Iranian Journal of Fuzzy Systems

Volume 22, Number 6, (2025), pp. 67-81

Original paper

University of
Sistan and Baluchestan

Existence and uniqueness of mild solution for initial value problem of a class of fuzzy evolution equations with delay

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Abstract

In this paper, we consider a class of fuzzy delay evolution equations under generalized differentiability. By using the operator semigroup theory, the upper and lower solutions and the monotone iterative technique, the existence and uniqueness of mild solutions for the initial value problem of fuzzy delay evolution equations are obtained. The continuous dependence of two kinds of mild solutions on the initial value is also proved. In particular, the first innovation of this paper is to introduce the delay term into the fuzzy evolution equation, and the second innovation is that when discussing the continuous dependence of the solution on the initial value, it completely eliminates the restriction of the nonlinear function on the Lipschitz condition coefficients and optimizes the conditions for the existence of the solutions, which is relatively rare. Corresponding examples are added at the end of the article to make the conclusion better applied to practice.

Keywords: Fuzzy delay evolution equations, monotone iterative technique, fuzzy strongly continuous semigroup, fuzzy mild solution, existence and uniqueness.

1 Introduction

The evolution equations are a hot topic in the field of differential equations, which are applied in various fields such as electrical engineering, medicine, ecology and biology. There have been a lot of works investigating evolution equations in Banach space, it can refer to [9, 10, 11, 12, 17] and its references. Because the properties of fuzzy number space are not as favorable as those of Banach space, solving differential equations in fuzzy number space is very difficult. Therefore, in recent years, many scholars have begun to pay attention to related problems of differential equations in fuzzy number space[1, 2, 5, 6, 18, 20, 21, 24, 25, 28]. In reference [29], Zhang studied the continuous dependence of the mild solution on the initial value of fractional fuzzy evolution equations in fuzzy number space. In reference [27], Nguyen Thi Kim Son provided the definition of strongly continuous semigroups of fuzzy-valued operators in triangular fuzzy number spaces. In the sense of the generalized Hukuhara difference, they constructed new concepts of fuzzy infinitesimal generators and fuzzy resolvent operators for fuzzy semigroups. In reference [26], Shao Yabin discussed the existence theorem of solutions to fuzzy functional differential equations under compact and dissipative conditions using the properties of embedded mappings of fuzzy numbers into Banach spaces. In 2015[3], Robab Alikhani introduced a new concept of upper and lower solutions for fuzzy integro-differential equations. By employing the upper and lower solutions method, they established the existence and uniqueness of solutions for the initial value problem

$$\begin{cases} u'(t) = f(t, u(t), Tu), \quad t \in J, \\ u(a) = u_0 \end{cases}$$

Corresponding Author: Y. W. Jia

Received: September 2024; Revised: September 2025; Accepted: October 2025.

https://doi.org/10.22111/ijfs.2025.49723.8779

of first-order nonlinear fuzzy integro-differential equations, where $J = [a, b], f \in C(J \times \mathbb{R}^2_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}}), u_0 \in \mathbb{R}_{\mathcal{F}}, \mathbb{R}_{\mathcal{F}}$ is a fuzzy number space and

$$(Tu)(t) = \int_a^t k(t, s)u(s)ds,$$

here $k \in C(I, \mathbb{R}^+)$ and $\mathbb{R}^+$ denotes the set of all nonnegative numbers.

This study introduces several significant advancements to the field of fuzzy evolution equations. First, we propose a novel framework for fuzzy delay evolution equations by incorporating time-delay terms (e.g., $\mu_t(s) = (t + s)$), which realistically model memory-dependent dynamics in uncertain systems, an aspect entirely unexplored in prior works like Alikhani [3]. Second, leveraging the theory of fuzzy strongly continuous semigroups, we establish existence and uniqueness results under generalized differentiability, extending classical semigroup methods to fuzzy-valued operator spaces. Third, we develop two classes of fuzzy upper-lower solutions coupled with monotone iterative techniques, enabling the construction of convergent sequences for extremal solutions without relying on Lipschitz constraints for nonlinear terms a marked improvement over existing approaches. Fourth, we systematically classify mild solutions into two types, and rigorously prove their continuous dependence on initial values, eliminating restrictive Lipschitz conditions. Finally, our work is substantiated by a detailed case study on a fuzzy delayed population model, complete with numerical simulations, bridging theoretical rigor to practical applications in environmental science and engineering. These innovations collectively advance the mathematical toolkit for analyzing uncertain dynamical systems with memory effects, offering broader applicability and weaker assumptions than conventional methods.

In this paper, we focus on the following initial value problem of fuzzy delay evolution equations

$$\begin{cases} \mu'(t) = A\mu(t) + f(t, \mu(t), \mu_t), & t \in I, \\ \mu(t) = \psi(t), & t \in [-r, 0] \end{cases} \tag{1}$$

where $A : \mathcal{D}(A) \subset \mathcal{P} \to \mathcal{P}$ is a closed linear operator and A is a generator of positive fuzzy strongly continuous semigroup $\{\mathcal{G}(t), t \geq 0\}$ on $\mathcal{P}$, the function $u(\cdot)$ is valued in the triangular fuzzy number space $\mathcal{P}$, the nonlinear term $f : I \times \mathcal{P} \times C([-r, 0], \mathcal{P}) \to \mathcal{P}$ is continuous, $I = [0, a]$, $a > 0$ is a constant. μ_t is a function in $C([-r, 0], \mathcal{P})$. For $s \in [-r, 0]$, it is defined as $\mu_t(s) = \mu(t + s)$ and $\mu_t(\cdot)$ is the time history of the state from time $t - r$ to the current time t. $C_t := C([-r, t], \mathcal{P})$ denotes the triangular fuzzy number space of all the continuous functions from $[-r, t]$ into $\mathcal{P}$ endowed with distance $D_t(\mu, \nu) = \sup\limits_{t \in [-r, t]} D(\mu(s), \nu(s))$.

The structure of this paper is organized as follows: In Section 1 (Introduction), we present the background and context of the research. Section 2 (Preliminaries) introduces the necessary definitions, concepts, and foundational knowledge to facilitate readers, understanding. Section 3 (Upper and lower solution method and monotone iterative technique) establishes the existence and uniqueness of two types of mild solutions for fuzzy delay evolution equations by employing the upper and lower solutions method combined with the monotone iterative technique. In Section 4 (Continuous dependence of mild solution on initial value), we investigate the continuous dependence of two types of mild solutions on the initial value. In section 5 (Examples and applications), an example is provided to illustrate the feasibility of the results. In section 6 (Summary and prospect), the paper is summarized, and future potential breakthroughs and challenges are discussed.

2 Preliminaries

Below we will introduce some symbols, definitions, and some basic concepts as necessary.

Let $\mathbb{R}_{\mathcal{F}}$ be fuzzy subsets of the real axis $\mu : \mathbb{R} \to [0, 1]$ and
(i) μ is upper semi-continuous;
(ii) μ is fuzzy convex, i.e. ,for all $c \in (0, 1]$, $a, b \in \mathbb{R}$, $\mu(ca + (1 - c)b) \geq \min\{\mu(a), \mu(b)\}$;
(iii) $[\mu]^0 = \overline{\{y \in \mathbb{R} \mid \mu(y) > 0\}}$ is compact;
(iv) μ is normal, i.e. $\exists\, y_0 \in \mathbb{R}$ such that $\mu(y_0) = 1$.

We denote by $\mu_{\alpha l}$ and $\mu_{\alpha r}$ the lower and upper bifurcations of fuzzy numbers $\mu \in \mathbb{R}_{\mathcal{F}}$, respectively. For any $\alpha \in (0, 1)$, set $[\mu]^\alpha = \{y \in \mathbb{R} \mid \mu(y) \geq \alpha\} := [\mu_{\alpha l}, \mu_{\alpha r}]$ is the α-level set of the fuzzy number μ. The diameter of α- level set is calculated as $[\mu]^\alpha = \mu_{\alpha r} - \mu_{\alpha l}$. The distance from for every $w \in \mathbb{R}_{\mathcal{F}}$ to any $e \in \mathbb{R}_{\mathcal{F}}$ is defined as

$$D(w, e) = \sup_{\alpha \in [0,1]} d_{\mathcal{H}}([w]^\alpha, [e]^\alpha),$$

where $d_{\mathcal{H}}$ denotes the Hausdorff distance.

The following properties can be found in [8, 29] for the metric D defined on $\mathbb{R}_{\mathcal{F}}$:
(1) $D(m \oplus z, n \oplus z) = D(m, n)$, for all $m, n, z \in \mathbb{R}_{\mathcal{F}}$;
(2) $D(m \oplus n, w \oplus e) \leq D(m, w) \oplus D(n, e)$, for all $m, n, w, e \in \mathbb{R}_{\mathcal{F}}$;
(3) $D(\lambda m, \lambda n) = |\lambda| D(m, n)$, for all $\lambda \in \mathbb{R}, m, n \in \mathbb{R}_{\mathcal{F}}$;
and $(\mathbb{R}_{\mathcal{F}}, D)$ has completeness.

Definition 2.1. [13] *Let $w_1, w_2 \in \mathbb{R}_{\mathcal{F}}$, and they have the following operations in $\mathbb{R}_{\mathcal{F}}$*

$$[w_1 \oplus w_2]^\alpha = [w_1]^\alpha + [w_2]^\alpha = \{x_1 + y_1 | x_1 \in [w_1]^\alpha, y_1 \in [w_2]^\alpha\},$$

$$[k \cdot w_1]^\alpha = k[w_1]^\alpha = \{ky | y \in [w]^\alpha\}.$$

Definition 2.2. [8] *The generalized Hukuhara differential(hereafter referred to as $g\mathcal{H}$ difference) of fuzzy numbers $w, z \in \mathbb{R}_{\mathcal{F}}$ is defined as follows:*

$$w \ominus_{g\mathcal{H}} z = m \Leftrightarrow \begin{cases} (i)\ w = z \oplus m, \quad or \\ (ii)\ z = w \oplus (-1)m. \end{cases}$$

Suppose that the set of all triangular fuzzy numbers in $\mathbb{R}_{\mathcal{F}}$ is $\mathcal{P}$. Then $(\mathcal{P}, D)$ is a complete metric space and subspace of space $(\mathbb{R}_{\mathcal{F}}, D)$. As everyone knows, Bede gave in [27] that if $x, y \in \mathcal{P}$, then $x \ominus_{g\mathcal{H}} y = (-1) \odot (y \ominus_{g\mathcal{H}} x)$ always exists in $\mathcal{P}$ which is called $g\mathcal{H}$ difference $x \ominus_{g\mathcal{H}} y$.

Definition 2.3. [8] *Suppose that fuzzy function $h \in C(I, \mathcal{P})$, $y_0 \in I$, and k is a real number such that $y_0 + k \in I$. If there is $\mathcal{D}_{g\mathcal{H}} h(y_0) \in \mathcal{P}$ such that*

$$\mathcal{D}_{g\mathcal{H}} h(y_0) = \lim_{k \to 0} \frac{h(y_0 + k) \ominus_{g\mathcal{H}} h(x_0)}{k},$$

then h is $g\mathcal{H}$-differentiable(Hereinafter referred to as $g\mathcal{H}d$) at y_0.

Definition 2.4. [8] *Suppose that $f \in C^1_{g\mathcal{H}}(I, \mathcal{P})$, for any $t \in I$, $\alpha \in [0, 1]$, have form $[f(t)]^\alpha = [f_{\alpha l}, f_{\alpha r}]$. It can be said*
(i) *f is $[g\mathcal{H}d]^{(i)}$ at $y_0 \in I$ if $[D_{g\mathcal{H}} f(y_0)]^\alpha = [(f_{\alpha l})'(y_0), (f_{\alpha r})'(y_0)]$,*
we denote this derivative by $D^i_{g\mathcal{H}} f(y_0)$;
(ii) *f is $[g\mathcal{H}d]^{(ii)}$ differentiable at $y_0 \in I$ if $[D_{g\mathcal{H}} f(y_0)]^\alpha = [(f_{\alpha r})'(y_0), (f_{\alpha l})'(y_0)]$,*
we denote this derivative by $D^{ii}_{g\mathcal{H}} f(t_0)$.

Definition 2.5. [8] *Let $g : [a, b] \to \mathcal{P}$ for every $x \in I$, $\alpha \in [0, 1]$, g has a form $[g(x)]^\alpha = [g_{\alpha l}, g_{\alpha r}]$, where $g_{\alpha l}, g_{\alpha r}$ is Lebesgue integrable on $[a, b]$ and measurable. The Lebesgue integral of g be expressed as $\int_a^b g(x)dt$, it is can be defined as the following formula*

$$\Big[\int_a^b g(x)dt\Big]^\alpha = \Big(\int_a^b g_{\alpha l}(x)dt, \int_a^b g_{\alpha r}(x)dt\Big) \text{ for all } \alpha \in [0, 1].$$

Lemma 2.6. [19] *Suppose that $h, g : [0, T] \to [0, \infty)$ be continuous functions and δ is a normal constants such that $h(t) \leq \delta + \int_0^t g(s)h(s)ds$, $t \in [0, T]$, then $h(t) \leq \delta \exp(\int_0^t g(s)ds)$.*

In the following, we introduce some basic facts and definitions of strongly continuous operator semigroups defined on the triangular fuzzy number set. Let $\mathfrak{L}(\mathcal{P})$ be a space composed of all bounded linear operators $Y : \mathcal{P} \to \mathcal{P}$, denote

$$\| Y \| = \inf\{N : D(Ym, \hat{0}) \leq ND(x, \hat{0}), \text{ for all } x, m \in \mathbb{R}_{\mathcal{F}}\},$$

$$(\alpha Y_1 + \beta Y_2)(m) = \alpha \odot (Y_1 m) \oplus \beta \odot (Y_2 m), \text{ for all } m \in \mathcal{P},$$

where α, $\beta \in \mathbb{R}$, Y_1, $Y_2 \in \mathfrak{L}(\mathcal{P})$.

Definition 2.7. [16] *Let $\{\mathcal{G}(t), t \geqslant 0\} \subset \mathfrak{L}(\mathcal{P})$ be fuzzy operator semigroup with the following properties*
(i) *$\mathcal{G}(0) = Id_{\mathcal{P}}$;*
(ii) *$\mathcal{G}(\mu + \nu) = \mathcal{G}(\mu)\mathcal{G}(\nu)$, for all μ, ν are non-negative.*

Definition 2.8. [27] *Let $\{\mathcal{G}(t), t \geqslant 0\} \subset \mathfrak{L}(\mathcal{P})$ be semigroup satisfying*

$$\lim_{t \to 0^+} \mathcal{G}(t)m = m \text{ for all, } m \in \mathcal{P},$$

then $\{\mathcal{G}(t), t \geqslant 0\}$ is called a fuzzy strongly continuous semigroup on a triangular fuzzy set $\mathcal{P}$.

Definition 2.9. [16] *Suppose that* $\{\mathcal{G}(t), t \geqslant 0\} \subset \mathfrak{L}(\mathcal{P})$. *The definition of generator* $A : \mathcal{D}(A) \subset \mathcal{P} \to \mathcal{P}$ *is as follows*

$$Ay = \lim_{k \to 0^+} \frac{1}{k} \odot (\mathcal{G}(h)y \ominus_{g\mathcal{H}} y),$$

$$\mathcal{D}(A) = \big\{y \in \mathcal{P} : \lim_{k \to 0^+} \frac{1}{k} \odot (\mathcal{G}(h)y \ominus_{g\mathcal{H}} y) \text{ exists in } \mathcal{P}\big\}.$$

From [27, Lemma 5.1], we can see

$$M := \sup_{t \in I} \|\mathcal{G}(t)\| \geq 1. \tag{2}$$

For more properties of operator semigroups, please refer to [27].

The following will introduce the related properties of the order relation "≤" in fuzzy number space.

Definition 2.10. *For every* $y \geq \hat{0}$, $y \in \mathcal{P}$ *and* $t \geq 0$, *if the order inequality* $\mathcal{G}(t)y \geq \hat{0}$ *holds then the* $\{\mathcal{G}(t), t \geqslant 0\} \subset \mathfrak{L}(\mathcal{P})$ *on* $\mathcal{P}$ *is positive.*

Definition 2.11. [22] *Suppose* $a, b \in \mathcal{P}$. *For every* $\alpha \in [0, 1]$, *we say that* $a \leq b$ *if and only if* $a_{\alpha l} \leq b_{\alpha l}$, $a_{\alpha r} \leq b_{\alpha r}$.

The interval of fuzzy number $[x, y]$ is defined as $\{z \in \mathcal{P} : x \leq z \leq y\}$. Let $g_1, g_2 \in C(I, \mathcal{P})$ be two fuzzy functions, for every $x \in I$, we can say that $g_1 \leq g_2$ if $g_1(x) \leq g_2(x)$.

Lemma 2.12. [22] *Suppose that* $u, q, e, b \in \mathcal{P}$ *and* $c \in \mathbb{R}^+$, *then there are the following order relations*
(1) $u = q$ *if and only if* $u \leq q$ *and* $q \leq u$.
(2) *If* $u \leq q$, *then* $u \oplus e \leq q \oplus e$.
(3) *If* $u \leq q$ *and* $e \leq b$, *then* $u \oplus e \leq q \oplus b$.
(4) *If* $u \leq q$, *then* $cu \leq cq$.

Lemma 2.13. [22] *Suppose that* $f_1, f_2 \in C(I, \mathcal{P})$ *and* $f_1 \leq f_2$, *then* $\int_0^t f_1(s)ds \leq \int_0^t f_2(s)ds$, *for all* $t \in I$.

Lemma 2.14. [23] *If* $\{f_n\} \subseteq C(I, \mathcal{P})$ *and* $h \in C(I, \mathcal{P})$ *such that* $f_n \leq h$, *for all* $n \in \mathbb{N}$, *and for every* $x \in I$, $f_n(x)$ *converges to* $f(x)$ *in* $\mathbb{R}_{\mathcal{F}}$, *then* $f \leq h$.

For any $t \in [0, a]$, $C_t := C([-r, t], \mathcal{P})$ denotes a triangular fuzzy number space consisting of all continuous functions from $[-r, t]$ into $\mathcal{P}$ endowed with the distance $D_t(\mu, \nu) = \sup\limits_{-r \leq s \leq t} D(\mu(s), \nu(s))$.

Let $\hat{F}[\mu](t) := \mathcal{G}(t)\mu_0 \ominus (-1) \int_0^t \mathcal{G}(t - \tau) \odot f\big(\tau, \mu(\tau), \mu_\tau\big)d\tau$, $t \in [0, a]$, and let $\hat{C}_a := \hat{C}([-r, a], \mathcal{P}) = \{\mu \in C_a\}$: $\hat{F}[\mu](t)$ exists for all $t \in [0, a]$.

Definition 2.15. [4] $\mu \in C_a$ *is a mild solution to the initial value problem* (1) *if and only if* $\mu \in C_a$, *for any* $t \geq 0$, $\mu(t) \in \mathcal{D}(A)$ *and* μ *satisfies*

$$\mu(t) \ominus_{g\mathcal{H}} \mathcal{G}(t)\varphi(0) = \int_0^t \mathcal{G}(t - \tau) \odot f\big(\tau, \mu(\tau), \mu_\tau\big)d\tau, \quad t \in I. \tag{3}$$

$$\mu(t) = \psi(t), \quad t \in [-r, 0].$$

In addition, if the mild solution μ *is* $[g\mathcal{H}d]^{(i)}$ *differentiable, then* (1) *is equivalent to*

$$\mu(t) = \begin{cases} \mathcal{G}(t)\mu_0 \oplus \int_0^t \mathcal{G}(t - \tau) \odot f\big(\tau, \mu(\tau), \mu_\tau\big)d\tau, & t \in I, \\ \psi(t), & t \in [-r, 0], \end{cases} \tag{4}$$

if the μ *is* $[g\mathcal{H}d]^{(ii)}$ *differentiable, then* (1) *is equivalent to*

$$\mu(t) = \begin{cases} \mathcal{G}(t)\mu_0 \ominus (-1) \int_0^t \mathcal{G}(t - \tau) \odot f\big(\tau, \mu(\tau), \mu_\tau\big)d\tau, & t \in I, \\ \psi(t), & t \in [-r, 0]. \end{cases} \tag{5}$$

Definition 2.16. *Sppose that* $\omega^{(0)}$, $\upsilon^{(0)}$, $\hat{\omega}^{(0)}$, $\hat{\upsilon}^{(0)} \in C_a$, *we can say*
(i) $\omega^{(0)}$ *and* $\upsilon^{(0)}$ *denote the first kind of fuzzy upper solution and fuzzy lower solution of the* (1)

$$\overline{\mu}^{(1)}(t) \geq \begin{cases} \mathcal{G}(t)\mu_0 \oplus \int_0^t \mathcal{G}(t - \tau) \odot f\big(\tau, \overline{\mu}^{(1)}(\tau), (\overline{\mu}^{(1)})_\tau\big)d\tau, & t \in I, \\ \psi(t), & t \in [-r, 0]. \end{cases} \tag{6}$$

and

$$\underline{\mu}^{(1)}(t) \le \begin{cases} \mathcal{G}(t)\mu_0 \oplus \int_0^t \mathcal{G}(t-\tau) \odot f\big(\tau, \underline{\mu}^{(1)}(\tau), (\underline{\mu}^{(1)})_\tau\big) d\tau, & t \in I, \\ \psi(t), & t \in [-r, 0]. \end{cases} \tag{7}$$

(ii) $\hat{\omega}^{(0)}$ *and* $\hat{\upsilon}^{(0)}$ *denote the second kind of fuzzy upper solution and fuzzy lower solution of the* (1)

$$\overline{\mu}^{(2)}(t) \ge \begin{cases} \mathcal{G}(t)(t)\mu_0 \ominus (-1) \int_0^t \mathcal{G}(t-\tau) \odot f\big(\tau, \overline{\mu}^{(2)}(\tau), (\overline{\mu}^{(2)})_\tau\big) d\tau, & t \in I, \\ \psi(t), & t \in [-r, 0]. \end{cases} \tag{8}$$

and

$$\underline{\mu}^{(2)}(t) \le \begin{cases} \mathcal{G}(t)\mu_0 \ominus (-1) \int_0^t \mathcal{G}(t-\tau) \odot f\big(\tau, \underline{\mu}^{(2)}(\tau), (\underline{\mu}^{(2)})_\tau\big) d\tau, & t \in I, \\ \psi(t), & t \in [-r, 0]. \end{cases} \tag{9}$$

3 Upper and lower solution method and monotone iterative technique

Theorem 3.1. *Assume that* $A : \mathcal{D}(A) \subset \mathcal{P} \to \mathcal{P}$ *is an infinitesimal generator of a positive fuzzy strongly continuous semigroup* $\mathcal{G}(t)$*. Let* f *map a bounded set in* $I \times \mathcal{P} \times C_0$ *to the bounded set in* $\mathcal{P}$ *and satisfy the hypothesis:*
(H1) The fuzzy upper solution $\omega^{(0)}$ *and fuzzy lower solution* $\upsilon^{(0)}$ *of the first kind of* (1) *exist and*

$$f(t, \alpha_1, \beta_1) \le f(t, \alpha_2, \beta_2), \quad t \in I, \tag{10}$$

for $\nu^{(0)} \le \alpha_1 \le \alpha_2 \le \omega^{(0)}, \nu_t^{(0)} \le \beta_1 \le \beta_2 \le \omega_t^{(0)}$*. Then* (1) *has maximal and minimal mild solutions of the first kind.*

Proof. Define the sequence $\{\nu^{(n)}\}_{n=1}^\infty$ as

$$\nu^{(n)}(t) = \mathcal{G}(t)\varphi(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f(s, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_s) d\tau, \quad t \in I, \tag{11}$$

$$\nu^{(n)}(t) = \psi(t),\ t \in [-r, 0]. \tag{12}$$

Define the sequence $\{\omega^{(n)}\}_{n=1}^\infty$ as

$$\omega^{(n)}(t) = \mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f(s, \omega^{(n-1)}(\tau), (\omega^{(n-1)})_\tau) d\tau, \quad t \in I, \tag{13}$$

$$\omega^{(n)}(t) = \omega(t),\ t \in [-r, 0]. \tag{14}$$

Then

$$\underline{\mu}^{(1)} = \nu^{(0)} \le \nu^{(1)} \le \nu^{(2)} \le \cdots \le \nu^{(n)} \le \cdots \le \omega^{(n)} \le \cdots \le \omega^{(2)} \le \omega^{(1)} \le \omega^{(0)} = \overline{\mu}^{(1)}. \tag{15}$$

To prove it, let $n = 1$ for (11), we have

$$\nu^{(1)}(t) = \mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f(\tau, \nu^{(0)}(\tau), (\nu^{(0)})_\tau) d\tau, \quad t \in I.$$

According to Definition 2.16, we get $\nu^{(0)}(t) \le \nu^{(1)}(t) \le \omega^{(0)}(t)$, $t \in [-r, a]$. Therefore $(\nu^{(0)})_t \le (\nu^{(1)})_t \le (\omega^{(0)})_t$. Suppose that $n = k$, $\nu^{(k-1)}(t) \le \nu^{(k)}(t) \le \omega^{(0)}(t)$, $t \in [-r, a]$, therefore $(\nu^{(k)})_t \le (\nu^{(k+1)})_t$. Form (11) we have

$$\nu^{(k+1)}(t) = \mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f\big(\tau, \nu^{(k)}(\tau), (\nu^{(k)})_\tau\big) d\tau, \quad t \in I,$$

and

$$\nu^{(k)}(t) = \mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f\big(\tau, \nu^{(k-1)}(\tau), (\nu^{(k-1)})_\tau\big) d\tau, \quad t \in I. \tag{16}$$

By condition $(H1)$ and the formula (16), we arrive at

$$\nu^{(k)}(t) \le \nu^{(k+1)}(t) \le \omega^{(0)}(t),\ t \in [-r, a].$$

Thus, $\nu^{(n)}(t) \le \nu^{(n+1)}(t) \le w^{(0)}(t)$ by mathematical induction for all $t \in [-r, a]$. Using the same method of proof, we get $\nu^{(0)} \le \omega^{(n+1)} \le \omega^{(n)} \le \omega^{(0)}$.

Then we will prove that $\nu^{(n)} \leq \omega^{(n)}$, $n \in \mathbb{N}^+$. Form, $\nu^{(0)} \leq \omega^{(0)}$. Combining (11) and (13) we get $\nu^{(1)}(t) \leq \omega^{(1)}(t)$. Suppose that $n = k$, we have $\nu^{(k)}(t) \leq \omega^{(k)}(t)$, $t \in I$. Let $n = k+1$, then

$$\nu^{(k+1)}(t) = \mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f\big(\tau, \nu^{(k)}(\tau), (\nu^{(k)})_\tau\big) d\tau, \quad t \in I,$$

$$\omega^{(k+1)}(t) = \mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f\big(\tau, \omega^{(k)}(\tau), (\omega^{(k)})_\tau\big) d\tau, \quad t \in I. \tag{17}$$

Using the nondecreasing property of f and (17), it is possible to obtain that $\nu^{(k+1)}(t) \leq \omega^{(k+1)}(t)$, $t \in [-r, a]$. Thus, for any $n \in \mathbb{N}^+$, $\nu^{(n+1)}(t) \leq \omega^{(n+1)}(t)$, by mathematical induction for all $t \in [-r, a]$. By condition $(H1)$

$$\nu^{(0)} \leq \nu^{(1)} \leq \nu^{(2)} \cdots \leq \nu^{(n)} \leq \cdots \leq \omega^{(n)} \leq \cdots \leq \omega^{(2)} \leq \omega^{(1)} \leq \omega^{(0)}, \tag{18}$$

This completes the proof of inference.

Next, we explain the equicontinuity of $\{\nu^{(n)}\}$ and $\{\omega^{(n)}\}$. Owing to the continuous function f maps bounded sets in $I \times \mathcal{P} \times C_0$ to bounded sets into $\mathcal{P}$ and the properties of distance D. Let $N := \sup_{t \in I} D(f(t, u(t)), \hat{0})$ and $t_1 \leq t_2$, denote $t_1 + \delta = t_2$, $\delta \in [0, a]$. We conclude

$$\begin{aligned}
& D\big(\nu^{(n)}(t_1), \nu^{(n)}(t_2)\big) \\
=& D\Big(\mathcal{G}(t_1)\psi(0) \oplus \int_0^{t_1} \mathcal{G}(t_1-\tau) \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau, \mathcal{G}(t_2)\psi(0) \\
& \oplus \int_0^{t_2} \mathcal{G}(t_2-\tau) \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau\Big) \\
\leq& D\big(\mathcal{G}(t_1)\psi(0), \mathcal{G}(t_2)\psi(0)\big) + D\Big(\int_0^{t_1} \mathcal{G}(t_1-\tau) \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau, \\
& \int_0^{t_2} \mathcal{G}(t_2-s) \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau\Big) \\
=& D\big(\mathcal{G}(t_1)\psi(0), \mathcal{G}(t_2)\psi(0)\big) + D\Big(\int_0^{t_1} \mathcal{G}(t_1-\tau) \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau \oplus \hat{0}, \\
& \int_0^{t_2} \mathcal{G}(t_2-\tau) \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau \oplus \hat{0}\Big) \\
\leq& D\big(\mathcal{G}(t_1)\psi(0), \mathcal{G}(t_2)\psi(0)\big) \\
& + D\Big(\int_0^{t_1} \mathcal{G}(t_1-\tau) \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau \oplus \hat{0}, \int_0^{t_1} \mathcal{G}(t_2-s) \\
& \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau \oplus \int_{t_1}^{t_2} \mathcal{G}(t_2-\tau) \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau\Big) \\
\leq& D\big(\mathcal{G}(t_1)\psi(0), \mathcal{G}(t_2)\psi(0)\big) \\
& + D\Big(\int_0^{t_1} \mathcal{G}(t_1-\tau) \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau, \int_0^{t_1} \mathcal{G}(t_2-\tau) \\
& \odot f\big(s, v^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau\Big) + D\Big(\int_{t_1}^{t_2} \mathcal{G}(t_2-\tau) \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau, \hat{0}\Big) \\
=& D\big(\mathcal{G}(t_1)\psi(0), \mathcal{G}(t_2)\psi(0)\big) \\
& + D\Big(\int_0^{t_1} \mathcal{G}(t_1-\tau) \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau, \int_0^{t_1} \mathcal{G}(t_1-\tau+t_2-t_1) \\
& \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau\Big) + d\Big(\int_{t_1}^{t_2} \mathcal{G}(t_2-\tau) \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau, \hat{0}\Big) \\
=& D\big(\mathcal{G}(t_1)\psi(0), \mathcal{G}(t_2)\psi(0)\big) \\
& + D\Big(\int_0^{t_1} \mathcal{G}(t_1-\tau) \odot f\big(\tau, \nu^{(n-1)}(\tau), (\nu^{(n-1)})_\tau\big) d\tau, \int_0^{t_1} \mathcal{G}(t_1-s)\mathcal{G}(t_2-t_1)
\end{aligned}$$

$$
\begin{aligned}
&\odot f\big(\tau,\nu^{(n-1)}(\tau),(\nu^{(n-1)})_\tau\big)d\tau\Big)+D\Big(\int_{t_1}^{t_2}\mathcal{G}(t_2-\tau)\odot f\big(\tau,\nu^{(n-1)}(\tau),(\nu^{(n-1)})_\tau\big)d\tau,\hat{0}\Big)\\
\le& D\big(\mathcal{G}(t_1)\psi(0),\mathcal{G}(t_2)\psi(0)\big)\\
&+MD\Big(\int_0^{t_1}f\big(\tau,\nu^{(n-1)}(\tau),(\nu^{(n-1)})_\tau\big)d\tau,\int_0^{t_1}\mathcal{G}(t_2-t_1)\\
&\odot f\big(\tau,\nu^{(n-1)}(\tau),(\nu^{(n-1)})_\tau\big)d\tau\Big)+D\Big(\int_{t_1}^{t_2}\mathcal{G}(t_2-\tau)\odot f\big(\tau,\nu^{(n-1)}(\tau),(\nu^{(n-1)})_\tau\big)d\tau,\hat{0}\Big)\\
\le& D\big(\mathcal{G}(t_1)\psi(0),\mathcal{G}(t_2)\psi(0)\big)+\int_{t_1}^{t_2}D\Big(\mathcal{G}(t_2-\tau)\odot f\big(\tau,\nu^{(n-1)}(\tau),(\nu^{(n-1)})_\tau\big),\hat{0}\Big)d\tau\\
&+M\int_0^{t_1}D\Big(f\big(\tau,\nu^{(n-1)}(\tau),(\nu^{(n-1)})_\tau\big),\mathcal{G}(t_2-t_1)\odot f\big(\tau,\nu^{(n-1)}(\tau),(\nu^{(n-1)})_\tau\big)\Big)d\tau\\
\le& D\big(\mathcal{G}(t_1)\psi(0),\mathcal{G}(t_2)\psi(0)\big)+M\int_0^{t_1}D\Big(f\big(\tau,\nu^{(n-1)}(\tau),(\nu^{(n-1)})_\tau\big),\mathcal{G}(t_2-t_1)\\
&\odot f\big(\tau,\nu^{(n-1)}(\tau),(\nu^{(n-1)})_\tau\big)\Big)ds+MN(t_2-t_1)\\
\le& D\big(\mathcal{G}(t_1)\psi(0),\mathcal{G}(t_2-t_1)\psi(0)\big)+M\int_0^{t_1}D\Big(f\big(\tau,\nu^{(n-1)}(\tau),(\nu^{(n-1)})_\tau\big),\mathcal{G}(t_2-t_1)\\
&\odot f\big(\tau,\nu^{(n-1)}(\tau),(\nu^{(n-1)})_\tau\big)\Big)d\tau+MN(t_2-t_1).\\
\le& MD\big(\psi(0),\mathcal{G}(t_2-t_1)\odot\psi(0)\big)\\
&+M\int_0^{t_1}D\Big(f\big(\tau,\nu^{(n-1)}(\tau),(\nu^{(n-1)})_\tau\big),\mathcal{G}(t_2-t_1)\odot f\big(\tau,\nu^{(n-1)}(\tau),(\nu^{(n-1)})_\tau\big)\Big)d\tau\\
&+MN(t_2-t_1).
\end{aligned}
$$

Because the fuzzy operator semigroup $\{\mathcal{G}(t)(t\geqslant 0)\}$ has strong continuity, we arrive at $D(\nu^{(n+1)}(t_1),\nu^{(n+1)}(t_2))$ tends to 0 as $t_1\to t_2$. Therefore, $\{\nu^{(n)}\}$ is equicontinuous.

The following prove the uniform boundedness of $\{\nu^{(n)}(t)\}$

$$
\begin{aligned}
D\big(\nu_{n+1})(t),\hat{0}\big)=&D\Big(\mathcal{G}(t)\psi(0)\oplus\int_0^t\mathcal{G}(t-\tau)\odot f\big(\tau,\nu^{(n)}(\tau),(\nu^{(n)})_\tau\big)d\tau,\hat{0}\Big)\\
\le& D\big(\mathcal{G}(t)\psi(0),\hat{0}\big)+D\Big(\int_0^t\mathcal{G}(t-\tau)\odot f\big(\tau,\nu^{(n)}(\tau),(\nu^{(n)})_\tau\big)d\tau,\hat{0}\Big)\\
\le& MD(\psi(0),\hat{0})+aMD(f(\tau,\nu^{(n)}(\tau),(\nu^{(n)})_\tau),\hat{0})\\
\le& MD(\psi(0),\hat{0})+aMN.
\end{aligned}
$$

Because f maps bounded sets in $I\times\mathcal{P}\times C_0$ to bounded sets in $\mathcal{P}$, so $\{\nu^{(n)}(t)\}$ is uniformly bounded.

Therefore, $\{\nu^{(n)}\}$ is sequentially compact in C_a by the Arzela-Ascoli theorem. Repeating the methods and steps in the proof about $\{\omega^{(n)}\}$, one can obtain that $\{\omega^{(n)}\}$ is sequentially compact in C_a. Thus, $\{\nu^{(n)}\}$, $\{\omega^{(n)}\}$ have subsequences that converge to $\underline{u}$, $\overline{\mu}$, respectively. Through the monotonicity of sequences $\{\nu^{(n)}\}$ and $\{\omega^{(n)}\}$, the existence continuous function $\underline{\mu}$ and $\overline{\mu}$ such that $\{\nu^{(n)}\}$ converges to $\underline{\mu}$ and $\{\omega^{(n)}\}$ to $\overline{\mu}$ on I. Making limits as $n\to\infty$ in the formulas (11) and (13), we get

$$
\underline{\mu}(t)=\mathcal{G}(t)\psi(0)\oplus\int_0^t\mathcal{G}(t-\tau)\odot f(\tau,\underline{\mu}(\tau),\underline{\mu}_\tau)d\tau,\quad \overline{\mu}(t)=\mathcal{G}(t)\psi(0)\oplus\int_0^t\mathcal{G}(t-\tau)\odot f(\tau,\overline{\mu}(\tau),\overline{\mu}_\tau)d\tau,
$$

and we have $\nu^{(0)}\le\underline{\mu}\le\overline{\mu}\le\omega^{(0)}$. By using the function f and the formula (11), we get $\nu^{(n)}\le\mu$. Similarly, we arrive at $\mu\le\omega^{(n)}$. For any n, we have

$$
\nu^{(n)}(t)\le\mu(t)\le\omega^{(n)}(t),\quad t\in[-r,a],
$$

making limits to $n\to\infty$, we arrive at

$$
\nu^{(0)}\le\underline{\mu}\le\mu\le\overline{\mu}\le\omega^{(0)}.
$$

Therefore, $\overline{\mu}$ and $\underline{\mu}$ are maximal and minimal fuzzy mild solutions of the first kind for (1) in C_a. □

Theorem 3.2. *Assume that $A : \mathcal{D}(A) :\subset \mathcal{P} \to \mathcal{P}$ is an infinitesimal generator of a positive fuzzy strongly continuous semigroup $\mathcal{G}(t)$. Let f map bounded sets in $I \times \mathcal{P} \times C_0$ to bounded sets in $\mathcal{P}$ and $\hat{C}_a := \hat{C}([-r, a], \mathcal{P}) \neq \emptyset$. If satisfies the following assumptions:*
(H2) There exist fuzzy upper solution $\widehat{\omega}^{(0)}$ and fuzzy lower solution $\widehat{\nu}^{(0)}$ of the second kind for (1) and (3.1) is established for $\widehat{\nu}^{(0)} \le \alpha_1 \le \alpha_2 \le \widehat{\omega}^{(0)}, \widehat{\nu}_t^{(0)} \le \beta_1 \le \beta_2 \le \widehat{\omega}_t^{(0)}$. Then (1) has maximal and minimal mild solutions of the second kind.

Proof. Proof similar to Theorem 3.1 and will not be described here. ☐

Theorem 3.3. *Suppose that all assumptions of Theorem 3.1 are satisfied. Let there exist numbers $L \ge 0$ such that*

$$f(t, \alpha_1, \beta_1) + L\alpha_1 \le f(t, \alpha_2, \beta_2) + L\alpha_2, \quad t \in I \tag{19}$$

for $\alpha_1, \alpha_2 \in \mathcal{P}$, $\beta_1, \beta_2 \in C_0$ with $\nu^{(0)} \le \alpha_2 \le \alpha_1, (\nu^{(0)})_t \le \beta_2 \le \beta_1 \le (\omega^{(0)})_t$, then (1) has a unique mild solution of the first kind.

Proof. Because all the assumptions of Theorem 3.1 are true, there exists a maximum and minimum mild solution $\overline{\mu}, \underline{\mu}$ of (1). According to Definition 2.10, because $\overline{\mu} \le \underline{\mu}$, thus

$$(\underline{\mu})_{\alpha l} \le (\overline{\mu})_{\alpha l}, \quad \text{and} \quad (\underline{\mu})_{\alpha r} \le (\overline{\mu})_{\alpha r}, \quad \forall\, \alpha \in [0, 1],$$

There are two crisp functions $R_l(t, \alpha), R_r(t, \alpha) \in C(I, \mathbb{R}^+)$ and make $\alpha \in [0, 1]$ fixed such that

$$(\overline{\mu})_{\alpha l} = (\underline{\mu})_{\alpha l} + R_l(t, \alpha), \quad \text{and} \quad (\overline{\mu})_{\alpha r} = (\underline{\mu})_{\alpha r} + R_r(t, \alpha), \quad \forall\, \alpha \in [0, 1].$$

The following proof process shows that $\overline{\mu} = \underline{\mu}$. First, consider $\overline{\mu}, \underline{\mu}$ to be in S_1, so it satisfies the equation (3). Therefore, using the hypothesis (19), we have

$$\begin{aligned}
(\underline{\mu}(t))_{\alpha l} + R_l(t, \alpha) =& (\overline{\mu}(t))_{\alpha l} \\
=& (\mathcal{G}(t)\psi(0))_{\alpha l} \oplus \int_0^t (\mathcal{G}(t - \tau) \odot f(\tau, \overline{\mu}(\tau), \overline{\mu}_\tau))_{\alpha l} d\tau \\
\le& (\mathcal{G}(t)\psi(0))_{\alpha l} \oplus \int_0^t (\mathcal{G}(t - \tau) \odot f(\tau, \underline{\mu}(\tau), \underline{\mu}_\tau))_{\alpha l} d\tau + ML \int_0^t R_l(\tau, \alpha) d\tau \\
=& (\underline{\mu}(t))_{\alpha l} + ML \int_0^t R_l(\tau, \alpha) d\tau.
\end{aligned}$$

Then $R_l(t, \alpha) \le ML \int_0^t R_l(\tau, \alpha) d\tau$. From Lemma 2.6, we can obtain $R_l(t, \alpha) \le 0$ and using the same method, $R_r(t, \alpha) \le 0$. Because $R_l(t, \alpha), R_r(t, \alpha) \in C(I, \mathbb{R}^+)$, so $R_l(t, \alpha) = 0$, $R_r(t, \alpha) = 0$. Therefore $\overline{\mu} = \underline{\mu}$, then (1) has a unique mild solution of the first kind. ☐

Theorem 3.4. *Suppose that all assumptions of Theorem 3.1 are satisfied. Let there exist numbers $\hat{L} \ge 0$ such that*

$$f(t, \alpha_1, \beta_1) + \hat{L}\alpha_1 \le f(t, \alpha_2, \beta_2) + \hat{L}\alpha_2, \quad t \in I \tag{20}$$

for $\alpha_1, \alpha_2 \in \mathcal{P}$, $\beta_1, \beta_2 \in \hat{C}_0$ with $\hat{\nu}^{(0)} \le \alpha_2 \le \alpha_1, (\hat{\nu}^{(0)})_t \le \beta_2 \le \beta_1 \le (\hat{\omega}^{(0)})_t$, then (1) has a unique mild solution of the second kind.

Proof. Proof similar to Theorem 3.3 and will not be described here. ☐

4 Continuous dependence of mild solution on initial value

Theorem 4.1. *Assume that $A : \mathcal{D}(A) :\subset \mathcal{P} \to \mathcal{P}$ is an infinitesimal generator of fuzzy strongly continuous semigroup $\mathcal{G}(t)$. If the condition*

(H3) Let $f \in C(I \times \mathcal{P} \times C_0, \mathcal{P})$, there exists are positive constants L_1, L_2, such that

$$D(f(t, \mu, \bar{\mu}), f(t, \nu, \bar{\nu})) \le L_1 D(\mu, \nu) + L_2 D_0(\bar{\mu}, \bar{\nu}), \quad \forall\, \mu, \nu \in \mathcal{P}, \bar{\mu}, \bar{\nu} \in C_0.$$

holds, then (1) has a unique fuzzy mild solution $u \in C_a$ of the first kind. For any $\varepsilon > 0$, there is a positive constant such that for arbitrary $\tilde{\psi}(t) \in \mathcal{P}$ with $D(\tilde{\psi}(t), \psi(t)) < \delta$, the initial value problem of fuzzy delay evolution equations

$$\begin{cases} \mu'(t) = A\mu(t) + f(t, \mu(t), \mu_t), & t \in I, \\ \tilde{\mu}(t) = \tilde{\psi}(t), & t \in [-r, 0] \end{cases} \tag{21}$$

has a unique mild solution $\tilde{\mu} \in C_a$ of the first kind and $D(\tilde{\mu}(t), \mu(t)) < \varepsilon$.

Proof. First of all, we need to prove the existence of the mid solution of (1). To this end, we consider the following operator $Q : C_a \to C_a$

$$(Q\mu)(t) = \begin{cases} \mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f\big(\tau, \mu(\tau), \mu_\tau\big)d\tau, \quad t \in I, \\ \psi(t), \quad t \in [-r, 0]. \end{cases} \tag{22}$$

Obviously, the operator Q defined in (22) has continuity. Let $R_1 = 2MD(\psi(0), \hat{0})$, $\hat{N} = \max\limits_{t\in[-r,a]} D_a(f(t, \hat{0}, \hat{0}), \hat{0})$. Define $\Omega_{R_1} = \{\mu \in C_a : D_a(\mu(t), \hat{0}) \le R_1, t \in [-r, a]\}$, then Ω_{R_1} is centered at $\hat{0}$ and radius is the closed ball of R_1 in C_a. Let $\mathcal{H}(\mu, \nu) = \{\sup\limits_{t\in[-r,a]} d\big(\mu(t), \nu(t)\big)e^{-\lambda t}\}, \mu, \nu \in \mathcal{P},\ \lambda > 0$. When $t \in I$

$$\begin{aligned}
&D_a((Q\mu)(t), \hat{0}) \\
=&D_a(\mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f\big(\tau, \mu(\tau), \mu_\tau\big)d\tau, \hat{0}) \\
\le&D_a(\mathcal{G}(t)\psi(0), \hat{0}) + D_a(\int_0^t \mathcal{G}(t-\tau) \odot f(\tau, \mu(\tau), \mu_\tau)d\tau, \hat{0}) \\
\le&MD_a(\psi(0), \hat{0}) + M\int_0^t D_a(f(\tau, \mu(\tau), \mu_\tau), \hat{0})d\tau \\
\le&MD_a(\psi(0), \hat{0}) + M\int_0^t D_a(f(\tau, \mu(\tau), \mu_\tau), f(\tau, \hat{0}, \hat{0})) + D_a(f(\tau, \hat{0}, \hat{0}), \hat{0})d\tau \\
\le&MD_a(\psi(0), \hat{0}) + M\int_0^t L_1 D(\mu(\tau), \hat{0}) + L_2 D_0(\mu_\tau, \hat{0}) + \hat{N}d\tau \\
\le&MD_a(\psi(0), \hat{0}) + M((L_1 + L_2)D(\mu, \hat{0}) + \hat{N}) \\
\le&MD_a(\psi(0), \hat{0}) + M((L_1 + L_2)R_1 + \hat{N}) \\
\le&2MD_a(\psi(0), \hat{0}) = R_1.
\end{aligned}$$

When $t \in [-r, 0]$, $D((Q\mu)(t), \hat{0}) = D(\varphi(t), \hat{0}) < R_1$, it shows that $Q\mu \in \Omega_{R_1}$. Therefore, we prove the continuity of the operator $Q : \Omega_{R_1} \to \Omega_{R_1}$. Next, explain the Q(by definition (22))is contraction in Ω_{R_1}. For every $\mu, \nu \in C(I, \mathcal{P})$ and $t \in I$, it follows from condition $(H3)$

$$\begin{aligned}
&D_a((Q\mu)(t), (Q\nu)(t)) \\
=&D_a(\mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f(\tau, \mu(\tau), \mu_\tau)d\tau, \mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f(\tau, \nu(\tau), \nu_\tau)d\tau) \\
\le&D_a(\int_0^t \mathcal{G}(t-\tau) \odot f(\tau, \mu(\tau), \mu_\tau)d\tau, \int_0^t \mathcal{G}(t-\tau) \odot f(\tau, \nu(\tau), \nu_\tau)d\tau) \\
\le&M\int_0^t D_a(f(\tau, \mu(\tau), \mu_\tau), f(\tau, \nu(\tau), \nu_\tau))d\tau \\
\le&M\int_0^t L_1 D(\mu(\tau), \nu(\tau)) + L_2 D_0(\mu_\tau, \nu_\tau)d\tau \\
\le&M\int_0^t L_1 D_a(\mu, \nu) + L_2 D_a(\mu, \nu)d\tau \\
\le&M\int_0^t (L_1 + L_2)D_a(\mu, \nu).
\end{aligned}$$

From this, we can get

$$\begin{aligned}
\sup_{t\in[0,a]} D\big(A\mu(\tau), A\nu(\tau)\big)e^{-\lambda t} &\le M\int_0^t (L_1 + L_2)\mathcal{H}(\mu, \nu)e^{\lambda\tau}d\tau e^{-\lambda t} \\
&\le M(L_1 + L_2)\mathcal{H}(\mu, \nu)\int_0^t e^{\lambda\tau}d\tau e^{-\lambda t}
\end{aligned}$$

$$\leq M(L_1+L_2)\mathcal{H}(\mu,\nu)\frac{1-e^{-\lambda t}}{\lambda},$$

Using that

$$\lim_{\lambda\to\infty}\frac{1-e^{-\lambda t}}{\lambda}=0,$$

we choose $\lambda>0$

$$M(L_1+L_2)\frac{1-e^{-\lambda t}}{\lambda}<1,$$

when $t\in[-r,0]$, $D_a((Q\mu)(t),(Q\nu)(t))=D_a(\psi(t),\psi(t))=0$. Therefore

$$\mathcal{H}((Qu),(Qv))<\mathcal{H}(u,v),$$

this shows that the operator Q (given by definition (22)) has a unique fixed point $\mu\in\Omega_{R_1}$, which is also the only bubble temperature solution of the first kind of (1) in Ω_{R_1}. Next, we prove that the existence and uniqueness of mild solutions of the first kind for (21). Consider the operator $F:C_a\to C_a$ defined by

$$(F\mu)(t)=\begin{cases}\mathcal{G}(t)\tilde{\psi}(0)\oplus\int_0^t\mathcal{G}(t-\tau)\odot f\big(\tau,\mu(\tau),\mu_\tau\big)d\tau, & t\in I,\\ \psi(t), & t\in[-r,0].\end{cases}\tag{23}$$

Make $\tilde{R}_1=2MD(\psi(\tilde{0}),\hat{0})$, and define $\Omega_{\tilde{R}_1}=\{\mu\in C_a: D(\mu(t),\hat{0})\leq\tilde{R}_1, t\in[-r,a]\}$. From the above argument and formula (21) for operator Q, it is known that F has a unique fixed point $\tilde{\mu}\in\Omega_{\tilde{R}_1}$, and the fixed point is the unique fuzzy mild solution of the first kind of problem (21) in $\Omega_{\tilde{R}_1}$.

Let $R=\min\{R_1,\tilde{R}_1\}$, $\Omega_R=\{\mu\in C_a: D_a(u(t),\hat{0})\leq R, t\in[-r,a]\}$. Thus, $u\in\Omega_R$ and $\tilde{\mu}\in\Omega_{\tilde{R}_1}$ is the unique mild solution of the first kind of (1) and formula (21), respectively, which are equivalent to the following formula

$$\mu(t)=\begin{cases}\mathcal{G}(t)\psi(0)\oplus\int_0^t\mathcal{G}(t-\tau)\odot f\big(\tau,\mu(\tau),\mu_\tau\big)d\tau, & t\in I,\\ \psi(t), & t\in[-r,0].\end{cases}\tag{24}$$

and

$$\tilde{\mu}(t)=\begin{cases}\mathcal{G}(t)\tilde{\psi}(0)\oplus\int_0^t\mathcal{G}(t-\tau)\odot f\big(\tau,\tilde{\mu}(\tau),\tilde{\mu}_\tau\big)d\tau, & t\in I,\\ \tilde{\psi}(t), & t\in[-r,0].\end{cases}\tag{25}$$

Then, we will prove the continuous dependence of the first kind of mild solution of (1). For any $\varepsilon>0$, making $\delta=\varepsilon\cdot\frac{1-Ma(L_1+L_2)}{M}$, $D(\psi(0),\psi(\tilde{0}))\leq\delta$, when $t\in I$,

$$\begin{aligned}&D_a(\mu(t),\hat{\mu}(t))\\ =&D_a(\mathcal{G}(t)\psi(0)\oplus\int_0^t\mathcal{G}(t-\tau)\odot f(\tau,\mu(\tau),\mu_\tau)d\tau,\mathcal{G}(t)\tilde{\psi}(0)\oplus\int_0^t\mathcal{G}(t-\tau)\odot f(\tau,\tilde{\mu}(\tau),\tilde{\mu}_\tau)d\tau)\\ \leq&MD_a(\psi(0),\psi(\tilde{0}))+D_a(\int_0^t\mathcal{G}(t-\tau)\odot f(\tau,\mu(\tau),\mu_\tau)d\tau,\int_0^t\mathcal{G}(t-\tau)\odot f(\tau,\tilde{\mu}(\tau),\tilde{\mu}_\tau)d\tau)\\ \leq&MD_a(\psi(0),\psi(\tilde{0}))+M\int_0^t D_a(f(\tau,\mu(\tau),\mu_\tau),f(\tau,\tilde{\mu}(\tau),\tilde{\mu}_\tau))d\tau\\ \leq&MD_a(\psi(0),\psi(\tilde{0}))+M\int_0^t L_1D(\mu(\tau),\tilde{\mu}(\tau))+L_2D_0(\mu_\tau,\tilde{\mu}_\tau)d\tau\\ \leq&MD_a(\psi(0),\psi(\tilde{0}))+M\int_0^t(L_1+L_2)D_a(\mu,\tilde{\mu})d\tau\\ \leq&M(\delta+D_a(\mu,\tilde{\mu})a(L_1+L_2)),\end{aligned}$$

when $t\in[-r,0]$, $D(\mu(t),\hat{\mu}(t))=D(\varphi(t),\tilde{\varphi}(t))<\delta<\varepsilon$, $D_a(\mu,\tilde{u})=\sup\limits_{-r\leq t\leq a}d(\mu(t),\tilde{\mu}(t))\leq M(\delta_1+D_a(\mu,\tilde{\mu})a(L_1+L_2))$. Obviously, $D_a(\mu,\tilde{\mu})\leq\frac{M\delta}{1-Ma(L_1+L_2)}<\varepsilon$. The proof of Theorem 4.1 is over. □

We will prove the continuous dependence of the second kind of mild solution on the initial value. To prove this conclusion, suppose that

$$\hat{Q}(\mu)(t)=\begin{cases}\mathcal{G}(t)\mu_0\ominus(-1)\int_0^t\mathcal{G}(t-\tau)\odot f\big(\tau,\mu(\tau),\mu_\tau\big)d\tau, & t\in I,\\ \psi(t), & t\in[-r,0].\end{cases}\tag{26}$$

Theorem 4.2. *Suppose that $A : \mathcal{D}(A) :\subset \mathcal{P} \to \mathcal{P}$ be an infinitesimal generator of a fuzzy strongly continuous semigroup $\mathcal{G}(t)$. If the condition $(H3)$ and*
$(H4)$ $\hat{C}_a = \{\mu \in C_a : \hat{Q}(\mu)(t) exists\ for\ all\ \ t \in [-r, a]\} \neq \emptyset$ and $\hat{C}_a = \{\mu \in C_a : \hat{F}(u)(t) exists\ for\ all\ \ t \in [-r, a]\} \neq \emptyset$,
$(H5)$ if $\mu \in \hat{C}_a$, then $\hat{Q}(\mu)(t)$, and if $\mu \in \hat{C}(a)$, then $\hat{F}(\mu)(t) \in \hat{C}_a$ and fulfilled. Then (1) has a unique fuzzy mild solution $\hat{\mu} \in \hat{C}_a$ of the second kind. In addition, for every $\varepsilon > 0$, there is a normal constant δ such that for any $\tilde{\psi}(t) \in \mathcal{P}$ satisfying $D(\tilde{\psi}(t), \psi(t)) < \delta$, the initial value problem has a unique fuzzy mild solution of the second kind $\tilde{\hat{\mu}} \in \hat{C}_a$ and $D(\tilde{\hat{\mu}}(t), \hat{\mu}(t)) < \varepsilon$.

Proof. From the similar proof of Theorem 4.1, we arrive at there is a constant L_1, L_2 and $R > 0$, such that (1) and (21) has a unique fuzzy mild solution of the second kind $\hat{\mu} \in \hat{\Omega}_R := \{\mu \in C_a : D_a(\mu(t), \hat{0}) \leq R, t \in [-r, a]\}$ and $\hat{\hat{\mu}} \in \hat{\hat{\Omega}}_R := \{\mu \in \hat{C}_a : D_a(\mu(t), \hat{0}) \leq R, t \in [-r, a]\}$ of the second kind, respectively, and they are given in the following formula

$$\hat{\mu}(t) = \begin{cases} \mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f\big(\tau, \hat{\mu}(\tau), \hat{\mu}_\tau\big)d\tau, & t \in I, \\ \psi(t), \quad t \in [-r, 0]. \end{cases} \tag{27}$$

and

$$\hat{\hat{\mu}}(t) = \begin{cases} \mathcal{G}(t)\tilde{\psi}(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f\big(\tau, \hat{\hat{\mu}}(\tau), \hat{\hat{\mu}}_\tau\big)d\tau, & t \in I, \\ \tilde{\psi}(t), \quad t \in [-r, 0]. \end{cases} \tag{28}$$

For every $\varepsilon > 0$, choosing $\delta = \varepsilon \cdot \frac{1-Ma(L_1+L_2)}{M}$, such that $D(\psi(t), \hat{\psi}(t)) \leq \delta$. From the condition $(H3)$, when $t \in I$,

$$\begin{aligned}
&D_a(\hat{\mu}(t), \hat{\hat{\mu}}(t)) \\
=&D_a(\mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f(\tau, \hat{\mu}(\tau), \hat{\mu}_\tau)d\tau, \mathcal{G}(t)\tilde{\psi}(0) \oplus \int_0^t \mathcal{G}(t-\tau) \odot f(\tau, \hat{\hat{\mu}}(\tau), \hat{\hat{\mu}}_\tau)d\tau) \\
\leq&MD_a(\psi(0), \tilde{\psi}(0)) + D_a(\int_0^t \mathcal{G}(t-\tau) \odot f(\tau, \hat{\mu}(\tau), \hat{\mu}_\tau)d\tau, \int_0^t \mathcal{G}(t-\tau) \odot f(\tau, \hat{\hat{\mu}}(\tau), \hat{\hat{\mu}}_\tau)d\tau) \\
\leq&MD_a(\psi(0), \tilde{\psi}(0)) + M\int_0^t D_a(f(\tau, \hat{\mu}, \hat{\mu}_\tau), f(\tau, \hat{\hat{\mu}}(\tau), \hat{\hat{\mu}}_\tau))d\tau \\
\leq&MD_a(\psi(0), \tilde{\psi}(0)) + M\int_0^t L_1D(\hat{\mu}(\tau), \hat{\hat{\mu}}(\tau)) + L_2D_0(\hat{\mu}_\tau, \hat{\hat{\mu}}_\tau)d\tau \\
\leq&MD_a(\psi(0), \tilde{\psi}(0)) + M\int_0^t (L_1 + L_2)D_a(\hat{\mu}, \hat{\hat{\mu}})d\tau \\
\leq&M(\delta + D_a(\hat{\mu}, \hat{\hat{\mu}})a(L_1 + L_2)).
\end{aligned}$$

Since the rest of this theorem can be obtained by using the proof method of Theorem 4.1, we will not repeat it here. □

5 Examples and applications

To demonstrate the practical significance of our theoretical results, we consider the following fuzzy delay population growth model with environmental carrying capacity:

$$\begin{cases} \mu'(t) = A\mu(t) \oplus \pi[\frac{r}{K} \odot \mu(t) \odot (K \ominus \mu_t)], & t \in [0, T], \\ \mu(t) = \psi(t), \quad t \in [-r, 0], \end{cases} \tag{29}$$

where $\mu(t) \in \mathcal{P}$ represents the fuzzy population density, $A = -\sigma \odot I$ generates a fuzzy contraction semigroup $G(t) = e^{-\sigma t} \odot I$. Let $\tilde{K}$ be a non-degenerate fuzzy number(e.g.,a triangular fuzzy number $\tilde{k} = (K - \varepsilon, K, K + \varepsilon)$, where $\varepsilon > 0$), and $r \in \mathbb{R}^+$. μ_t is a function in $C([-r, 0], \mathcal{P})$. For $s \in [-r, 0]$, it is defined as $\mu_t(s) = \mu(t + s)$ and $\mu_t(\cdot)$ is the time history of the state from time $t - r$ to current time t.

Step 1: Verification of Operator Semigroup Properties
Definition of Fuzzy Contraction Semigroup:

$$\mathcal{G}(t)\mu = e^{-\sigma t} \odot \mu, \quad t \geq 0,$$

where σ is a positive constant, $e^{-\sigma t}$ is the scalar exponential function, and $\odot$ denotes fuzzy scalar multiplication. Verification Conditions:

1. Semigroup Property:
(1). $\mathcal{G}(0) = \mu$.
(2). $\mathcal{G}(t+s)\mu = e^{-\sigma(t+s)} \odot \mu = \mathcal{G}(t)\mathcal{G}(s)\mu$.
2. Strong Continuity:
For any $\mu \in \mathcal{P}$,

$$\lim_{t\to 0^+} D(G(t)\mu, \mu) = \lim_{t\to 0^+} D(e^{-\sigma t} \odot \mu, \mu) = \lim_{t\to 0^+} |1 - e^{-\sigma t}| \cdot D(\mu, 0) = 0.$$

3. Positivity:
If $\mu \geq \hat{0}$, then $\mathcal{G}(t)\mu = e^{-\sigma t} \odot \mu \geq \hat{0}$, since the scalar $e^{\sigma t} > 0$.

Step 2: Construction and Verification of Upper-Lower Solutions
Definition of Upper-Lower Solutions:
Upper solution: $\overline{\mu}(t) = \tilde{K} \oplus \epsilon$.(where ϵ is small fuzzy perturbation).
Lower solution: $\underline{\mu}(t) = \hat{0}$. Verify for $t \in [0, T]$:
Verification of Upper Solution Condition:
Require:

$$\overline{\mu}'(t) = \hat{0} \geq A\overline{\mu}(t) \oplus f(t, \overline{\mu}, \overline{\mu}_t),$$

Since $\overline{\mu}(t)$ is constant, its generalized derivative is $\overline{\mu}'(t) = \hat{0}$. Substituting the right-hand side:

$$A\overline{\mu}(t) + f(t, \overline{\mu}(t), \overline{\mu}_t) = -\sigma \odot \tilde{K} \oplus \frac{r}{\tilde{K}} \odot (\tilde{K} \ominus \tilde{K}) = -\sigma \odot \tilde{K} \oplus \hat{0} = -\sigma \odot \tilde{K}.$$

Thus, the condition reduces to:

$$\hat{0} \geq -\sigma \odot \tilde{K} \Rightarrow \sigma \odot \tilde{K} \geq \hat{0}.$$

Since $\sigma > 0$ and $\tilde{K} > \hat{0}$, this holds.

Verification of Lower Solution Condition:
Similarly, $\underline{\mu}'(t) = \hat{0} \leq A\underline{\mu}(t) \oplus f(t, \overline{\mu}, \overline{\mu}_t)$, which trivially holds.

Step 3: Construction and Convergence of Monotone Iterative Sequences Iterative Scheme:
Construct iterative sequences:

$$\mu^{(n)}(t) = \mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-s) \odot \big[\frac{r}{\tilde{K}} \odot \mu^{(n-1)}(s) \odot \big(\tilde{K} \ominus \mu_s^{(n-1)}\big)\big] ds.$$

Monotonicity via Mathematical Induction:
Base Case: $\mu^{(0)} = \hat{0} \leq \mu^{(1)}(t) \leq \tilde{K} \oplus \epsilon$.
Inductive Hypothesis: Assume $\mu^{(k-1)}(t) \leq \mu^{(k)}t \leq \tilde{K} \oplus \epsilon$.
Inductive Step: By the monotonicity of f with respect to μ(when $\mu \leq \tilde{K}$):

$$\begin{aligned}\mu^{(k)}(t) &= \mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-s) f(s, \mu^{(k-1)}(s), \mu_s^{(k-1)}) ds \\ &\leq \mathcal{G}(t)\psi(0) \oplus \int_0^t \mathcal{G}(t-s) f(s, \mu^{(k)}(s), \mu_s^{(k)}) ds \\ &= \mu^{(k+1)}(t).\end{aligned}$$

By induction, the sequence $\mu^{(n)}(t)$ is monotonically increasing and uniformly bounded, hence converging to a fuzzy function μ^*. When $t \in [-r, 0]$, it is obviously reasonable.

Step 4: Introduce the Nonlinear Term (Theorem 3.3)
For the nonlinear term $f(t, \mu, \mu_t) = \frac{r}{\tilde{K}} \odot \mu \odot (\tilde{K} \ominus \mu_t)$: For any $\alpha \in [0, 1], t \in [0, T]$:

$$[f(t, \mu, \mu_t)]^\alpha = [\frac{r_{\alpha l}}{\tilde{K}_{\alpha r}} \mu_{\alpha l}(\tilde{K}_{\alpha r} - (\mu_t)_{\alpha r}), \frac{r_{\alpha r}}{\tilde{K}_{\alpha l}} \mu_{\alpha r}(\tilde{K}_{\alpha l} - (\mu_t)_{\alpha l})]^\alpha.$$

To enforce monotonicity of $\hat{f} = f + L\mu$, we require:

$$\frac{r_{\alpha r}}{\tilde{K}_{\alpha l}}(\tilde{K}_{\alpha l} - (\mu_t)_{\alpha l}) + L \leq \frac{r_{\alpha r}}{\tilde{K}_{\alpha l}}(\tilde{K}_{\alpha l} - (\mu_t)_{\alpha l}) + L.$$

However, to bound the nonlinear growth, choose:

$$L = \sup_{\alpha\in[0,1]} \frac{r_{\alpha r}}{\tilde{K}_{\alpha l}} \cdot \sup_{\mu\leq\tilde{K}} (\tilde{K}_{\alpha l} - (\mu_t)_{\alpha l}).$$

Since $(\mu_t)_{\alpha l} \leq \tilde{K}_{\alpha l}$, this simplifies to:

$$L = \sup_{\alpha\in[0,1]} r_{\alpha r}.$$

Define $\hat{f}(t,\mu,\mu_t) = f(t,\mu,\mu_t) + L\cdot\mu$. For $L>0$, $\mu_1 \leq \mu_2$: Compare $\alpha\in[0,1]$:

$$\frac{r_{\alpha l}}{\tilde{K}_{\alpha r}}(\mu_{1_t})_{\alpha l}(\tilde{K}_{\alpha r} - (\mu_{1_t})_{\alpha r}) + L(\mu_{1_t})_{\alpha r} \leq \frac{r_{\alpha l}}{\tilde{K}_{\alpha r}}(\mu_{2_t})_{\alpha r}(\tilde{K}_{\alpha r} - (\mu_{2_t})_{\alpha r}) + L(\mu_{2_t})_{\alpha l}.$$
$$\frac{r_{\alpha r}}{\tilde{K}_{\alpha l}}(\mu_{1_t})_{\alpha r}(\tilde{K}_{\alpha l} - (\mu_{1_t})_{\alpha l}) + L(\mu_{1_t})_{\alpha r} \leq \frac{r_{\alpha}r}{\tilde{K}_{\alpha l}}(\mu_{2_t})_{\alpha r}(\tilde{K}_{\alpha l} - (\mu_{2_t})_{\alpha l}) + L(\mu_{2_t})_{\alpha r}.$$

Simplify inequalities: Since $\mu_1 \leq \mu_2$, we have $(\mu_{1_t})_{\alpha l} \leq (\mu_{2_t})_{\alpha l}$, and $(\mu_{1_t})_{\alpha r} \leq (\mu_{2_t})_{\alpha r}$. Additionally, $(\mu_{1_t})_{\alpha r} \leq (\mu_{2_t})_{\alpha r}$ and $(\mu_{1_t})_{\alpha l} \leq (\mu_{2_t})_{\alpha l}$. Global fuzzy order preservation: By verifying monotonicity at every $\alpha-$ level, we ensure $\hat{f}(t,\mu_1,\mu_{1_t}) \leq \hat{f}(t,\mu_2,\mu_{2_t})$ under the fuzzy order $\leq$.

Assume two solutions μ_1,μ_2. Define the distance function:

$$D(\mu_1(t),\mu_2(t)) \leq M\int_0^t (L_1 D(\mu_1(s),\mu_2(s)) + L_2 D(\mu_{1_s},\mu_{2_s}))ds,$$

where $L_1 = ||-\sigma+L||, L_2 = \sup\frac{r_\alpha^r}{K_\alpha^l}$.

If $L_1 + L_2 < \gamma$ for some $\gamma > 0$, then:

$$D(\mu_1(t),\mu_2(t)) \leq D(0)e^{\gamma t}.$$

Since $D(0) = 0$(identical initial conditions), we have $D(\mu_1(t),\mu_2(t)) \equiv 0$, implying $\mu_1 \equiv \mu_2$. When $t\in[-r,0]$, it is obviously reasonable.

Step 5: Continuous Dependence Verification (Theorem 4.1)

Let $\tilde{\psi}(t)$ be a perturbed initial condition with $D(\psi(t),\tilde{\psi}) < \delta$, and $\tilde{\mu}(t)$ the corresponding solution. Then, for $\forall\, t\in[0,T]$:

$$D(\mu(t),\tilde{\mu}(t)) \leq MD(\psi(0),\tilde{\psi}(0)) + M\int_0^t L_1 D(\mu(s),\tilde{\mu}(s)) + L_2 D(\mu_t,\tilde{\mu}_t)ds.$$

Choosing $\delta = \frac{\varepsilon(1-MT(L_1+L_2))}{M}$, Gronwall's inequality yields:

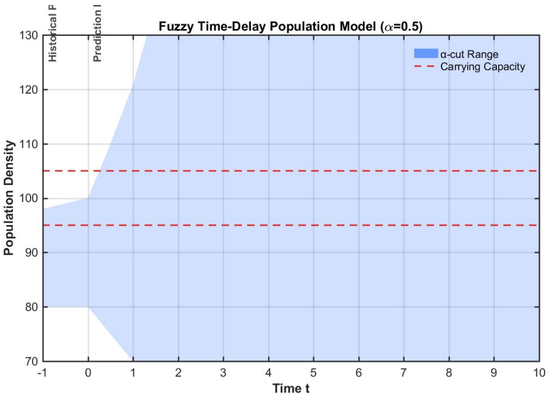
$$D(\mu,\tilde{\mu}) \leq \frac{M\delta}{1-MT(L_1+L_2)} \leq \varepsilon,$$

when $t\in[-r,0]$, $D(\mu(t),\tilde{\mu}(t)) = D(\psi(t),\tilde{\psi}(t)) < \delta < \varepsilon$.

Step 6: Numerical Illustration (Enhanced Intuition)

Parameter Settings: $\tilde{K} = (90,100,110)$(triangular fuzzy number for carrying capacity), $r = (0.8, 1.0, 1.2)$, $\sigma = 0.1, T = 10$, Initial history $\psi(t) = (80,90,100)(t\in[-1,0])$;

Iterative Computation: Approximate the solution via finite iterations (e.g.,$n = 5$) and plot the fuzzy solution curves(α-cuts over time), demonstrating convergence from $\hat{0}$ to the equilibrium $\tilde{K}$.



Through the above steps:

Verified the strong continuity and positivity of the operator semigroup, satisfying Theorem 3.1. Constructed upper-lower solutions and demonstrated the convergence of monotone iterative sequences, aligning with Theorem 3.1. Proved uniqueness via Gronwall's inequality, corresponding to Theorem 3.3. Analyzed solution sensitivity to initial perturbations, validating Theorem 4.1. This example is theoretically rigorous and can be visualized through numerical simulations to illustrate the dynamics of fuzzy delayed systems, robustly supporting the core theorems of the paper.

6 Summary and prospect

This paper studies the evolution equation with time delay, which is different from some literature. Because the fuzzy number space is merely a complete metric space, many fixed point theorems cannot be applied to evolution equations and differential equations in the fuzzy number space. Additionally, this creates difficulties in applying the upper and lower solution method and the monotone iterative method to the problems studied in this paper. Secondly, we discuss the continuous dependence of the mild solution on the initial value and find that subtle changes in the initial value lead to corresponding changes in the mild solution. This characteristic is illustrated through an example.

We attempt to introduce a monotone iterative method of quasi-upper and lower solutions into the fuzzy number space to solve the evolution equation. However, due to the limitations of the fuzzy number space, the existing framework does not support this method. We will continue to pursue efforts in this direction in the future, and we welcome readers interested in this topic to exchange ideas and engage in discussions with us.

Acknowledgement

This study was supported by the Graduate Research Support Project of Northwest Normal University, China (Grant No. KYZZ2025-LXB029).

The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the referees.

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