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Original paper

Introduction to q-fractional fuzzy graphs and their energy analysis

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Abstract

Over the years, traditional fuzzy graph models such as intuitionistic fuzzy graphs (IFGs), Pythagorean fuzzy graphs (PyFGs), and q-rung orthopair fuzzy graphs (q-ROFGs) have emerged as powerful tools for modeling uncertainty in complex networked structures. Despite their advancements, these models exhibit critical limitations when both membership and non-membership degrees approach extreme values. Such restrictions prevent these frameworks from adequately modeling many real-world scenarios, where this constraint is frequently violated. Moreover, existing models often struggle to eliminate the inherent dependency between the membership and non-membership grades. To overcome these limitations, we propose a novel framework called the q-Fractional Fuzzy Graph (q-FFG), which extends fuzzy graph theory, allows greater flexibility and independence between membership and non-membership grades. Unlike previous models, q-fractional fuzzy graphs can handle extreme degrees of uncertainty by letting both membership and non-membership values approach 1 independently, without violating consistency. We formally define the structure of q-fractional fuzzy graphs and some operations on these graphs. Additionally, we discuss the degree and adjacency matrices of these graphs. These fundamental matrices serve as essential tools for understanding the structure and behavior of these graphs. The study presents an extensive review of the spectral properties and energy of q-FFGs. It also validates their comparative energy analysis with existing models through experimental and graphical illustrations. Additionally, an algorithm based on Principal Component Analysis (PCA) technique is proposed which effectively reduces the dimensionality of data while preserving the essential features.

Keywords: q-fractional fuzzy graphs, P-union and R-union, energy of q-fractional fuzzy graphs.

1 Introduction

Graph theory is a fascinating branch of mathematics that explores the intricate web of connections that shape the very fabric of our world. From social systems to communication networks, graph theory serves as a foundational tool that bridges theoretical mathematics with real-world applications, making it an essential component of modern scientific and technological advancements.

Zadeh [20], in 1965, introduced the concept of a fuzzy set, marking a pivotal shift from the rigid binary logic of classical set theory to a more flexible framework that allows partial membership. The concept of fuzzy graphs was initially proposed by Kauffman, and later, Rosenfeld formalized fuzzy relations and fuzzy graphs [11]. Atanassov [6], expanded on the concept of fuzzy set (FS) and introduced an intuitionistic fuzzy set (IFS), which incorporates not only the degree of membership but also the degree of non-membership and the inherent hesitation. Unlike the traditional fuzzy set, which accounts solely for membership, IFS offers a more refined representation of uncertainty, making it a powerful tool in domains where vagueness is inherent. It is particularly useful in decision-making problems where information is uncertain, resulting in more realistic and accurate outcomes. IFS is highly versatile and can be applied to various domains where decision-making under uncertainty is crucial. The concept of intuitionistic fuzzy graph (IFG) was introduced by Atanassov [7] as an extension of his earlier notion of IFS. An IFG offers a more precise approach to

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modeling uncertainty in network relationships. In addition to the membership degree, it incorporates non-membership and hesitation degrees, thereby enabling a better representation of ambiguity in real-world problems.

Atanassov [5] introduced the concept of intuitionistic fuzzy set of the second type (IFS2), which was later redefined by Yager [18] as the Pythagorean fuzzy set (PyFS), where the sum of the squares of the membership and non-membership degrees is constrained to be less than or equal to one. Owing to this relaxed condition, PyFS provides greater flexibility in representing uncertainty. Building on this idea, Naz et al. [16] introduced the concept of Pythagorean fuzzy graph (PyFG). Compared to the traditional fuzzy and intuitionistic fuzzy models, the Pythagorean fuzzy framework offers greater flexibility and enhanced practical applicability. However, this extension does have its limitations. To address these limitations, the q-rung orthopair fuzzy set (q-ROFS) was introduced [19]. This modification provides a more adaptable mechanism for encoding information, catering to various preferences and requirements in decision-making contexts. Later, Habib and Akram [12] introduced q-rung orthopair fuzzy graph (q-ROFG) theory as a further a further generalization of IFGs and PyFGs. The q-ROFGs provide a more flexible and expressive framework for modeling uncertainty and imprecise information than IFGs and PyFGs as they allow higher degrees of membership and non-membership.

The notion of energy in fuzzy graphs was first introduced in [15], establishing a link between graph energy and the spectral properties of fuzzy graphs. This concept has found wide-ranging applications in modeling social and communication networks, image processing and decision-making systems. Extending this framework, [8] investigated the Laplacian energy of intuitionistic fuzzy graphs. As a refinement of classical fuzzy graph energy, the Laplacian energy of an intuitionistic fuzzy graph is an important measure that extends the traditional graph theory to better reflect the complexities of real-world systems. By integrating uncertainty and hesitation, it offers a more accurate depiction of network structures. In a recent study, Mudrić-Staniškovski et al. [14] further advanced the study of fuzzy set theory by defining the energy and λ-energy of fuzzy soft sets, together with limiting energies. Later, Djurovic et al. [9] explored the properties of interval-valued soft sets by integrating interval-valued fuzzy sets and soft sets. They also introduced the notions of energy along with pessimistic and optimistic energies. Building on this line of research, Stojanovic et al. [17] developed numerical characteristics and the energy of interval-valued hesitant fuzzy soft sets, highlighting the role of these energies in constructing effective decision-making algorithms. More recently, Alcantud et al. [2] proposed the concept of scored energy by combining the scores of hesitant fuzzy elements with matrix singular values. They also discussed some properties and applications of scored energy.

Problem statement of the study

Although fuzzy graph theory has witnessed significant advancement, existing models such as IFG, PyFG, or even q-ROFG exhibit limitations, particularly in scenarios where either the membership or non-membership degrees (or both) approach extremes (equal to 1), a situation inadequately addressed by current generalizations. This leaves out many important cases that occur in real-world settings. To address this limitation, the concept of the q-fractional fuzzy graph is introduced. This framework relaxes these constraints and enables the modeling of independent levels of agreement (membership) and disagreement (non-membership) between individuals, groups, or countries. The notion of the q-fractional fuzzy graph opens new possibilities for modeling complex relationships with greater expressiveness, making it highly valuable for real-world applications in various fields.

Research gap and motivation of the work

In the real world, interaction among individuals, groups and communities plays a key role in shaping social dynamics. With the advent of technology, particularly with the rise of social media platforms such as WhatsApp, Facebook and Twitter, the traditional face-to-face interactions have been significantly reduced. Some individuals are highly active on social media platforms but less engaged in real life (face-to-face) interactions, while others show the opposite pattern. There are also some individuals who maintain a high level of activity in both domains, as well as those who are largely inactive in both domains. The scenario can be modeled by a pair $(a,\ b)$ such that $a, b \in [0,1]$. Traditional fuzzy graph models fail to accurately represent such situations due to their restrictive interdependence between membership and non-membership degrees. This leaves out many important cases that occur in real-world settings. To address this limitation, the concept of the q-fractional fuzzy graph is introduced. This framework relaxes these constraints and enables the modeling of independent levels of agreement (membership) and disagreement (non-membership) between individuals, groups, or countries.

Objectives and novelty of the study

The primary objectives of this study are as follows:

1. To introduce and formalize the framework of the q-fractional fuzzy graph (q-FFG) as a generalization of q-ROFG.
2. To establish the theoretical foundations of q-FFGs, including structural properties supported by graphical comparisons with existing fuzzy graph frameworks.
3. To investigate the spectral properties and energy of q-FFGs, deriving new results and validating them through graphical and experimental illustrations.

The proposed model aims to relax the restrictive dependence between membership and non-membership degrees in traditional fuzzy graphs, thereby providing a more flexible and expressive tool to represent independent levels of agreement and disagreement in real-world systems. The study particularly focuses on capturing complex interaction patterns among individuals, groups, and communities. Unlike earlier generalizations, the q-FFG framework provides the flexibility to represent real-world dual-domain interactions that cannot be accurately modeled by conventional approaches.

This paper is organized as follows: In Section 2, some basic definitions and results are provided to lay the groundwork for the proposed framework. In Section 3, we formally present the q-fractional fuzzy graph models, along with their key operations and results. We further explore the structural properties and energy of these graphs. Additionally, an algorithm based on the Principal Component Analysis (PCA) technique is proposed, which effectively reduces the dimensionality of the data while preserving its key features.

2 Preliminaries

This section provides a concise overview of the fundamental definitions. We start with the definition of q-fractional fuzzy set.

Definition 2.1. [10] *A q-fractional fuzzy set over a non-empty set X is defined as $\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x), \widehat{\mu}_{\mathcal{A}}(x)) \mid x \in X\}$. Where $\mu_{\mathcal{A}}(x), \widehat{\mu}_{\mathcal{A}}(x) \in [0,1]$ are independent membership and non-membership grades, satisfying the condition $0 \leq \frac{\mu_{\mathcal{A}}(x)}{q} + \frac{\widehat{\mu}_{\mathcal{A}}(x)}{q} \leq 1$, where $q \geq 2$.*

Definition 2.2. [10] *Let $\mathcal{A}_1 = (\mu_{\mathcal{A}_1}(x), \widehat{\mu}_{\mathcal{A}_1}(x))$ and $\mathcal{A}_2 = (\mu_{\mathcal{A}_2}(x), \widehat{\mu}_{\mathcal{A}_2}(x))$ be two q-fractional fuzzy sets such that $0 \leq \frac{\mu_{\mathcal{A}_1}(x)}{q} + \frac{\widehat{\mu}_{\mathcal{A}_1}(x)}{q} \leq 1$ and $0 \leq \frac{\mu_{\mathcal{A}_2}(x)}{q} + \frac{\widehat{\mu}_{\mathcal{A}_2}(x)}{q} \leq 1$ where $q \geq 2$. The following operations can be defined*

i) $\mathcal{A}_1 \subseteq_P \mathcal{A}_2$ *if* $\mu_{\mathcal{A}_1}(x) \leq \mu_{\mathcal{A}_2}(x)$ *and* $\widehat{\mu}_{\mathcal{A}_1}(x) \leq \widehat{\mu}_{\mathcal{A}2}(x)$,
ii) $\mathcal{A}_1 \subseteq_R \mathcal{A}_2$ *if* $\mu_{\mathcal{A}_1}(x) \leq \mu_{\mathcal{A}_2}(x)$ *and* $\widehat{\mu}_{\mathcal{A}_1}(x) \geq \widehat{\mu}_{\mathcal{A}2}(x)$,
iii) $\mathcal{A}_1 = \mathcal{A}_2$ *if* $\mu_{\mathcal{A}_1}(x) = \mu_{\mathcal{A}_2}(x)$ *and* $\widehat{\mu}_{\mathcal{A}_1}(x) = \widehat{\mu}_{\mathcal{A}2}(x)$,
iv) $\mathcal{A}_1 \cup_P \mathcal{A}_2 = \{\max\{\mu_{\mathcal{A}_1}(x), \mu_{\mathcal{A}_2}(x)\}, \max\{\widehat{\mu}_{\mathcal{A}_1}(x), \widehat{\mu}_{\mathcal{A}2}(x)\}\}$,
v) $\mathcal{A}_1 \cup_R \mathcal{A}_2 = \{\max\{\mu_{\mathcal{A}_1}(x), \mu_{\mathcal{A}_2}(x)\}, \min\{\widehat{\mu}_{\mathcal{A}_1}(x), \widehat{\mu}_{\mathcal{A}2}(x)\}\}$,
vi) $\mathcal{A}_1 \cap_P \mathcal{A}_2 = \{\min\{\mu_{\mathcal{A}_1}(x), \mu_{\mathcal{A}_2}(x)\}, \min\{\widehat{\mu}_{\mathcal{A}_1}(x), \widehat{\mu}_{\mathcal{A}2}(x)\}\}$,
vii) $\mathcal{A}_1 \cap_R \mathcal{A}_2 = \{\min\{\mu_{\mathcal{A}_1}(x), \mu_{\mathcal{A}_2}(x)\}, \max\{\widehat{\mu}_{\mathcal{A}_1}(x), \widehat{\mu}_{\mathcal{A}2}(x)\}\}$,
viii) $\mathcal{A}^c = (1 - \mu_{\mathcal{A}}(x), 1 - \widehat{\mu}_{\mathcal{A}}(x))$.

Proposition 2.3. [10] *Let $\mathcal{A}_1 = (\mu_{\mathcal{A}_1}(x), \widehat{\mu}_{\mathcal{A}_1}(x))$ such that $0 \leq \frac{\mu_{\mathcal{A}_1}(x)}{q} + \frac{\widehat{\mu}_{\mathcal{A}_1}(x)}{q} \leq 1, \mathcal{A}_2 = (\mu_{\mathcal{A}_2}(x), \widehat{\mu}_{\mathcal{A}_2}(x))$ such that $0 \leq \frac{\mu_{\mathcal{A}_2}(x)}{q} + \frac{\widehat{\mu}_{\mathcal{A}_2}(x)}{q} \leq 1$ and $\mathcal{A}_3 = (\mu_{\mathcal{A}_3}(x), \widehat{\mu}_{\mathcal{A}_3}(x))$ such that $0 \leq \frac{\mu_{\mathcal{A}_3}x)}{q} + \frac{\widehat{\mu}_{\mathcal{A}_3}(x)}{q} \leq 1$, where $q \geq 2$ be any three q-fractional fuzzy sets. Then*

i) $If \mathcal{A}_1 \subseteq_P \mathcal{A}_2 \subseteq_P \mathcal{A}_3$, *then* $\mathcal{A}_1 \subseteq_P \mathcal{A}_3$,
ii) $If \mathcal{A}_1 \subseteq_R \mathcal{A}_2 \subseteq_R \mathcal{A}_3, then \mathcal{A}_1 \subseteq_R \mathcal{A}_3$,
iii) $(\mathcal{A}^c)^c = \mathcal{A}$,
iv) $(\mathcal{A}_1 \cup_P \mathcal{A}_2)^c = (\mathcal{A}_1)^c \cap_P (\mathcal{A}_2)^c$,
v) $(\mathcal{A}_1 \cap_P \mathcal{A}_2)^c = (\mathcal{A}_1)^c \cup_P (\mathcal{A}_2)^c$,
vi) $(\mathcal{A}_1 \cup_R \mathcal{A}_2)^c = (\mathcal{A}_1)^c \cap_R (\mathcal{A}_2)^c$,
vii) $(\mathcal{A}_1 \cap_R \mathcal{A}_2)^c = (\mathcal{A}_1)^c \cup_R (\mathcal{A}_2)^c$.

Theorem 2.4. [8] *Let $IG = (V, E, \sigma, \mu)$ be an intuitionistic fuzzy graph with n vertices and $(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_n)$ be the membership Laplacian eigenvalues, $(\widetilde{\lambda}_1 \geq \widetilde{\lambda}_2 \geq \widetilde{\lambda}_3 \geq ... \geq \widetilde{\lambda}_n)$ be non-membership Laplacian eigenvalues, then*

i) $\sum_{i=1}^{n} \lambda_i = 2 \sum_{1 \leq i < j \leq n} \sigma_{ij}$,
ii) $\sum_{i=1}^{n} \lambda_i^2 = 2 \sum_{1 \leq i < j \leq n} \sigma_{ij}^2 + \sum_{i=1}^{n} d^2(v_i)$,
iii) $\sum_{i=1}^{n} \widetilde{\lambda}_i = 2 \sum_{1 \leq i < j \leq n} \mu_{ij}$,
iv) $\sum_{i=1}^{n} \widetilde{\lambda}_i^2 = 2 \sum_{1 \leq i < j \leq n} \mu_{ij}^2 + \sum_{i=1}^{n} \widetilde{d}^2(v_i)$.

3 The q-fractional fuzzy graph

In this section, we formally define q-FFGs and their associated matrices. We also introduce the P-union and R-union of q-FFG and derive related results.

Definition 3.1. *Let $G = (V, E)$ be a simple graph with V as a non-empty set of vertices and $E \subseteq V \times V$ as a set of edges. A q-fractional fuzzy graph is defined as $G_q = (\mathcal{A}, \mathcal{B})$, where:*

$\mathcal{A} = \{(v_i, \mu_{\mathcal{A}}(v_i), \widehat{\mu}_{\mathcal{A}}(v_i)) \mid v_i \in V\}$ is a fuzzy q-fractional vertex set.

$\mathcal{B} = \{((v_i, v_j), \mu_{\mathcal{B}}(v_i, v_j), \widehat{\mu}_{\mathcal{B}}(v_i, v_j)) \mid (v_i, v_j) \in V \times V\}$ is a fuzzy q-fractional edge set. Such that the following conditions are satisfied:

1) For vertex grade of membership and non-membership mappings $\mu_{\mathcal{A}}, \widehat{\mu}_{\mathcal{A}} : V \to [0, 1]$,

$$0 \leq \frac{\mu_{\mathcal{A}}(v_i)}{q} + \frac{\widehat{\mu}_{\mathcal{A}}(v_i)}{q} \leq 1.$$

With $q \geq 2$, for all $v_i \in V$ $(i = 1, 2, 3..., n)$.

2) For edge grade of membership and non-membership mappings $\mu_{\mathcal{B}}, \widehat{\mu}_{\mathcal{B}} : V \times V \to [0, 1]$,

$$0 \leq \frac{\mu_{\mathcal{B}}(v_i, v_j)}{q} + \frac{\widehat{\mu}_{\mathcal{B}}(v_i, v_j)}{q} \leq 1.$$

With $q \geq 2$, for all $(v_i, v_j) \in V \times V$ $(i, j = 1, 2, 3..., n)$. Also,

$$\begin{aligned} \mu_{\mathcal{B}}(v_i, v_j) &= m_{ij} \leq \min\{\mu_{\mathcal{A}}(v_i), \mu_{\mathcal{A}}(v_j)\}, \\ \widehat{\mu}_{\mathcal{B}}(v_i, v_j) &= \widehat{m}_{ij} \geq \max\{\widehat{\mu}_{\mathcal{A}}(v_i), \widehat{\mu}_{\mathcal{A}}(v_j)\}. \end{aligned}$$

Note: for $q = 1$, it becomes intuitionistic fuzzy graph.

Consider the q-fractional fuzzy graph G_q comprising four vertices v_1, v_2, v_3 and v_4.

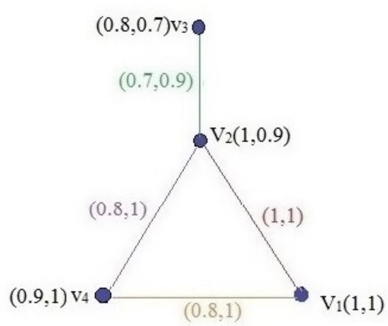

Figure 1: q-fractional fuzzy graph G_q

Each vertex has membership and non-membership degrees that satisfy the condition

$$0 \leq \frac{\mu_{\mathcal{A}}(v_i)}{q} + \frac{\widehat{\mu}_{\mathcal{A}}(v_i)}{q} \leq 1.$$

Similarly, each edge has membership and non-membership degrees that satisfy the q-fractional constraint

$$0 \leq \frac{\mu_{\mathcal{B}}(v_i, v_j)}{q} + \frac{\widehat{\mu}_{\mathcal{B}}(v_i, v_j)}{q} \leq 1.$$

Definition 3.2. *The order of a q-fractional fuzzy graph $G_q = (\mathcal{A}, \mathcal{B})$ is defined as:*

$$O(G_q) = \sum_{v_i \in V} (\mu_{\mathcal{A}}(v_i), \widehat{\mu}_{\mathcal{A}}(v_i)).$$

Where, $0 \leq \frac{\mu_{\mathcal{A}}(v_i)}{q} + \frac{\widehat{\mu}_{\mathcal{A}}(v_i)}{q} \leq 1$.

Example 3.3. *Consider a q-fractional fuzzy graph G_q as shown in Figure 1,*

$$O\left(G_q\right) = (3.7, 3.6)\,.$$

Definition 3.4. *The degree of a vertex $v_i \in V$ in a q-fractional fuzzy graph $G_q = (\mathcal{A}, \mathcal{B})$ is defined as:*

$$D\left(v_i\right) = \sum_{(v_i, v_j) \in E} \left(\mu_{\mathcal{B}}(v_i, v_j), \widehat{\mu}_{\mathcal{B}}(v_i, v_j)\right).$$

Where, $0 \leq \frac{\mu_{\mathcal{A}}(v_i,v_j)}{q} + \frac{\widehat{\mu}_{\mathcal{A}}(v_i,v_j)}{q} \leq 1$. Since $\mu_{\mathcal{B}}$ and $\widehat{\mu}_{\mathcal{B}}$ values are bounded but not interdependent, the degree of a vertex in q-FFG can reflect a wider range of interactions as compared to IFG, PyFG and q-ROFG.

Example 3.5. *Consider a q-fractional fuzzy graph in Figure 1, degree of each vertex is*

$$D(v_1) = (1.8, 2)\,,\ D(v_2) = (2.5, 2.9)\,,$$

$$D(v_3) = (0.7, 0.9)\,,\ D(v_4) = (1.6, 2)\,.$$

Definition 3.6. *Let $G_{q_1} = (\mathcal{A}_1, \mathcal{B}_1)$ and $G_{q_2} = (\mathcal{A}_2, \mathcal{B}_2)$ be two q-fractional fuzzy graphs, then their Cartesian product is defined as:*

$$G_{q_1} \times G_{q_2} = \left(\mathcal{A}_1 \times \mathcal{A}_2, \mathcal{B}_1 \times \mathcal{B}_2\right),$$

such that the following conditions are satisfied.

For all $v_1, v_2 \in V$:
1.$\mu_{\mathcal{A}_1 \times \mathcal{A}_2}(v_1, v_2) = \min\left\{\mu_{\mathcal{A}_1}(v_1), \mu_{\mathcal{A}_2}(v_2)\right\},$
$\widehat{\mu}_{\mathcal{A}_1 \times \mathcal{A}_2}(v_1, v_2) = \max\left\{\widehat{\mu}_{\mathcal{A}_1}(v_1), \widehat{\mu}_{\mathcal{A}_2}(v_2)\right\}.$
2. For all $v \in V_1, v_2u_2 \in E_2$:
$\mu_{\mathcal{B}_1 \times \mathcal{B}_2}\left((v, v_2)(v, u_2)\right) = \min\left\{\mu_{\mathcal{A}_1}(v), \mu_{\mathcal{B}_2}(v_2u_2)\right\},$
$\widehat{\mu}_{\mathcal{B}_1 \times \mathcal{B}_2}\left((v, v_2)(v, u_2)\right) = \max\left\{\widehat{\mu}_{\mathcal{A}_1}(v), \widehat{\mu}_{\mathcal{B}_2}(v_2u_2)\right\}.$
3. For all $v \in V_2, v_1u_1 \in E_1$:
$\mu_{\mathcal{B}_1 \times \mathcal{B}_2}\left((v_1, v)(u_1, v)\right) = \min\left\{\mu_{\mathcal{B}_1}(v_1u_1), \mu_{\mathcal{A}_2}(v)\right\},$
$\widehat{\mu}_{\mathcal{B}_1 \times \mathcal{B}_2}\left((v_1, v)(u_1, v)\right) = \max\left\{\widehat{\mu}_{\mathcal{B}_1}(v_1u_1), \widehat{\mu}_{\mathcal{A}_2}(v)\right\}.$

Proposition 3.7. *The Cartesian product of two q-fractional fuzzy graphs is a q-fractional fuzzy graph.*

Definition 3.8. *Let $G_{q_1} = (\mathcal{A}_1, \mathcal{B}_1)$ and $G_{q_2} = (\mathcal{A}_2, \mathcal{B}_2)$ be two q-fractional fuzzy graphs, then their P-union is defined as*

$G_{q_1} \cup_P G_{q_2} = \left(\mathcal{A}_1 \cup_P \mathcal{A}_2, \mathcal{B}_1 \cup_P \mathcal{B}_2\right),$ *where*
1. *For $v \in V_1$ and $v \notin V_2$:*
$\mu_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(v) = \left\{\mu_{\mathcal{A}_1}(v), \widehat{\mu}_{\mathcal{A}_1}(v)\right\}.$
2. *For $v \in V_2$ and $v \notin V_1$:*
$\mu_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(v) = \left\{\mu_{\mathcal{A}_2}(v), \widehat{\mu}_{\mathcal{A}_2}(v)\right\}.$
3. *For $v \in V_1 \cap V_2$:*
$\mu_{\mathcal{A}_1 \cup_P \mathcal{A}_2}(v) = \left\{\max\left(\mu_{\mathcal{A}_1}(v), \mu_{\mathcal{A}_2}(v)\right), \max\left(\widehat{\mu}_{\mathcal{A}1}(v), \widehat{\mu}_{\mathcal{A}_2}(v)\right)\right\}.$
4. *For $vw \in E_1$ and $vw \notin E_2$:*
$\mu_{\mathcal{B}_1 \cup_P \mathcal{B}_2}(vw) = \left\{\mu_{\mathcal{B}_1}(vw), \widehat{\mu}_{\mathcal{B}_1}(vw)\right\}.$
5. *For $vw \in E_2$ and $vw \notin E_1$,*
$\mu_{\mathcal{B}_1 \cup_P \mathcal{B}_2}(vw) = \left\{\mu_{\mathcal{B}_2}(vw), \widehat{\mu}_{\mathcal{B}_2}(vw)\right\}.$
6. *For $vw \in E_1 \cap E_2$.*
$\mu_{\mathcal{B}_1 \cup_P \mathcal{B}_2}(vw) = \left\{\max\left\{\mu_{\mathcal{B}_1}(vw), \mu_{\mathcal{B}_2}(vw)\right\}, \max\{\widehat{\mu}_{\mathcal{B}_1}(vw), \widehat{\mu}_{\mathcal{B}_2}(vw)\}\right\}.$

Example 3.9. *Consider two q-fractional fuzzy graphs G_{q_1} and G_{q_2} in Figure 2.*

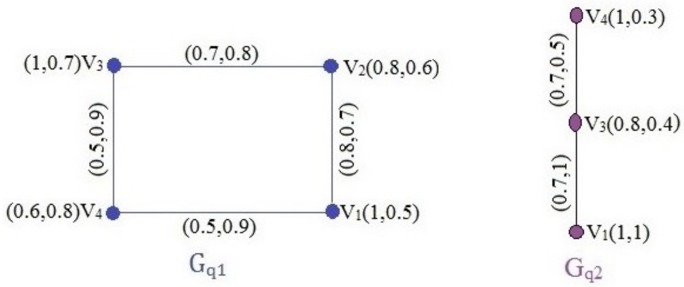

Figure 2: q-fractional fuzzy graphs G_{q_1} and G_{q_2}

Then, their P-union $G_{q_1} \cup_P G_{q_2}$ is shown in Figure 3. In this operation, the vertices and edges of G_{q_1} and G_{q_2} are combined under q-fractional framework where membership and non-membership degrees of common elements are determined using the P-union rule.

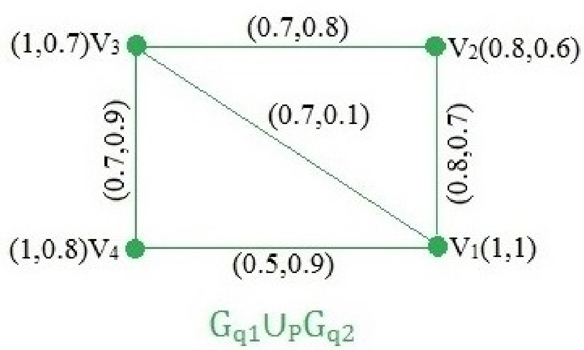

Figure 3: P-Union of q-fractional fuzzy graphs

Definition 3.10. *Let $G_{q_1} = (\mathcal{A}_1, \mathcal{B}_1)$ and $G_{q_2} = (\mathcal{A}_2, \mathcal{B}_2)$ be two q-fractional fuzzy graphs, then their R-union is defined as*

$$G_{q_1} \cup_R G_{q_2} = (\mathcal{A}_1 \cup_R \mathcal{A}_2, \mathcal{B}_1 \cup_R \mathcal{B}_2),$$

where

1. $\mu_{\mathcal{A}_1 \cup_R \mathcal{A}_2}(v) = \{\mu_{\mathcal{A}_1}(v), \widehat{\mu}_{\mathcal{A}_1}(v)\}$, *for* $v \in V_1$ *and* $v \notin V_2$,
2. $\mu_{\mathcal{A}_1 \cup_R \mathcal{A}_2}(v) = \{\mu_{\mathcal{A}_2}(v), \widehat{\mu}_{\mathcal{A}_2}(v)\}$, *for* $v \in V_2$ *and* $v \notin V_1$,
3. $\mu_{\mathcal{A}_1 \cup_R \mathcal{A}_2}(v) = \{\max(\mu_{\mathcal{A}_1}(v), \mu_{\mathcal{A}_2}(v)), \min(\widehat{\mu}_{\mathcal{A}_1}(v), \widehat{\mu}_{\mathcal{A}_2}(v))\}$,
   for $v \in V_1 \cap V_2$,
4. $\mu_{\mathcal{B}_1 \cup_R \mathcal{B}_2}(vw) = \{\mu_{\mathcal{B}_1}(vw), \widehat{\mu}_{\mathcal{B}_1}(vw)\}$,*for* $vw \in E_1$ *and* $vw \notin E_2$,
5. $\mu_{\mathcal{B}_1 \cup_R \mathcal{B}_2}(vw) = \{\mu_{\mathcal{B}_2}(vw), \widehat{\mu}_{\mathcal{B}_2}(vw)\}$,*for* $vw \in E_2$ *and* $vw \notin E_1$,
6. $\mu_{\mathcal{B}_1 \cup_R \mathcal{B}_2}(vw) = \{\max\{\mu_{\mathcal{B}_1}(vw), \mu_{\mathcal{B}_2}(vw)\}, \min\{\widehat{\mu}_{\mathcal{B}_1}(vw), \widehat{\mu}_{\mathcal{B}_2}(vw)\}\}$,
   for $vw \in E_1 \cap E_2$.

Example 3.11. *Consider two q-fractional fuzzy graphs G_{q_1} and G_{q_2} as shown in Figure 2, then their R-union is shown in Figure 4.*

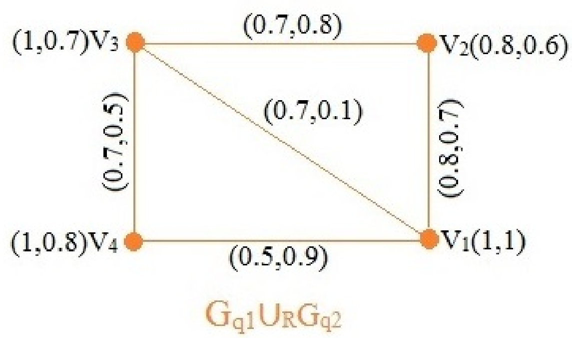

Figure 4: R-union of q-fractional fuzzy graphs

By applying the R-union operation, the two graphs are merged to form $G_{q_1} \cup_R G_{q_2}$. In this construction, the vertices and edges of both graphs are combined using R-union rule.

4 Energy of q-fractional fuzzy graphs

In this section, we explore the intricate world of q-fractional fuzzy graphs by providing innovative definitions and examples. To uncover foundational patterns, we analyze their membership and non-membership degree matrices along with their associated adjacency matrices. We then examine the membership and non-membership Laplacian matrices of these graphs, revealing their distinctive characteristics. A key focus of this discussion is to analyze their Laplacian energies, offering deeper understanding of the energy dynamics within complex graph structures.

Definition 4.1. *Degree matrix $D(G_q)$ of a q-fractional fuzzy graph $G_q = (\mathcal{A}, \mathcal{B})$ consists of two components: the membership degree matrix and the non-membership degree matrix, defined as follows:*

$$D(G_q) = [d_{ij}] = \left\{ \begin{array}{c} \left(d(v_i), \widehat{d}(v_i)\right) \text{ if } i = j \\ (0,0) \text{ otherwise} \end{array} \right\},$$

where $d(v_i)$ represents the membership degree and $\widehat{d}(v_i)$ the non-membership degree of vertex v_i.

In q-FFG, both membership and non-membership adjacency matrices are defined with independent entries under q-fractional bounds. This results in better control over the structure and more expressive representation of uncertainty, particularly useful in data-driven applications where relationship strengths vary significantly.

Example 4.2. *Consider the q-fractional fuzzy graph shown in Figure 1. The degree matrix $D(G_q)$ is defined as follows:*

$$D(G_q) = \begin{bmatrix} (1.8,2) & (0,0) & (0,0) & (0,0) \\ (0,0) & (2.5,3) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0.7,1) & (0,0) \\ (0,0) & (0,0) & (0,0) & (1.6,2) \end{bmatrix}.$$

Definition 4.3. *The adjacency matrix $A(G_q)$ of q-fractional fuzzy graph $G_q = (\mathcal{A}, \mathcal{B})$ consists of two components: the membership terms adjacency matrix and the non-membership terms adjacency matrix, defined as follows:*

$$A(G_q) = [(m_{ij}, \widehat{m}_{ij})].$$

Here, $m_{ij} = \sigma_{\mathcal{B}}(v_i, v_j)$ represents the membership edge degree and $\widehat{m}_{ij} = \widehat{\sigma}_{\mathcal{B}}(v_i, v_j)$ represents the non-membership edge degree.

Example 4.4. *For the q-fractional fuzzy graph shown in Figure 1, the adjacency matrix $A(G_q)$ is defined as follows:*

$$A(G_q) = \begin{bmatrix} (0,0) & (1,1) & (0,0) & (0.8,1) \\ (1,1) & (0,0) & (0.7,0.9) & (0.8,1) \\ (0,0) & (0.7,0.9) & (0,0) & (0,0) \\ (0.8,1) & (0.8,1) & (0,0) & (0,0) \end{bmatrix}.$$

Definition 4.5. *The energy $E(G_q)$ of a q-fractional fuzzy graph $G_q = (\mathcal{A}, \mathcal{B})$ is defined as the pair of the sum of the absolute eigenvalues of membership and non-membership terms adjacency matrices, that is*

$$E(G_q) = \left(\sum_{i=1}^{n} |\delta_i|, \sum_{i=1}^{n} |\widehat{\delta}_i| \right),$$

δ_i and $\widehat{\delta}_i$ for $(i = 1, 2, 3, ..., n)$ are the eigenvalues of the membership and non-membership terms adjacency matrices respectively. Among the various fuzzy graph models, q-fractional fuzzy graphs offer independent and the highest level of energy representation.

Example 4.6. *Consider the adjacency matrix $A(G_q)$ associated with graph in Figure 1. The eigenvalues of membership terms adjacency matrix are:*

$$1.8368, 0.17755, -0.77728, -1.2371.$$

The eigenvalues of non-membership terms adjacency matrix are:

$$2.1374, 0.26939, -1, -1.4068.$$

The energy of membership terms adjacency matrix is:

$$\sum_{i=1}^{4} |\delta_i| = 1.8368 + 0.17755 + 0.77728 + 1.2371 = 4.0287.$$

The energy of non-membership terms adjacency matrix is:

$$\sum_{i=1}^{4} |\widehat{\delta}_i| = 2.1374 + 0.26939 + 1 + 1.4068 = 4.8136.$$

The energy of graph is:

$$E(G_q) = \left(\sum_{i=1}^{4} |\delta_i|, \sum_{i=1}^{4} |\widehat{\delta}_i| \right) = (4.0287, 4.8136).$$

Theorem 4.7. *Let $G_q = (\mathcal{A}, \mathcal{B})$ be a q-fractional fuzzy simple graph and $\left(\delta_i, \widehat{\delta}_i\right)_{i=1}^{n}$ be the eigenvalues associated with the membership and non-membership terms adjacency matrices, then $\sum_{i=1}^{n} \left(\delta_i, \widehat{\delta}_i\right) = (0, 0)$.*

Proof. The sum of the eigenvalues of a graph's adjacency matrix, including those of fuzzy graph is equal to trace of adjacency matrix. Moreover, $A(G_q)$ is a symmetric matrix consisting of the membership and non-membership terms entries with zero trace thus, the eigenvalues of adjacency matrix have sum equal to zero. □

Theorem 4.8. *Let $G_q = (\mathcal{A}, \mathcal{B})$ be a q-fractional fuzzy simple graph and $\left(\delta_i, \widehat{\delta}_i\right)_{i=1}^{n}$ be the eigenvalues associated with the membership and non-membership terms adjacency matrices, then $\sum_{i=1}^{n} \left(\delta_i^2, \widehat{\delta}_i^2\right) = 2 \sum_{1 \leq i < j \leq n} \left(m_{ij}^2, \widehat{m}_{ij}^2\right)$.*

Proof. By general property of matrices, the sum of square of eigenvalues of adjacency matrix gives the trace of square of that matrix. Therefore, we have

$$\begin{aligned}
\sum_{i=1}^{n} \left(\delta_i^2, \widehat{\delta}_i^2\right) &= tr\left(A^2(G_q)\right) \\
&= \left((0 + m_{12}^2 + m_{13}^2 + ... + m_{1n}^2), (0 + \widehat{m}_{12}^2 + \widehat{m}_{13}^2 + ... + \widehat{m}_{1n}^2)\right) \\
&\quad + \left((m_{21}^2 + 0 + m_{23}^2 + ... + m_{2n}^2), (\widehat{m}_{21}^2 + 0 + \widehat{m}_{23}^2 + ... + \widehat{m}_{2n}^2)\right) \\
&\quad + \left((m_{n1}^2 + m_{n2}^2 + ... + 0), (\widehat{m}_{n1}^2 + \widehat{m}_{n2}^2 + ... + 0)\right) \\
&= 2 \sum_{1 \leq i < j \leq n} \left(m_{ij}^2, \widehat{m}_{ij}^2\right).
\end{aligned}$$

□

Example 4.9. *Consider the eigenvalues* $\left(\delta_i,\widehat{\delta}_i\right)_{i=1}^{4}$ *of the membership and non-membership terms adjacency matrices calculated in above example, then*

$$\begin{aligned}\sum_{i=1}^{4}\left(\delta_i^2,\widehat{\delta}_i^{\,2}\right) &= \left(\left(\delta_1^2,\widehat{\delta}_1^2\right)+\left(\delta_2^2,\widehat{\delta}_2^2\right)+\left(\delta_3^2,\widehat{\delta}_3^2\right)+\left(\delta_4^2,\widehat{\delta}_4^2\right)\right)\\ &= \begin{pmatrix} \left((1.8368)^2,(2.1374)^2\right)+\left((0.17755)^2,(0.26939)^2\right)\\ +\left((-0.77728)^2,(-1)^2\right)+\left((-1.2371)^2,(-1.4068)^2\right)\end{pmatrix}\\ &= (5.54,7.62)\,.\end{aligned}$$

Also,

$$\begin{aligned}2\sum_{1\le i<j\le 4}\left(m_{ij}^2,\widehat{m}_{ij}^2\right) &= 2\begin{pmatrix}\left(m_{12}^2,\widehat{m}_{12}^2\right)+\left(m_{13}^2,\widehat{m}_{13}^2\right)+\left(m_{14}^2,\widehat{m}_{14}^2\right)\\ +\left(m_{23}^2,\widehat{m}_{23}^2\right)+\left(m_{24}^2,\widehat{m}_{24}^2\right)+\left(m_{34}^2,\widehat{m}_{34}^2\right)\end{pmatrix}\\ &= 2\begin{pmatrix}\left(1^2,1^2\right)+\left(0^2,0^2\right)+\left((0.8)^2,1^2\right)\\ +\left(0.7^2,(0.9)^2\right)+\left((0.8)^2,1^2\right)+\left(0^2,0^2\right)\end{pmatrix}\\ &= 2(2.77,3.81)=(5.54,7.62).\end{aligned}$$

Hence $\sum_{i=1}^{4}\left(\nu_i^2,\widehat{\nu}_i^2\right)=2\sum_{1\le i<j\le 4}\left(m_{ij}^2,\widehat{m}_{ij}^2\right)$.

Definition 4.10. *Let* $D(G_q)$ *be the degree matrix and* $A\left(G_q\right)$ *be the adjacency matrix of q-fractional fuzzy graph* $G_q=(\mathcal{A},\mathcal{B})$*, then the Laplacian matrix* $L\left(G_q\right)$ *is defined as:*

$$L\left(G_q\right)=D\left(G_q\right)-A\left(G_q\right).$$

Example 4.11. *Consider the q-fractional fuzzy graph in Figure* 1*, we can find the Laplacian matrix as follows:*

$$\begin{aligned}L\left(G_q\right) &= D\left(G_q\right)-A\left(G_q\right)\\ &= \begin{bmatrix}(1.8,2) & (-1,-1) & (0,0) & (-0.8,-1)\\ (-1,-1) & (2.5,3) & (-0.7,-0.9) & (-0.8,-1)\\ (0,0) & (-0.7,-0.9) & (0.7,1) & (0,0)\\ (-0.8,-1) & (-0.8,-1) & (0,0) & (1.6,2)\end{bmatrix}.\end{aligned}$$

Definition 4.12. *The Laplacian polynomial of a q-fractional fuzzy graph* $G_q=(\mathcal{A},\mathcal{B})$ *is the characteristic polynomial of Laplacian matrix* $L\left(G_q\right)$ *and is defined as:*

$$P\left(L\left(G_q\right),\gamma\right)=\det((\gamma,\widehat{\gamma})I_n-L\left(G_q\right).$$

The roots of the Laplacian polynomial $(\gamma_i,\widehat{\gamma}_i)_{i=1}^{n}$ *are called the Laplacian eigenvalues of* G_q.

Theorem 4.13. *Let* $G_q=(\mathcal{A},\mathcal{B})$ *be a q-fractional fuzzy graph with n vertices and* $(\gamma_i,\widehat{\gamma}_i)_{i=1}^{n}$ *be the membership and non-membership terms eigenvalues associated with Laplacian matrix* $L\left(G_q\right)$*, then* $\sum_{i=1}^{n}(\gamma_i,\widehat{\gamma}_i)=2\sum_{1\le i<j\le n}(m_{ij},\widehat{m}_{ij})$.

Proof. Laplacian matrix comprising the membership and non-membership terms as entries are symmetric and positive semi-definite by definition, also their respective eigenvalues are non-negative, such that

$$\begin{aligned}\sum_{i=1}^{n}(\gamma_i,\widehat{\gamma}_i) &= \left(trL\left(G_q\right)\right)\\ &= \left(\sum_{i=1}^{n}d\left(v_i\right),\sum_{i=1}^{n}\widehat{d}(v_i)\right)\\ &= \left(2\sum_{1\le i<j\le n}m_{ij},2\sum_{1\le i<j\le n}\widehat{m}_{ij}\right)\\ &= 2\sum_{1\le i<j\le n}(m_{ij},\widehat{m}_{ij})\,.\end{aligned}$$

□

Theorem 4.14. *Let $G_q = (\mathcal{A}, \mathcal{B})$ be a q-fractional fuzzy graph with n vertices and $(\gamma_i, \widehat{\gamma}_i)_{i=1}^{n}$ be the eigenvalues associated with the Laplacian matrix $L(G_q)$, then $\sum_{i=1}^{n}\left(\gamma_i^2, \widehat{\gamma}_i^2\right) = 2\sum_{1\le i<j\le n}(m_{ij}^2, \widehat{m}_{ij}^2) + \sum_{i=1}^{n}\left(d^2(v_i), \widehat{d}^2(v_i)\right)$.*

Proof. Membership and non-membership Laplacian matrices are defined as:

$$
\begin{aligned}
(L(G_q)) &= \left(\begin{bmatrix} \left(d(v_1), \widehat{d}(v_1)\right) & (-m_{12}, -\widehat{m}_{12}) & \dots & (-m_{1n}, -\widehat{m}_{1n}) \\ (-m_{21}, -\widehat{m}_{21}) & \left(d(v_2), \widehat{d}(v_2)\right) & \dots & (-m_{2n}, -\widehat{m}_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ (-m_{n1}, -\widehat{m}_{n1}) & (-m_{n2}, -\widehat{m}_{n2}) & \dots & \left(d(v_n), \widehat{d}(v_n)\right) \end{bmatrix}\right) \\
&= \left(\begin{bmatrix} d(v_1) & -m_{12} & \dots & -m_{1n} \\ -m_{21} & d(v_2) & \dots & -m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -m_{n1} & -m_{n2} & \dots & d(v_n) \end{bmatrix}, \begin{bmatrix} \widehat{d}(z_1) & -\widehat{m}_{12} & \dots & -\widehat{m}_{1n} \\ -\widehat{m}_{21} & \widehat{d}(v_2) & \dots & -\widehat{m}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -\widehat{m}_{n1} & -\widehat{m}_{n2} & \dots & \widehat{d}(v_n) \end{bmatrix}\right).
\end{aligned}
$$

We obtain,

$$
\begin{aligned}
\left(tr(L(G_q)^2)\right) &= \begin{pmatrix} \left((d^2(v_1) + m_{12}^2 + \dots + m_{1n}^2), (\widehat{d}_a^2(v_1) + \widehat{m}_{12}^2 + \dots + \widehat{m}_{1n}^2)\right) \\ + \left((m_{21}^2 + d^2(v_2) + \dots + m_{2n}^2), (\widehat{m}_{21}^2 + \widehat{d}^2(v_2) + \dots + \widehat{m}_{2n}^2)\right) \\ \vdots \\ \left((m_{n1}^2 + m_{n2}^2 + \dots + d^2(v_n)), (\widehat{m}_{n1}^2 + \widehat{m}_{n2}^2 + \dots + \widehat{d}^2(v_n))\right) \end{pmatrix} \\
&= \left(2\sum_{1\le i<j\le n} m_{ij}^2 + \sum_{i=1}^{n} d^2(v_i), 2\sum_{1\le i<j\le n} \widehat{m}_{ij}^2 + \sum_{i=1}^{n} \widehat{d}^2(v_i)\right) \\
&= 2\sum_{1\le i<j\le n} (m_{ij}^2, \widehat{m}_{ij}^2) + \sum_{i=1}^{n} \left(d^2(v_i), \widehat{d}^2(v_i)\right).
\end{aligned}
$$

□

5 Applications

In real-world scenarios, we often encounter situations where a person has to maintain a relationship with an individual, group or community despite having minor or major reservations. Conversely, there are scenarios where a person might have no specific concerns about a community but still does not engage actively with it .

Example 5.1. *Social Interaction*

Consider a community-based organization with three individuals: Ali, Saira and Tahir. Their membership and non-membership behaviors towards the organization are described as follows: Ali is an active contributor, regularly involved in decision-making, without any concerns about the organization's operations. His attitude is modeled as $(1, 0)$. Saira is also an active member, who often participates in activities, but she frequently voices strong reservations about the organization's transparency. Her profile can be modeled as $(1, 0.8)$. Tahir is a member of the organization and holds deep skepticism about its motives, leading to a profile of $(1, 1)$. This scenario can't be modeled using IFG, PyFG or q-rung models, where the membership and non-membership degrees are mutually dependent.

In the modern governance system, governments often launch policy initiatives such as environmental regulations, healthcare reforms or tax reforms that generate mixed reactions from citizens. Traditional fuzzy models like IFGs, PyFGs typically assume a strict dependency between membership and non-membership values, making them inadequate for capturing sentiments where support and distrust may coexist. A q-fractional fuzzy graph can represent varying sentiments.

Example 5.2. *Public Sentiment Towards Policy Initiatives*

Suppose a citizen X supports the policy wholeheartedly without any concerns, then this situation can be modeled as $(1,0)$. Citizen Y supports the policy goals but completely distrusts the implementation process. This attitude can be represented by $(1,1)$. Citizen Z rejects the policy and distrusts the governing body. This behavior corresponds to $(0,1)$. Citizen W does not support the policy due to personal priorities but believes it is fair. This can be expressed as $(0,0)$. This level of expressiveness is not achievable using existing models that restrict the sum of membership and non-membership values.

In international relations, countries often maintain formal relationships based on mutual interests while simultaneously harboring ideological disagreements and geopolitical concerns. Such relationships involve a duality, with cooperation and conflict coexisting. Traditional fuzzy graph models, which typically assume a dependency between degrees of membership and non-membership, lack the expressive capacity to represent such refined dualities.

Example 5.3. *Diplomatic Relations with Ideological Disparity*

Consider three countries, Country A, Country B and Country C. Country A and Country B maintain strong military and intelligence cooperation. However, Country A has persistent and public concerns over B's human rights violations and governance practices. Their relationship can be represented by $(1,1)$. Country B and Country C are tied by significant economic interdependence. At the same time, Country B regularly expresses discontent with C's foreign policy and military expansion, situation can be modeled as $(0.85, 0.75)$. Country A and Country C maintain minimal diplomatic engagement. However, due to long-standing historical and territorial disputes, mutual distrust remains very high, resulting in the pair $(0.5, 1)$.

Graphical Comparison of q-FFG with Existing Models

The q-fractional fuzzy graph is a useful generalization of existing models such as IFGs, PyFGs and q-ROFGs. It accommodates extreme values that are not addressed by existing models. Here, we provide a graphical comparison of the q-FFG space with q-ROFGs and other existing spaces.

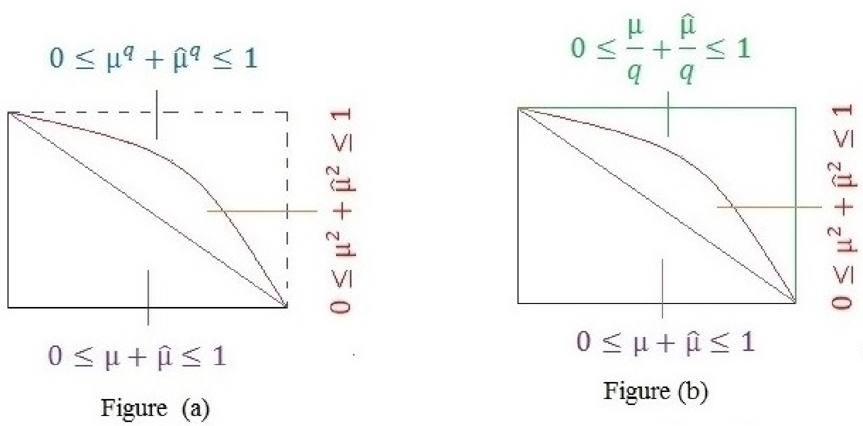

Figure 5: graphical comparison of q-ROFG and q-FFG spaces

The above figure illustrates the graphical comparison between q-ROFG and q-FFG spaces. Dotted lines in q-ROFG space indicate the region where the sum of q-th powers of membership and non-membership grades exceeds 1, showing that q-ROFG model fails to accommodate this region. In contrast, the q-FFG space (solid lines) admits this region, highlighting its ability to represent independent and extreme grades of uncertainty. This clearly demonstrates that the q-FFG model extends beyond the limitations of q-ROFG and other existing models like IFG and PyFG.

The q-FFG: An Illustrative Scenario

Consider a group of four individuals: Alex, Bob, Chris, and David. Their degrees of real life and social media interactions are modeled using the pair $(\mu, \widehat{\mu})$, where $\mu, \widehat{\mu} \in [0,1]$ respectively represent the degree of activeness in real interactions (membership) and inactiveness on social media (non-membership) platforms. It is important to note that a low value of $\widehat{\mu}$ indicates high activity on social media, and vice versa. The behavioral profile and interaction strengths are illustrated in Figure 6.

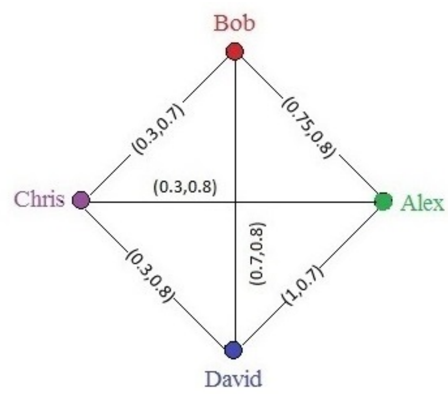

Figure 6: q-fractional fuzzy graph

1. Spectral Analysis under the IFG Constraint

The adjacency matrix comprising the membership and non-membership degrees under the IFG constraint is:

$$A=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & (0.3,0.7) & 0\\ 0 & (0.3,0.7) & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues associated with the adjacency matrix A are:

$$\{(0.3,0.7),(0,0),(0,0),(-0.3,-0.7)\}.$$

Hence, the energy of the IFG is: $(0.6,1.4)$.

This shows that under the IFG setting, only a limited number of interactions are captured, resulting in lower spectral energy values.

2. Spectral Analysis under the PyFG Constraint

The adjacency matrix under the PyFG constraint is:

$$B=\begin{bmatrix} 0 & 0 & (0.3,0.8) & 0\\ 0 & 0 & (0.3,0.7) & 0\\ (0.3,0.8) & (0.3,0.7) & 0 & (0.3,0.8)\\ 0 & 0 & (0.3,0.8) & 0 \end{bmatrix}.$$

The eigenvalues associated with the adjacency matrix B are:

$$\left\{\begin{array}{c} (0.51962,1.3304),\left(2.7367\times10^{-30},-5.3085\times10^{-31}\right),\\ \left(-1.2622\times10^{-30},-2.1803\times10^{-29}\right),(-0.51962,-1.3304) \end{array}\right\}.$$

Thus, the energy of the PyFG is: $(1.0392,2.6608)$.

Compared to IFG, PyFG accommodates a richer relational structure, reflecting a higher energy due to enhanced tolerance in the membership and non-membership values.

3. Spectral Analysis under the q-ROFG Constraint

The adjacency matrix under the q-ROFG constraint is:

$$C=\begin{bmatrix} 0 & (0.75,0.8) & (0.3,0.8) & 0\\ (0.75,0.8) & 0 & (0.3,0.7) & (0.7,0.8)\\ (0.3,0.8) & (0.3,0.7) & 0 & (0.3,0.8)\\ 0 & (0.7,0.8) & (0.3,0.8) & 0 \end{bmatrix}.$$

The eigenvalues associated with the adjacency matrix C are:

$$\left\{\begin{array}{c} (1.2382,1.9878),\left(8.5834\times10^{-4},-5.0968\times10^{-30}\right),(0.20467,-0.7),\\ (-1.0344,-1.2878) \end{array}\right\}.$$

Hence, the energy of the q-ROFG is: $(2.4781,3.9756)$.

This indicates a further expansion of the expressive power, capturing more complex relationships compared to IFG and PyFG.

4. Spectral Analysis under the q-FFG Constraint

Finally, the adjacency matrix under the q-FFG constraint is:

$$D = \begin{bmatrix} 0 & (0.75, 0.8) & (0.3, 0.8) & (1, 0.7) \\ (0.75, 0.8) & 0 & (0.3, 0.7) & (0.7, 0.8) \\ (0.3, 0.8) & (0.3, 0.7) & 0 & (0.3, 0.8) \\ (1, 0.7) & (0.7, 0.8) & (0.3, 0.8) & 0 \end{bmatrix}.$$

The eigenvalues associated with the adjacency matrix D are:

$$\{(1.7911, 2.3), (-0.14861, -0.7), (-0.63986, -0.7), (-1.0027, -0.9)\}.$$

Therefore, the energy of the q-FFG is: $(3.5823, 4.6)$.

The highest energy values demonstrate that the q-FFG framework provides the most comprehensive representational capacity, capturing all possible interaction dynamics between the individuals.

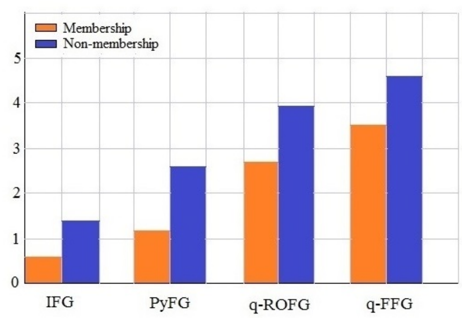

Figure 7: Comparative Energy Analysis

Dimension reduction in q-fractional fuzzy graph using PCA

Principal component analysis (PCA) is a well known statistical technique for dimensionality reduction. It transforms high dimensional data into a lower dimensional space while preserving the most significant information. In this work, we propose an algorithm that applies PCA to a q-fractional fuzzy graphs (q-FFG), where edges are characterized by independently defined membership and non-membership degrees. The goal is to reduce the complexity of the graph while retaining key structural and informational patterns, especially in real world scenarios like medical diagnosis systems. The algorithm consists of the following steps.

Step 1.

Consider a q-fractional fuzzy graph $G_q = (\mathcal{A}, \mathcal{B})$ with N vertices. Each edge $(v_i, v_j) \in E$ is assigned a membership degree $a_{ij} \in [0, 1]$, non-membership degree $b_{ij} \in [0, 1]$, both independently chosen. Construct the adjacency matrix A, where each entry is a pair

$$A = \begin{bmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) & ... & (a_{1N}, b_{1N}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) & ... & (a_{2N}, b_{2N}) \\ \vdots & \vdots & \vdots & \vdots \\ (a_{N1}, b_{N1}) & (a_{N2}, b_{N2}) & ... & (a_{NN}, b_{NN}) \end{bmatrix}.$$

Step 2.

For each vertex v_i compute the average membership and non-membership values across all adjacent vertices.

$$(A_i, B_i) = \left(\frac{a_{i1} + a_{i2} + ... + a_{iN}}{N}, \frac{b_{i1} + b_{i2} + ... + b_{iN}}{N} \right), i = 1, 2, 3, ..., N.$$

This step combines the information from all the vertices joining a particular vertex into single scalar values, representing the strength of average membership and non-membership terms across the source. Here, A_i represents the

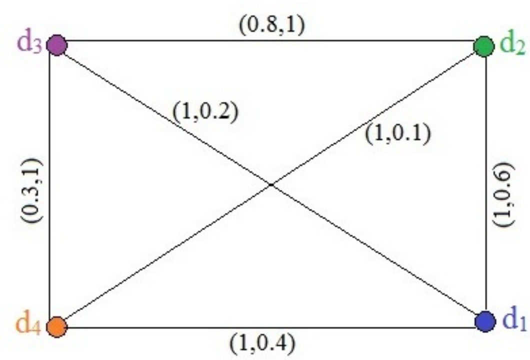

Figure 8: q-fractional fuzzy graph

average membership values and B_i represents the average non-membership term values of each column. This gives us two vectors $X_1 = \{A_1, A_2, A_3, ..., A_n\}$ and $X_2 = \{B_1, B_2, B_3, ..., B_n\}$.

Step 3.

In this step, first we find the mean of each feature vector, then we find covariance matrix whose entries are calculated using relation:

$$cov(X_j, X_k) = \frac{1}{n-1}\sum_{i=1}^{n}(A_i - \overline{X}_1)(B_i - \overline{X}_2). \text{ Where } j, k = 1, 2.$$

Step 4.

In this step, we compute the eigenvalues and eigenvectors of covariance matrix obtained in step 4.

Step 5.

In last step, we first normalize the eigenvectors and find principal component.

A case study:

In healthcare decision support system, we often deal with symptom overlap, partial test results, and conflicting indicators across diseases. Consider a q-fractional fuzzy graph in Figure 4, constructed from four respiratory diseases: d_1 =COVID-19, d_2 =Flu, d_3 =Pneumonia and d_4 =Asthma. Aim of this algorithm is to simplify and visualize a medical diagnosis network modeled as a q-fractional fuzzy graph, where both membership (diagnostic evidence) and non-membership (diagnostic contradiction) degrees are allowed to vary independently. The algorithm reduces high-dimensional fuzzy data to a lower-dimensional space using Principal Component Analysis (PCA) technique, preserving essential diagnostic patterns.

Step 1.

The q-fractional fuzzy graph of the diseases is shown in Figure 8.

First we find the associated adjacency matrix

$$A = \begin{bmatrix} 0 & (1, 0.6) & (1, 0.2) & (1, 0.4) \\ (1, 0.6) & 0 & (0.8, 1) & (1, 0.1) \\ (1, 0.2) & (0.8, 1) & 0 & (0.3, 1) \\ (1, 0.4) & (1, 0.1) & (0.3, 1) & 0 \end{bmatrix}.$$

Step 2.

Next, we find the average of entries of each column of adjacency matrix obtained in step 1.

$$(A_1, B_1) = \left(\frac{0+1+1+1}{4}, \frac{0+0.6+0.2+0.4}{4}\right) = (0.75, 0.3).$$

$$(A_2, B_2) = \left(\frac{1+0+0.8+1}{4}, \frac{0.6+0+1+0.1}{4}\right) = (0.7, 0.425).$$

$$(A_3, B_3) = \left(\frac{1+0.8+0+0.3}{4}, \frac{0.2+1+0+1}{4}\right) = (0.525, 0.55).$$

$$(A_4, B_4) = \left(\frac{1+1+0.3+0}{4}, \frac{0.4+0.1+1+0}{4}\right) = (0.575, 0.375)\,.$$

Step 4.
Let $X_1 = \{A_1, A_2, A_3, A_4\}$ and $X_2 = \{B_1, B_2, B_3, B_4\}$.

X_1	0.75	0.7	0.525	0.575
X_2	0.3	0.425	0.55	0.375

$\bar{X}_1 = 0.6375,\ \bar{X}_2 = 0.4125.$

$$\begin{aligned} cov(X_1, X_1) &= \frac{1}{N-1}\sum_{k=1}^{N}(A_k - \bar{X}_1)(A_k - \bar{X}_1) \\ &= \frac{1}{3}\left[\begin{array}{c} (0.75-0.6375)^2 + (0.7-0.6375)^2 \\ +(0.525-0.6375)^2 + (0.575-0.6375)^2 \end{array}\right] \\ &= \frac{1}{3}\left[1.2656\times10^{-2} + 3.9063\times10^{-3} + 1.2656\times10^{-2} + 3.9063\times10^{-3}\right] \\ &= 0.011042. \end{aligned}$$

$$\begin{aligned} cov(X_1, X_2) &= \frac{1}{N-1}\sum_{k=1}^{N}(A_k - \bar{X}_1)(B_k - \bar{X}_2) \\ &= \frac{1}{3}\left[\begin{array}{c} (0.75-0.6375)\,(0.3-0.4125) + (0.7-0.6375)\,(0.425-0.4125) + \\ (0.525-0.6375)\,(0.55-0.4125) + (0.575-0.6375)\,(0.375-0.4125) \end{array}\right] \\ &= \frac{1}{3}\left[-1.2656\times10^{-2} + 7.8125\times10^{-4} - 1.5469\times10^{-2} + 2.3438\times10^{-3}\right] \\ &= \frac{1}{3}\left[-2.5000\times10^{-2}\right] = -8.3333\times10^{-3}. \end{aligned}$$

$$\begin{aligned} cov(X_2, X_2) &= \frac{1}{N-1}\sum_{k=1}^{N}(B_k - \bar{X}_2)(B_k - \bar{X}_2) \\ &= \frac{1}{3}\left[\begin{array}{c} (0.3-0.4125)^2 + (0.425-0.4125)^2 \\ +(0.55-0.4125)^2 + (0.375-0.4125)^2 \end{array}\right] \\ &= \frac{1}{3}\left[1.2656\times10^{-2} + 1.5625\times10^{-4} + 1.8906\times10^{-2} + 1.4063\times10^{-3}\right] \\ &= \frac{3.3125\times10^{-2}}{3} = 0.011042. \end{aligned}$$

$$cov(X_2, X_1) = cov(X_1, X_2) = -8.3333\times10^{-3}.$$

Thus, the covariance matrix is:

$$\begin{pmatrix} cov(X_1,X_1) & cov(X_1,X_2) \\ cov(X_2,X_1) & cov(X_2,X_2) \end{pmatrix} = \begin{pmatrix} 0.011042 & -8.3333\times10^{-3} \\ -8.3333\times10^{-3} & 0.011042 \end{pmatrix}.$$

Step 5.
Now we move on to eigenvalues calculations. The eigenvalues obtained for the covariance matrix are $\lambda_1 = 0.019375, \lambda_2 = 0.0027087$, corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 0.70711 \\ -0.70711 \end{pmatrix}, v_2 = \begin{pmatrix} 0.70711 \\ 0.70711 \end{pmatrix}.$$

Step 6.
In our last step, we move on to principal component calculation. To normalize the eigenvectors, we first find magnitudes of eigenvectors

$$\sqrt{(0.70711)^2 + (-0.70711)^2} = 1, \sqrt{(0.70711)^2 + (0.70711)^2} = 1.$$

Normalized eigenvectors are:

$$e_1 = \left\{ \begin{pmatrix} 0.70711 \\ -0.70711 \end{pmatrix} \right\}, e_2 = \begin{pmatrix} 0.70711 \\ 0.70711 \end{pmatrix}.$$

Transpose of normalized eigenvectors are:

$$e_1^T = \left\{ \begin{pmatrix} 0.70711 & -0.70711 \end{pmatrix} \right\},\ e_2^T = \left\{ \begin{pmatrix} 0.70711 & 0.70711 \end{pmatrix} \right\}.$$

Now, we iterate through different values of k to compute principal component values.
Consider

$$e_1^T \begin{pmatrix} A_k - \bar{X}_1 \\ B_k - \bar{X}_2 \end{pmatrix}, k = 1, 2, 3, 4.$$

$$\begin{aligned} e_1^T \begin{pmatrix} A_1 - \bar{X}_1 \\ B_1 - \bar{X}_2 \end{pmatrix} &= \begin{pmatrix} 0.70711 & -0.70711 \end{pmatrix} \begin{pmatrix} 0.75 - 0.6375 \\ 0.3 - 0.4125 \end{pmatrix} \\ &= \begin{pmatrix} 0.70711 & -0.70711 \end{pmatrix} \begin{pmatrix} 0.1125 \\ -0.1125 \end{pmatrix} \\ &= 0.15910. \end{aligned}$$

$$\begin{aligned} e_1^T \begin{pmatrix} A_2 - \bar{X}_1 \\ B_2 - \bar{X}_2 \end{pmatrix} &= \begin{pmatrix} 0.70711 & -0.70711 \end{pmatrix} \begin{pmatrix} 0.7 - 0.6375 \\ 0.425 - 0.4125 \end{pmatrix} \\ &= \begin{pmatrix} 0.70711 & -0.70711 \end{pmatrix} \begin{pmatrix} 0.0625 \\ 0.0125 \end{pmatrix} \\ &= 0.035356. \end{aligned}$$

$$\begin{aligned} e_1^T \begin{pmatrix} A_3 - \bar{X}_1 \\ B_3 - \bar{X}_2 \end{pmatrix} &= \begin{pmatrix} 0.086025 & 0.996\,29 \end{pmatrix} \begin{pmatrix} 0.525 - 0.6375 \\ 0.55 - 0.4125 \end{pmatrix} \\ &= \begin{pmatrix} 0.086025 & 0.99629 \end{pmatrix} \begin{pmatrix} -0.1125 \\ 0.1375 \end{pmatrix} = 0.12731. \end{aligned}$$

$$\begin{aligned} e_1^T \begin{pmatrix} A_4 - \bar{X}_1 \\ B_4 - \bar{X}_2 \end{pmatrix} &= \begin{pmatrix} 0.70711 & -0.70711 \end{pmatrix} \begin{pmatrix} 0.575 - 0.6375 \\ 0.375 - 0.4125 \end{pmatrix} \\ &= \begin{pmatrix} 0.70711 & -0.70711 \end{pmatrix} \begin{pmatrix} -0.0625 \\ -0.0375 \end{pmatrix} \\ &= -0.017678. \end{aligned}$$

Iteration	**Iteration** 1	**Iteration** 2	**Iteration 3**	**Iteration 4**
First Principal component	0.15910	0.035356	0.12731	−0.017678

Table 1: **PCA Findings**

The first principal component captures the dominant fuzzy behavior across the graph by projecting each node's membership and non-membership profile onto a single axis of maximum variance. Nodes with high scores (e.g., COVID−19 and Pneumonia) exhibit a stronger fuzzy relationship structure, while those with lower or negative scores (e.g., Asthma) deviate from this dominant pattern. This transformation enables direct comparison of node behavior using a single scalar value, simplifying downstream analysis and visualization.

6 Comparison

The introduction of q-FFG marks a significant advancement in fuzzy graph theory. It eliminates the rigid interdependence between membership and non-membership grades found in existing models such as IFGs, PyFGs, and q-ROFGs.

Unlike these traditional approaches, which impose strict constraints on combined values of the membership and non-membership degrees to stay within strict bounds, q-FFGs enable each degree to independently attain any value in [0,1], including the maximum value of 1. This unique flexibility enables the modeling of vertex and edge states such as (1,1), (1,0.5), (0.2,1), or similar combinations without violating any structural rule. Existing models cannot accommodate such configurations due to their inherent mathematical constraints. The resulting adjacency and degree matrices exhibit greater diversity, leading to a broader and more dynamic range of graph energy. These enhanced spectral properties allow for a deeper understanding of network behavior, making q-FFGs an indispensable tool for modeling highly uncertain and conflicting scenarios in various domains.

Fuzzy Graphs	**Limitations**	**Graph Flexibility**	**Energy Trend**
Intuitionistic Fuzzy Graph (IFG)	$0 \leq \mu(v_i, v_j) + \nu(v_i, v_j) \leq 1$	Less flexible: μ and ν are strongly coupled	Lower energy (more conservative)
Pythagorean Fuzzy Graph (PyFG)	$0 \leq \mu^2(v_i, v_j) + \nu^2(v_i, v_j) \leq 1$	Moderate flexibility	Slightly higher than IFG
q-Rung Ortho Pair Fuzzy Graph (q-ROFG)	$0 \leq \mu^q(v_i, v_j) + \nu^q(v_i, v_j) \leq 1$	High flexibility for larger q	Increases with q
q-Fractional Fuzzy Graph	$0 \leq \frac{\mu(v_i,v_j)}{q} + \frac{\nu(v_i,v_j)}{q} \leq 1$	Highest flexibility: μ and ν independent, and scaling fractional	Highest energy among all (most expressive)

Table 2: **Comparative Energy Analysis**

7 Conclusion

In this study, we introduced the novel concept of the q-fractional fuzzy graph (q-FFG) as a generalization of q-rung orthopair fuzzy graphs (q-ROFGs). The proposed framework overcomes key limitations of existing models by allowing both membership and non-membership degrees to attain extreme values independently. We established a series of theoretical results for the proposed model and presented graphical comparisons with existing frameworks to demonstrate the enhanced representational capacity of q-FFGs. The study mainly focused on the energy analysis of q-FFG structures, where several new results were derived and validated through graphical and experimental illustrations. Furthermore, we incorporated Principal Component Analysis (PCA), a well-known dimensionality reduction technique, in conjunction with q-FFG to efficiently reduce the dimensionality of data in complex fuzzy environments.

Beyond the scope of this paper, several promising research avenues emerge. Recent works on (p, q)-fractional fuzzy sets [3], q-fractional hesitant fuzzy sets [4], and fractional fuzzy similarity measures [13] point to direct extensions of the present framework into multi-criteria decision making, power system evaluation, and supply chain management. Just like this work takes advantage of previous works on spectral theory of fuzzy graphs and intuitionistic fuzzy graphs to study, reference [1] inspires the investigation of graph energy for complex q-fractional fuzzy sets. Furthermore, while the energy of q-ROFGs remains unexplored, our results indicate that investigating graph energy within the q-fractional fuzzy setting may yield fruitful outcomes for both theory and applications.

Conflict of Interest
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