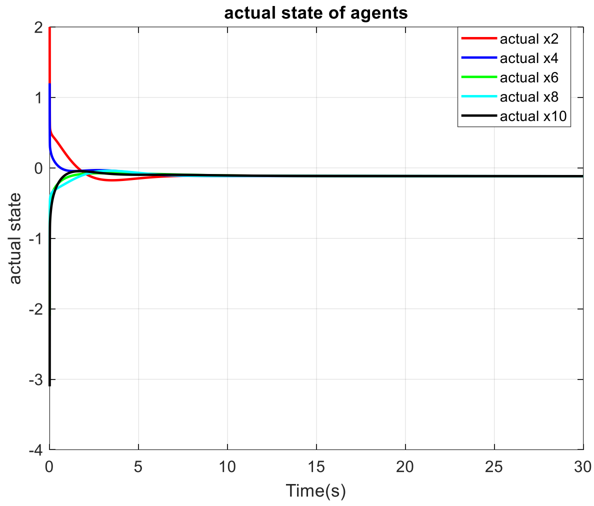

Figure 5. Consensus for the second actual state

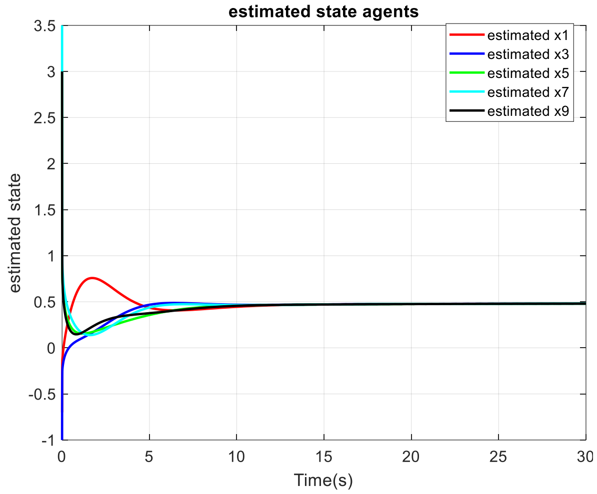

Figure 6. Consensus for the first estimated state

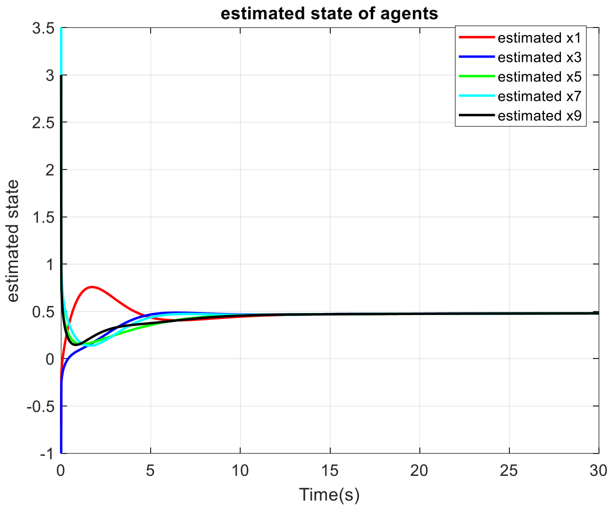

Figure 7. Consensus for the second estimated state

As illustrated in Figure 4 through Figure 7, the simulation results clearly demonstrate that the states of all agents exhibit a consistent trend of convergence over time, ultimately aligning with the average of the initial conditions of the states. This behavior is indicative of the system's inherent consensus dynamics, which drive the agents toward a unified state. The convergence rate, as observed in the figures, is influenced by the network topology and the interaction weights among agents, as described by the proposed control protocol. The results validate the theoretical analysis presented earlier, confirming that the designed control strategy effectively achieves consensus in a distributed multi-agent system.

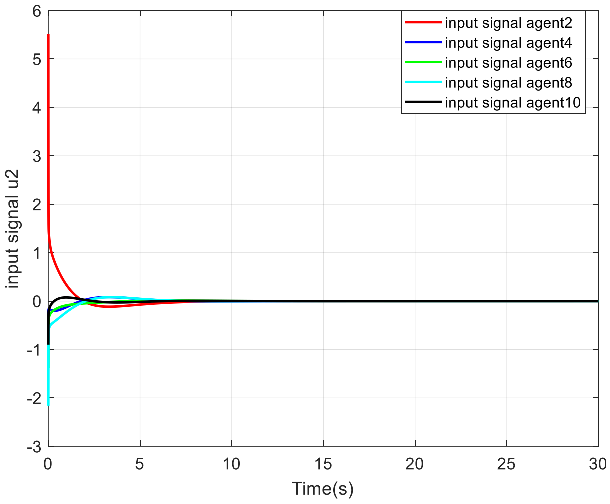

Figure 8. Input signal u2

Figure 8 shows the second input signal for five agents. As it is clear from the relationships, the first input signal is always zero because the value of zero in the first row of matrix B is zero. The control effort in the proposed method is optimized through the use of a state feedback law and a distributed consensus protocol. The control inputu_i is designed as:

$$u_i = -ck \sum_{j \in N_i} w_{ij} (\hat{x}_j - \hat{x}_i).$$

where c is a positive scalar, k is the feedback matrix, and w_{ij} represents the interaction weights between agents. This design ensures that the control effort is distributed efficiently among agents, minimizing unnecessary energy consumption.

Traditional methods often rely on centralized control or simpler feedback mechanisms, which may require higher control effort, especially in systems with large Lipschitz constants or complex nonlinearities. The proposed method, by leveraging fractional-order dynamics and one-sided Lipschitz conditions, reduces the control effort while maintaining stability and performance. The proposed method achieves global Mittag-Leffler stability, which ensures that the system states converge to consensus asymptotically. The synchronization time is influenced by the network topology and the interaction weights among agents. The use of fractional-order dynamics allows for faster convergence in systems with memory and hereditary effects, which are common in real-world applications. The norm of the error signals $\| e \|$, where $e = \hat{x} - x$, is minimized through the design of a stable observer and a distributed consensus protocol. The error dynamics are analyzed using Lyapunov's second method, ensuring that the error converges to zero over time. The proposed method guarantees that the norm of the error signals remains bounded and converges to zero, even in the presence of nonlinearities and uncertainties.

V. Limitations of the Study

While this study presents a robust observer-based controller design for fractional-order multi-agent systems within the Lipschitz and one-sided Lipschitz frameworks, several limitations should be acknowledged. First, the proposed approach assumes ideal communication conditions among agents, neglecting potential communication delays that are prevalent in real-world multi-agent systems. The presence of delays can significantly impact the stability and performance of the consensus protocol, and future work should address this limitation by incorporating delay compensation mechanisms. Additionally, the study focuses on fixed network topologies, which may not fully capture the dynamic nature of real-world systems where network topologies can change over time due to agent mobility or link failures. Extending the proposed framework to handle stochastic switching topologies would enhance its applicability to more complex and realistic scenarios. Furthermore, the analysis is confined to Mittag-Leffler stability, and the concept of fixed-time stability, which guarantees convergence within a predefined time regardless of initial conditions, is not explored. Investigating fixed-time stability could provide stronger guarantees for time-critical applications. Lastly, while the one-sided Lipschitz condition broadens the applicability of the proposed method to a wider class of nonlinear systems, the design and implementation complexity may increase for systems with highly nonlinear dynamics or exogenous disturbances. Future research should explore these aspects to further improve the robustness and practicality of the proposed approach.

VI. Conclusion

Many existing research methods are limited to stabilizing dynamical systems with small Lipschitz constants and often fail to provide effective solutions when the Lipschitz constant becomes large. To address this limitation, this paper leverages the concept of one-sided Lipschitz continuity, which was initially introduced in mathematical literature and is capable of encompassing a broader family of nonlinear systems. By utilizing this framework, we propose an observer-based distributed consensus control design tailored for nonlinear fractional-order multi-agent systems within both Lipschitz and one-sided Lipschitz classes. The proposed controller, designed using a state feedback law, ensures global Mittag-Leffler stability of the system, thereby offering a robust solution for a wider range of nonlinear dynamics. This advancement not only overcomes the constraints of traditional methods but also provides a foundation for further exploration in more complex scenarios, such as systems with communication delays, stochastic switching topologies, or exogenous disturbances. Future research could extend this work to incorporate fixed-time stability guarantees or address time-varying network conditions, further enhancing the applicability and robustness of the proposed approach in real-world multi-agent systems.

REFERENCES

[1] Javan, Ali Akbar Kekha, and Assef Zare. "Images encryption based on robust multi-mode finite time synchronization of fractional-order hyper-chaotic Rikitake systems." Multimedia Tools and Applications vol.83, no.1, pp. 1103-1123, 2024.

[2] M. Nazifi, M. Pourgholi, "Adaptive Fractional-Order Consensus Control of Cyber-Physical Power Systems in The Presence of Unbounded Perturbations." International Journal of Industrial Electronics Control and Optimization (2024).

[3] F. Aazam Manesh, M. Pourgholi, and E. Amini Boroujeni, "Fractional-Order Multi-agent Formation Using Distributed NMPC Design with Obstacles and Collision Avoidance and Connectivity Maintenance," *Journal of Control, Automation and Electrical Systems,* pp. 1-11, 2022.

[4] M. Sallah and C. Margeanu, "Effect of fractional parameter on neutron transport in finite disturbed reactors with quadratic scattering," 2016.

[5] A. El-Sayed, S. Rida, and A. Arafa, "On the solutions of time-fractional bacterial chemotaxis in a diffusion gradient chamber," *International Journal of Nonlinear Science,* vol. 7, no. 4, pp. 485-492, 2009.

[6] E. Balci, S. Kartal, and I. Ozturk, "Comparison of dynamical behavior between fractional order delayed and discrete conformable fractional order tumor-immune system," *Mathematical Modelling of Natural Phenomena,* vol. 16, p. 3, 2021.

[7] A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*. elsevier, 2006.

[8] M. Shahvali, M.-B. Naghibi-Sistani, and J. Askari, "Adaptive fault compensation control for nonlinear uncertain fractional-order systems: static and dynamic event generator approaches," *Journal of the Franklin Institute,* vol. 358, no. 12, pp. 6074-6100, 2021.

[9] Varga, Bence, J.K. Tar, and R. Horváth, "Fractional order inspired iterative adaptive control," *Robotica*, vol.42, no. 2, pp. 482-509, 2024.

[10] Chendrayan, J. Hoon Jeong, and Y. H. Joo, "Stochastic exponential stabilization and optimal control results for a class of fractional order equations," *Chaos, Solitons & Fractals*, vol. 185, no. 1, pp.15087, 2024.

[11] H. Kheloufi, A. Zemouche, F.Bedouhene, and M. Boutayeb, "On LMI conditions to design observer-based controllers for linear systems with parameter uncertainties," *Automatica,* vol. 49, no. 12, pp. 3700-3704, 2013.

[12] A. Boroujeni and H. R. Momeni, "Observer based control of a class of nonlinear fractional order systems using LMI," *International Journal of Science and Engineering Investigations,* vol. 1, no. 1, pp. 48-52, 2012.

[13] M. Pourgholi and E. A. Boroujeni, "An iterative LMI-based reduced-order observer design for fractional-order chaos synchronization," *Circuits, Systems, and Signal Processing,* vol. 35, no. 6, pp. 1855-1870, 2016.

[14] S. Ahmad and M. Rehan, "On observer-based control of one-sided Lipschitz systems," *Journal of the Franklin Institute,* vol. 353, no. 4, pp. 903-916, 2016.

[15] H. Chu, X. Liu, W. Zhang, and Y. Cai, "Observer-based consensus tracking of multi-agent systems with one-sided Lipschitz nonlinearity," *Journal of the Franklin Institute,* vol. 353, no. 7, pp. 1594-1614, 2016.

[16] T. Kaczorek, "Reduced-order perfect nonlinear observers of fractional descriptor discrete-time nonlinear systems," *International Journal of Applied Mathematics and Computer Science,* vol. 27, no. 2, 2017.

[17] A. Jmal, O. Naifar, A. Ben Makhlouf, N. Derbel, and M. A. Hammami, "On observer design for nonlinear Caputo fractional-order systems," *Asian journal of control,* vol. 20, no. 4, pp. 1533-1540, 2018.

[18] M. Karkhaneh and M. Pourgholi, "Adaptive observer design for one-Sided Lipschitz class of nonlinear systems," *The Modares Journal of Electrical Engineering,* vol. 11, no. 4, pp. 45-51, 2012.

[19] Y. Bai, A. Marino, and J. Wang, "Observer-based distributed fault detection and isolation for second-order multi-agent systems using relative information," *Journal of the Franklin Institute,* vol. 358, no. 7, pp. 3779-3802, 2021.

[20] X. Xue, X. Yue, and J. Yuan, "Connectivity Preservation and Collision Avoidance Control for Spacecraft Formation Flying with Bounded Actuation," in *Advances in Guidance, Navigation and Control*: Springer, 2022, pp. 3685-3697.

[21] C. Thompson and P. Reverdy, "Multi-Agent Recurrent Rendezvous Using Drive-Based Motivation," *arXiv preprint arXiv:2109.13408,* 2021.

[22] Y. Xu, T. Li, and S. Tong, "Event-triggered adaptive fuzzy bipartite consensus control of multiple autonomous underwater vehicles," *IET Control Theory & Applications,* vol. 14, no. 20, pp. 3632-3642, 2020.

[23] M. Rodriguez, "Intelligent Intersection Management through Gradient-Based Multi-Agent Coordination of Traffic Lights and Vehicles," University of Maryland, College Park, 2021.

[24] H. Bai and J. T. Wen, "Cooperative load transport: A formation-control perspective," *IEEE Transactions on Robotics,* vol. 26, no. 4, pp. 742-750, 2010.

[25] W. Ren and R. W. Beard, *Distributed consensus in multi-vehicle cooperative control* (no. 2). Springer, 2008.

[26] Z. Li, Z. Duan, and G. Chen, "Dynamic consensus of linear multi-agent systems," *IET Control Theory & Applications,* vol. 5, no. 1, pp. 19-28, 2011.

[27] H. Zhao, S. Xu, and D. Yuan, "An LMI approach to consensus in second-order multi-agent systems," *International Journal of Control, Automation and Systems,* vol. 9, no. 6, pp. 1111-1115, 2011.

[28] W. Yu, G. Chen, M. Cao, and J. Kurths, "Second-order consensus for multiagent systems with directed topologies and nonlinear dynamics," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics),* vol. 40, no. 3, pp. 881-891, 2009.

[29] M. Rehan, A. Jameel, and C. K. Ahn, "Distributed consensus control of one-sided Lipschitz nonlinear multiagent systems," *IEEE Transactions on Systems, Man, and Cybernetics: Systems,* vol. 48, no. 8, pp. 1297-1308, 2017.

[30] M. Doostinia, M. T. Beheshti, S. A. Alavi, and J. M. Guerrero, "Distributed control strategy for DC microgrids based on average consensus and fractional-order local controllers," *IET Smart Grid,* 2021.

[31] L. Chen, X. Li, Y. Chen, R. Wu, A. M. Lopes, and S. Ge, "Leader-follower non-fragile consensus of delayed fractional-order nonlinear multi-agent systems," *Applied Mathematics and Computation,* vol. 414, p. 126688, 2022.

[32] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: Algorithms and theory," *IEEE Transactions on automatic control,* vol. 51, no. 3, pp. 401-420, 2006.

[33] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on automatic control,* vol. 50, no. 5, pp. 655-661, 2005.

[34] I. Podlubny, "Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications", Elsevier, vol 198, 1998.

[35] B. Duan and Z. Zhang, "A Rational Approximation Scheme for Computing Mittag-Leffler Function with Discrete Elliptic Operator as Input," *Journal of Scientific Computing,* vol. 87, no. 3, pp. 1-20, 2021.

[36] M. A. Duarte-Mermoud, N. Aguila-Camacho, J. A. Gallegos, and R. Castro-Linares, "Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems," *Communications in Nonlinear Science and Numerical Simulation,* vol. 22, no. 1-3, pp. 650-659, 2015.

[37] M. N. Soorki and M. S. Tavazoei, "Adaptive robust control of fractional-order swarm systems in the presence of model uncertainties and external disturbances," *IET control theory & applications,* vol. 12, no. 7, pp. 961-969, 2018.

[38] Y. Li, Y. Chen, and I. Podlubny, "Mittag–Leffler stability of fractional order nonlinear dynamic systems," *Automatica,* vol. 45, no. 8, pp. 1965-1969, 2009.

Farshid Aazam Manesh received his Master’s degree in Electrical Control Engineering from Shahid Beheshti University, Iran in 2021. He is currently involved as a lecturer at Ilam Technical and Vocational University. His research interests include Multi-agent systems, Model predictive control, Adaptive control, Quadrotor dynamics, Robotics, Fault diagnosis and tolerance, Fractional calculus, Nonlinear Systems.

Elham Amini Boroujeni is an Assistant Professor at Kharazmi University, Tehran, Iran. She received her B.Sc. from the University of Tehran in 2006 and her M.Sc. and Ph.D. in Control Engineering from Tarbiat Modares University in 2009 and 2014, respectively. Her research focuses on nonlinear control, robust control, fractional-order systems, and multi-agent systems control.

Fateme Bazarkhak received her bachelor's degree from Qom University of Technology and her master's degree from Kharazmi University in 2021. Her interests include fractional-order systems, multi-agent systems, nonlinear systems, and fault diagnosis.

Mahdi Pourgholi is an Associate Professor at Shahid Beheshti University (SBU), Tehran, Iran. He received his PhD in Control Engineering from Tarbiat Modares University in 2010. His research interests include nonlinear control, fuzzy systems, and cooperative control of multi-agent systems.