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Numerical assessment of optimal control problems with variable-order fractional integro-differential equation based on Laguerre wavelets functions

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ABSTRACT

This paper deals with a general form of fractional optimal control problems involving variable-order fractional integro-differential equation using orthonormal Laguerre wavelets expansions. By effectively employing these functions, product variable-order operational matrices have been obtained. By using these fractional operational matrices and collocation points, the study transforms the original continuous-time optimal control problems of variable-order fractional integro-differential equations into a system of linear or non-linear algebraic equations. Attempts have been made to use the collocation method with a joint application of Lagrange multiplier technique, to obtain the approximate cost function based on determining the state and control functions. The main components for applying these wavelets is to have viable solutions due to their orthogonality. In addition, the convergence analysis is presented with respect to the operational matrices of this scheme. Simulation results indicate that the proposed method works well and provides satisfactory results with regard to accuracy and computational effort.

I. Introduction

Variable-order fractional differential equations (V-FDEs) are considered as a growing research field which have extensive action on interactive inhibitor, application of electric circuit analysis and development of excitatory neurons models [1, 37]. The V-FDEs provide a flexible framework for modeling and analyzing signals with varying degrees of regularity and complexity [38]. Due to the significant increase of nonlinear mixed V-FDEs and optimal control problems (OCPs) [25, 26], in the last decades, many approximation techniques are developed for these types of problems. Zheng and Wang [39] proved the well-possedness and smoothing property of an OCP governed by a hidden-memory V-FDE. Cardinal functions based on Chebyshev polynomials have been used in [14] for a common type of OCPs generated by nonlinear dynamical systems involved with V-FDE. Transcendental Bernstein functions were proposed in [15, 16] for the numerical solutions of these problems. Authors in [2] applied a robust method for nonlinear variable-order fractional systems to obtain the approximate values of optimal control. Wavelet method used for some type of OCPs governed by V-FDEs in [12,32].

Integral equations have attracted extensive research in the field of applicable models [8,18]. To obtain the approximate solutions of these equations, the wavelet and spectral methods have been applied in [24]. In the modeling of many non-linear physical phenomena and engineering applications [5], a generalization of OCPs has been presented, whose dynamic system is described by an integro-differential equation [19]. Considerable researches have been done on developing the valid and efficient techniques for solving these problems [3, 33]. Fractional frameworks make it possible to tackle the challenges associated with environmental changes which is favorable in physical scenarios [31], where the objects are subject to dynamic changes. Researchers and practitioners in fields like

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engineering, physics, economics, and applied mathematics often study and address fractional integro-differential equations to analyze and control complex systems with fractional dynamics. Touchard wavelets are employed to obtain approximation values of OCPs involving Volterra integral equation with fractional derivative [27]. A direct method based on piece-wise Bernoulli functions is applied to find the optimal control of these Volterra equations [17]. Authors in [10] design the Gegenbauer approach for evaluating the approximate outputs of OCP with fractional integro-differential equation. A robust method has been used in [20] to obtain the optimal control of fractional Volterra integro-differential equation. A generalized fractiona Chebyshevl wavelets [13], Fibonacci wavelets [28] and Lucas wavelets [29] have been suggested to solve variable-order fractional OCPs. Some approaches for these problems can be found in [34, 35].

An OCP governed by variable-order fractional integro-differential equations (V-FIDEs), refer to a class of mathematical models that involve both fractional calculus and control theory. Although various type of fractional OCPs have been solved by many researchers [40], but to the best of our knowledge, the literature dealing with OCPs influenced by nonlinear V-FIDEs is seemingly lacking and has not yet been employed. Here, we want to determine an optimal control strategy that optimizes a given performance index governed by V-FIDEs in the sense of Volterra-Fredholm integro-differential equations and manage the computational complexity associated with analytical solutions. By generalizing these concepts, an approximation scheme is here proposed to solve the following OCP:

$$Min_u \; J(x,u) = \int_{t_0}^{t_f} Q(t,x(t),u(t))dt, \qquad (1)$$

$$s.t. \; {}_0D_t^{\beta(t)}x(t) = \int_{t_0}^{t} G(t,s,x(s),u(s))ds + \int_{t_0}^{t_f} K(t,s,x(s),u(s))ds + g(t), \qquad (2)$$

$$x(t_0) = x_0, \quad u(t) \in U, \qquad (3)$$

in which $t \in [0,1]$, $x_0 \in \mathbb{R}$, Q, G, K and g are continuously differentiable, (x,u) is the couple of state and control variables, respectively. The set $U \subset \mathbb{R}^m$ indicates the allowed inputs that are continuous functions, and ${}_0D_t^{\beta(t)}$ is the variable-order fractional derivative with the arbitrary order $\beta(t)$ in the sense of variable-order Liouville-Caputo derivative. Throughout this paper, we assume that the optimal control problem is well-posed. In particular, in this case, $0 < \beta(t) \leq 1$ is continuous and bounded, ensuring the existence and uniqueness of the solution to the integral constraint. We want to acquire the control function $u^*(t)$ that optimizes the outlay of the controlling effort (1) by extracting the differential system (2) through conditions (3). This necessitates the development of efficient numerical methods to address these equations and manage the computational complexity associated with analytical solutions. To assess this concept, an approximate idea is here proposed based on the Laguerre polynomials and Laguerre wavelets functions [30]. The prominent feature of this approximation is the expression of functions as series expansion using orthogonal polynomials. Orthogonality significantly reduces the complexity of the problem by re-scaling the variable-order fractional OCP (1)-(3) up to a simpler system that reveals the important characteristics of fractional OCPs and, comes from the direct methods for solving OCPs. Indeed, the investigated method performed very well when employing some algebraic equations for detecting the solution of OCP (1)-(3). Throughout the paper, we will build upon the existing body of knowledge while introducing novel methodologies and insights. By structuring our paper into the following main sections, we will provide readers with a cohesive and logically organized presentation of our research on V-FIDEs and their applications in optimal control problems. We continue the discussion as described in follow. The essential concepts of fractional calculus and also, the major specifications of the Laguerre wavelets functions and their properties are introduced in Section II. The proposed method for the expressed problem and its convergence analysis are investigated in Section III and Section IV, respectively. The validity of this technique is investigated by solving several examples in Section V. Furthermore, Section VI includes a brief conclusion.

II. Preliminary Tools

In the last section, we provided a thorough overview of the topic, including the fundamental concepts and key theories related to V-FIDEs and their applications. This background will serve as a foundation for understanding the subsequent sections of the paper. In the second section, some preparations regarding fractional calculus and Laguerre wavelets functions are provided.

A. Variable-order fractional calculus

The characteristic of fractional operators is the conquest of memory and scale transformations. However, if there is a permanent (or static) order in the fractional models, it can be potentially caused the heterogeneity of the system to be neglected over time [11]. Accordingly, to cover this shortcoming, models are introduced with variable-order derivatives that potentially capture time-dependent system properties [4]. Regarding this pivotal characteristic, the operators of these calculations can be expressed as follows.

Definition 1 The variable-order fractional Riemann-Liouville integral of order $\beta(t) > 0$ for a given function $x(t)$ is determined as:

$$I^{\beta(t)}x(t) = \frac{1}{\Gamma(\beta(t))}\int_0^t (t-\tau)^{\beta(t)-1}x(\tau)d\tau,$$

and satisfies the following useful property:

$$I^{\beta(t)}t^b = \begin{cases} \frac{\Gamma(b+1)}{\Gamma(b+1+\beta(t))}t^{b+\beta(t)}, & b > -1, \\ 0, & \text{elsewhere.} \end{cases}$$

Definition 2 Riemann-Liouville fractional derivatives of order $\beta(t) > 0$, $n - 1 < \beta(t) \le n$, $n \in \mathbb{N}$, are given by:

$${}_0^{RL}D_t^{\beta(t)}x(t) = \frac{1}{\Gamma(n-\beta(t))}\left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\beta(t)-1}x(\tau)d\tau,$$

and

$${}_t^{RL}D_{t_f}^{\beta(t)}x(t) = \frac{(-1)^n}{\Gamma(n-\beta(t))}\left(\frac{d}{dt}\right)^n \int_t^{t_f} (\tau-t)^{n-\beta(t)-1}x(\tau)d\tau.$$

Definition 3 Liouville-Caputo fractional derivatives of order $\beta(t) > 0$, $n - 1 < \beta(t) \le n$, are given by:

$${}_0D_t^{\beta(t)}x(t) = \frac{1}{\Gamma(n-\beta(t))}\int_0^t (t-\tau)^{n-\beta(t)-1}x^{(n)}(\tau)d\tau,$$

and

$${}_tD_{t_f}^{\beta(t)}x(t) = \frac{(-1)^n}{\Gamma(n-\beta(t))}\int_t^{t_f} (\tau-t)^{n-\beta(t)-1}x^{(n)}(\tau)d\tau.$$

Since, Liouville-Caputo's definition has some useful properties in dealing with initial value problems, specially in tackling natural models, we selected it in this work. This operator also holds the linearity property as follows:

$${}_0D_t^{\beta(t)}(\sigma g(t) + \kappa f(t)) = \sigma\, {}_0D_t^{\beta(t)}g(t) + \kappa\, {}_0D_t^{\beta(t)}f(t),$$

when σ and κ are constants. Besides, for $n \le b \in \mathbb{N}$, it has the following property:

$${}_0D_t^{\beta(t)}t^b = \begin{cases} \frac{\Gamma(b+1)}{\Gamma(b+1-\beta(t))}t^{b-\beta(t)}, & n-1 < \beta(t) \le n, \\ 0, & \text{elsewhere,} \end{cases}$$

and ${}_0D_t^{\beta(t)}C = 0$, when C is a constant.

B. Properties of Laguerre wavelets

Consider the following differential equation in $(0,\infty)$:

$$tx'' + (\alpha + 1 - t)x' + nx = 0,$$

or

$$(t^{\alpha+1}\exp(-t)x')' + nt^{\alpha}\exp(-t)x = 0,$$

in which $\alpha > -1$. The polynomial solutions of these equations are known as Laguerre polynomials [23] and, introduced as follows:

$$\mathbf{L}_0^{\alpha}(t) = 1,$$
$$\mathbf{L}_1^{\alpha}(t) = 1 + \alpha - t,$$

and for any $k \ge 1$, we have:

$$\mathbf{L}_{k+1}^{\alpha}(t) = \frac{2k+1+\alpha-t}{k+1}\mathbf{L}_k^{\alpha}(t) - \frac{k+\alpha}{k+1}\mathbf{L}_{k-1}^{\alpha}(t),$$

in which $\mathbf{L}_0^{\alpha}(t) = 1$ and $\mathbf{L}_1^{\alpha}(t) = \alpha + 1 - t$. The Rodriguez formula for them is:

$$\mathbf{L}_k^{\alpha}(t) = \exp(t)\frac{t^{-\alpha}}{k!}\frac{d^k}{dt^k}(t^{k+\alpha}\exp(-t)).$$

Furthermore,

$$\mathbf{L}_k^{\alpha}(t) = \sum_{i=0}^{k}\binom{\alpha-\gamma+k-i}{k-i}\mathbf{L}_i^{\gamma}(t), \quad \alpha,\gamma \in \mathbb{R}.$$

Also, by differentiation of these polynomials we obtain:

$$\frac{d^n}{dt^n}\mathbf{L}_k^{\alpha}(t) = (-1)^k\mathbf{L}_{k-n}^{\alpha+n}(t), \qquad k \ge n.$$

Moreover, the following equation holds:

$$\frac{1}{n!}\frac{d^n}{dt^n}(t^{\alpha}\mathbf{L}_k^{\alpha}(t)) = \binom{k+\alpha}{n}t^{\alpha-n}\mathbf{L}_k^{\alpha-n}(t).$$

Considering the weighting function $t^{\alpha}\exp(-t)$, we have:

$$\int_0^{\infty} t^{\alpha}\exp(-t)\mathbf{L}_m^{\alpha}(t)\mathbf{L}_k^{\alpha}(t)dt = \frac{(k+\alpha)!}{k!}\delta_{m,k},$$

that means the orthogonality of Laguerre polynomials in $[0,\infty)$, in which $\delta_{m,k}$ is the Dirac delta function [23]. Particularly, for $\alpha = 0$, it is true that

$$\mathbf{L}_k(t) = \sum_{i=0}^{k}\frac{(-1)^i}{i!}\frac{k!}{i!\,(k-i)!}t^i,$$

in which the Laguerre polynomials $\mathbf{L}_k^0(t) = \mathbf{L}_k(t)$ satisfy $\mathbf{L}_k(0) = 1$ and are orthogonal in $[0,\infty)$. Also, there is a well-known classical global uniform for Laguerre polynomials that estimates by [23]:

$$|\mathbf{L}_k^{\alpha}(t)| \le \exp\left(\frac{t}{2}\right)\frac{(\alpha+1)k}{k!}, \alpha \ge 0, \quad t \ge 0, k = 0,1,2,\cdots.$$

Therefore, the Laguerre wavelets are defined as follows in interval [0,1):

$$\phi_{n,m}(x) = \begin{cases} \frac{2^{\frac{k}{2}}}{m!}\mathbf{L}_m^{\alpha}(2^k t - 2n + 1), \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ \qquad n = 1,\cdots,2^{k-1}, k \in \mathbb{Z}^+, \\ 0, \qquad \text{elsewhere.} \end{cases} \tag{4}$$

III. Method of Solution

In this paper, we aim to address the shortcomings of the previous researches and highlight the significance of our study. In the comprehensive background section we emphasized the significance of our study by discussing the limitations and gaps in the existing literature. We identified the specific challenges and unresolved issues in the field of V-FIDEs and OCPs, highlighting the need for novel approaches and techniques. By doing so, we will establish the rationale for our research and demonstrate its potential contributions to the field. Finally, we will explicitly state the aim of our study, which is to propose a novel approximation method for solving OCPs governed by V-FIDEs. We will outline the key objectives and research questions that will guide our investigation. By developing an efficient and accurate numerical method, we aim to provide a valuable tool for analyzing and solving complex optimal control problems in diverse applications.

Herein, the fractional derivative of variable-order $\beta(t)$ is extracted to an operational matrix by employing the Laguerre wavelet expansions. They provide strong interpolation properties and achieve higher accuracy with fewer collocation points. It is observed that the proposed method is fully compatible with the complexity of such problems and is very user-friendly. Accordingly, to evaluate the dynamical system (2), we consider the introduced operational matrix. The desired method with the help of the collocation method is present to solve problem (1)-(3). Next, the collocation points should be considered to generate some algebraic equations. Additionally, the initial condition (3) is utilized to extract an algebraic equation. After applying the Lagrange multipliers algorithm for this system, the constrained extremum method was employed to obtain the best-fitted parameters for state and control variables.

A function $f(t) \in C[0,1]$ can be expanded in terms of Laguerre wavelets as follows:

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \gamma_{n,m} \phi_{n,m}(t),$$

where $\gamma_{n,m} =< f(t), \phi_{n,m}(t) >$, $\phi_{n,m}(t)$ for $n = 1,2,\cdots$, $m = 0,1,2,\cdots$ are the Laguerre wavelets which has been defined in Equation (4) (in which for simplicity we assumed that $\alpha = 0$) and $<.,.>$ signifies the inner produce in $L^2[0,1]$. If these infinite series are truncated, then we have:

$$f(t) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \gamma_{n,m} \phi_{n,m}(t) = \sigma^T \Phi(t), \tag{5}$$

where $\Phi(t)$ and σ are two $2^{k-1}M$ column vectors, given by$[\phi_{1,0}(t), \phi_{1,1}(t), \cdots, \phi_{2^{k-1},(M-1)}(t)]$, and σ is $[\gamma_{1,0}, \gamma_{1,1}, \cdots, \gamma_{1,(M-1)}, \gamma_{2,0}, \cdots, \gamma_{2^{k-1},(M-1)}]^T$, in which $\gamma_{n,m} = \int_0^1 \phi_{n,m}(t) f(t) dt$. For solving our problem, by attention to relation (4) and considering the collocation points $t_s = \frac{s}{q-1}$, $s = 0,1,\cdots, q-1$, $q = 2^{k-1}(M-1)$, we obtain:

$$\mathbf{L}_{q\times q} = [\Phi(0), \Phi(\frac{1}{q-1}), \cdots, \Phi(1)],$$

which is called the Laguerre wavelets matrix. Accordingly, the Liouville-Caputo variable-order fractional derivative of Laguerre polynomials is given by:

$$\begin{aligned} &{}_0D_t^{\beta(t)} \mathbf{L}_n(t) \\ &= \sum_{k=[\beta(t)]}^{n} \frac{(-1)^k}{k!} \frac{n!}{(n-k)!} \frac{t^{k-\beta(t)}}{\Gamma(k-\beta(t)+1)}. \end{aligned}$$

So, for ${}_0D_t^{\beta(t)}\Phi(t)$ we get:

$$({}_0D_t^{\beta(t)}\Phi)(t) \cong (\mathbf{L}_{q\times q} B^{\beta(t)} \mathbf{L}_{q\times q}^{-1})\Phi(t), \tag{6}$$

wherein B^{α} is obtained as follows:

$$\frac{1}{(q-1)^{\beta(t)}\Gamma(\beta(t)+2)} \times \begin{bmatrix} 0 & b_{1,1} & \binom{1}{0} b_{1,2} + \binom{2}{0} b_{2,2} & \cdots & \sum_{j=1}^{q} \binom{j}{0} b_{j,q} \\ 0 & -b_{1,1} & -[\binom{1}{1} b_{1,2} + \binom{2}{1} b_{2,2}] & \cdots & -\sum_{j=2}^{q} \binom{j}{1} b_{j,q} \\ 0 & 0 & \binom{2}{2} b_{2,2} & \cdots & \sum_{j=3}^{q} \binom{j}{2} b_{j,q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (-1)^q \binom{q}{q} b_{q,q} \end{bmatrix},$$

in which:

$$\begin{aligned} &b_{r,s} \\ &= \begin{cases} (-1)^r \dfrac{r!}{\Gamma\big(r+1-\beta(t)\big)} \dbinom{s}{r}, \text{if } s \geq r \geq [\beta(t)] \\ 0, \qquad \text{otherwise.} \end{cases} \end{aligned}$$

Now, we want to solve the OCP (1)-(3). Utilizing this approach, we first expand the state and the control variables as follows:

$$x(t) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \phi_{n,m}(t) x_{n,m} = X^T \Phi(t), \tag{7}$$

$$u(t) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \phi_{n,m}(t) u_{n,m} = U^T \Phi(t),$$

in which

$$X = [x_{1,0}, x_{1,1}, \cdots, x_{2^{k-1},(M-1)}]^T,$$
$$U = [u_{1,0}, u_{1,1}, \cdots, u_{2^{k-1},(M-1)}]^T.$$

Substitution Equations (7) into the performance index (1), holds approximately the subsequent relation:

$$J(X,U) = \int_{t_0}^{t_f} Q(t, X^T\Phi(t), U^T\Phi(t)) dt, \tag{8}$$

where X and U are the same as before. Accordingly, Mathematica software can be computed the integral in Equation (8). Therefore, it can be approximated by $J(X,U) \cong \tilde{F}(X,U)$. In a similar way, substituting from Equations (7) and (6) into the variable-order fractional system (2) yields:

$$\begin{aligned} &(t-t_0)^{-\beta(t)} X^T S^{\beta(t)} \Phi(t) - g(t) \\ &- \int_{t_0}^{t} G(t, s, X^T\Phi(s), U^T\Phi(s)) ds \\ &- \int_{t_0}^{t_f} K(t, s, X^T\Phi(s), U^T\Phi(s)) ds \\ &\cong E(t, X, U) \simeq 0. \end{aligned} \tag{9}$$

The algebraic system of equations are generated by the following scheme. First, we approximate the dynamical system (9) with the above operational matrix (7). Applying the introduced points t_s, Equation (9) can be easily implemented to obtain a system of nonlinear algebraic equations. Then, we get:

$$\Lambda_s \cong E(t_s, X, U) = 0, \tag{10}$$

for $s = 1, \ldots, q-1$. Meanwhile, from the initial condition (3) and using approximation (7), we get an algebraic equation as follows:

$$\Lambda_0 \cong X^T \Phi(t_0) - \mathbf{x}_0 = 0. \tag{11}$$

Now, in order to acquire the solutions of the generated algebraic system, we need to optimize the behavior of cost function (8) in the presence of conditions (10)-(11). So, we pursue the Lagrange multipliers method as follows:

$$J^*(X, U, \lambda) = J(X, U) + \Lambda\lambda,$$

in which $\lambda = [\lambda_0 \ \lambda_1 \ \ldots \ \lambda_{2^{k-1}(M-1)}]^T$ are the unknown Lagrange multipliers and $\Lambda = [\Lambda_0 \ \Lambda_1 \ \ldots \ \Lambda_{2^{k-1}(M-1)}]$. Thus, in order to assess an extremum for problem (1)-(3), we employed the modified Newton's method to solve the following necessary optimality conditions:

$$\frac{\partial J^*}{\partial X} = 0, \quad \frac{\partial J^*}{\partial U} = 0, \quad \frac{\partial J^*}{\partial \lambda} = 0. \tag{12}$$

Accordingly, X and U should be successfully computed from Equation (12) to obtain an approximate solution of problem (1)-(3) using Laguerre wavelet functions.

A. The Iterative Algorithm

To combine the scheme developed in the previous section, here we apply an iterative algorithm.

Step 1. Take a positive pre assigned error bound $\epsilon > 0$ and set $i = 1$.

Step 2. Consider positive integer values for the constants k and M, and determine the collocation points.

Step 3. Approximate $x(t)$ and $u(t)$ using Equation (7) and compute the approximation of ${}_0D_t^{\beta(t)}x(t)$ from the method developed in Section 3.

Step 4. Use the approximated values in step 3 and decompose problem (1)-(3) using relations (8)-(11).

Step 5. Construc the system of linear or nonlinear equations given in (12) using the Lagrange multipliers technique. By determining X and U via the modified Newton's method, we have determined the approximated values of the state and control functions.

Step 6. Terminate the algorithm if the expression $e_i = |\frac{J_i - J_{i-1}}{J_i}|$ satisfies $e_i < \epsilon$, else, replace i by $i+1$ and go to step 2.

IV. Convergence Analysis and Error Estimate

Here, we perform the error analysis of the Laguerre wavelet expansion. For this purpose, we assume $|f(t)| \leq C$ in which C is a real positive number.

Theorem 1 If $f(t) \in L^2[0,1]$ is an arbitrary function which can be approximated by an infinite series of Laguerre wavelet functions, then this series converges uniformly to $f(t)$.

Proof. We know, $f(t)$ can be defined as Equation (5) in such a way that $\gamma_{n,m} = \langle f(t), \phi_{n,m}\rangle$. Hence, we have for $n, m > 0$ we have $\gamma_{n,m} = \langle f(t), \phi_{n,m}\rangle = \int_0^1 \phi_{n,m} f(t)dt = \frac{2^{\frac{k}{2}}}{m!}\int_{I_{nk}} f(t)\mathbf{L}_m(2^k t - 2n + 1)dt$, in which $I_{nk} = [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}})$. Upon assigning the change of variable $2^k t - 2n + 1 = p$, yields to:

$$\gamma_{n,m} = \frac{2^{\frac{-k}{2}}}{m!}\int_{-1}^{1} f(\frac{p-1+2n}{2^k})\mathbf{L}_m(p)dp.$$

Then we will have $|\gamma_{n,m}| \leq \frac{1}{2^{\frac{k}{2}}m!}\int_{-1}^{1} |f(\frac{p-1+2n}{2^k})|\,|\mathbf{L}_m(p)|dp \leq \frac{C}{2^{\frac{k}{2}}m!}\int_{-1}^{1} |\mathbf{L}_m(p)|dp$.
According to the properties of continuity and integrability of $\mathbf{L}_m(.)$ on $(-1,1)$, let $\int_{-1}^{1} |\mathbf{L}_m(p)|dp = A$. Then, we get: $|\gamma_{n,m}| \leq \frac{CA}{2^{\frac{k}{2}}m!}$. So, the series $\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\gamma_{n,m}$ is absolutely convergent. This means that the presented series in relation (5) is uniformly convergent. Suppose that $P_{nm} = \{\phi_{1,0}, \cdots, \phi_{2^{k-1},(M-1)}\}$ is the set of Laguerre wavelets of nm degree and $\Upsilon = span(P_{nm})$ in which $n = 2^{k-1}$ and $m = M-1$.

Theorem 2 Let $\tilde{f}(t) = \sigma^T\Phi(t)$ be the best approximation of $f(t)$ from Υ. We will have:

$$\| f - \tilde{f} \|_2 \leq \frac{R}{M!\sqrt{2M+1}},$$

in which $R = max_{\tau\in[0,1]}|f^{(M)}(\tau)|$.

Proof. We know that $\{1, t, t^2, \cdots, t^{M-1}\}$ is a basis set for polynomials space with $M-1$ degree. Applying Taylor's formula we have $\hat{f}(t) = f(\tau) + f'(\tau)(t-\tau) + \frac{f''(\tau)}{2!}(t-\tau)^2 + \cdots + \frac{f^{(M-1)}(\tau)}{(M-1)!}(t-\tau)^{M-1} + R_{M-1}(t), \quad \tau \in [0,1]$, in which $R_{M-1} = \frac{f^{(M)}(\tau)}{M!}(t-\tau)^M$, $\tau \in [0,1]$. Hence, it is clear that $\left|f(t) - \hat{f}(t)\right| = \frac{t^M f^{(M)}(\tau)}{M!}, \quad \tau \in [0,1]$. Now, due to the error of the Taylor expansion, and using the best approximation concept we obtain $\| f(t) - \tilde{f}(t) \|_2^2 = \| f(t) - \sigma^T\Phi(t) \|_2^2 \leq \sum_{n=1}^{2^{k-1}} \| f(t) - \hat{f}(t) \|_{I_{n,k}}^2 \leq \sum_{n=1}^{2^{k-1}} \int_{I_{n,k}} (f(t) - \hat{f}(t))^2 dt \leq \sum_{n=1}^{2^{k-1}} \int_{I_{n,k}} (\frac{t^M max_{\tau\in I_{n,k}}|f^{(M)}(\tau)|}{M!})^2 dt \leq \int_0^1 (\frac{t^M R}{M!})^2 dt = \frac{R^2}{(M!)^2(2M+1)}$. According to the computing procedure, we have the absolute error function for this case as follows:

$$E(t) = |f(t) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\gamma_{n,m}\phi_{n,m}(t)|,$$

wherein $f(t)$ is a continuous bounded function defined on interval [0,1] which can be approximated by the Laguerre wavelet expansion defined in Equation (5). Therefore, based on the above analysis, when M approaches to ∞, the obtained approximate solution with this method converges to the exact solution of problem (1)-(3).

V. Applications

Three examples of OCPs governed by V-FIDE are considered in this section. The calculations related to the examples were done using Mathematica 14.1 from a 2.30 MHz Core i7 PC with 16 GB RAM. Also, we used the default number of digits in Mathematica, that this digit is equal to 10^{-16}. Also, we have been used the following notations:

$$\| E_u \|_2^2 = \| u(t) - u^*(t) \|_2^2 = \int_{t_0}^{t_f} (u(t) - u^*(t))^2 dt,$$

$$\| E_x \|_2^2 = \| x(t) - x^*(t) \|_2^2 = \int_{t_0}^{t_f} (x(t) - x^*(t))^2 dt,$$

to get a fair comparison of performances in which (x, u) and (x^*, u^*) are the couples of approximate and exact solutions, respectively.

Example 1 Consider the following OCP:

$$\underset{u}{\text{Min}} J(x, u) = \int_0^1 (u(t) - t^2)^2 + (x(t) - \text{sin}t)^2 dt,$$

subject to the following Volterra integro-differential equation:

$$ {}_0D_t^{\beta(t)} x(t) =$$

$$g(t) + t^2 x^3(t) + t^2 u^2(t) + \int_0^t s x^3(s) ds,$$

where $x(0) = 0$ and, $g(t) = -t^6 + \text{cos}t + \frac{2}{3} t\text{cos}t - \frac{2}{3}\text{sin}t - \frac{2}{3} t^2 \text{sin}t + \frac{2}{3} t^2 \text{cos}^2 t\text{sin}t + \frac{1}{3} t\text{cos}t\text{sin}^2 t - \frac{1}{9}\text{sin}^3 t - \frac{1}{3} t^2 \text{sin}^3 t$. The analytical solution for this problem is $(x^*(t), u^*(t)) = (\text{sin}t, t^2)$. The Laguerre wavelet polynomials approximation are given as $x(t) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \phi_{n,m}(t) x_{n,m}$, and $u(t) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \phi_{n,m}(t) u_{n,m}$. Considering $\alpha = 0$, $k = 1$ and $M = 4$, we have:

$$\phi_{1,0}(t) = 2\sqrt{2},$$

$$\phi_{2,0}(t) = 2\sqrt{2},$$

$$\phi_{1,1}(t) = 4\sqrt{2} - 16\sqrt{2}t,$$

$$\phi_{2,1}(t) = 8\sqrt{2} - 16\sqrt{2}t,$$

$$\phi_{1,2}(t) = \frac{11}{\sqrt{2}} - 32\sqrt{2}t + 32\sqrt{2}t^2,$$

$$\phi_{1,3}(t) = \frac{40\sqrt{2}}{9} - \frac{284\sqrt{2}t}{9} t + \frac{512\sqrt{2}t^2}{9} - \frac{256\sqrt{2}t^3}{9}.$$

By taking the collocation points $\{t_0 = 0, t_1 = \frac{1}{6}, \ldots, t_6 = 1\}$ and following the proposed approximation method, the generated algebraic system for deriving the extremum of J^* can be derived. In order to perform the proposed method, the first stage is executed. The results of this stage, which has been shown in Table 1, are the evolution of J_M^* for different choices of $\beta(t)$ and M, and the values of CPU time required by the proposed scheme for $M = 6$. Detected errors of approximate solutions with $\beta(t) = 1$ are reported in Fig. 1 and Table 2. Also, these results are compared with the presented approach in [36]. With this comparison, we also want to verify the incremental performance of the Laguerre wavelet polynomials in contrast of other literature. These results allow our approach to reach more accurate behaviors than the other literature by selecting fewer iteration.

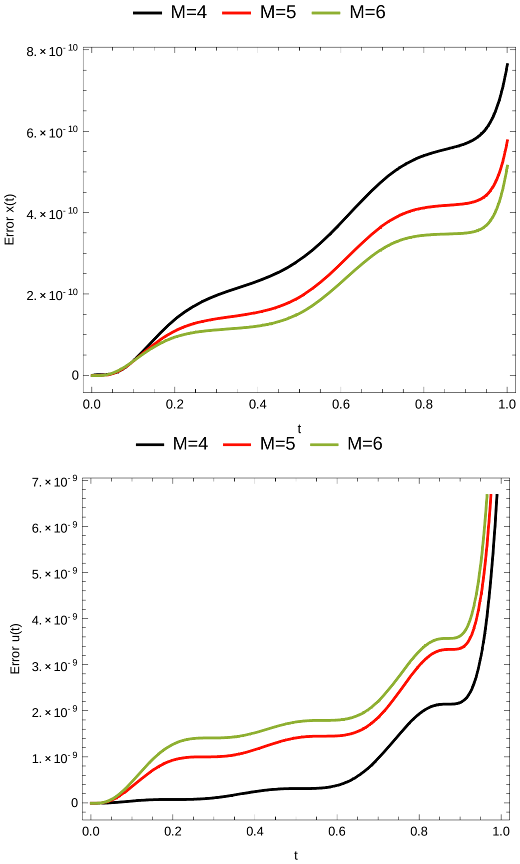

Fig. 1. Absolute errors with $k = 1, \beta(t) = 1$ for Example 1.

TABLE I Evaluated results of J_M^* with $k = 1$ for Example 1.

This study	$\beta(t)$				
	1	$1 - 0.001t$	$1 - 0.05t$	$1 - \sin 0.001t$	$1 - \sin 0.05t$
$M = 3, k = 1$	8.03644×10^{-7}	7.51393×10^{-7}	1.06269×10^{-5}	7.51393×10^{-7}	1.06193×10^{-5}
$M = 4, k = 1$	9.30188×10^{-9}	8.54271×10^{-9}	1.20137×10^{-5}	8.54255×10^{-9}	1.20053×10^{-5}
$M = 5, k = 1$	1.21444×10^{-8}	1.15061×10^{-8}	1.15228×10^{-5}	1.15063×10^{-8}	1.14577×10^{-5}
$M = 6, k = 1$	1.46575×10^{-8}	1.30573×10^{-8}	1.20035×10^{-5}	1.33206×10^{-8}	1.1949×10^{-5}

CPU Time	31.0781	32.0938	30.7969	31.7031	31.1281

TABLE II Absolute errors with $\beta(t) = 1$ for Example 1.

Method	Number of basis functions	$\parallel E_x \parallel_2^2$	$\parallel E_u \parallel_2^2$
This study	$M = 3, k = 1$	1.37303×10^{-8}	7.89914×10^{-7}
	$M = 4, k = 1$	7.64273×10^{-10}	8.53761×10^{-9}
	$M = 5, k = 1$	5.76936×10^{-10}	1.15675×10^{-8}
Legendre polynomials [36]	$M = 4$	9.5×10^{-7}	1.2×10^{-7}

Example 2 Consider the OCP:

$$\underset{u}{\text{Min}}\ J(x,u) = \int_0^1 ((u(t) - e^t)^2 + (x(t) - e^t)^2)dt,$$

subject to the following Volterra integro-differential equation:

$${}_0D_t^{\beta(t)}x(t) = e^t(1 - t - \frac{e^t}{2}) - e^t(\frac{e^t}{2} + 1) + 1 + \big(t + x(t)\big)u(t) + \int_0^t u(s)ds,$$

where $x(0) = 1$. It should be noted that the analytical solution for this problem is $(x^*(t), u^*(t)) = (e^t, e^t)$. The behavior of cost functional J_M^* using different choices of $\beta(t)$ and M have been shown in Table 3. Also, this table shows we achieve the accurate values with the small number of M in comparision with Haar wavelet and hat functions methods [21, 22]. Also, we obtain the reasonable CPU time for $M = 5$ in this table. Further, we present the absolute errors of control function and optimal trajectory for $\beta(t) = 1$ in Table 4 and Fig. 2. Moreover, Fig. 3 explores the best-fitted evolution results for different choices of $\beta(t)$, $k = 1$ and $M = 5$. This behavior indicates an agreement between the exact and approximate solutions. In addition, this figure illustrates the convergence pattern with depicting solutions for various choices of $\beta(t)$ using a small number of M. This accurate behavior can be improved by increasing the number of M. These excellent evaluations allow us to claim that the variable-order fractional derivatives have been used as benchmark performances in dynamical models.

TABLE III Evaluated results of J_M^* for Example 2.

Number of basis function	$\beta(t)$				
	1	$1 - 0.001t$	$1 - 0.05t$	$1 - \sin 0.001t$	$1 - \sin 0.05t$
$M = 3, k = 1$	1.57744×10^{-4}	1.57154×10^{-4}	1.60185×10^{-4}	1.57154×10^{-4}	1.60173×10^{-4}
$M = 4, k = 1$	1.57939×10^{-6}	1.66966×10^{-6}	3.6558×10^{-5}	1.66967×10^{-6}	3.65479×10^{-5}
$M = 5, k = 1$	3.42939×10^{-7}	7.23057×10^{-7}	3.07637×10^{-5}	7.23057×10^{-7}	3.0768×10^{-5}
$M = 6, k = 1$	3.3273×10^{-7}	3.38541×10^{-7}	3.06707×10^{-5}	3.38541×10^{-7}	3.06618×10^{-5}
$M = 32$ [21]	4.4380×10^{-4}	---	---	---	---
$M = 32$ [22]	1.168×10^{-7}	---	---	---	---
CPU Time	12.0156	13.1406	12.7188	13.5	11.7969

TABLE IV Absolute errors with $\beta(t) = 1$ for Example 2.

Number of basis functions	$\parallel E_x \parallel_2^2$	$\parallel E_u \parallel_2^2$
$M = 2, k = 1$	7.22747×10^{-5}	8.54695×10^{-5}
$M = 3, k = 1$	1.57554×10^{-7}	1.42184×10^{-6}
$M = 4, k = 1$	1.69442×10^{-8}	3.25995×10^{-7}
$M = 5, k = 1$	1.7371×10^{-8}	3.15359×10^{-7}

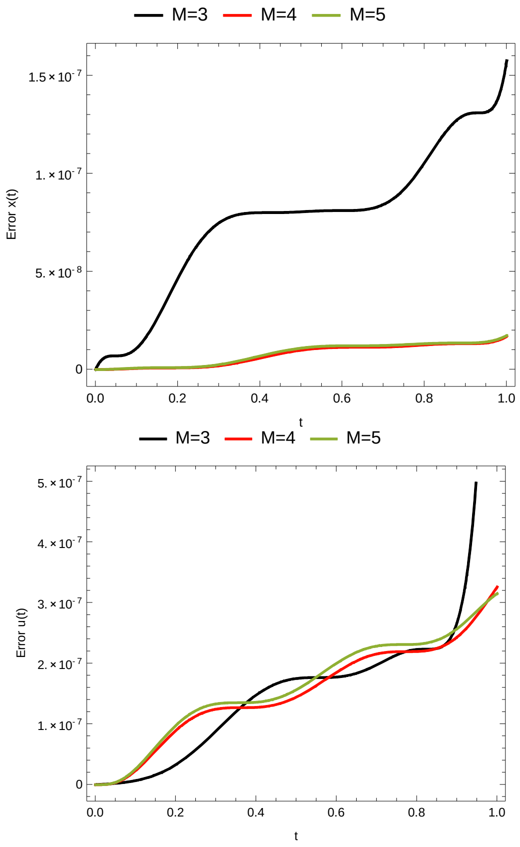

Fig. 2. Evaluated errors with $\beta(t) = 1$ and $k = 1$ for Example 2.

Example 3 Consider the following performance index:

$$\underset{u}{\text{Min}} J(x,u) = \int_0^1 (u(t) - x(t))^2 dt,$$

subject to the following Fredholm integro-differential equation with variable-order fractional derivative:

$$
{}_0D_t^{\beta(t)}x(t) = e^t - \frac{1}{3}t + \int_0^1 \left(e^{-2s}s^2tu(s)x'(s)\right)ds,
$$

and condition $x(0) = 1$. The behavior of J_M^* using different choices of $\beta(t)$ and M in comparison with other literature have been listed in Table 5. Comparing the results utilize that the Laguerre wavelet functions create good conditions to receive more accurate approaches in the presence of CPU time for $M = 6$. The analytical solutions of this problem are $u^*(t) = e^t$ and $x^*(t) = e^t$. Thus, we present the absolute errors of control function and optimal trajectory for $\beta(t) = 1$ in Fig. 4 and Table 6. In addition, a comparison has been made with the obtained results in [36] in this table. The presented results in Table 6 demonstrate a good agreement with the reported results in other literature. The Evaluated results of $x(t)$ and $u(t)$ for $\beta(t) = 1$ and various choices of M with $k = 1$ are depicted in Fig. 5. This behavior indicates that variable-order models have the capacity to attract the attention of researchers.

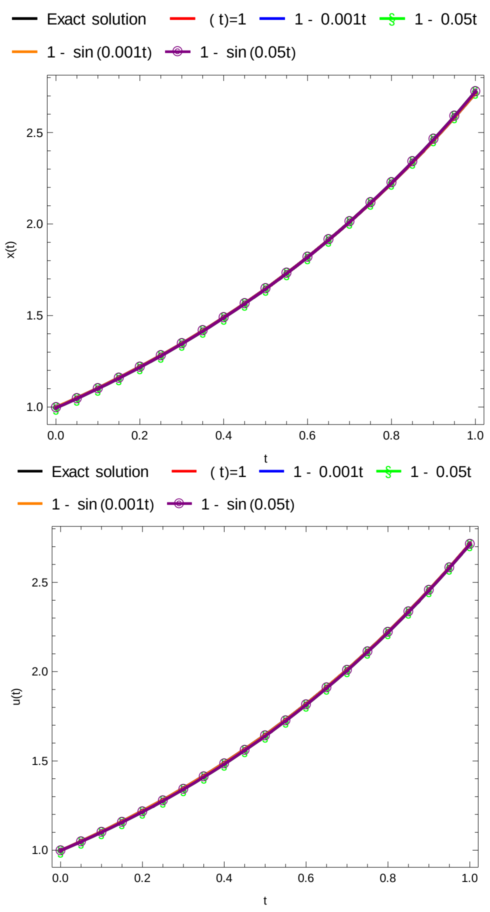

Fig. 3. The state and control functions with some variable-orders $\beta(t)$ and $M = 5$ for Example 2.

TABLE V Evaluated results of J_M^* with various choices of $\beta(t)$ for Example 3.

This study	$\beta(t)$				
	1	$1 - 0.001t$	$1 - 0.05t$	$1 - \sin 0.001t$	$1 - \sin 0.05t$
$M = 4, k = 1$	3.77958×10^{-7}	3.78572×10^{-7}	3.78573×10^{-7}	3.73113×10^{-7}	8.032×10^{-3}
$M = 5, k = 1$	4.90204×10^{-7}	4.90968×10^{-7}	4.90969×10^{-7}	4.8358×10^{-7}	9.5512×10^{-3}
$M = 6, k = 1$	5.92307×10^{-9}	5.12684×10^{-7}	6.10464×10^{-9}	7.08212×10^{-9}	1.09014×10^{-2}
VIMP with $M = 2, k = 3$ [6]	4.0135×10^{-4}	---	---	---	---
HAM with $M = 2, k = 3$ [7]	1.09746×10^{-7}	---	---	---	---
PMP with $M = 2, k = 3$ [9]	1.09746×10^{-7}	---	---	---	---
CPU Time	11.4219	11.2969	11.2188	12.3594	10.7969

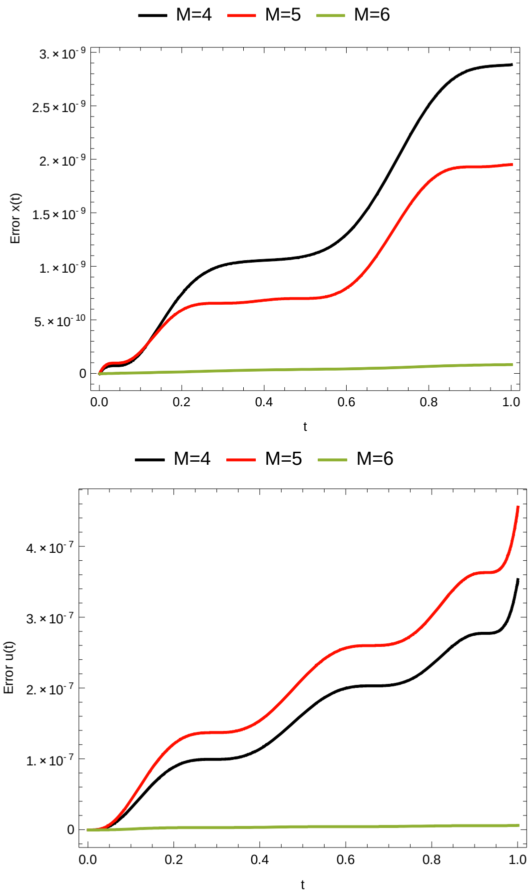

Fig. 4. Absolute errors with $\beta(t) = 1$ and $k = 1$ for Example 3.

TABLE VI Absolute errors with $\beta(t) = 1$ for Example 3.

Method	Number of basis functions	$\parallel E_x \parallel_2^2$	$\parallel E_u \parallel_2^2$
This study	$M = 4, k = 1$	2.88592×10^{-9}	3.534×10^{-7}
	$M = 5, k = 1$	1.95403×10^{-9}	4.55891×10^{-7}
	$M = 6, k = 1$	8.58198×10^{-11}	6.39124×10^{-9}
Legendre polynomials [36]	$M = 2, k = 3$	8.88256×10^{-8}	1.98494×10^{-7}

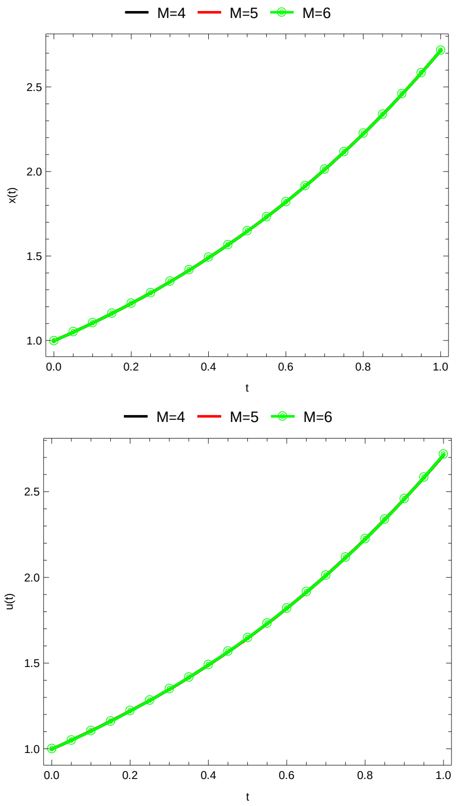

Fig. 5. The Evaluated results of $x(t)$ and $u(t)$ for $\beta(t) = 1$ and various choices of M for Example 3.

Example 4 In this example, we consider the variable-order fractional model of population growth in the form below:

$$ {}_0D_t^{\beta(t)}x(t) = rx(t)\left(1 - \frac{x(t)}{K}\right) - \int_0^t M(t - s)x(s)ds - u(t)x(t), \quad (13) $$

in which, $x(t)$ and $u(t)$ signify, respectively, the population biomass and the harvesting effort (or control variable) at time t and the parameters r and K are the intrinsic growth rate and the environmental carrying capacity, respectively. In addition, $M(.)$ is the delayed effects in memory core modeling (e.g. diseases, maturation delay). The harvest rate can be controlled to maximize the total harvested biomass while avoiding population collapse as follows:

$$ J(\mathrm{x}, u) = \int_0^T [pu(t)x(t) - cu^2(t)]dt, \quad (14) $$

wherein p and c are the price per unit biomass harvested and the cost coefficient for harvesting effort, respectively. The objective is to design the optimal tracking control $u^*(t)$ which moves the system (13) from the initial population $x(0) = x_0 > 0$ and harvesting effort limits $u(t) \leq u_{\max}$ to maximize (14) over time T. In order to solve the optimal tracking control problem, here we consider the mentioned parameters take the values $r = 0.5$(per year), $K = 1000$(tons), $M(t) = 0.1e^{-0.3t}$, $p = 10$ (price per ton), $c = 1$, $u_{\max} = 2$ and $T = 5$ (years). Simulation results including the cost values are summarized in Table 7. This table indicates that for different values of $\beta(t)$, the numerical achievements decrease as the value of M increases. Fig. 6 shows the plots of $x(t)$ and $u(t)$ after different iterations for $\beta(t) = 1 - 0.001t$ and $k = 1$. The perspective view of this figure confirms that the proposed approach works well to solve the optimal tracking control problem (13)-(14).

TABLE VII Evaluated results of J_M^* with various choices of $\beta(t)$ for Example 4.

Number of basis function	$\beta(t)$		
	1	$1 - 0.001t$	$1 - \sin 0.01t$
$M = 4, k = 1$	22.1772	22.1722	22.1269
$M = 5, k = 1$	18.8498	18.8459	18.8113
$M = 6, k = 1$	17.0781	17.0747	17.0452
CPU Time for $M = 6$	4.42188	4.70313	4.75

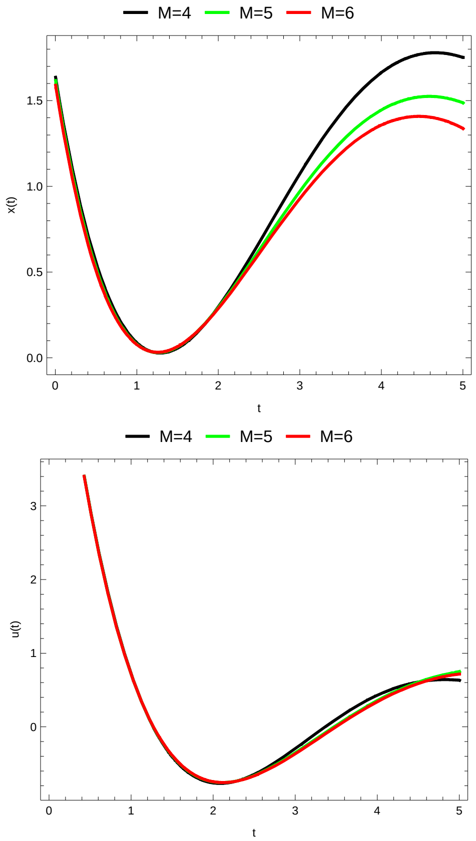

Fig. 6. The Evaluated results of $x(t)$ and $u(t)$ for $\beta(t) = 1 - 0.001t$ and various choices of M for Example 4.

VI. Conclusion

A joint application of variable-order fractional calculus and optimal control theory was presented to obtain the efficient control of V-FIDE. Through using Laguerre wavelets, the state and control variables are expanded. Also, the operational matrix of Liouville-Caputo variable-order fractional derivative is deduced for the mentioned polynomials. Consequently, the procedure of the constrained extremum problem supported by the collocation method is utilized to convert this OCP into some nonlinear algebraic equations that can be solved via any arbitrary iteration method. Approximate solutions utilize a high accuracy with a small number of iterations. Simulations with various costs of variable-order $\beta(t)$ and iteration M, support the effectiveness of this method.

Since each wavelet component is actually a differently scaled bandpass filter in the frequency domain, the signal analysis capability of the wavelet transform is limited in the time frequency plane and, therefore, the wavelet transform is inefficient for processing signals whose energy is not well concentrated in the frequency domain.

The study of variable-order fractional-integro OCPs represents an advanced frontier in fractional calculus. Continued research and development in this area hold the promise of significantly enhancing the effectiveness and applicability of optimal control strategies across various scientific and engineering domains. The results of this paper present an encouraging view for future discussion. Indeed, for the recently developed of fractional equations, the theoretical justification on the merit of this method will enhance the engineers, physicists and other scientists to use these results for their practical objectives.

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