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Original paper

On pseudo-irreducibility and Boolean lifting property of filters in residuated lattices

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Abstract

This paper advances residuated lattice theory by introducing pseudo-irreducible filters and establishing their fundamental connections to the Boolean lifting property (BLP). Also, key structural properties of these filters are established, and new characterizations of the BLP using pseudo-irreducible filters and the residuated lattice of fractions are derived. Further, we investigate the BLP of the radical of a filter by introducing weak MTL-algebras and the transitional property of radical decomposition (TPRD) as a unifying framework that generalizes Boolean algebras, MV-algebras, BL-algebras, MTL-algebras, and Stonean residuated lattices. By addressing an open question in the literature concerning the BLP of the radical of a residuated lattice, we provide algebraic and topological solutions grounded in the TPRD and the space of maximal filters. Complementary results deepening 0n the understanding of BLP in residuated lattices are also established.

Keywords: Pseudo-irreducible filter, Boolean lifting property, residuated lattice, weak MTL-algebra, transitional property of radicals decomposition (TPRD).

1 Introduction

Classical logic is fundamentally modeled by Boolean algebras, so the study of non-classical logics requires corresponding algebraic structures. Significant research focuses on fuzzy logic, where the conjunction of truth values is non-commutative. Developing algebraic models for such logics is central to fuzzy systems, hence residuated lattices that provide a key framework for this purpose are important.

Residuated lattices, introduced by Ward and Dilworth in 1939 [23] as a generalization of the lattice of ideals in rings, offer a unifying framework for algebraic structures arising in both algebra and logic. Important subclasses of residuated lattices include MV-algebras, MTL-algebras, and BL-algebras, which are central to fuzzy logic and non-classical propositional logics such as Łukasiewicz logic, Hájek's Basic Logic, and substructural logics (see [8, 12]).

Inspired by ring theory, where pseudo-irreducible ideals play an important role for comaximal factorizations [15], this concept was extended to De Morgan residuated lattices [20]. Crucially, Birkhoff's subdirect representation theorem implies that every algebra has a subdirect representation with subdirectly irreducible (and thus directly indecomposable) factors. Furthermore, in residuated lattice theory, Theorem 3.7 establishes that a directly indecomposable residuated lattice is precisely a quotient modulo a pseudo-irreducible filter. This underscores the importance of studying pseudo-irreducible filters. Also, under some conditions on a non-empty family $\{F_i\}_{i\in I}$ of pseudo-irreducible filters in a residuated lattice L, the family $\{L/F_i\}_{i\in I}$ forms the family of stalks of a sheaf over L, revealing global properties from local quotients [6]. Furthermore, the study of pseudo-irreducible filters can lead to new insights into the classification of residuated lattices based on their filter structures, potentially aiding in the development of new logical systems or the refinement

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of existing ones by providing deeper structural insights, which is particularly relevant to the context of fuzzy logic and related fields [6, 9, 11].

Research on residuated lattices has advanced through the study of structural properties like the Boolean Lifting Property (BLP). Originally defined for MV-algebras [7], BLP was later generalized to BL-algebras and residuated lattices [5, 10, 18]. In [9], this concept was further extended to universal algebras, where they investigated lifting idempotent and Boolean elements from quotients modulo filters. This property concerns the lifting of Boolean elements modulo filters and bridges algebraic and topological perspectives, aiding in the characterization of residuated lattices [7, 10]. Also, in the study of residuated lattices, the BLP can be used to characterize when a residuated lattice can be decomposed into a product of simpler structures, such as local residuated lattices. This decomposition is particularly useful in understanding the global structure of the algebra from its local properties, as highlighted in [11].

Important unresolved problems concern, particularly regarding BLP of filter radicals. While the ability to lift Boolean or idempotent elements from the quotient of an algebra to the original algebra, has proven to be a powerful tool in ring theory and MV-algebras [7, 21], the conditions under which radical of a filter or a residuated lattice satisfies BLP are unresolved [11]. Thus, studying pseudo-irreducible filters and their role in lifting properties is crucial. This paper addresses these gaps by introducing pseudo-irreducible filters, analyzing their relationship with BLP, and characterizing filters and radicals satisfying BLP.

2 Preliminaries

In this section, we review some definitions and results which will be used throughout this paper.

Definition 2.1. [8] *A residuated lattice is an algebra* $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *satisfying the following axioms:*
(RL1) $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice (whose partial order is denoted by* $\leq$*);*
(RL2) $(L, \odot, 1)$ *is a commutative monoid;*
(RL3) *For every* $x, y, z \in L$, $x \odot z \leq y$ *if and only if* $z \leq x \rightarrow y$ *(residuation).*

For $x, y \in L$ and $n \in \mathbb{N}$, we define:

$$x^* := x \rightarrow 0, x^{**} := (x^*)^*, x^0 := 1, x^n := x^{n-1} \odot x, \text{ and } x \leftrightarrow y := (x \rightarrow y) \wedge (y \rightarrow x).$$

An element x of a residuated lattice L is called *complemented* if there is an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$, if such an element y exists it is unique and it is called *the complement of* x. We denote the complement of x by x'. The set of all complemented elements in L is denoted by $\mathrm{B}(L)$ and is called *the Boolean center of* L.

In the following proposition, we collect some main and well-known properties of residuated lattices, and throughout the paper, we frequently use them without referring, see [8, 12, 19, 22] for more information.

Proposition 2.2. *Let* L *be a residuated lattice,* $x, y, z \in L$ *and* $e, f \in \mathrm{B}(L)$. *Then we have the following statements:*

1. $x \leq y$ *if and only if* $x \rightarrow y = 1$*;*
2. *If* $x \leq y$, *then* $y^* \leq x^*$*;*
3. $x \odot x^* = 0$*;*
4. $x \odot y = 0$ *if and only if* $x \leq y^*$*;*
5. $x \leq x^{**}$, $x^{***} = x^*$*;*
6. $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) = (x \odot y) \rightarrow z$*;*
7. $1 \rightarrow x = x$ *and* $x \rightarrow 1 = 1$*;*
8. $x \odot (x \rightarrow y) \leq y$ *and* $y \leq x \rightarrow y$*;*
9. *Since* $1 \rightarrow x = x$ *and* $x \rightarrow 1 = 1$, *we have* $x = (1 \rightarrow x) \wedge (x \rightarrow 1) = 1 \leftrightarrow x$. *Also, Since* $0 \rightarrow x = 1$ *and* $x \rightarrow 0 = x^*$, *we have* $x^* = (0 \rightarrow x) \wedge (x \rightarrow 0) = 0 \leftrightarrow x$*;*
10. $(x \vee y)^* = x^* \wedge y^*$*;*
11. $x \vee (y \odot z) \geq (x \vee y) \odot (x \vee z)$*;*

12. $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$;
13. $e' = e^*$, $e \wedge e^* = e \odot e^* = 0$, *and* $e^{**} = e$;
14. $e \odot f = e \wedge f \in \mathrm{B}(L)$;
15. $e \odot x = e \wedge x$;
16. $e \wedge (x \odot y) = (e \wedge x) \odot (e \wedge y)$;
17. $e \vee (x \odot y) = (e \vee x) \odot (e \vee y)$;
18. $e \wedge (x \vee y) = (e \wedge x) \vee (e \wedge y)$. *In particular,* $e \wedge (x \vee e^*) = e \wedge x$;
19. $e \vee (x \wedge y) = (e \vee x) \wedge (e \vee y)$. *In particular,* $e \vee (x \wedge e^*) = e \vee x$;
20. $(x \wedge e)^* = x^* \vee e^*$;
21. $e \to x = e^* \vee x$ *and* $x \to e = x^* \vee e$;
22. $e \odot (e \to x) = e \wedge x$ *and* $x \odot (x \to e) = e \wedge x$;
23. *For each* $n \in \mathbb{N}$, $e^n = e$;
24. *If* $x \vee x^* = 1$, *then* $x \in \mathrm{B}(L)$.

Definition 2.3. [3, 19] *A residuated lattice L is called*

1. *semi-G-algebra, if* $(x^2)^* = x^*$ *for all* $x \in L$;
2. *Gödel algebra (G-algebra for short), if* $x^2 = x$ *for all* $x \in L$;
3. *involutive, if* $x^{**} = x$ *for all* $x \in L$;
4. *MTL-algebra, if* $(x \to y) \vee (y \to x) = 1$ *for all* $x, y \in L$;
5. *De Morgan, if* $(x \wedge y)^* = x^* \vee y^*$ *for all* $x, y \in L$;
6. *Stonean, if* $x^* \vee x^{**} = 1$ *for all* $x \in L$;
7. *hyperarchimedean, if for any* $x \in L$ *there is an* $n \in \mathbb{N}$ *such that* $x^n \in \mathrm{B}(L)$.

Proposition 2.4. [3, Proposition 9] *The following conditions are equivalent for a residuated lattice L.*

1. *L is a semi-G-algebra;*
2. *For every* $x \in L$, $x \wedge x^* = 0$.

Definition 2.5. [8] *A non-empty subset F of a residuated lattice L is called a filter of L if the following conditions hold:*
(F1) *If* $x, y \in F$, *then* $x \odot y \in F$;
(F2) *If* $x \leq y$ *and* $x \in F$, *then* $y \in F$.

Definition 2.6. [8] *A non-empty subset F of a residuated lattice L is called a deductive system of L if the following conditions hold:*
(D1) $1 \in F$;
(D2) *If* $x, x \to y \in F$, *then* $y \in F$.

A non-empty subset F of a residuated lattice L is a filter if and only if it is a deductive system. We denote by $\mathrm{Filt}(L)$ the set of all filters of L. A filter F is called *proper* if $F \neq L$. Clearly, a filter F of L is proper if and only if $0 \notin F$, equivalently, if for any $x \in L$ we have either $x \notin F$ or $x^* \notin F$, see [19] for more details.

For a non-empty subset S of a residuated lattice L, we set $[S) := \bigcap\{F \in \mathrm{Filt}(L) \mid S \subseteq F\}$, called *the filter of L generated by S*. We denote by $[x)$ *the filter of L generated by* $\{x\}$. For $F \in \mathrm{Filt}(L)$ and $x \in L$, we set $F(x) := [F \cup \{x\})$. It is well-known that the lattice $(\mathrm{Filt}(L), \subseteq)$ is distributive and complete. Actually, for a family $\{F_i\}_{i\in A}$ of filters of L we have $\bigwedge_{i\in A} F_i = \bigcap_{i\in A} F_i$ and $\bigvee_{i\in A} F_i = [\bigcup_{i\in A} F_i)$. If $F, G \in \mathrm{Filt}(L)$ we set $F \to G := \{x \in L \mid F \cap [x) \subseteq G\}$, see [19, 22].

Proposition 2.7. [19] *Let S be a non-empty subset of a residuated lattice L, $x, y \in L$ and $F, G \in \mathrm{Filt}(L)$. Then*

1. $[S) = \{x \in L \mid s_1 \odot \cdots \odot s_n \leq x$ *for some* $n \geqslant 1$ *and* $s_1, ..., s_n \in S\}$;
2. $[x) = \{z \in L \mid x^n \leq z$ *for some* $n \geqslant 1\}$. *In particular, if* $e \in \mathrm{B}(L)$, *then* $[e) = \{z \in L \mid e \leq z\}$;
3. $F(x) = \{z \in L \mid i \odot x^n \leq z$ *for some* $i \in F$ *and* $n \geqslant 1\}$;
4. $F(x) \vee F(y) = F(x \wedge y) = F(x \odot y)$. *In particular,* $[x) \vee [y) = [x \wedge y) = [x \odot y)$;
5. $F(x) \cap F(y) = F(x \vee y)$. *In particular,* $[x) \cap [y) = [x \vee y)$;
6. $F \vee G = [F \cup G) = \{x \in L \mid a \odot b \leq x$ *for some* $a \in F$ *and* $b \in G\}$;
7. $F \rightarrow G \in \mathrm{Filt}(L)$.

Let F be a filter of a residuated lattice L. Consider the binary relation θ_F on L defined by $(x, y) \in \theta_F$ if and only if $x \leftrightarrow y \in F$. Then θ_F is a congruence on L, and let L/F denote the quotient set L/θ_F. Clearly, L/F becomes a residuated lattice with the natural operations induced by those of L. Also, for $x \in L$, we denote by x/F the class of x concerning to θ_F, see [19] for more information.

For a non-empty subset X of a residuated lattice L, we set $X/F := \{x/F \mid x \in X\}$. Clearly, for $x \in L$; $x/F = 1/F$ if and only if $x \in F$, and $x/F = 0/F$ if and only if $x^* \in F$. Also, $x/F \leq y/F$ if and only if $x \rightarrow y \in F$ for each $x, y \in L$. Furthermore, if G is a filter of L containing F, then G/F is a filter of L/F and for $x \in L$ we have $x/F \in G/F$ if and only if $x \in G$, see [19] for more information.

Recall that a proper filter $P \in \mathrm{Filt}(L)$ is called *prime* if for $x, y \in L$, $x \vee y \in P$ implies either $x \in P$ or $y \in P$. We denote by $\mathrm{SpecF}(L)$ the set of all prime filters of L. An easy argument shows that a proper filter P is prime if and only if for $F, G \in \mathrm{Filt}(L)$ if $F \cap G \subseteq P$ then we have either $F \subseteq P$ or $G \subseteq P$. Also, a proper filter $M \in \mathrm{Filt}(L)$ is called *maximal* if M is not strictly contained in a proper filter of L. We denote by $\mathrm{MaxF}(L)$ the set of all maximal filters of L. Clearly, every maximal filter of a residuated lattice is prime. Also, every proper filter is contained in a maximal filter, see [12, 19, 22] for more details.

For every subset X of a residuated lattice L, we set $V(X) := \{P \in \mathrm{SpecF}(L) \mid X \subseteq P\}$, and for each $x \in L$, we set $V(x) := V(\{x\})$. The family $\{V(X)\}_{X \subseteq L}$ satisfies the axioms for closed sets for a topology over $\mathrm{SpecF}(L)$. This topology is called *the Stone topology*. Since every maximal filter is prime, we can consider $\mathrm{MaxF}(L)$ as a subspace of $\mathrm{SpecF}(L)$. For each $X \subseteq L$, we define $V_{Max}(X) := V(X) \cap \mathrm{MaxF}(L)$; the family $\{V_{Max}(X)\}_{X \subseteq L}$ satisfies the axioms for closed sets for a topology over $\mathrm{MaxF}(L)$, for more information see [9].

Also, recall that for a proper filter F of a residuated lattice L, the intersection of all maximal filters of L containing F is called *the radical of* F and is denoted by $\mathrm{Rad}(F)$, that is, $\mathrm{Rad}(F) = \bigcap V_{Max}(F)$. Clearly, $\mathrm{Rad}(F)$ is a filter of L, $F \subseteq \mathrm{Rad}(F)$, and $\mathrm{Rad}(\mathrm{Rad}(F)) = \mathrm{Rad}(F)$. In the special case, $\mathrm{Rad}(\{1\})$ is the intersection of all maximal filters of L, which is always denoted by $\mathrm{Rad}(L)$ and is called *the radical of* L, see [19] for more information.

Proposition 2.8. [19] *Let L be a residuated lattice, $F \in \mathrm{Filt}(L)$ and $a \in L$. Then we have the following statements:*

1. *If $a \in L \setminus F$, then there exists a prime filter P of L containing F such that $a \notin P$;*
2. *F is the intersection of all prime filters which contain F, that is, $F = \bigcap V(F)$;*
3. *If F is proper, then F is maximal if and only if for any $x \in L \setminus M$ there is an $n \in \mathbb{N}$ such that $(x^n)^* \in M$;*
4. *L is hyperarchimedean if and only if* $\mathrm{SpecF}(L) = \mathrm{MaxF}(L)$.

Proposition 2.9. [9] *Let L be a residuated lattice, $F, G \in \mathrm{Filt}(L)$ and $x, y \in L$. Then the following statements hold:*

1. *$F = L$ if and only if $V_{Max}(F) = \varnothing$ if and only if $V(F) = \varnothing$;*
2. *$V(F) \cup V(G) = V(F \cap G)$ and $V_{Max}(F) \cup V_{Max}(G) = V_{Max}(F \cap G)$;*
3. *$V(x) \cup V(y) = V(x \vee y)$ and $V_{Max}(x) \cup V_{Max}(y) = V_{Max}(x \vee y)$;*
4. *If $\{X_k\}_{k \in K}$ is a family of subsets of L, then $V(\bigcup_{k \in K} X_k) = \bigcap_{k \in K} V(X_k)$ and $V_{Max}(\bigcup_{k \in K} X_k) = \bigcap_{k \in K} V_{Max}(X_k)$.*

For a topological space X, let $\mathrm{Clop}(X)$ be the set of all closed and open subsets of X.

Proposition 2.10. [9, Proposition 6.8] *For a residuated lattice L, we have* $\mathrm{Clop}(\mathrm{SpecF}(L)) = \{V(e) \mid e \in \mathrm{B}(L)\}$.

Recall that a $\wedge$-*closed system* of a residuated lattice L is a non-empty subset S of L such that $1 \in S$ and if $x, y \in S$, then $x \wedge y \in S$. For a $\wedge$-closed system S of a residuated lattice L, we consider the binary relation θ_S defined by $(x, y) \in \theta_S$ if and only if there exists $e \in S \cap \mathrm{B}(L)$ such that $x \wedge e = y \wedge e$. By [2, Lemma 4], θ_S is a congruence on L, and hence, $L[S] := \frac{L}{\theta_S}$, the set of all equivalence class of L with respect to θ_S, becomes a residuated lattice with the natural operations induced from those of L. $L[S]$ is called *the residuated lattice of fractions of L relative to S*. For each $x \in L$, $\frac{x}{S}$ denotes the equivalence class of x relative to θ_S, see [2] for more details.

Proposition 2.11. [2, Remark 6 and Proposition 4] *For a $\wedge$-closed system S of a residuated lattice L we have the following statements:*

1. $\frac{e}{S} = \frac{1}{S}$ *for each* $e \in S \cap \mathrm{B}(L)$*;*
2. $\frac{x}{S} \in \mathrm{B}(L[S])$ *if and only if* $e \leq x \vee x^*$ *for some* $e \in S \cap \mathrm{B}(L)$.

3 Pseudo-irreducible filters in residuated lattices

Consider a lattice A with a greatest element 1. Recall that an element $p \neq 1$ of A is called *irreducible* (more precisely, *finitely meet-irreducible*) if whenever $p = x \wedge y$, it necessarily follows that $p = x$ or $p = y$. Within the context of a residuated lattice L, a proper filter is defined as *irreducible* if it is an irreducible element of the lattice $(\mathrm{Filt}(L), \subseteq)$.

In this section, we introduce "pseudo-irreducible" filters to extend the idea of irreducible filters, then we explore their main properties.

Definition 3.1. *A pseudo-irreducible filter of a residuated lattice L is a proper filter F of L such that for any $G, H \in \mathrm{Filt}(L)$, if $F = G \cap H$ and $G \vee H = L$, then we have either $G = L$ or $H = L$.*

Example 3.2. *Let $L = \{0, a, b, c, 1\}$ denote a residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\rightarrow$ via the accompanying tables, see* [16, Example 4.4]*:*

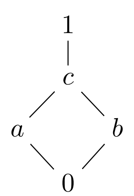

Figure 1: Hasse diagram

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

Table 1: Operation $\odot$

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Table 2: Operation $\rightarrow$

Set $G := [a) = \{a, c, 1\}$ and $H := [b) = \{b, c, 1\}$. Since $[c) = \{c, 1\} = G \cap H$ and $G \vee H = L$, the filter $[c)$ is not a pseudo-irreducible filter of L.

Recall that a residuated lattice L is called *local* if L has a unique maximal filter.

In the following proposition, we consider conditions under which all proper filters of a residuated lattice are pseudo-irreducible.

Proposition 3.3. *Every proper filter of a residuated lattice L is pseudo-irreducible if and only if L is local.*

Proof. $\Rightarrow$). Assume that every proper filter of L is pseudo-irreducible. If M and N are two distinct maximal filters of L, then $F := M \cap N$ is pseudo-irreducible by our assumption. Now since $M \vee N = L$, we deduce that either $M = L$ or $N = L$, which is impossible. Hence, L has a unique maximal filter.

$\Leftarrow$). Let L be a local residuated lattice with the unique maximal filter M. If $G, H \in \mathrm{Filt}(L)$ such that $G \vee H = L$, then we have either $G \nsubseteq M$ or $H \nsubseteq M$, or equivalently, we have either $G = L$ or $H = L$. Therefore, every proper filter of L is pseudo-irreducible. □

Proposition 3.4. *Every prime (and hence every maximal) filter of a residuated lattice is pseudo-irreducible.*

Proof. Let F be a prime filter of a residuated lattice L. Then F is a proper filter. If there are $G, H \in \mathrm{Filt}(L)$ with $F = G \cap H$ and $G \vee H = L$, then $G \cap H \subseteq F$. Now since F is a prime filter, we have either $G \subseteq F$ or $H \subseteq F$. Assume that $G \subseteq F$. Hence, $G \subseteq H$, and we have $H = G \vee H = L$, that is, $H = L$. Similarly, if $H \subseteq F$, then we have $G = L$. Therefore, F is pseudo-irreducible. Finally, since every maximal filter is prime, we deduced that every maximal filter is also pseudo-irreducible. □

In the following example, we show that pseudo-irreducible filters need not be prime in general.

Example 3.5. *Let $L = \{0, a, b, n, c, d, 1\}$ denote a residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\rightarrow$ via the accompanying tables, see* [17, Example, Page 190]*:*

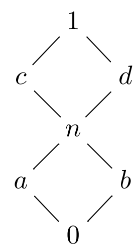

Figure 2: Hasse diagram

$\odot$	0	a	b	n	c	d	1
0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a
b	0	0	b	b	b	b	b
n	0	a	b	n	n	n	n
c	0	a	b	n	c	n	c
d	0	a	b	c	c	d	d
1	0	a	b	n	c	d	1

Table 3: Operation $\odot$

$\rightarrow$	0	a	b	n	c	d	1
0	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1
b	a	a	1	1	1	1	1
n	0	a	b	1	1	1	1
c	0	a	b	d	1	d	1
d	0	a	b	n	n	d	1
1	0	a	b	n	c	d	1

Table 4: Operation $\rightarrow$

Since $c \vee d = 1 \in \{1\}$ while $c, d \notin \{1\}$, the trivial filter $\{1\}$ is not prime in L. Now if $G, H \in \mathrm{Filt}(L)$ such that $G \vee H = L$, then by Proposition 2.7(6), there are $x \in G$ and $y \in H$ such that $x \odot y = 0$. If G and H are proper filters, then by the above tables, we have $x \neq y$ and $x, y \in \{a, b\}$. It follows that either ($a \in G$ and $b \in H$) or ($b \in G$ and $a \in H$). Now since $a \vee b = n$, in both cases, we have $n \in G \cap H$. Therefore, there are no proper filters $G, H \in \mathrm{Filt}(L)$ such that $\{1\} = G \cap H$ and $G \vee H = L$, and so the filter $\{1\}$ is pseudo-irreducible.

We recall that a residuated lattice L is *directly indecomposable* if $L \cong L_1 \times L_2$ implies either L_1 or L_2 is trivial, where L_1 and L_2 are two residuated lattices and $L_1 \times L_2$ is their direct product. It is well-known that a non-trivial residuated lattice L is directly indecomposable if and only if $\mathrm{B}(L) = \{0, 1\}$. For more details, we refer the reader to [4, 8].

Proposition 3.6. *Let L be a residuated lattice. Then L is directly indecomposable if and only if $\mathrm{SpecF}(L)$ is a connected topological space with respect to the Stone topology.*

Proof. Let $e, f \in \mathrm{B}(L)$. By Proposition 2.7(2), we have $e = f$ if and only if $[e) = [f)$. Also by Proposition 2.8(2), $[e) = [f)$ if and only if $V(e) = V(f)$. Consequently, $e = f$ if and only $V(e) = V(f)$.

On the other hand, we know that L is directly indecomposable if and only if $\mathrm{B}(L) = \{0, 1\}$. Now from Proposition 2.10 and the fact that a topological space is connected if and only if it has no non-trivial clopen subsets, we conclude that:

$$\begin{aligned} L \text{ is directly indecomposable} &\Leftrightarrow \mathrm{B}(L) = \{0, 1\} \\ &\Leftrightarrow \mathrm{Clop}(\mathrm{SpecF}(L)) = \{V(0) = \varnothing, V(1) = \mathrm{SpecF}(L)\} \\ &\Leftrightarrow \mathrm{SpecF}(L) \text{ is connected.} \end{aligned}$$

Therefore, L is directly indecomposable if and only if $\mathrm{SpecF}(L)$ is a connected topological space with respect to the Stone topology. □

In the following theorem, we provide some conditions that are equivalent to a filter being pseudo-irreducible.

Theorem 3.7. *The following statements are equivalent for a proper filter F of a residuated lattice L:*

1. *F is a pseudo-irreducible filter of L;*
2. *For $x, y \in L$, if $x \vee y \in F$ and $x \odot y = 0$, then we have either $x \in F$ or $y \in F$;*
3. *For $x \in L$, if $x \vee x^* \in F$, then we have either $x \in F$ or $x^* \in F$;*
4. *The residuated lattice L/F is non-trivial and directly indecomposable;*

5. *$V(F)$ is connected as a subspace of* $\mathrm{SpecF}(L)$ *with respect to the Stone topology.*

Proof. (1) $\Rightarrow$ (2). Assume that $x \vee y \in F$ and $x \odot y = 0$ for some $x, y \in L$. By Proposition 2.7(4-5), we have $F = F(x \vee y) = F(x) \cap F(y)$ and $F(x) \vee F(y) = F(x \odot y) = F(0) = L$. By assumption, we conclude that either $F(x) = L$ or $F(y) = L$. If $F(x) = L$, then $y \in L = F(x)$. Since $y \in F(y)$, we have $y \in F(x) \cap F(y) = F$. Similarly, if $F(y) = L$, then we have $x \in F$.

(2) $\Rightarrow$ (1). Suppose that $G, H \in \mathrm{Filt}(L)$ such that $F = G \cap H$ and $G \vee H = L$. Since $0 \in L = G \vee H$, we have $a \odot b = 0$ for some $a \in G$ and $b \in H$ by Proposition 2.7(6). So $a \vee b \in G \cap H = F$. Our assumption implies that either $a \in F$ or $b \in F$. Suppose that $a \in F$. Since $F \subseteq H$, we have $a \in H$. Thus we conclude that $a \odot b \in H$. Therefore, $0 \in H$ and so $H = L$. Similarly, if $b \in F$, then $G = L$. Therefore, F is a pseudo-irreducible filter of L.

(2) $\Rightarrow$ (3). The implication holds because $x \odot x^* = 0$ for all $x \in L$, satisfying the condition of (2) with $y = x^*$.

(3) $\Rightarrow$ (2). Assume that $x \vee y \in F$ and $x \odot y = 0$ for some $x, y \in L$. Then, $x \leq y^*$ and $y \leq x^*$. It follows that $(x \vee y) \leq (x \vee x^*)$ and $(x \vee y) \leq (y \vee y^*)$, and consequently $x \vee x^*, y \vee y^* \in F$. By assumption, we have (either $x \in F$ or $x^* \in F$) and (either $y \in F$ or $y^* \in F$). If $x^*, y^* \in F$, then $(x \vee y)^* = x^* \wedge y^* \in F$. Hence, $x \vee y, (x \vee y) \to 0 \in F$. This implies $0 \in F$, a contradiction. Therefore, we have either $x \in F$ or $y \in F$.

(3) $\Rightarrow$ (4). Since F is a proper filter of L, the residuated lattice L/F is non-trivial. Now we want to prove that L/F is directly indecomposable. For this purpose, we show that $\mathrm{B}(L/F) = \{0/F, 1/F\}$. Let $x/F \in \mathrm{B}(L/F)$. Then $x/F \vee x^*/F = 1/F$. Hence, $(x \vee x^*)/F = 1/F$, and so $x \vee x^* \in F$. By our assumption, we have either $x \in F$ or $x^* \in F$, or equivalently, we have either $x/F = 1/F$ or $x/F = 0/F$. It follows that $\mathrm{B}(L/F) = \{0/F, 1/F\}$.

(4) $\Rightarrow$ (3). Suppose $x \vee x^* \in F$ for some $x \in L$. Thus $x/F \vee x^*/F = (x \vee x^*)/F = 1/F$, and so $x/F \in \mathrm{B}(L/F)$ by Proposition 2.2(24). Now since L/F is directly indecomposable, we conclude that either $x/F = 1/F$ or $x/F = 0/F$, or equivalently, we have either $x \in F$ or $x^* \in F$.

(4) $\Leftrightarrow$ (5). We know that the prime filters of L/F are exactly of the form $P/F := \{x/F \mid x \in P\}$, where P is a prime filter of L containing F. Hence, the space $V(F)$ is homeomorphic to the space $\mathrm{SpecF}(L/F)$ by the natural homeomorphism. Using Proposition 3.6 and the above argument, we have L/F is directly indecomposable if and only if $\mathrm{SpecF}(L/F)$ is a connected topological space with respect to the Stone topology if and only if $V(F)$ is connected as a subspace of $\mathrm{SpecF}(L)$ with respect to the Stone topology. □

The following corollary is a direct consequence of Theorem 3.7.

Corollary 3.8. *If F is a pseudo-irreducible (e.g., maximal or prime by Proposition 3.4) filter of L, then for each $e \in \mathrm{B}(L)$ we have either $e \in F$ or $e^* \in F$.*

The following proposition considers when the intersection of two pseudo-irreducible filters is pseudo-irreducible.

Proposition 3.9. *Let F and G be two pseudo-irreducible filters of a residuated lattice L. Then $F \vee G \neq L$ if and only if $F \cap G$ is pseudo-irreducible.*

Proof. $\Rightarrow$). Let $x \vee x^* \in F \cap G$. Then $x \vee x^* \in F$ and $x \vee x^* \in G$. Thus by Theorem 3.7, we have (either $x \in F$ or $x^* \in F$) and (either $x \in G$ or $x^* \in G$). If $x \in F$ and $x^* \in G$, then $0 = x \odot x^* \in F \vee G$. This implies $G \vee H = L$, contradicting the hypothesis that $G \vee H \neq L$. Similarly, if $x^* \in F$ and $x \in G$, then we have $F \vee G = L$, which is impossible. Hence, we have either $x \in F \cap G$ or $x^* \in F \cap G$. Therefore, $F \cap G$ is pseudo-irreducible by Theorem 3.7

$\Leftarrow$). Assume that $F \cap G$ is pseudo-irreducible. If $F \vee G = L$, then we have either $F = L$ or $G = L$. It is impossible since F and G are pseudo-irreducible filters of L and they must be proper. □

To conclude this section, we compare pseudo-irreducible filters with other types of filters in a residuated lattice. We begin with some key definitions.

Definition 3.10. *A non-empty subset F of a residuated lattice L is defined as follows:*

1. ***Boolean**: F is Boolean if it is a filter of L and for all $x \in L$, $x \vee x^* \in F$;*

2. ***Positive Implicative**: F is positive implicative if $1 \in F$, and for all $x, y, z \in L$, $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$;*

3. ***Implicative**: F is implicative if $1 \in F$, and for all $x, y, z \in L$, $x \to (y \to z) \in F$ and $x \to y \in F$ imply that $x \to z \in F$;*

4. ***Fantastic**: F is fantastic if $1 \in F$, and for all $x, y, z \in L$, $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$;*

5. ***Obstinate***: *F is obstinate if F is a proper filter of L and for all $x, y \in L$, $x, y \notin F$ imply either $x \to y \in F$ and $y \to x \in F$.*

Example 3.11. *Let $L = \{0, a, b, c, 1\}$ denote a residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\to$ via the accompanying tables:*

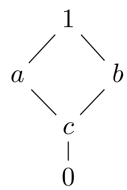

Figure 3: Hasse diagram

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	c	c	a
b	0	c	b	c	b
c	0	c	c	c	c
1	0	a	b	c	1

Table 5: Operation $\odot$

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

Table 6: Operation $\to$

The filter $F := \{b, 1\}$ is not fantastic, since $1 \in F$ and $1 \to (0 \to a) = 1 \in F$, but $((a \to 0) \to 0) \to a = a \notin F$. However, a straightforward calculation shows that F is a prime filter, and thus it is pseudo-irreducible by Proposition 3.4.

Remark 3.12. *In Example 6.10, the filter $[c)$ is not pseudo-irreducible, yet it is Boolean. Since a filter in a residuated lattice is Boolean if and only if it is positive implicative, see* [13]*, this implies that neither Boolean nor positive implicative filters are necessarily pseudo-irreducible. Additionally, as every positive implicative filter is fantastic, see* [13]*, it follows that fantastic filters are not always pseudo-irreducible.*

Conversely, the filter $\{1\}$ in Example 6.10 is pseudo-irreducible but not Boolean, since $c \vee c^* = c \vee 0 = c \notin \{1\}$. Thus, a pseudo-irreducible filter is not necessarily Boolean or positive implicative.

Example 3.13. *Let $L = \{0, a, b, 1\}$ denote a residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\to$ via the accompanying tables, see* [13, Example 3.3]*:*

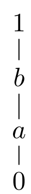

Figure 4: Hasse diagram

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Table 7: Operation $\odot$

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Table 8: Operation $\to$

The set $F := \{b, 1\}$ is a filter of L. However, since $a \to (a \to 0) \in F$, $a \to a \in F$, but $a \to 0 \notin F$, F is not implicative. It can be shown that F is a maximal filter, and hence, it is pseudo-irreducible by Proposition 3.4.

By [1, Theorem 3.13]*, a filter is obstinate if and only if it is both maximal and implicative, thus, F is not obstinate. This shows that a pseudo-irreducible filter need not be implicative or obstinate.*

Example 3.14. *Let $L = \{0, a, b, 1\}$ denote a residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\to$ via the accompanying tables:*

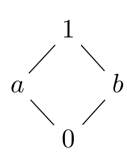

Figure 5: Hasse diagram

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Table 9: Operation $\odot$

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Table 10: Operation $\to$

The set $F := \{1\}$ is a filter of L. Since $a \vee a^* = a \vee b = 1 \in F$, but $a \notin F$ and $a^* \notin F$, by Theorem 3.7, F is not pseudo-irreducible. As L is a Boolean algebra (and hence it is a BL-algebra) and thus a Gödel algebra, [13, Theorem 3.7] *implies that F is implicative. This shows that an implicative filter is not necessarily pseudo-irreducible.*

Proposition 3.15. *Let F be a poper filter of a residuated lattice L. Then if F is obstinate, then it is pseudo-irreducible.*

Proof. Since every obstinate filter is maximal, see [1]. Hence, every obstinate filter is pseudo-irreducible by Proposition 3.4. □

The preceding examples demonstrate no general logical relationship between pseudo-irreducible filters and implicative, positive implicative, fantastic,or Boolean filters. Thus, pseudo-irreducible filters constitute a distinct class. By Propositions 3.4 and 3.15 and [1, Theorem 3.13], the implications among filter types are:

$$\text{Obstinate filter} \Rightarrow \text{Maximal filter} \Rightarrow \text{Prime filter} \Rightarrow \text{pseudo-irreducible filter.}$$

By Proposition 2.8(4), a residuated lattice is hyperarchimedean if and only if its prime and maximal filters are the same. In the following theorems, we extend this result and consider the converses of the above implications.

Theorem 3.16. *Let L be a residuated lattice. Then the following statements are equivalent:*

1. *L is hyperarchimedean;*
2. $\mathrm{MaxF}(L) = \mathrm{SpecF}(L) = \{F \in \mathrm{Filt}(L) \mid F$ *is pseudo-irreducible*$\}$.

Proof. (1) ⇒ (2). By Proposition 3.4, $\mathrm{MaxF}(L) \subseteq \mathrm{SpecF}(L) \subseteq \{F \in \mathrm{Filt}(L) \mid F \text{ is pseudo-irreducible}\}$. Now if F is a pseudo-irreducible filter of L and $x \in L \setminus F$. By assumption, there is an $n \in \mathbb{N}$ such that $x^n \in \mathrm{B}(L)$, that is, $x^n \vee (x^n)^* = 1$. Thus $x^n \vee (x^n)^* \in F$. Now by Theorem 3.7, we have either $x^n \in F$ or $(x^n)^* \in F$. If $x^n \in F$, then $x \in F$ since $x^n \leq x$, which is a contradiction. Hence, $(x^n)^* \in F$, and so F is maximal by Proposition 2.8(3). Therefore, $\mathrm{MaxF}(L) = \mathrm{SpecF}(L) = \{F \in \mathrm{Filt}(L) \mid F \text{ is pseudo-irreducible}\}$.

(2) ⇒ (1). It is clear, by Proposition 2.8(4). □

Theorem 3.17. *Let L be a G-algebra. Then maximal and obstinate filters coincide:*

Proof. By [1, Theorem 3.6], every obstinate filter is maximal. If M is a maximal filter and $x \notin M$, then by Proposition 2.8(3) there is an $n \in \mathbb{N}$ such that $(x^n)^* \in M$. Since L is a G-algebra, we have $x^n = x$ for all $n \in \mathbb{N}$. Hence, we have $x^* \in M$. Thus for each $x \in L$ we have either $x \in M$ or $x^* \in M$, and so M is an obstinate filter of L by [1, Theorem 3.10]. □

4 Boolean lifting property and pseudo-irreducibility

In this section, we consider the Boolean lifting property for a filter and its connection with pseudo-irreducibility. We begin with the following definition.

Definition 4.1. *A (not necessarily proper) filter F of a residuated lattice L has the Boolean lifting property (BLP), whenever each $\alpha \in \mathrm{B}(L/F)$ can be lifted, that is, there exists $e \in \mathrm{B}(L)$ such that $\alpha = e/F$.*

Example 4.2. *An easy argument shows that the improper filter and the trivial filter of a residuated lattice have BLP, see also* [9, Corollary 4.5].

First of all, in the following theorem, we provide some conditions which are equivalent to a filter having BLP. Later, we will present additional equivalent conditions.

Theorem 4.3. *Let F be a filter of a residuated lattice L. Then the following statements are equivalent:*

1. *F has BLP;*
2. *For each $x \in L$, if $x \vee x^* \in F$, then there exists $e \in \mathrm{B}(L)$ such that $x \leftrightarrow e \in F$ (or equivalently, $x \vee e^*, x^* \vee e \in F$). In this case, $x^* \leftrightarrow e^* \in F$;*
3. *If there are $G, H \in \mathrm{Filt}(L)$ such that $F = G \cap H$ and $G \vee H = L$, then there exists $e \in \mathrm{B}(L)$ such that $e \in G$ and $e^* \in H$;*
4. *If there are $G, H \in \mathrm{Filt}(L)$ such that $G \cap H \subseteq F$ and $G \vee H = L$, then there exists $e \in \mathrm{B}(L)$ such that $e \in F \vee G$ and $e^* \in F \vee H$.*

Proof. (1) ⇒ (2). Assume that $x \vee x^* \in F$. Thus $x/F \vee x^*/F = (x \vee x^*)/F = 1/F$, and so $x/F \in \mathrm{B}(L/F)$ by Proposition 2.2(24). By assumption, there exists $e \in \mathrm{B}(L)$ such that $x/F = e/F$. This implies that $x \leftrightarrow e \in F$, or equivalently, $x \vee e^*, x^* \vee e \in F$ (see Proposition 2.2(21)). If this case happens, since $x/F = e/F$, we have $x^*/F = e^*/F$ and so $x^* \leftrightarrow e^* \in F$.

(2) ⇒ (1). If $x/F \in \mathrm{B}(L/F)$, then $(x \vee x^*)/F = x/F \vee x^*/F = 1/F$, and so $x \vee x^* \in F$. Hence by our assumption, there exists $e \in \mathrm{B}(L)$ such that $x \leftrightarrow e \in F$. It follows that $x/F = e/F$, thus F has BLP.

(2) ⇒ (3). Assume that there are $G, H \in \mathrm{Filt}(L)$ such that $F = G \cap H$ and $G \vee H = L$. Since $G \vee H = L$, $0 \in G \vee H$. Thus by Proposition 2.7(6), there exist $x \in G$ and $y \in H$ such that $x \odot y = 0$ and $x \vee y \in G \cap H = F$. We conclude from $x \odot y = 0$ that $x \leq y^*$ and $y \leq x^*$. This gives $x \vee y \leq x \vee x^*$, and consequently $x \vee x^* \in F$. By assumption, there exists $e \in \mathrm{B}(L)$ such that $x \leftrightarrow e \in F = G \cap H$ and $x^* \leftrightarrow e^* \in F = G \cap H$. From $x \leftrightarrow e \in G$ and $x \in G$, we have $e \in G$. Also, since $y \leq x^*$, we have $x^* \in H$. Now since $x^* \leftrightarrow e^* \in H$, we conclude that $e^* \in H$. Hence, $e \in G$ and $e^* \in H$.

(3) ⇒ (2). Assume that $x \vee x^* \in F$ for some $x \in L$. Using Proposition 2.7(4-5), we have $F = F(x \vee x^*) = F(x) \cap F(x^*)$ and $F(x) \vee F(x^*) = F(x \odot x^*) = F(0) = L$. By assumption, there exists $e \in \mathrm{B}(L)$ such that $e \in F(x)$ and $e^* \in F(x^*)$. We conclude from $e \leq x \to e$ and $x \leq e \to x$ that $x \to e, e \to x \in F(x)$, or equivalently, $x \leftrightarrow e \in F(x)$. Also, from $x^* \leq x \to e$ and $e^* \leq e \to x$, we have $x \to e, e \to x \in F(x^*)$, or equivalently, $x \leftrightarrow e \in F(x^*)$. Therefore, $x \leftrightarrow e \in F(x) \cap F(x^*) = F$.

(3) ⇒ (4) Assume that there are $G, H \in \mathrm{Filt}(L)$ such that $G \cap H \subseteq F$ and $G \vee H = L$. Since the lattice $(\mathrm{Filt}(L), \subseteq)$ is distributive, $(F \vee G) \cap (F \vee H) = F \vee (G \cap H) = F$. Also, since $G \vee H \subseteq (F \vee G) \vee (F \vee H)$, we have $(F \vee G) \vee (F \vee H) = L$. Thus by assumption, there exists $e \in \mathrm{B}(L)$ such that $e \in F \vee G$ and $e^* \in F \vee H$.

(4) ⇒ (3) Assume that $G, H \in \mathrm{Filt}(L)$ such that $F = G \cap H$ and $G \vee H = L$. Hence, $G \cap H \subseteq F$ and $G \vee H = L$, and by our assumption, there exists $e \in \mathrm{B}(L)$ such that $e \in F \vee G$ and $e^* \in F \vee H$. Now since $F = G \cap H$, we have $F \subseteq G$ and $F \subseteq H$. Therefore, $e \in F \vee G = G$ and $e^* \in F \vee H = H$. □

A quick consequence of Theorems 3.7 and 4.3, is the following corollary.

Corollary 4.4. *Every pseudo-irreducible filter has BLP.*

Proof. By Theorem 3.7, the residuated lattice L/F is non-trivial and directly indecomposable for a pseudo-irreducible filter F. Hence $\mathrm{B}(L/F) = \{0/F, 1/F\}$. Thus F has BLP. □

The converse of the above corollary is not true in general, see the following two examples.

Example 4.5. *Consider a directly decomposable residuated lattice L, i.e., $L = L_1 \times L_2$ for non-trivial residuated lattices L_1 and L_2. Since $L/F \cong L$ for the trivial filter $F := \{(1_{L_1}, 1_{L_2})\}$, we have that the trivial filter $F = \{(1_{L_1}, 1_{L_2})\}$ is not pseudo-irreducible by Theorem 3.7((1) ⇔ (4)). However, Example 4.2 shows that it has BLP.*

Example 4.6. *Consider the filter F in Example 3.14, which fails to be pseudo-irreducible. Since L is a Boolean algebra and all filters in Boolean algebras have BLP, it follows that F has BLP despite not being pseudo-irreducible.*

Proposition 4.7. *Let L be a non-trivial residuated lattice. Then L is directly indecomposable if and only if every proper filter that has BLP is a pseudo-irreducible filter.*

Proof. ⇒). Let F be a proper filter of L that has BLP and $x \vee x^* \in F$ for some $x \in L$. By assumption and Theorem 4.3, there exists $e \in \mathrm{B}(L)$ such that $x \leftrightarrow e \in F$, or equivalently, $x \vee e^*, x^* \vee e \in F$. Since L is directly indecomposable, we have $\mathrm{B}(L) = \{0, 1\}$. It follows that either $e = 0$ or $e = 1$. Hence, we have either $x \in F$ or $x^* \in F$. Consequently, F is a pseudo-irreducible filter of L by Theorem 3.7.

⇐). By Example 4.2, the filter $\{1\}$ is a proper filter of L that has BLP. Hence, it is pseudo-irreducible. Now since $L \cong L/\{1\}$, we conclude that L is directly indecomposable by Theorem 3.7. □

We end this section with the following propositions, which consider some relationship between our two main concepts.

Proposition 4.8. *Let F and G be two filters of L such that $F \vee G \neq L$. If F is a pseudo-irreducible filter and G has BLP, then $F \cap G$ has BLP.*

Proof. Suppose $x \vee x^* \in F \cap G$ for some $x \in L$. Since $x \vee x^* \in G$, by Theorem 4.3 there exists $e \in \mathrm{B}(L)$ such that $x \leftrightarrow e, x^* \leftrightarrow e^* \in G$. On the other hand, since $x \vee x^* \in F$, by Theorem 3.7 we have either $x \in F$ or $x^* \in F$. Also, using Corollary 3.8 and pseudo-irreducibility of F, we have either $e \in F$ or $e^* \in F$. Now we consider the following cases:

Case 1: If $x, e \in F$, then $x \leftrightarrow e \in F$ and so $x \leftrightarrow e \in F \cap G$.

Case 2: If $x^*, e^* \in F$, then since $x^* \leq x \to e$ and $e^* \leq e \to x$, we have $x \leftrightarrow e \in F$ and so $x \leftrightarrow e \in F \cap G$.

Case 3: If $x, e^* \in F$, then $x \odot (x \to e) \in F \vee G$. From $x \odot (x \to e) \leq e$, we conclude that $e \in F \vee G$. Also, since $e^* \in F \subseteq F \vee G$, we have $0 = e \odot e^* \in F \vee G$. Thus $F \vee G = L$, which is impossible.

Case 4: If $x^*, e \in F$, then $x^* \odot (x^* \to e^*) \in F \vee G$. From $x^* \odot (x^* \to e^*) \leq e^*$, we deduce that $e^* \in F \vee G$. Also, since $e \in F \subseteq F \vee G$, we have $0 = e \odot e^* \in F \vee G$. Thus $F \vee G = L$, which is a impossible.

Therefore, $F \cap G$ has BLP by Theorem 4.3. □

Proposition 4.9. *If F and G are two proper filters of L with $F \subseteq G$. Then we have the following statements:*

1. *If F is pseudo-irreducible and G has BLP, then G is pseudo-irreducible;*
2. *G is a pseudo-irreducible filter of L if and only if G/F is a pseudo-irreducible filter of L/F;*
3. *If G has BLP in L, then G/F has BLP in L/F. The converse is true if F has BLP.*

Proof. (1). Suppose that $x \vee x^* \in G$ for some $x \in L$. By assumption and Theorem 4.3, there exists $e \in \mathrm{B}(L)$ such that $x \leftrightarrow e \in G$ and $x^* \leftrightarrow e^* \in G$. Using Corollary 3.8, we have either $e \in F$ or $e^* \in F$. If $e \in F$, then $e \in G$. Hence $e, e \to x \in G$, and we have $x \in G$. Similarly, if $e^* \in F$, then $x^* \in G$. Therefore, G is a pseudo-irreducible filter of L by Theorem 3.7.

(2). Clearly G is a proper filter of L if and only if G/F is a proper filter of L/F. By the second isomorphism theorem, we have $L/G \cong (L/F)/(G/F)$. Hence, by Theorem 3.7 we have:

$$\begin{aligned} G \text{ is a pseudo-irreducible filter of } L &\Leftrightarrow L/G \text{ is directly indecomposable} \\ &\Leftrightarrow (L/F)/(G/F) \text{ is directly indecomposable} \\ &\Leftrightarrow G/F \text{ is a pseudo-irreducible filter of } L/F. \end{aligned}$$

(3). Assume that G has BLP in L and $x/F \vee x^*/F \in G/F$ for some $x \in L$. Hence $(x \vee x^*)/F \in G/F$, and so $x \vee x^* \in G$. Now using Theorem 4.3, there exists $e \in \mathrm{B}(L)$ such that $x \leftrightarrow e \in G$. Thus $(x \leftrightarrow e)/F \in G/F$. This shows that $x/F \leftrightarrow e/F \in G/F$ and $e/F \in \mathrm{B}(L/F)$. Therefore, G/F has BLP in L/F by Theorem 4.3.

Conversely, suppose that F has BLP in L and G/F has BLP in L/F. If $x \vee x^* \in G$, then $x/F \vee x^*/F = (x \vee x^*)/F \in G/F$. By assumption and Theorem 4.3, there exists $\alpha \in \mathrm{B}(L/F)$ such that $x/F \leftrightarrow \alpha \in G/F$. Now since $\alpha \in \mathrm{B}(L/F)$ and F has BLP, there exists $e \in \mathrm{B}(L)$ such that $\alpha = e/F$. Thus we conclude that $x/F \leftrightarrow e/F \in G/F$, or equivalently, $(x \leftrightarrow e)/F \in G/F$. Hence $x \leftrightarrow e \in G$. This shows that G has BLP in L by Theorem 4.3. □

5 Boolean lifting property and the residuated lattice of fractions

First of all, in this section, we extend some properties of the residuated lattice of fractions of a residuated lattice relative to a $\wedge$-closed system from the viewpoint of filter theory. Then, we use the obtained results to provide some additional characterizations for filters that have BLP.

Remark 5.1. *In the following, to avoid any misunderstanding between the symbols used for the two concepts "the residuated lattice of fractions relative to a $\wedge$-closed system" and "the quotient of a residuated lattice with respect to a filter", for a residuated lattice L, we will use the symbol $L[S]$ (and $\frac{x}{S}$ for its equivalence class) for the residuated lattice of fractions of L relative to a $\wedge$-closed system S, and we will use the symbol L/F (and x/F for its equivalence class) for the quotient of the residuated lattice L with respect to a filter F.*

In the following proposition, we consider the filters of the residuated lattice of fractions relative to a $\wedge$-closed system.

Proposition 5.2. *Let F be a filter and S be a $\wedge$-closed system of a residuated lattice L. Then $F[S] := \{\frac{x}{S} \mid x \in F\}$ is a filter of $L[S]$. In fact, every filter of $L[S]$ is of the form $F[S]$ for some filter F of L.*

Proof. Clearly, $F[S] \neq \varnothing$. Let $\frac{x}{S}, \frac{y}{S} \in F[S]$, where $x, y \in F$. Since $x \odot y \in F$, we have $\frac{x}{S} \odot \frac{y}{S} = \frac{x \odot y}{S} \in F[S]$. Now let $x \in F$ and $y \in L$ such that $\frac{x}{S} \leq \frac{y}{S}$. It follows that $\frac{x \to y}{S} = \frac{x}{S} \to \frac{y}{S} = \frac{1}{S}$. By definition, there exists $e \in S \cap \mathrm{B}(L)$ such that $(x \to y) \wedge e = 1 \wedge e = e$. So $e \leq x \to y$, or equivalently, $e \to (x \to y) = 1$. From Proposition 2.2(6), we have $x \to (e \to y) = 1 \in F$, and so $e \to y \in F$ since $x \in F$. As $e \wedge (e \to y) = e \odot (e \to y) = e \wedge y$, we have $\frac{y}{S} = \frac{e \to y}{S} \in F[S]$. Hence, $\frac{y}{S} \in F[S]$. Therefore, $F[S]$ is a filter of $L[S]$.

Now assume that T is a filter of $L[S]$. Set $F := \{x \in L \mid \frac{x}{S} \in T\}$. It is easily seen that F is a filter of L and $T = F[S]$. □

Lemma 5.3. *Let F be a filter and S be a $\wedge$-closed system of a residuated lattice L. Then $\frac{x}{S} \in F[S]$ if and only if there exist $e \in S \cap \mathrm{B}(L)$ and $a \in F$ such that $x \wedge e = a \wedge e$, or equivalently, $x \vee e^* \in F$. In particular, $\frac{x}{S} = \frac{0}{S}$ if and only if there exists $e \in S \cap \mathrm{B}(L)$ such that $x \leq e^*$.*

Proof. By definition, $\frac{x}{S} \in F[S]$ holds if and only if there exists $a \in F$ such that $\frac{x}{S} = \frac{a}{S}$, which is equivalent to the existence of $e \in S \cap \mathrm{B}(L)$ satisfying $x \wedge e = a \wedge e$. Hence, $\frac{x}{S} \in F[S]$ if and only if there exist $e \in S \cap \mathrm{B}(L)$ and $a \in F$ such that $x \wedge e = a \wedge e$.

Suppose $x \wedge e = a \wedge e$ for some $e \in S \cap \mathrm{B}(L)$ and $a \in F$. By Theorem 2.2(19), we obtain $x \vee e^* = (x \wedge e) \vee e^* = (a \wedge e) \vee e^* = a \vee e^*$, yielding $x \vee e^* = a \vee e^*$. Since $a \leq a \vee e^*$ and F is a filter, it follows that $a \vee e^* \in F$. Consequently, $x \vee e^* \in F$. Conversely, let $x \vee e^* \in F$. Theorem 2.2(19) implies $e \wedge (x \vee e^*) = e \wedge x$. Thus, $\frac{x}{S} = \frac{x \vee e^*}{S} \in F[S]$, which completes the proof. The particular case follows immediately from the definitions. □

Proposition 5.4. *Let F be a filter and S be a $\wedge$-closed system of a residuated lattice L. Then $F[S] = L[S]$ if and only if there exists $e \in S \cap \mathrm{B}(L)$ such that $e^* \in F$.*

Proof. If $F[S] = L[S]$, then $\frac{0}{S} \in F[S]$. Hence $\frac{0}{S} = \frac{x}{S}$ for some $x \in F$, or equivalently, there is $e \in S \cap \mathrm{B}(L)$ such that $x \wedge e = 0 \wedge e = 0$. It follows that $x \odot e = x \wedge e = 0$, and so $x \leq e^*$. Now since $x \in F$, we have $e^* \in F$.

Conversely, if $e^* \in F$ for some $e \in S \cap \mathrm{B}(L)$, then $e^* \wedge e = 0 \wedge e$. Thus $\frac{0}{S} = \frac{e^*}{S} \in F[S]$. Therefore, $F[S] = L[S]$. □

Proposition 5.5. *Let F and G be two filters and S be a $\wedge$-closed system of a residuated lattice L. Then $F[S] \cap G[S] = (F \cap G)[S]$.*

Proof. Clearly, $(F \cap G)[S] \subseteq F[S] \cap G[S]$. If $\frac{x}{S} \in F[S] \cap G[S]$, then by Lemma 5.3 there exist $a \in F$, $b \in G$ and $e, f \in S \cap \mathrm{B}(L)$ such that $x \wedge e = a \wedge e$ and $x \wedge f = b \wedge f$. Hence we have

$$\begin{aligned}(a \vee b) \wedge (e \wedge f) &= (a \vee b) \odot (e \odot f)\\ &= (a \odot e \odot f) \vee (b \odot e \odot f)\\ &= (a \wedge e \wedge f) \vee (b \wedge e \wedge f)\\ &= (x \wedge e \wedge f) \vee (x \wedge e \wedge f)\\ &= (x \wedge e \wedge f)\\ &= x \wedge (e \wedge f).\end{aligned}$$

Now since $e \wedge f \in S \cap \mathrm{B}(L)$, we have $\frac{x}{S} = \frac{a \vee b}{S}$. Thus from $a \vee b \in F \cap G$ we deduce that $\frac{x}{S} \in (F \cap G)[S]$. Therefore, $F[S] \cap G[S] \subseteq (F \cap G)[S]$, and so $F[S] \cap G[S] = (F \cap G)[S]$. □

Proposition 5.6. *Let F be a proper filter and S be a $\wedge$-closed system of a residuated lattice L such that $F[S] \neq L[S]$. Then the following statements are equivalent:*

1. *$F[S]$ is a pseudo-irreducible filter of $L[S]$;*
2. *If $e^* \vee x \vee x^* \in F$ for some $e \in S \cap \mathrm{B}(L)$ and $x \in L$, then there exists $f \in S \cap \mathrm{B}(L)$ such that we have either $f^* \vee x \in F$ or $f^* \vee x^* \in F$.*

Proof. By definition and Lemma 5.3, we first establish some facts about elements of residuated lattices L and $L[S]/F[S]$.

Fact 1: For $x \in L$ we have:

$$\begin{aligned}\frac{x}{S}/F[S] \in \mathrm{B}(L[S]/F[S]) &\Leftrightarrow \frac{x \vee x^*}{S}/F[S] = \frac{1}{S}/F[S]\\ &\Leftrightarrow \frac{(x \vee x^*)}{S} \in F[S]\\ &\Leftrightarrow (x \vee x^*) \wedge e = a \wedge e \text{ for some } e \in S \cap \mathrm{B}(L) \text{ and } a \in F\\ &\Leftrightarrow e^* \vee x \vee x^* \in F \text{ for some } e \in S \cap \mathrm{B}(L).\end{aligned}$$

Fact 2: For $x \in L$ we have:

$$\begin{aligned}\frac{x}{S}/F[S] = \frac{0}{S}/F[S] &\Leftrightarrow \frac{x^*}{S} \in F[S]\\ &\Leftrightarrow x^* \wedge f = a \wedge f \text{ for some } f \in S \cap \mathrm{B}(L) \text{ and } a \in F\\ &\Leftrightarrow x^* \vee f^* \in F \text{ for some } f \in S \cap \mathrm{B}(L).\end{aligned}$$

Fact 3: For $x \in L$ we have:

$$\begin{aligned}\frac{x}{S}/F[S] = \frac{1}{S}/F[S] &\Leftrightarrow \frac{x}{S} \in F[S] \\ &\Leftrightarrow x \wedge f = a \wedge f \text{ for some } f \in S \cap \mathrm{B}(L) \text{ and } a \in F \\ &\Leftrightarrow x \vee f^* \in F \text{ for some } f \in S \cap \mathrm{B}(L).\end{aligned}$$

Now by Theorems 3.7, we know that $F[S]$ is a pseudo-irreducible filter of $L[S]$ if and only if $L[S]/F[S]$ is directly indecomposable if and only if $\mathrm{B}(L[S]/F[S]) = \{\frac{0}{S}/F[S], \frac{1}{S}/F[S]\}$. Therefore, the above facts complete the proof. □

Proposition 5.7. *Let F be a proper filter and S be a $\wedge$-closed system of a residuated lattice L. Then we have the following statements:*

1. *If F is a pseudo-irreducible filter of L and $F[S] \neq L[S]$, then $F[S]$ is a pseudo-irreducible filter of $L[S]$;*
2. *If F has BLP in L, then $F[S]$ has BLP in $L[S]$.*

Proof. (1). Assume that $\frac{x}{S} \vee \frac{x^*}{S} \in F[S]$, then $\frac{x \vee x^*}{S} \in F[S]$. By Lemma 5.3, there exist $a \in F$ and $e \in S \cap \mathrm{B}(L)$ such that $(x \vee x^*) \wedge e = a \wedge e$. Now by Corollary 3.8 either $e \in F$ or $e^* \in F$. Since $F[S] \neq L[S]$, we have $e \in F$ by Proposition 5.4, hence $a \wedge e \in F$, and so $x \vee x^* \in F$ since $(a \wedge e) \leq (x \vee x^*)$. By assumption and Theorem 3.7, we have either $x \in F$ or $x^* \in F$, and hence we have either $\frac{x}{S} \in F[S]$ or $\frac{x^*}{S} \in F[S]$. Therefore, $F[S]$ is a pseudo-irreducible filter of $L[S]$.

(2). Assume that $\frac{x}{S} \vee \frac{x^*}{S} \in F[S]$. Hence $\frac{x \vee x^*}{S} \in F[S]$. By Lemma 5.3, there exist $a \in F$ and $e \in S \cap \mathrm{B}(L)$ such that $(x \vee x^*) \wedge e = a \wedge e$. It follows that

$$\begin{aligned}a \odot e = a \wedge e \leq x \vee x^* &\Rightarrow a \leq e \to (x \vee x^*) \\ &\Rightarrow a \leq e^* \vee (x \vee x^*).\end{aligned}$$

Hence, we have

$$\begin{aligned}(x \vee e^*) \vee (x \vee e^*)^* &= (x \vee e^*) \vee (x^* \wedge e) \\ &= (x \vee e^*) \vee (x^* \odot e) \\ &\geq (x \vee e^* \vee x^*) \odot (x \vee e^* \vee e) \\ &= (x \vee e^* \vee x^*) \odot 1 \\ &= (x \vee e^* \vee x^*) \geq a.\end{aligned}$$

Consequently, $a \leq (x \vee e^*) \vee (x \vee e^*)^*$, and since $a \in F$, we have $(x \vee e^*) \vee (x \vee e^*)^* \in F$. By hypothesis and Theorem 4.3, there exists $f \in \mathrm{B}(L)$ such that $(x \vee e^*) \leftrightarrow f \in F$. Thus, $\frac{(x \vee e^*) \leftrightarrow f}{S} \in F[S]$, or equivalently, $(\frac{x}{S} \vee \frac{e^*}{S}) \leftrightarrow \frac{f}{S} \in F[S]$. From $e \in S \cap \mathrm{B}(L)$, we have $\frac{e^*}{S} = \frac{0}{S}$ by Lemma 5.3. Consequently,

$$\frac{x}{S} \leftrightarrow \frac{f}{S} = (\frac{x}{S} \vee \frac{0}{S}) \leftrightarrow \frac{f}{S} = (\frac{x}{S} \vee \frac{e^*}{S}) \leftrightarrow \frac{f}{S} \in F[S].$$

From $f \in \mathrm{B}(L)$, we have $\frac{f}{S} \in \mathrm{B}(L[S])$, and so $F[S]$ has BLP in $L[S]$ by Theorem 4.3. □

Theorem 5.8. *Let F be a proper filter and S be a $\wedge$-closed system of a residuated lattice L. Then we have a residuated lattice isomorphism*

$$L/(F \vee [S \cap \mathrm{B}(L))) \cong L[S]/F[S].$$

Proof. Define $\psi : L \to L[S]/F[S]$ by $\psi(x) := \frac{x}{S}/F[S]$. This map is a surjective morphism of residuated lattices, as it preserves operations and covers all cosets in $L[S]/F[S]$. If $x \in Ker(\psi)$, then $\frac{x}{S}/F[S] = \frac{1}{S}/F[S]$, or equivalently, $\frac{x}{S} \leftrightarrow \frac{1}{S} \in F[S]$. It follows that $\frac{x}{S} = \frac{x \leftrightarrow 1}{S} \in F[S]$. Using Lemma 5.3, there exist $e \in S \cap \mathrm{B}(L)$ and $a \in F$ such that $x \wedge e = a \wedge e$. Now we have

$$\begin{aligned}a \odot e = a \wedge e \leq x &\Rightarrow a \leq e \to x \\ &\Rightarrow e \to x \in F.\end{aligned}$$

Now since $e \odot (e \to x) \leq x$ and $e \in S \cap \mathrm{B}(L)$, we have $x \in F \vee [S \cap \mathrm{B}(L))$ by Proposition 2.7(6).

If $y \in F \vee [S \cap \mathrm{B}(L))$, then by Proposition 2.7(6) there are $a \in F$ and $e \in S \cap \mathrm{B}(L)$ such that $a \odot e \leq y$. We conclude from Proposition 2.11 that $\frac{e}{S} = \frac{1}{S}$, hence we have

$$\begin{aligned} a \odot e \leq y &\Rightarrow a \leq e \rightarrow y \\ &\Rightarrow e \rightarrow y \in F \\ &\Rightarrow \frac{e \rightarrow y}{S} \in F[S] \\ &\Rightarrow \frac{e}{S} \rightarrow \frac{y}{S} \in F[S] \\ &\Rightarrow \frac{1}{S} \rightarrow \frac{y}{S} \in F[S] \\ &\Rightarrow \frac{y}{S} \in F[S] \\ &\Rightarrow \psi(y) = \frac{y}{S}/F[S] = \frac{1}{S}/F[S] \\ &\Rightarrow y \in Ker(\psi). \end{aligned}$$

Thus $Ker(\psi) = F \vee [S \cap \mathrm{B}(L))$. Therefore, we have $L/(F \vee [S \cap \mathrm{B}(L))) \cong L[S]/F[S]$ by the first isomorphism theorem. □

Lemma 5.9. *If $x \in L$ and $e \in \mathrm{B}(L)$ such that $e \leq x \vee x^*$, then $e \wedge x, e \wedge x^*, e^* \vee x^*, e^* \vee x^{**} \in \mathrm{B}(L)$.*

Proof. We shall only prove that $e \wedge x \in \mathrm{B}(L)$. The other cases can similarly be proved. Since $e \leq x \vee x^*$, we have

$$\begin{aligned} e &= e \wedge (x \vee x^*) = (e \wedge x) \vee (e \wedge x^*) \\ &= (e \wedge e \wedge x) \vee (e \wedge x^*) = e \wedge ((e \wedge x) \vee x^*). \end{aligned}$$

Hence $e \leq ((e \wedge x) \vee x^*)$. It follows that

$$(e \wedge x) \vee (e \wedge x)^* = (e \wedge x) \vee (e^* \vee x^*) = ((e \wedge x) \vee x^*) \vee e^* \geq e \vee e^* = 1.$$

Therefore, $(e \wedge x) \vee (e \wedge x)^* = 1$, and so $e \wedge x \in \mathrm{B}(L)$ by Proposition 2.2(24). □

Theorem 5.10. *Let F be a proper filter of a residuated lattice L such that for each $e \in \mathrm{B}(L)$ we have either $e \in F$ or $e^* \in F$. Set $S_F := F \cap \mathrm{B}(L)$. Then S_F is a $\wedge$-closed system of L and the residuated lattice $L[S_F]$ is directly indecomposable.*

Proof. Since F is a filter, S_F is clearly a $\wedge$-closed system of L. If $\frac{x}{S_F} \in \mathrm{B}(L[S_F])$, then by Proposition 2.11, there exists $e \in S_F$ such that $e \leq x \vee x^*$. By Proposition 2.11 and Lemma 5.9, we have $\frac{e}{S_F} = \frac{1}{S_F}$ and $x \wedge e \in \mathrm{B}(L)$. It follows that

$$\frac{x}{S_F} = \frac{x}{S_F} \wedge \frac{1}{S_F} = \frac{x}{S_F} \wedge \frac{e}{S_F} = \frac{x \wedge e}{S_F}.$$

By assumption, we have either $x \wedge e \in S_F$ or $(x \wedge e)^* \in S_F$. If $x \wedge e \in S_F$, then $\frac{x}{S_F} = \frac{e \wedge x}{S_F} = \frac{1}{S_F}$, and if $(x \wedge e)^* \in S_F$, then $\frac{x}{S_F} = \frac{e \wedge x}{S_F} = \frac{0}{S_F}$ by Proposition 2.11 and Lemma 5.3. Therefore, $\mathrm{B}(L[S_F]) = \{\frac{0}{S_F}, \frac{1}{S_F}\}$, or equivalently, the residuated lattice $L[S_F]$ is directly indecomposable. □

For a filter F of a residuated lattice L, set $F' := [F \cap \mathrm{B}(L)) = [\{e \mid e \in F \cap \mathrm{B}(L)\})$. If $F = L$, then $0 \in F \cap \mathrm{B}(L)$ and so $F' = L$. Thus $F' = L$ if and only if $F = L$.

Lemma 5.11. *Let $F, G \in \mathrm{Filt}(L)$ for a residuated lattice L. Then $F' \vee G' = L$ if and only if there exists $e \in \mathrm{B}(L)$ such that $e \in F$ and $e^* \in G$.*

Proof. Assume that $F' \vee G' = L$. Thus $0 \in F' \vee G' = L$, and so by Proposition 2.7(6) there are $a \in F'$ and $b \in G'$ such that $a \odot b = 0$. Since $a \in F'$ and $b \in G'$, there are $e \in F \cap \mathrm{B}(L)$ and $f \in G \cap \mathrm{B}(L)$ such that $e \leq a$ and $f \leq b$ by Proposition 2.7(1-2). Hence $e \odot f \leq a \odot b = 0$, and so $e \odot f = 0$. Thus $f \leq e^*$, and since $f \in G$, we have $e^* \in G$. Therefore, $e \in F$ and $e^* \in G$. The converse is clear. □

The following theorem provides some additional conditions that are equivalent to a filter having BLP.

Theorem 5.12. *Let F be a proper filter of a residuated lattice L. Then the following statements are equivalent:*

1. *F has BLP;*
2. *For each maximal (or prime) filter M of L, the filter $F[S_M]$ has BLP in $L[S_M]$;*
3. *For each maximal (or prime) filter M of L, if $F[S_M] \neq L[S_M]$, then $F[S_M]$ is a pseudo-irreducible filter of $L[S_M]$;*
4. *For each maximal (or prime) filter M of L, if $F \cap (L \setminus S_M) = \varnothing$, then $F \vee [S_M)$ is a pseudo-irreducible filter of L;*
5. *For each maximal (or prime) filter M of L, if $F \vee [S_M) \neq L$ and $e^* \vee x \vee x^* \in F$ for some $e \in S_M$ and $x \in L$, then there exists $f \in S_M$ such that we have either $f^* \vee x \in F$ or $f^* \vee x^* \in F$.*

Proof. (1) $\Rightarrow$ (2). It follows from Proposition 5.7.
(2) $\Rightarrow$ (3). Let $F[S_M]$ be a proper filter of the residuated lattice $L[S_M]$. By assumption, $F[S_M]$ has BLP. Now by Theorem 5.10, the residuated lattice $L[S_M]$ is directly indecomposable, and hence $F[S_M]$ is a pseudo-irreducible filter of $L[S_M]$ by Proposition 4.7.

(3) $\Rightarrow$ (1). Let $x \vee x^* \in F$. By Proposition 2.7(7), $[x) \to F, [x^*) \to F \in \mathrm{Filt}(L)$. Now set $T := ([x) \to F)' \vee ([x^*) \to F)'$. Let $M \in \mathrm{MaxF}(L)$ be such that $T \subseteq M$. If $F[S_M] = L[S_M]$, then by Proposition 5.4 there exists $e \in S_M \subseteq M$ such that $e^* \in F$, and so $e^* \in F'$. By definition of $[x) \to F$, we have $F \subseteq [x) \to F$. Hence, $F' \subseteq ([x) \to F)' \subseteq T$, and so $F' \subseteq T$. It follows that $e^* \in T \subseteq M$, which is a contradiction. Hence, $F[S_M] \neq L[S_M]$, and so $F[S_M]$ is a pseudo-irreducible filter of $L[S_M]$ by our assumption. From $x \vee x^* \in F$, we have $\frac{x}{S_M} \vee \frac{x^*}{S_M} = \frac{x \vee x^*}{S_M} \in F[S_M]$. Using Theorem 3.7, we have either $\frac{x}{S_M} \in F[S_M]$ or $\frac{x^*}{S_M} \in F[S_M]$. If $\frac{x}{S_M} \in F[S_M]$, then by Lemma 5.3 there exists $a \in F$ and $e \in S_M$ such that $x \wedge e = a \wedge e$. Thus we have

$$\begin{aligned} a \wedge e \leq x \Rightarrow & a \odot e \leq x \\ \Rightarrow & a \leq e \to x \\ \Rightarrow & a \leq e^* \vee x. \end{aligned}$$

Now since $a \in F$, we have $e^* \vee x \in F$. Hence by Proposition 2.7(5), we have

$$\begin{aligned} e^* \vee x \in F &\Rightarrow [e^* \vee x) \subseteq F \Rightarrow [x) \cap [e^*) \subseteq F \\ &\Rightarrow e^* \in [x) \to F \\ &\Rightarrow e^* \in ([x) \to F)'. \end{aligned}$$

But from $([x) \to F)' \subseteq T \subseteq M$, we have $e^* \in M$, which is a contradiction. Similarly, if $\frac{x^*}{S_M} \in F[S_M]$, we have a contradiction. Therefore, for each maximal filter M of L, we have $T \nsubseteq M$. Hence $T = L$. Thus by Lemma 5.11, there exists $e \in \mathrm{B}(L)$ such that $e \in ([x) \to F)'$ and $e^* \in ([x^*) \to F)'$. Consequently, $e \in ([x) \to F)$ and $e^* \in ([x^*) \to F)$, that is, $[x \vee e) = [x) \cap [e) \subseteq F$ and $[x^* \vee e^*) = [x^*) \cap [e^*) \subseteq F$ by Proposition 2.7(5). Hence, $x \vee e, x^* \vee e^* \in F$, or equivalently, $x \leftrightarrow e^* \in F$. Therefore, F has BLP by Theorem 4.3.

(3) $\Leftrightarrow$ (4). First of all, note that for each $e \in \mathrm{B}(L)$ and $M \in \mathrm{SpecF}(L)$ we have either ($e \in M$ and $e^* \notin M$) or ($e^* \in M$ and $e \notin M$). Thus, by Proposition 5.4 $F[S_M] \neq L[S_M]$ if and only if $F \cap (L \setminus S_M) = \varnothing$. Now since $S_M \subseteq \mathrm{B}(L)$, we have $L[S_M]/F[S_M] \cong L/(F \vee [S_M))$ by Theorem 5.8. Using Theorem 3.7, we have:

$$\begin{aligned} F[S_M] \text{ is pseudo-irreducible in } L[S_M] &\Leftrightarrow L[S_M]/F[S_M] \text{ is non-trivial and directly indecomposable} \\ &\Leftrightarrow L/(F \vee [S_M)) \text{ is non-trivial and directly indecomposable} \\ &\Leftrightarrow F \vee [S_M) \text{ is pseudo-irreducible filter in } L. \end{aligned}$$

(3) $\Leftrightarrow$ (5). It follows from Proposition 5.6 and the fact that $F[S_M] \neq L[S_M]$ if and only if $F \vee [S_M) \neq L$. □

Proposition 5.13. *Let F be a filter of a residuated lattice L and $X \subseteq \mathrm{B}(L)$. Then if F has BLP, then $F \vee [X)$ has BLP.*

Proof. Let $M \in \mathrm{MaxF}(L)$. If $X \nsubseteq M$, then there exists $e \in X \setminus M$. So $e \in L \setminus S_M$, and thus $e \in (F \vee [X)) \cap (L \setminus S_M)$. Hence, $(F \vee [X)) \cap (L \setminus S_M) \neq \varnothing$.

Now assume that $X \subseteq M$. Then $X \subseteq S_M$. Hence $(F \vee [X)) \vee [S_M) = F \vee [S_M)$. Now since F has BLP, $(F \vee [X)) \vee [S_M) = F \vee [S_M)$ is a pseudo-irreducible filter of L when $(F \vee [X)) \cap (L \setminus S_M) = \varnothing$ by Theorem 5.12. Consequently, $F \vee [X)$ has BLP by Theorem 5.12. □

The following result is a useful consequence of Proposition 5.13 by setting $F = \{1\}$.

Corollary 5.14. *Let $X \subseteq \mathrm{B}(L)$. Then the filter $[X)$ has BLP.*

6 Boolean lifting property and the radical of filters

In this section, we consider the relation between a proper filter and its radical from the viewpoint of Boolean lifting property. We start with the following lemmas.

Lemma 6.1. *Let L be a residuated lattice. Then for $F, G, H \in \mathrm{Filt}(L)$ if $F = G \cap H$ and $G \vee H = L$, then* $\mathrm{Rad}(F) = \mathrm{Rad}(G) \cap \mathrm{Rad}(H)$ *and* $\mathrm{Rad}(G) \vee \mathrm{Rad}(H) = L$.

Proof. Since $F = G \cap H$ and $G \vee H = L$, we have $V(F) = V(G) \cup V(H)$, $V(G) \cap V(H) = \varnothing$, $V_{Max}(F) = V_{Max}(G) \cup V_{Max}(H)$, and $V_{Max}(G) \cap V_{Max}(H) = \varnothing$ by Proposition 2.9. Therefore, $\mathrm{Rad}(F) = \bigcap V_{Max}(F) = (\bigcap V_{Max}(G)) \cap (\bigcap V_{Max}(H)) = \mathrm{Rad}(G) \cap \mathrm{Rad}(H)$. Now since $G \subseteq \mathrm{Rad}(G)$ and $H \subseteq \mathrm{Rad}(H)$, we have $\mathrm{Rad}(G) \vee \mathrm{Rad}(H) = L$. □

Lemma 6.2. *For a proper filter F of a residuated lattice L and $e \in \mathrm{B}(L)$, if $e \in \mathrm{Rad}(F)$, then $e \in F$.*

Proof. Let $P \in V(F)$. Thus there is $M \in V_{Max}(F)$ with $P \subseteq M$. From $e \in \mathrm{Rad}(F)$, we have $e \in M$. Now since $e \vee e^* = 1 \in P$, we have either $e \in P$ or $e^* \in P$. If $e^* \in P$, then $e^* \in M$, which is a contradiction. Thus $e \in P$ for any $P \in V(F)$. Therefore, $e \in \bigcap V(F)$. By Proposition 2.8(2), $F = \bigcap V(F)$. Therefore, $e \in F$. □

Theorem 6.3. *For a filter F of a residuated lattice L, if* $\mathrm{Rad}(F)$ *has BLP, then F has BLP.*

Proof. Let $F = G \cap H$ and $G \vee H = L$ for some $G, H \in \mathrm{Filt}(L)$. By Lemma 6.1, $\mathrm{Rad}(F) = \mathrm{Rad}(G) \cap \mathrm{Rad}(H)$ and $\mathrm{Rad}(G) \vee \mathrm{Rad}(H) = L$. By Theorem 4.3 since $\mathrm{Rad}(F)$ has BLP, there exists $e \in \mathrm{B}(L)$ such that $e \in \mathrm{Rad}(G)$ and $e^* \in \mathrm{Rad}(H)$. By lemma 6.2 $e \in G$ and $e^* \in H$. Consequently, F has BLP by Theorem 4.3. □

In [11, Open question 3.4], the authors posed the question "Can sufficient, or even necessary and sufficient conditions be provided for a residuated lattice L to be such that $\mathrm{Rad}(L)$ has BLP?". In the following, we answer this question.

Definition 6.4. *We say that a residuated lattice L has the transitional property of radicals decomposition (TPRD), whenever for $F, G, H \in \mathrm{Filt}(L)$ if $\mathrm{Rad}(F) = G \cap H$ and $G \vee H = L$, then there exist $G_0, H_0 \in \mathrm{Filt}(L)$ such that $G_0 \subseteq G$, $H_0 \subseteq H$, $\mathrm{Rad}(G_0) = G$, $\mathrm{Rad}(H_0) = H$, $F = G_0 \cap H_0$, and $G_0 \vee H_0 = L$.*

Theorem 6.5. *Let L be a residuated lattice that has TPRD and $F \in \mathrm{Filt}(L)$. Then F has BLP if and only if* $\mathrm{Rad}(F)$ *has BLP. In particular,* $\mathrm{Rad}(L)$ *has BLP.*

Proof. Assume that F has BLP and $\mathrm{Rad}(F) = G \cap H$ and $G \vee H = L$ for some $G, H \in \mathrm{Filt}(L)$. By assumption, there exist $G_0, H_0 \in \mathrm{Filt}(L)$ such that $G_0 \subseteq G$, $H_0 \subseteq H$, $\mathrm{Rad}(G_0) = G$, $\mathrm{Rad}(H_0) = H$, $F = G_0 \cap H_0$, and $G_0 \vee H_0 = L$. Now since F has BLP, by Theorem 4.3 there exists $e \in \mathrm{B}(L)$ such that $e \in G_0$ and $e^* \in H_0$. Therefore, $e \in G$ and $e^* \in H$, and so $\mathrm{Rad}(F)$ has BLP by Theorem 4.3. The converse follows from Theorem 6.3.

By Example 4.2, the trivial filter $\{1\}$ always has BLP, hence $\mathrm{Rad}(L) = \mathrm{Rad}(\{1\})$ has BLP if L has TPRD. □

Definition 6.6. *A residuated lattice L is called weak MTL-algebra, whenever for each $x \in L$ we have $(x^* \to x^{**}) \vee (x^{**} \to x^*) = 1$.*

Example 6.7. *Let $L = \{0, a, b, c, d, 1\}$ denote a residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\to$ via the accompanying tables, see* [17, Example 1, Page. 240]*:*

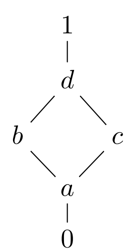

Figure 6: Hasse diagram

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	b	0	b	b
c	0	0	0	c	c	c
d	0	0	b	c	d	d
1	0	a	b	c	d	1

Table 11: Operation $\odot$

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	c	c	1	c	1	1
c	b	b	b	1	1	1
d	a	a	b	c	1	1
1	0	a	b	c	d	1

Table 12: Operation $\to$

Since $(b^* \to b^{**}) \vee (b^{**} \to b^*) = (c \to b) \vee (b \to c) = b \vee c = d \neq 1$, L is not a weak MTL-algebra. Also, note that an easy computation shows that L is involutive. Hence, an involutive residuated lattice need not be a weak MTL-algebra.

Proposition 6.8. *Every semi-G-algebra that is a De Morgan residuated lattice (or equivalently, Stonean residuated lattice, see* [3, Proposition 8 and Corollary 1]*), is a weak MTL-algebra.*

Proof. Let $x \in L$. By Proposition 2.4, we have $x \wedge x^* = 0$. Now by Proposition 2.2(8) gives

$$(x^* \to x^{**}) \vee (x^{**} \to x^*) \geq x^{**} \vee x^* = (x^* \wedge x)^* = 0^* = 1.$$

Therefore, $(x^* \to x^{**}) \vee (x^{**} \to x^*) = 1$ and so L is a weak MTL-algebra. □

Other examples of weak MTL-algebras are Boolean algebras, MV-algebras, BL-algebras, MTL-algebras. The class of weak MTL-algebras strictly contains the class of MTL-algebras and Stonean residuated lattices.

In the following examples and Example 6.14, we show that all assumptions of Proposition 6.8 are necessary. Also, the converse of Proposition 6.8 need not be true.

Example 6.9. *Let $L = \{0, n, a, b, , c, d, 1\}$ denote a residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\to$ via the accompanying tables, see* [14, Example 3.2]*:*

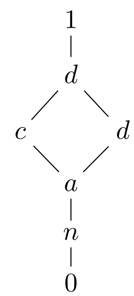

Figure 7: Hasse diagram

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	0	0	n	0	n	n
a	0	0	a	a	a	a	a
b	0	n	a	b	a	b	b
c	0	0	a	a	c	c	c
d	0	n	a	b	c	d	d
1	0	n	a	b	c	d	1

Table 13: Operation $\odot$

$\to$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	c	1	1	1	1	1	1
a	n	n	1	1	1	1	1
b	0	n	c	1	c	1	1
c	n	n	b	b	1	1	1
d	0	n	a	b	c	1	1
1	0	n	a	b	c	d	1

Table 14: Operation $\to$

An easy computation shows that $(x^* \to x^{**}) \vee (x^{**} \to x^*) = 1$ for each $x \in L$, that is, L is a weak MTL-algebra. But since $(b \to c) \vee (c \to b) = c \vee b = d \neq 1$, L is not an MTL-algebra. Also, since $(n^2)^* = 0^* = 1 \neq c = n^*$, L is neither semi-G-algebra nor Stonean residuated lattice, see [3, Proposition 8 and Corollary 1] *for more details. Also, since $a^{**} = c \neq a$, L is not an involutive residuated lattice.*

Example 6.10. *Let $L = \{0, n, a, b, c, d, e, f, m, 1\}$ denote a residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\to$ via the accompanying tables, see* [16, Example 4.2]*:*

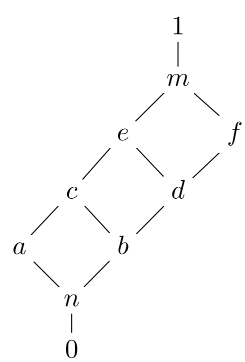

Figure 8: Hasse diagram

$\odot$	0	n	a	b	c	d	e	f	m	1
0	0	0	0	0	0	0	0	0	0	0
n	0	0	0	0	0	0	0	0	0	n
a	0	0	a	0	a	0	a	0	a	a
b	0	0	0	0	0	0	0	b	b	b
c	0	0	a	0	a	0	a	b	c	c
d	0	0	0	0	0	b	b	d	d	d
e	0	0	a	0	a	b	c	d	e	e
f	0	0	0	b	b	d	d	f	f	f
m	0	0	a	b	c	d	e	f	m	m
1	0	n	a	b	c	d	e	f	m	1

Table 15: Operation $\odot$

$\to$	0	n	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1	1	1
a	f	f	1	f	1	f	1	f	1	1
b	e	e	e	1	1	1	1	1	1	1
c	d	d	e	f	1	f	1	f	1	1
d	c	c	c	e	e	1	1	1	1	1
e	b	b	c	d	e	f	1	f	1	1
f	a	a	a	c	c	e	e	1	1	1
m	n	n	a	b	c	d	e	f	1	1
1	0	n	a	b	c	d	e	f	m	1

Table 16: Operation $\to$

By [16, Example 4.2] *and an easy argument we deduce that L is a De Morgan residuated lattice, but L is not a weak MTL-algebra since $(a^* \to a^{**}) \vee (a^{**} \to a^*) = (f \to a) \vee (a \to f) = a \vee f = m \neq 1$. Also, L is not a semi-G-algebra since $(b^2)^* = 0^* = 1 \neq e = b^*$.*

Proposition 6.11. *Let $F_1, ..., F_n$ be n proper filters of a residuated lattice L such that $F_i \vee F_j = L$ for each $i \neq j$. If $G_1, ..., G_n$ are n filters of L with $F_i \subseteq G_i$ for each $1 \leq i \leq n$ and $\cap_{i=1}^n F_i = \cap_{i=1}^n G_i$, then $F_i = G_i$ for all for $1 \leq i \leq n$.*

Proof. Since $(\text{Filt}(L), \subseteq)$ is a distributive lattice, we have $F_1 \vee (\cap_{i=2}^n F_i) = \cap_{i=2}^n (F_1 \vee F_i) = \cap_{i=2}^n L = L$. Then by Proposition 2.7(6), there exist $a \in F_1$ and $b \in \cap_{i=2}^n F_i$ such that $a \odot b = 0$. Let $x \in G_1$ be arbitrary. Then $x \vee b \in G_1 \cap (\cap_{i=2}^n F_i) \subseteq \cap_{i=1}^n G_i = \cap_{i=1}^n F_i \subseteq F_1$. Also, $x \vee a \in F_1$. Using Proposition 2.2(11), we have $x = x \vee 0 = x \vee (a \odot b) \geq (x \vee a) \odot (x \wedge b) \in F_1$. Hence $x \in F_1$. Consequently, $F_1 = G_1$. Similarly, we can prove $F_i = G_i$ for $i = 2, ..., n$. □

Corollary 6.12. *Let $F, G, H \in \mathrm{Filt}(L)$. If $\mathrm{Rad}(F) = G \cap H$ and $G \vee H = L$, then $G = \mathrm{Rad}(G)$ and $H = \mathrm{Rad}(H)$, or equivalently, $G = \bigcap V_{Max}(G)$ and $H = \bigcap V_{Max}(H)$.*

Proof. From Lemma 6.1 and the fact that $\mathrm{Rad}(\mathrm{Rad}(F)) = \mathrm{Rad}(F)$, we have $G \cap H = \mathrm{Rad}(G) \cap \mathrm{Rad}(H)$. Since $G \vee H = L$, by Proposition 6.11 and the fact that $G \subseteq \mathrm{Rad}(G)$ and $H \subseteq \mathrm{Rad}(H)$, we have $G = \mathrm{Rad}(G)$ and $H = \mathrm{Rad}(H)$, or equivalently, $G = \bigcap V_{Max}(G)$ and $H = \bigcap V_{Max}(H)$. □

In [9, Corollary 6.16], it was shown that $\mathrm{Rad}(L)$ has BLP for MV-algebras and BL-algebras; in the following proposition, we extend this result.

Theorem 6.13. *Every weak MTL-algebra (e.g., Boolean algebra, MV-algebra, BL-algebra, MTL-algebra, Stonean residuated lattice) L has TPRD. In particular, for every weak MTL-algebra (e.g., Boolean algebra, MV-algebra, BL-algebra, MTL-algebra, Stonean residuated lattice) L, $\mathrm{Rad}(L)$ has BLP.*

Proof. Let $F, G, H \in \mathrm{Filt}(L)$ such that $\mathrm{Rad}(F) = G \cap H$ and $G \vee H = L$. Since $G \vee H = L$, there exist $x \in G$ and $y \in H$ such that $x \odot y = 0$ by Proposition 2.7(6). Thus $y \leq x^*$, and so $x^* \in H$. Also, since $x \leq x^{**}$, we have $x^{**} \in G$.

Let $M \in V_{Max}(F)$. From $M \in V_{Max}(F)$ we conclude that $G \cap H = \mathrm{Rad}(F) \subseteq M$. Hence, we have either $G \subseteq M$ or $H \subseteq M$ from the fact that every maximal filter is prime. By assumption we have $(x^* \to x^{**}) \vee (x^{**} \to x^*) = 1$, and so $(x^* \to x^{**}) \vee (x^{**} \to x^*) \in M$. It follows that either $x^* \to x^{**} \in M$ or $x^{**} \to x^* \in M$.

Now if $G \subseteq M$ and $x^{**} \to x^* \in M$, then from $x^{**} \in G$ we have $x^*, x^{**} \in M$, which is impossible. Also, if $H \subseteq M$ and $x^* \to x^{**} \in M$, then from $x^* \in H$ we have $x^*, x^{**} \in M$, which is impossible. Therefore, $G \subseteq M$ if and only if $x^* \to x^{**} \in M$, and $H \subseteq M$ if and only if $x^{**} \to x^* \in M$, that is, $V_{Max}(G) = V_{Max}(x^* \to x^{**}) \cap V_{Max}(F)$ and $V_{Max}(H) = V_{Max}(x^{**} \to x^*) \cap V_{Max}(F)$.

Set $A := \{P \in V(F) \mid x^* \to x^{**} \in P\}$ and $B := \{P \in V(F) \mid x^{**} \to x^* \in P\}$. Let $P \in V(F)$. From $(x^* \to x^{**}) \vee (x^{**} \to x^*) = 1 \in P$, we have either $x^* \to x^{**} \in P$ or $x^{**} \to x^* \in P$. Consequently, we have $V(F) = A \cup B$. If $P \in A \cap B$, then $x^* \to x^{**}, x^{**} \to x^* \in P$. Since P is a proper filter of L containing F, there exists $M \in V_{Max}(F)$ such that $P \subseteq M$, and so $x^* \to x^{**}, x^{**} \to x^* \in M$, that is, $M \in V_{Max}(x^* \to x^{**}) \cap V_{Max}(x^{**} \to x^*) \cap V_{Max}(F) = V_{Max}(G) \cap V_{Max}(H) = \varnothing$, which is impossible. Therefore, $A \cap B = \varnothing$.

Set $G_0 := \bigcap A$ and $H_0 := \bigcap B$. Clearly, $G_0 \subseteq G$, $H_0 \subseteq H$. By the above argument, $V_{Max}(G) = V_{Max}(x^* \to x^{**}) \cap V_{Max}(F) = A \cap \mathrm{MaxF}(L)$ and $V_{Max}(H) = V_{Max}(x^{**} \to x^*) \cap V_{Max}(F) = B \cap \mathrm{MaxF}(L)$. Thus, by Corollary 6.12, $\mathrm{Rad}(G_0) = \bigcap(A \cap \mathrm{MaxF}(L)) = \bigcap V_{Max}(G) = G$ and $\mathrm{Rad}(H_0) = \bigcap(B \cap \mathrm{MaxF}(L)) = \bigcap V_{Max}(H) = H$. Moreover, since $V(F) = A \cup B$, we have $F = \bigcap V(F)$ by Proposition 2.8(2). Thus $F = \bigcap V(F) = (\bigcap A) \cap (\bigcap B) = G_0 \cap H_0$. Also, if $G_0 \vee H_0 \neq L$, then there exists a maximal filter M of L such that $G_0, H_0 \subseteq M$. Hence, $M \in A \cap B$. This contradics the fact that $A \cap B = \varnothing$. Thus, we have $G_0 \vee H_0 = L$. Therefore, $F = G_0 \cap H_0$ and $G_0 \vee H_0 = L$, and so L has TPRD. The rest of the theorem follows from Theorem 6.5. □

Example 6.14. *Let $L = \{0, a, b, c, d, 1\}$ denote a residuated lattice with Hasse diagram depicted in the following figure, and define the operations $\odot$ and $\to$ via the accompanying tables, see [9, Example 4.16]:*

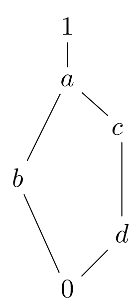

Figure 9: Hasse diagram

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	d	d	a
b	0	b	b	0	0	b
c	0	d	0	d	d	c
d	0	d	0	d	d	d
1	0	a	b	c	d	1

Table 17: Operation $\odot$

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	c	c	1
b	c	1	1	c	c	1
c	b	1	b	1	a	1
d	b	1	b	1	1	1
1	0	a	b	c	d	1

Table 18: Operation $\to$

By [9, Example 4.16], *$\mathrm{Rad}(L)$ has BLP (in fact, every filter of L has BLP), but L is neither a chain, nor local, nor a Boolean algebra, nor a G-algebra. Also, since $(b^* \to b^{**}) \vee (b^{**} \to b^*) = (c \to b) \vee (b \to c) = b \vee c = a \neq 1$ and $b^* \vee b^{**} = c \vee c^* = c \vee b = a \neq 1$, L is neither weak MTL-algebra, nor Stonean algebra. An easy computation shows that L is a semi-G-algebra.*

The preceding examples demonstrates no general logical relationships between weak MTL-algebras and related structures. Consequently, weak MTL-algebras constitute a distinct algebraic class. Proposition 6.8 and Theorem 6.13 establish the inclusion hierarchy, visually encoded through nested rectangular regions:

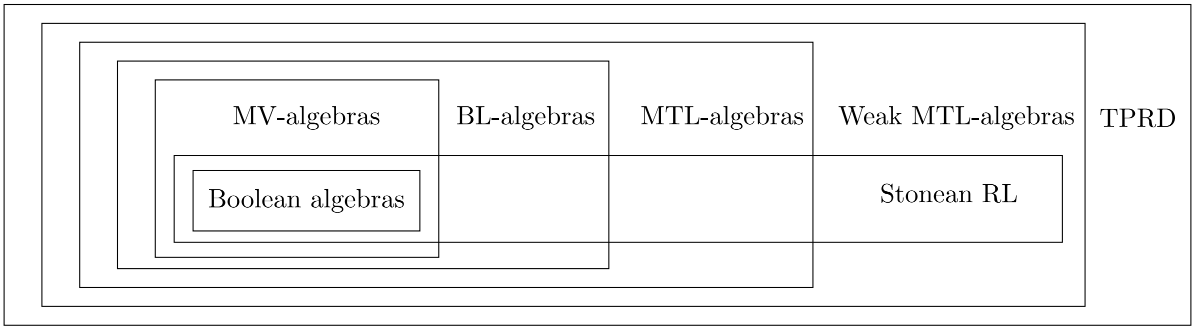

Figure 10: Inclusion hierarchy among some algebraic structures

We end this section with a topological characterization for residuated lattices whose radical has BLP. This characterization is a topological answer to [11, Open question 3.4].

Theorem 6.15. *Let L be a residuated lattice. Then* $\mathrm{Rad}(L)$ *has BLP if and only if*

$$\mathrm{Clop}(\mathrm{MaxF}(L)) = \{V_{Max}(e) \mid e \in \mathrm{B}(L)\},$$

where $\mathrm{Clop}(\mathrm{MaxF}(L))$ *is the set of all closed and open subsets of* $\mathrm{MaxF}(L)$ *with respect to the Stone topology.*

Proof. $\Rightarrow$). Clearly, $\{V_{Max}(e) \mid e \in \mathrm{B}(L)\} \subseteq \mathrm{Clop}(\mathrm{MaxF}(L))$. Let T be a clopen subset of $\mathrm{MaxF}(L)$. Then there exist $A, B \in \mathrm{Filt}(L)$ such that $T = V_{Max}(A)$ and $T^c := \mathrm{MaxF}(L) \setminus T = V_{Max}(B)$. Set $F := \bigcap T$ and $G := \bigcap T^c$. Hence, $\mathrm{Rad}(L) = F \cap G$. If $M \in \mathrm{MaxF}(L)$ such that $F, G \subseteq M$, then $A \subseteq F \subseteq M$ and $B \subseteq G \subseteq M$. Consequently, $M \in T \cap T^c = \varnothing$, which is impossible. Also, if $F \vee G \neq L$, then there exists a maximal filter M of L such that $F, G \subseteq M$. Hence, $M \in T \cap T^c$. This contradicts the fact that $T \cap T^c = \varnothing$. Therefore, $F \vee G = L$. Now by assumption and Theorem 4.3, there exists $e \in \mathrm{B}(L)$ such that $e \in F$ and $e^* \in G$. It follows that $T \subseteq V_{Max}(e)$ and $T^c \subseteq V_{Max}(e^*)$. From $\mathrm{MaxF}(L) = T \cup T^c$, $\mathrm{MaxF}(L) = V_{Max}(e) \cup V_{Max}(e^*)$, $T \cap T^c = \varnothing$, and $V_{Max}(e) \cap V_{Max}(e^*) = \varnothing$, we can assert that $T = V_{Max}(e)$ and $T^c = V_{Max}(e^*)$.

$\Leftarrow$). Let $x \vee x^* \in \mathrm{Rad}(L)$ for some $x \in L$. Hence for each $M \in \mathrm{MaxF}(L)$, we have $x \vee x^* \in M$. Thus for each $M \in \mathrm{MaxF}(L)$, we have either $x \in M$ or $x^* \in M$, equivalently, $\mathrm{MaxF}(L) = V_{Max}(x) \cup V_{Max}(x^*)$. Now since $x \odot x^* = 0$, we conclude that $V_{Max}(x) \cap V_{Max}(x^*) = \varnothing$. It follows that $V_{Max}(x)$ is a clopen subset of $\mathrm{MaxF}(L)$. Thus by our assumption, there exists $e \in \mathrm{B}(L)$ such that $V_{Max}(x) = V_{Max}(e)$ and $V_{Max}(x^*) = V_{Max}(e^*)$. Now let $M \in \mathrm{MaxF}(L)$. If $M \in V_{Max}(x) = V_{Max}(e)$, then $x, e \in M$, and so $x \leftrightarrow e = (x^* \vee e) \wedge (x \vee e^*) \in M$. If $M \in V_{Max}(x^*) = V_{Max}(e^*)$, then $x^*, e^* \in M$, and so $x \leftrightarrow e = (x^* \vee e) \wedge (x \vee e^*) \in M$. Therefore, for each $M \in \mathrm{MaxF}(L)$ we have $x \leftrightarrow e \in M$, or equivalently, $x \leftrightarrow e \in \mathrm{Rad}(L)$. It follows that $\mathrm{Rad}(L)$ has BLP by Theorem 4.3. □

7 Conclusions

This study advances the theory of residuated lattices by introducing pseudo-irreducible filters and establishing their essential connection to the Boolean lifting property (BLP). Then, the BLP of the radical of a filter is considered. Furthermore, by introducing weak MTL-algebras and the notion of TPRD (Transitional Property of Radicals Decomposition), we answer an open question in the literature concerning the BLP of the radical of a residuated lattice.

For future research, we propose: 1. A deeper investigation into the properties of weak MTL-algebras and the TPRD. 2. Extending the topological approach to other lifting properties. 3. Exploring sheaves of residuated lattices whose stalks possess the BLP, with a focus on utilizing pseudo-irreducible filters, particularly in the context of Pierce sheaf representations of residuated lattices. 4. Investigating the lifting problem using other topologies such as flat and constructible topology. 5. Defining new topologies using pseudo-irreducible filters and investigating the lifting problem within these topologies.

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References

[1] A. Borumand Saeid, M. Pourkhatoun, *Obstinate filters in residuated lattices*, Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie, **55**(103) (2012), 413-422. https://www.jstor.org/stable/43679273

[2] D. Buşneag, D. Piciu, *Residuated lattice of fractions relative to a ∧-closed system*, Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie, **49**(97) (2006), 13-24.

[3] D. Buşneag, D. Piciu, L. C. Holdon, *Some properties of ideals in Stonean residuated lattices*, Journal of Multiple-Valued Logic and Soft Computing, **24**(5-6) (2015), 529-546.

[4] L. C. Ciungu, *Directly indecomposable residuated lattices*, Iranian Journal of Fuzzy Systems, **6**(2) (2009), 7-18. https://doi.org/10.22111/IJFS.2009.204

[5] A. Di Nola, G. Georgescu, L. Leuştean, *Boolean products of BL-algebras*, Journal of Mathematical Analysis and Applications, **251**(1) (2000), 106-131. https://doi.org/10.1006/jmaa.2000.7024

[6] A. Filipoiu, G. Georgescu, *Compact and Pierce representations of MV-algebras*, Revue Roumaine de Mathématiques Pures et Appliquées, **40**(7) (1995), 599-618.

[7] A. Filipoiu, G. Georgescu, A. Lettieri, *Maximal MV-algebras*, Mathware, Soft Computing, **4** (1997), 53-62. http://eudml.org/doc/39098

[8] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, *Residuated lattices: An algebraic glimpse at substructural logics*, Elsevier, 2007.

[9] G. Georgescu, D. Cheptea, C. Mureşan, *Algebraic and topological results on lifting properties in residuated lattices*, Fuzzy Sets and Systems, **271** (2015), 102-132. https://doi.org/10.1016/j.fss.2014.11.007

[10] G. Georgescu, L. Leuştean, C. Mureşan, *Maximal residuated lattices with lifting Boolean center*, Algebra Universalis, **63** (2010), 83-99. https://doi.org/10.1007/s00012-010-0066-3

[11] G. Georgescu, C. Mureşan, *Boolean lifting property for residuated lattices*, Soft Computing, **18** (2014), 2075-2089. https://doi.org/10.1007/s00500-014-1318-5

[12] P. Hájek, *Mathematics of fuzzy logic*, Trends in Logic-Studia Logica, Kluwer Academic Publishers, Dordrecht/Boston/London, 1998.

[13] M. Haveshki, A. Borumand Saeid, E. Eslami, *Some types of filters in BL-algebras*, Soft Computing, **10**(8) (2006), 657-664. https://doi.org/10.1007/s00500-006-0098-y

[14] M. Haveshki, M. Mohamadhasani, *Essential extensions of filters in residuated lattices*, Journal of Multiple-Valued Logic and Soft Computing, **41** (2023), 85-113. https://doi.org/10.21203/rs.3.rs-185624/v1

[15] S. Hedayat, E. Rostami, *Decomposition of ideals into pseudo-irreducible ideals*, Communications in Algebra, **45**(4) (2017), 1711-1718. https://doi.org/10.1080/00927872.2016.1222410

[16] L. C. Holdon, *On ideals in De Morgan residuated lattices*, Kybernetika, **54**(3) (2018), 443-475. https://doi.org/10.14736/kyb-2018-3-0443

[17] A. Iorgulescu, *Algebras of logic as BCK-algebras*, Academy of Economic Studies, Bucharest, Editura ASE, 2008.

[18] L. Leuştean, *Representations of many-valued algebras*, Editura Universitariă, Craiova, 2010.

[19] D. Piciu, *Algebras of fuzzy logic*, Editura Universitaria, Craiova, 2007.

[20] E. Rostami, Sh. Ghorbani, *The complete comaximal decomposition in residuated lattices*, Filomat, **39**(3) (2025), 847-859. https://doi.org/10.2298/fil2503847r

[21] E. Rostami, S. Hedayat, R. Nekooei, S. Karimzadeh, *Comaximal factorization of lifting ideals*, Journal of Algebra and its Applications, **21**(3) (2022), 2250044. https://doi.org/10.1142/s021949882250044x

[22] E. Turunen, *Mathematics behind fuzzy logic*, Physica. Verlag, Wurzburg, 1999.

[23] M. Ward, R. P. Dilworth, *Residuated lattices*, Transactions of the American Mathematical Society, **45** (1939), 335-354. https://doi.org/10.1090/s0002-9947-1939-1501995-3