Iranian Journal of Fuzzy Systems

Volume 23, Number 1, (2026), pp. 75-94

Original paper

Strong robust similarity measures: A detailed analysis and application

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Abstract

Similarity measures are fundamental tools for comparing and evaluating data across various domains. Robust similarity measures extend classical similarities. However, when dealing with interval data, robust measures are often insufficient due to the intrinsic properties of intervals. In this study, we introduce the concept of strong robust similarity measures, which incorporate three additional axioms specifically considered to manage uncertainty represented by intervals. Furthermore, we characterize these measures through a novel class of functions, referred to as preinclusions. We also provide a comprehensive analysis of the proposed measures, examining their behaviour with respect to different axioms. Finally, we illustrate the applicability of our approach through a real-world case study using meteorological data collected by AEMET (the Spanish National Weather Service) in 2021.

Keywords: Similarity measure, robust similarity, strong robust similarity, interval, overlapping degree, meteorological data.

1 Introduction

In many real-world scenarios, determining the exact value of a variable is difficult or inherently uncertain. This challenge can be effectively addressed through the use of intervals, which allow us to represent a range of possible values rather than a single point. Intervals thus provide a powerful framework for modeling uncertainty and imprecision. The concept of interval-valued data has attracted considerable attention in recent years due to its broad applicability across diverse fields[1, 10, 21, 22, 25, 29].

In the literature, two main interpretations of intervals have been proposed: the epistemic and the ontic [8]. The ontic interpretation aims to provide an accurate description of reality, whereas the epistemic interpretation represents both reality and our knowledge about it, acknowledging the limited precision of available information. Generally speaking, the ontic model yields a precise, though potentially inaccurate, result, while the epistemic model produces an imprecise result that remains consistent with the underlying reality. In this study, we adopt the epistemic perspective.

Comparison is a fundamental topic in the study of fuzzy sets and their extensions, leading to the development of numerous measures for comparing such entities [2, 7, 17]. Similarity measures, in particular, have been extensively investigated due to their wide applicability in diverse domains, including fuzzy analysis [27], clustering [15], and decision-making [26], among others. The literature provides a wide variety of similarity measures, each exhibiting distinct advantages and limitations. In general, a similarity measure is defined as a real-valued function that quantifies the degree of resemblance between two entities, where a value of 0 denotes complete dissimilarity and a value of 1 indicates identical elements.

This paper specifically addresses similarity measures for interval data and the analysis of their properties. We introduce a new family of measures, called strong robust similarity measures, which extends the frameworks proposed by Huidobro et al. [11] and Kabir et al. [13]. The new family overcomes several of the limitations commonly associated

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Received: October 2024; Revised: November 2025; Accepted: December 2025.

https://doi.org/ 10.22111/ijfs.2025.49944.8826

with interval comparison measures, such as aliasing. Kabir et al. [13] identified aliasing as one of the most critical issues in computing similarity between closed intervals. Aliasing arises when distinct interval pairs yield identical similarity values, revealing that certain measures lack adequate sensitivity to variations in the intervals. Some similarity measures derived from the ratio model [24] are free from aliasing, providing more reliable comparisons.

Specifically, strong robust similarity measures are derived by incorporating three additional axioms into the existing framework of robust similarity measures. In addition to introducing this new class of measures, we also propose a general construction method applicable to both strong robust and (robust) similarity measures. This method is based on the composition of aggregation functions and a new class of functions, termed pre-inclusions, which are also introduced in this paper. It is worth recalling that this work focuses on similarity measures for interval data, under the assumption that the true value of a variable lies within the corresponding interval—an assumption consistent with the epistemic interpretation. Moreover, since all the proposed concepts are developed within the unit interval $[0,1]$, the framework can be naturally extended to interval-valued fuzzy sets [22].

Although numerous similarity measures for intervals have been proposed in the literature, most of them present important limitations. Classical measures such as Jaccard or Dice are not robust, since they suffer from aliasing and fail to capture differences in interval widths. Other measures based on the ratio model partially address these issues but do not fulfill all the robustness requirements. More recently, robust similarity measures have been introduced, but they still lack sensitivity with respect to width and scaling in some cases. This reveals a clear research gap: the need for similarity measures that jointly avoid aliasing, preserve robustness, and incorporate additional desirable properties.

The introduction of robustness is therefore crucial when comparing intervals. Without robustness, different interval pairs can yield the same similarity value (aliasing), or measures may ignore relevant structural differences such as scaling or width. Robust similarity measures address these issues by incorporating axioms that enforce sensitivity to overlap, inclusion, and scaling, ensuring that the similarity values faithfully reflect the underlying interval relationships. This makes robustness a key property that distinguishes our approach from many classical techniques in the literature.

The main contributions of this paper aim to fill this gap. First, the notion of strong robust similarity measures is introduced by extending the axiomatic framework of robust similarities with three new conditions. Second, a general construction method based on pre-inclusions and aggregation functions is provided, through which similarity measures with guaranteed properties are systematically generated. Third, a detailed theoretical analysis is conducted to identify which existing and newly proposed measures satisfy the full set of axioms. Finally, the usefulness of these measures is illustrated through an application to interval data derived from real temperature records. Altogether, both a theoretical advance and a practical tool for applications requiring reliable interval comparison are provided by these contributions.

The proposed framework enhances the reliability of interval comparison in both theoretical and applied contexts. Reliable similarity measures are crucial in areas such as decision-making, pattern recognition, clustering, and environmental data analysis, where uncertainty is naturally represented through intervals. In particular, applications involving meteorological or sensor data can benefit from measures that remain stable under noise, scaling, or partial overlap between intervals. By introducing strong robust similarity measures and their underlying pre-inclusion functions, this work provides a general and extensible foundation for constructing similarity models capable of capturing meaningful relationships between uncertain quantities.

2 Preliminaries

In this section, we recall the basic notions and notation used throughout the paper.

Let $\mathcal{L}([0,1]) = \{[\underline{a},\overline{a}] : \underline{a} \leq \overline{a} \text{ and } \underline{a},\overline{a} \in [0,1]\}$ be the family of closed subintervals of $[0,1]$ (the empty set is not in $\mathcal{L}([0,1])$). For $a = [\underline{a},\overline{a}] \in \mathcal{L}([0,1])$, its width is defined as $w(a) = \overline{a} - \underline{a} \in [0,1]$ [3]; by convention, $w(\emptyset) = 0$. Let us now introduce some basic operations related to intervals:

- **Inclusion:** $a \subseteq b$ iff $\underline{b} \leq \underline{a} \leq \overline{a} \leq \overline{b}$.
- **Intersection:** $a \cap b = \{x \in [0,1] : x \in a \wedge x \in b\}$.
- **Union:** $a \cup b = \{x \in [0,1] : x \in a \vee x \in b\}$.
- **Subtraction:** $a \setminus b = \{x \in [0,1] : x \in a \wedge x \notin b\}$.

When $a \cap b \neq \emptyset$, $a \cap b = [\max\{\underline{a},\underline{b}\}, \min\{\overline{a},\overline{b}\}]$; otherwise $a \cap b = \emptyset$.

Based on the inclusion relationship between the intervals, Zeng and Guo [28] define an inclusion measure as follows:

Definition 2.1. *A real function $I : \mathcal{L}([0,1]) \times \mathcal{L}([0,1]) \to [0,1]$ is called an inclusion measure if it fulfills the following properties:*

I0. $I([1,1],[0,0]) = 0$.

I1. $I(a,b) = 1$ *if and only if* $a \subseteq b$.

I2. *If* $a \subseteq b \subseteq c$*, then* $I(c,a) \leq I(c,b)$.

I3. *If* $a \subseteq b \subseteq c$*, then* $I(c,a) \leq I(b,a)$.

Another central concept in this paper is that of a similarity measure. In our context, similarities quantify the degree of resemblance between two intervals. Following Kabir et al. [13], we adopt a unified presentation in which robust similarities extend the usual axioms of similarity with additional robustness requirements:

Definition 2.2. *A similarity measure for intervals* $RS : \mathcal{L}([0,1]) \times \mathcal{L}([0,1]) \to [0,1]$ *is a robust similarity measure if the following properties hold for any* $a, b, c \in \mathcal{L}([0,1])$*:*

A1: $RS(a,b) = RS(b,a)$.

A2: $RS(a,b) = 1$ *if and only if* $a = b$.

A3: *If* $a \subseteq b \subseteq c$*, then* $RS(a,c) \leq RS(a,b)$ *and* $RS(a,c) \leq RS(b,c)$.

A4: $RS(a,b) = 0$ *if and only if* $w(a \cap b) = 0 \wedge a \neq b$.

A5: *Consider two interval pairs* $\{a_1, b_1\}$ *and* $\{a_2, b_2\}$ *with* $w(a_1 \cup b_1) = w(a_2 \cup b_2)$*, and* $w(a_1 \cap b_1) = w(a_2 \cap b_2) > 0$*. If* $w(a_1) \neq w(a_2)$*,* $w(a_1) \neq w(b_2)$*,* $w(b_1) \neq w(a_2)$ *and* $w(b_1) \neq w(b_2)$*, then* $RS(a_1, b_1) \neq RS(a_2, b_2)$.

A6: *If* $a \subset b$*, then* $RS(a,b) < 1$.

A7: *Consider two interval pairs* $\{a_1, b_1\}$ *and* $\{a_2, b_2\}$ *with* $w(a_1 \cap b_1) > 0$*,* $w(a_2 \cap b_2) > 0$*. If* $a_2 = n \cdot a_1 = [n \cdot \underline{a_1}, n \cdot \overline{a_1}]$ *and* $b_2 = n \cdot b_1 = [n \cdot \underline{b_1}, n \cdot \overline{b_1}]$ *where* $n > 0$ *is a scaling factor, then* $RS(a_1, b_1) = RS(a_2, b_2)$.

A8: *Consider two interval pairs* $\{a_1, b_1\}$ *and* $\{a_2, b_2\}$ *where* $w(a_1) = w(b_1) = w(a_2) = w(b_2)$*. If* $w(a_1 \cap b_1) < w(a_2 \cap b_2)$*, then* $RS(a_1, b_1) < RS(a_2, b_2)$.

When only *A1–A3* hold, we simply speak of a (non-robust) similarity measure [14, 19]. In this case, RS is symmetric (*A1*), so the similarity does not depend on the order of the intervals; it reaches its maximum value only when the intervals are identical (*A2*); and it decreases as the intervals become less nested within each other (*A3*), capturing a natural monotonicity with respect to inclusion.

The robustness property is ensured by the remaining five axioms, which refine the behavior of similarity measures under specific conditions. Axiom *A4* guarantees that the similarity between two disjoint, non-identical intervals is zero, ensuring a proper response to a lack of overlap. Axiom *A5* addresses the problem of aliasing, which occurs when two interval pairs—despite having different widths—produce identical intersection and union widths, and therefore the same similarity value. Such cases are undesirable, as they ignore structural differences between the compared intervals. Axiom *A6* requires that if one interval is strictly contained in another, the similarity must be lower than one. However, since *A2* already establishes that $RS(a,b) = 1$ if and only if $a = b$, this condition is redundant and is retained only for consistency with [13]. Axiom *A7* ensures that similarity remains invariant under uniform scaling of both intervals, reflecting robustness with respect to proportional size changes. Finally, Axiom *A8* focuses on the width of the intersection, guaranteeing that, for intervals of equal size, a larger overlap corresponds to a higher similarity value. Together, these axioms ensure that robust similarity measures faithfully capture structural, scaling, and overlap relations between intervals.

2.1 Measures derived from the ratio model

In this subsection, we introduce several interval comparison measures derived from the ratio model [24], as they will be used throughout the rest of the work.

Definition 2.3. [24] *The ratio model* $S_{RM}^{\alpha,\beta} : \mathcal{L}([0,1]) \times \mathcal{L}([0,1]) \to [0,1]$ *is defined as:*

$$S_{RM}^{\alpha,\beta}(a,b) = \begin{cases} 1 & \text{if } (\alpha = 0 \wedge w(b) = 0 \wedge b \in a) \vee (\beta = 0 \wedge w(a) = 0 \wedge a \in b) \vee \\ & \quad (w(a) = w(b) = 0 \wedge a = b) \\ 0 & \text{if } (\alpha = 0 \wedge w(b) = 0 \wedge b \notin a) \vee (\beta = 0 \wedge w(a) = 0 \wedge a \notin b) \vee \\ & \quad (w(a) = w(b) = 0 \wedge a \neq b) \vee (\alpha = \beta = 0 \wedge w(a \cap b) = 0) \\ \frac{w(a \cap b)}{w(a \cap b) + \alpha\, w(a \setminus b) + \beta\, w(b \setminus a)} & \text{otherwise} \end{cases}$$

where $\alpha, \beta \in [0,1]$.

In the original work, w was replaced by f, referred to as a cardinality function. In this paper, we adopt the most natural choice for measuring the cardinality of an interval. Let us also note that $S_{RM}^{0,0}(a,b)=0$ if the two intervals are disjoint and 1 otherwise, which makes it a rather uninformative measure. The ratio model is not a similarity measure, since it is asymmetric. However, specific choices of α and β do give rise to valid similarity measures.

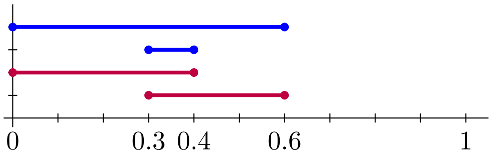
Example 2.4. *The well-known Jaccard and Dice similarities arise as particular cases of the ratio model* $S_{RM}^{\alpha,\beta}$ *for* $\alpha=\beta=1$ *and* $\alpha=\beta=0.5$*, respectively.*

Condition	$S_J(a,b)$ *(*$\alpha=\beta=1$*)* [12]	$S_D(a,b)$ *(*$\alpha=\beta=0.5$*)* [9]
$w(a)=w(b)=0 \wedge a=b$	1	1
$w(a)=w(b)=0 \wedge a\neq b$	0	0
otherwise	$\dfrac{w(a\cap b)}{w(a\cap b)+w(a\setminus b)+w(b\setminus a)}$	$\dfrac{w(a\cap b)}{w(a\cap b)+0.5\,w(a\setminus b)+0.5\,w(b\setminus a)}$

Although both functions satisfy the basic axioms of similarity, they are not robust. Consider the interval pairs $\{[0,0.6],[0.3,0.4]\}$ *and* $\{[0,0.4],[0.3,0.6]\}$*, then:*

$S_J([0,0.6],[0.3,0.4])=1/6$ $\quad S_D([0,0.6],[0.3,0.4])=2/7$

$S_J([0,0.4],[0.3,0.6])=1/6$ $\quad S_D([0,0.4],[0.3,0.6])=2/7$



So the Jaccard and Dice similarities do not fulfill Axiom 5, the one which is required for avoiding aliasing.

Another measure of comparison between intervals based on the ratio model is the subsethood degree, already introduced in [6], which is defined as follows:

Definition 2.5. *The subsethood degree* $S_h:\mathcal{L}([0,1])\times\mathcal{L}([0,1])\to[0,1]$ *is a measure that indicates the degree to which the interval* a *is a subset of* b*:*

$$S_h(a,b)=\begin{cases} 1 & \text{if } w(a)=0\wedge a\cap b\neq\emptyset\\ 0 & \text{if } w(a)=0\wedge a\cap b=\emptyset\\ \frac{w(a\cap b)}{w(a)} & \text{otherwise}\end{cases}$$

The subsethood degree is an asymmetric measure obtained when $\alpha=1$ and $\beta=0$ in the ratio model [18]. However, certain combinations of subsethood degrees generate similarity measures. Based on this idea, we introduce the following definitions. The first two of which are proposed by Kabir et.al. [13], the third is proposed by Huidobro et.al. [11], and the last two are proposed in this work based on the preceding ideas.

Definition 2.6. *Given* $a,b\in\mathcal{L}([0,1])$*, the following operators are defined:*

- *the subsethood minimum* S_{min} [13]*:*

$$S_{min}(a,b)=\min\{S_h(a,b),S_h(b,a)\}$$

- *the subsethood product* S_{prod} [13]*:*

$$S_{prod}(a,b)=S_h(a,b)\cdot S_h(b,a)$$

- *the Lukasievic subsethood* S_{LK}*:*

$$S_{LK}(a,b)=\max\{0,S_h(a,b)+S_h(b,a)-1\}$$

- *the subsethood mean* S_{mean} [11]*:*

$$S_{mean}(a,b)=\frac{S_h(a,b)+S_h(b,a)}{2}$$

- *the subsethood geometric mean* S_{g-mean}*:*

$$S_{g-mean}(a,b)=\sqrt{S_h(a,b)\cdot S_h(b,a)}$$

The proof that S_{min}, S_{prod}, and S_{mean} are similarity measures can be found in [11, 13]. Similarly, it is straightforward to show that S_{g-mean} and S_{LK} are also similarity measures.

2.2 Aggregation functions

In addition to the concepts already explained, we also introduce the basic notions about aggregation functions as they are an important tool in the rest of the paper.

Definition 2.7. [4, 5, 16] *An aggregation function* $\mathcal{A} : [0,1]^n \to [0,1]$ *is an increasing function in each argument that satisfies the boundary conditions* $\mathcal{A}(0,\ldots,0) = 0$ *and* $\mathcal{A}(1,\ldots,1) = 1$.

Some important properties related to aggregation functions are introduced below.

Definition 2.8. [4, 5, 20] *An aggregation function* $\mathcal{A} : [0,1]^n \to [0,1]$ *is called:*

- *symmetric if its value does not depend on the permutation of the arguments, i.e., for every permutation* $P = (P(1), P(2), \ldots, P(n))$ *of* $(1,2,\ldots,n)$*, then* $\mathcal{A}(x_1, x_2, \ldots, x_n) = \mathcal{A}(x_{P(1)}, x_{P(2)}, \ldots, x_{P(n)})$.
- *one-strict when* $\mathcal{A}(x_1, x_2, \ldots, x_n) = 1$ *only if* $x_i = 1$, $\forall i = 1, \ldots, n$.
- *zero-strict when* $\mathcal{A}(x_1, x_2, \ldots, x_n) = 0$ *only if* $x_i = 0$, $\forall i = 1, \ldots, n$.
- *homogeneous if for all* λ, $\mathcal{A}(\lambda\, x_1, \lambda\, x_2, \ldots, \lambda\, x_n) = \lambda\, \mathcal{A}(x_1, x_2, \ldots, x_n)$, $\forall x_i \in [0,1]$, $i = 1, \ldots, n$.

Examples of aggregation functions can be found in Definition 2.6. Any aggregation function may be employed to combine $S_h(a,b)$ and $S_h(b,a)$, however, for simplicity, we will focus on the examples considered in Definition 2.6.

3 Pre-inclusions

Before delving into the analysis of robust measures, we introduce a novel family of functions, denoted by E, that we name pre-inclusion functions, which are a relaxed variant of Zeng and Guo's inclusion measure (see Definition 2.1) [28].

Definition 3.1. *A function* $E : \mathcal{L}([0,1]) \times \mathcal{L}([0,1]) \to [0,1]$ *is called a pre-inclusion if it fulfills the properties* I1, I2, *and* I3 *in Definition 2.1.*

Note that pre-inclusion functions are not in general inclusion measures, as the following counterexample shows:

Example 3.2. *The function* $E(a,b) = \begin{cases} 1 & \text{if } a \subseteq b \\ 0.5 & \text{otherwise} \end{cases}$ *is a pre-inclusion as it verifies the Definition 3.1. However, it does not satisfy condition* I0 *of Definition 2.1, since* $E([1,1],[0,0]) = 0.5 \neq 0$. *Thus, it is not an inclusion measure.*

Not every pre-inclusion is an inclusion, but a pre-inclusion E can be transformed into an inclusion measure I_E, defined as:

$$I_E(a,b) = \frac{E(a,b) - E([1,1],[0,0])}{1 - E([1,1],[0,0])}.$$

However, for our purposes, Definition 3.1 is sufficient, and we have chosen to adopt the simplest definition.

In addition, not every ratio model is a pre-inclusion. For example, the Dice similarity is a case where property *I1* of Definition 2.1 is not satisfied. For $a = [0.2, 0.4]$ and $b = [0.1, 0.4]$, it is clear that $a \subseteq b$, but $S_D(a,b) = S_{RM}^{0.5,0.5}(a,b) = 0.8 \neq 1$. The following result provides the conditions on α and β under which $S_{RM}^{\alpha,\beta}$ can be considered a pre-inclusion.

Proposition 3.3. *If* $\alpha \neq 0$ *and* $\beta = 0$*, the ratio model* $S_{RM}^{\alpha,0}$ *is a pre-inclusion.*

Proof. The properties of Definition 3.1 must be proven:
I1: If $S_{RM}^{\alpha,0}(a,b) = 1$, then the possible cases are ($\beta = 0$, $w(a) = 0 \wedge a \in b$), ($w(a) = w(b) = 0 \wedge a = b$) and ($w(a \cap b) = w(a \cap b) + \alpha\, w(a \backslash b) + \beta\, w(b \backslash a)$). In the first two cases, this directly implies $a \subseteq b$. The last equality holds if and only if $\alpha\, w(a \backslash b) + \beta\, w(b \backslash a) = 0$. Since $\alpha \neq 0$ and $\beta = 0$, this is equivalent to $w(a \backslash b) = 0$ and so $a \subseteq b$.

If $a \subseteq b$, then:

- $\beta = 0$, $w(a) = 0$, that is, $S_{RM}^{\alpha,0}(a,b) = 1$.
- $w(a) = w(b) = 0$ and $a = b$ and so $S_{RM}^{\alpha,0}(a,b) = 1$.
- $S_{RM}^{\alpha,0}(a,b) = \dfrac{w(a \cap b)}{w(a \cap b) + \alpha\, w(a \backslash b)} = \dfrac{w(a)}{w(a)} = 1.$

I2: Let $a \subseteq b \subseteq c$. The cases where $(\beta = 0 \wedge w(c) = 0 \wedge c \notin b)$ or $(w(c) = w(b) = 0 \wedge c \neq b)$ would yield $S_{RM}^{\alpha,0}(c,b) = 0$ are not possible, since $b \subseteq c$.

If $(\beta = 0 \wedge w(c) = 0 \wedge c \in a)$ or $(w(c) = w(a) = 0 \wedge c = a)$, we have that $S_{RM}^{\alpha,0}(c,a) = 1$. Thus, $a = b = c$ and therefore $S_{RM}^{\alpha,0}(c,b) = 1$.

Otherwise, suppose $S_{RM}^{\alpha,0}(c,a) > S_{RM}^{\alpha,0}(c,b)$, then:

$$\frac{w(a)}{\alpha\ w(c) + w(a)\ \ (1-\alpha)} > \frac{w(b)}{\alpha\ w(c) + w(b)\ \ (1-\alpha)},$$

$$w(a)\ (\alpha\ w(c) + w(b)\ \ (1-\alpha)) > w(b)\ (\alpha\ w(c) + w(a)\ \ (1-\alpha))$$

$$\alpha\ w(c)\ w(a) + w(b)\ w(a) - \alpha\ w(b)\ w(a) > \alpha\ w(c)\ w(b) + w(a)\ w(b) - \alpha\ w(a)\ w(b),$$

$$0 > \alpha\ w(c)\ (w(b) - w(a)).$$

Therefore, we obtain a contradiction because α, $w(c)$ and $(w(b) - w(a))$ are greater or equal to zero. Hence, $S_{RM}^{\alpha,0}(c,a) \leq S_{RM}^{\alpha,0}(c,b)$.

I3: Let $a \subseteq b \subseteq c$. Since $a \subseteq b$, the cases $(\beta = 0 \wedge w(b) = 0 \wedge b \notin a)$ or $(w(a) = w(b) = 0 \wedge b \neq a)$ are not possible.

Moreover, for the cases $(\beta = 0 \wedge w(c) = 0 \wedge c \in a)$ or $(w(a) = w(c) = 0 \wedge a = c)$, we obtain that $a = b = c$ and therefore $S_{RM}^{\alpha,0}(b,a) = 1 = S_{RM}^{\alpha,0}(c,a) = 1$.

Otherwise, we have:

$$\frac{w(a)}{\alpha\ w(c) + w(a)\ \ (1-\alpha)} \leq \frac{w(a)}{\alpha\ w(b) + w(a)\ \ (1-\alpha)},$$

$$\frac{w(a \cap c)}{\alpha\ w(c) + w(a \cap c)\ \ (1-\alpha)} \leq \frac{w(a \cap b)}{\alpha\ w(b) + w(a \cap b)\ \ (1-\alpha)}.$$

So, $S_{RM}^{\alpha,0}(c,a) \leq S_{RM}^{\alpha,0}(b,a)$.

As *I1*, *I2* and *I3* are satisfied, then $S_{RM}^{\alpha,0}$ is a pre-inclusion. □

We will now consider a particular type of pre-inclusions, namely, those that are robust.

Definition 3.4. *A pre-inclusion* $E : \mathcal{L}([0,1]) \times \mathcal{L}([0,1]) \to [0,1]$ *is a robust pre-inclusion if it fulfills the following properties:*

- *Disjointness if it satisfies:*
  I4: $E(a,b) = E(b,a) = 0$ *iff* $w(a \cap b) = 0$ *except for the case* $a = b$.
- *Nonaliasing if it satisfies:*
  I5: *Consider two interval pairs* $\{a,b\}$ *and* $\{c,d\}$ *with* $w(a \cup b) = w(c \cup d)$, *and* $w(a \cap b) = w(c \cap d) > 0$. *If* $w(a) \neq w(c)$, $w(a) \neq w(d)$, $w(b) \neq w(c)$ *and* $w(b) \neq w(d)$ *then* $E(a,b) \neq E(c,d)$.
- *Scaling invariance if it satisfies:*
  I6: *If* $c = n \cdot a$ *and* $d = n \cdot b$, *then* $E(a,b) = E(c,d)$.
- *Increased overlapping if it satisfies:*
  I7: *If* $w(a) = w(b) = w(c) = w(d)$ *and* $w(a \cap b) < w(c \cap d)$, *then* $E(a,b) < E(c,d)$.

An essential family of pre-inclusions for our study is the one derived from the ratio model. We will show that, in most cases, this family is robust.

Proposition 3.5. *If* $\alpha \neq 0$ *and* $\beta = 0$, *the ratio model* $S_{RM}^{\alpha,0}$ *is a robust pre-inclusion.*

Proof. From Proposition 3.3, we know that $S_{RM}^{\alpha,0}$ is a pre-inclusion, and it only remains to prove that $S_{RM}^{\alpha,0}$ satisfies properties *I4–I7* stated in Definition 3.4.

I4: The proof is trivial.

I5: Consider two interval pairs $\{a,b\}$ and $\{c,d\}$ such that $w(a\cup b)=w(c\cup d)$ and $w(a\cap b)=w(c\cap d)>0$. As $w(a\cap b)=w(c\cap d)>0$, we can conclude that none of the four intervals a, b, c, and d is degenerate.

Assume also that $w(a)\neq w(c)$, $w(a)\neq w(d)$, $w(b)\neq w(c)$, and $w(b)\neq w(d)$.

For the pair $\{a,b\}$

$$S_{RM}^{\alpha,0}(a,b)=\frac{w(a\cap b)}{\alpha\, w(a)+(1-\alpha)\, w(a\cap b)}.$$

For the pair $\{c,d\}$

$$S_{RM}^{\alpha,0}(c,d)=\frac{w(c\cap d)}{\alpha\, w(c)+(1-\alpha)\, w(c\cap d)}.$$

Since $w(a\cap b)=w(c\cap d)$, it follows that

$$S_{RM}^{\alpha,0}(a,b)=\frac{w(c\cap d)}{\alpha\, w(a)+(1-\alpha)\, w(c\cap d)}.$$

Suppose now that $S_{RM}^{\alpha,0}(a,b)=S_{RM}^{\alpha,0}(c,d)$. Then $\alpha\, w(a)=\alpha\, w(c)$, which leads to $w(a)=w(c)$, which contradicts the assumption $w(a)\neq w(c)$. It is concluded that $S_{RM}^{\alpha,0}(a,b)\neq S_{RM}^{\alpha,0}(c,d)$.

I6: For the case where a or b are degenerate intervals, the proof is trivial. Otherwise, let us consider two interval pairs $\{a,b\}$ and $\{c,d\}$ such that $w(a\cap b)>0$ and $w(c\cap d)>0$. Let $c=n\cdot a=[n\cdot \underline{a},\, n\cdot \overline{a}]$ and $d=n\cdot b=[n\cdot \underline{b},\, n\cdot \overline{b}]$, where $n>0$ is a scaling factor. Then

$$S_{RM}^{\alpha,0}(c,d)=\frac{w(c\cap d)}{\alpha\, w(c)+(1-\alpha)\, w(c\cap d)}.$$

Since $w(c)=n\, w(a)$, $w(d)=n\, w(b)$, and $w(c\cap d)=n\, w(a\cap b)$, it is obtained that

$$S_{RM}^{\alpha,0}(c,d)=\frac{n\, w(a\cap b)}{n\big(\alpha\, w(a)+(1-\alpha)\, w(a\cap b)\big)}=S_{RM}^{\alpha,0}(a,b).$$

I7: Due to $w(a\cap b)<w(c\cap d)$, we can conclude that $w(c\cap d)>0$ and so c and d are not degenerate. As the widths of the intervals a, b, c and d are the same, none of them is degenerate. Let us assume that $w(a)=w(b)=w(c)=w(d)$ and that $w(a\cap b)<w(c\cap d)$. Define the function $h:[0,1]\to[0,1]$ by

$$h(x)=\frac{x}{\alpha\, w(a)+(1-\alpha)\, x}.$$

Since $w(a)=w(b)=w(c)=w(d)$, it can be written $S_{RM}^{\alpha,0}(a,b)=h\big(w(a\cap b)\big)$ and $S_{RM}^{\alpha,0}(c,d)=h\big(w(c\cap d)\big)$.

The derivative of h is

$$h'(x)=\frac{\alpha\, w(a)}{\big(\alpha\, w(a)+(1-\alpha)\, x\big)^2}.$$

As $h'(x)>0$ for all $x\in[0,1]$, the function h is strictly increasing. Hence, since $w(a\cap b)<w(c\cap d)$, it follows that $S_{RM}^{\alpha,0}(a,b)<S_{RM}^{\alpha,0}(c,d)$.

□

Additional properties can be incorporated into the robust pre-inclusion in order to strengthen it into a strong pre-inclusion.

Definition 3.6. *A robust pre-inclusion $E:\mathcal{L}([0,1])\times\mathcal{L}([0,1])\to[0,1]$ is a strong robust pre-inclusion if it fulfills the following properties:*

- *Width-symmetric if it satisfies:*
  I8*: If $w(a)=w(b)$ then $E(a,b)=E(b,a)$.*
- *Linear if it satisfies:*
  I9*: If $w(c\cap d)=n\ w(a\cap b)>0$ with $n>0$ a scaling factor and $w(a)=w(c)$, then $E(c,d)=n\ E(a,b)$.*

- *Width-ordered if it satisfies:*
  I10*: If* $w(a \cap b) = w(c \cap d)$ *and* $\max\{w(c), w(d)\} < \min\{w(a), w(b)\}$*, then* $E(a, b) < E(c, d)$*.*

So far, we have observed that both the ratio model (with $\alpha \neq 0$ and $\beta = 0$) and, as a particular case, the subsethood degree are robust pre-inclusions. Now it is examined whether they also satisfy the axioms required to qualify as strong robust pre-inclusions.

The ratio model fulfills properties *I8* and *I10*. However, as it is shown below, it does not fulfill *I9*.

Proposition 3.7. *If* $\alpha \neq 0$ *and* $\beta = 0$*, the ratio model* $S_{RM}^{\alpha,0}$ *is a robust pre-inclusion satisfying width-symmetric and width-ordered properties.*

Proof. If with $\alpha \neq 0$ and $\beta = 0$ by Proposition 3.5, $S_{RM}^{\alpha,0}$ is a robust pre-inclusion.

For property *I8*, if a and b are degenerate intervals, the proof is immediate. Otherwise, we have to check:

$$\frac{w(a \cap b)}{\alpha w(a) + w(a \cap b)(1-\alpha)} = \frac{w(b \cap a)}{\alpha w(b) + w(b \cap a)(1-\alpha)}.$$

that is fulfilled if $\alpha\, w(a) = \alpha\, w(b)$. As $w(a) = w(b)$, then it is always fulfilled. Thus, $S_{RM}^{\alpha,0}$ is width-symmetric.

For property *I10*, as $\max\{w(c), w(d)\} < \min\{w(a), w(b)\}$, a and b are non-degenerate intervals. If c and d are degenerate intervals, then $w(a \cap b) = w(c \cap b) = 0$ and the proof is immediate. Otherwise, since $\alpha > 0$, then

$$\alpha\, w(a) > \alpha\, w(c),$$

and

$$\frac{w(a \cap b)}{\alpha\, w(a) + w(a \cap b)(1-\alpha)} < \frac{w(c \cap d)}{\alpha\, w(c) + w(c \cap d)(1-\alpha)}.$$

Therefore, $S_{RM}^{\alpha,0}(a, b) < S_{RM}^{\alpha,0}(c, d)$, being that $S_{RM}^{\alpha,0}$ fulfills the width-ordered property. ☐

In the next example, we will see that $S_{RM}^{\alpha,0}$ is not a strong robust pre-inclusion as the linearity property fails.

Example 3.8. *Let us consider the following intervals* $a = [0.1, 0.4]$*,* $b = [0.3, 0.8]$*,* $c = [0.3, 0.6]$ *and* $d = [0.4, 0.9]$*. It is clear that* $w(a) = w(c) = 0.3$*,* $w(b) = w(d) = 0.5$*,* $w(a \cap b) = 0.1$ *and* $w(c \cap d) = 0.2$*. Despite the fact that* $w(c \cap d) = 2\, w(a \cap b)$*, we have that* $S_{RM}^{0.5,0}(c, d) \neq 2\, S_{RM}^{0.5,0}(a, b)$ *as* $S_{RM}^{0.5,0}(c, d) = 0.8$ *and* $S_{RM}^{0.5,0}(a, b) = 0.5$*.*

Although the ratio model $S_{RM}^{\alpha,0}$ satisfies key properties like width-symmetry and width-ordering, it fails to meet the linearity property, which limits its consistency when intervals are proportionally scaled, as shown in Example 3.8. This shortcoming can hinder its applicability in contexts where scaling invariance is crucial. In contrast, the subsethood degree S_h fulfills all the requirements of a strong robust pre-inclusion, including linearity, making it a more reliable and consistent measure for representing interval relationships across a wider range of applications. Therefore, S_h is the better choice when robustness and scaling behaviour are considered.

Proposition 3.9. *The subsethood degree* S_h *is a strong robust pre-inclusion.*

Proof. The operator S_h is a robust pre-inclusion as it is the ratio model $S_{RM}^{\alpha,\beta}$ with $\alpha = 1$ and $\beta = 0$. For the same reason, by applying Proposition 3.7, we know that it satisfies the width-symmetric and width-ordered properties; thus, we only need to prove linearity:

I9: Consider $w(c \cap d) = n\, w(a \cap b) > 0$ with $n > 0$ and $w(a) = w(c)$. As the width of the intersection is greater than 0, none of the intervals is degenerate. Then, $S_h(c, d) = \frac{w(c \cap d)}{w(c)} = \frac{n w(a \cap b)}{w(c)} = \frac{n w(a \cap b)}{w(a)} = n S_h(a, b)$.

☐

4 Construction method for similarity measure

In this section, we provide a characterization of similarity measures, robust similarity measures and our proposal, strong robust similarity measures as a composition of pre-inclusions and aggregation functions.

4.1 Similarity measures

This proposition asserts that under specific conditions, a similarity measure can be constructed using an aggregation and a pre-inclusion function.

Proposition 4.1. *Let E be a pre-inclusion and let $\mathcal{A}$ be a symmetric one-strict aggregation function. The operator $\mathcal{S}_{\mathcal{A}}^{E} : \mathcal{L}([0,1]) \times \mathcal{L}([0,1]) \to [0,1]$ defined by $\mathcal{S}_{\mathcal{A}}^{E}(a,b) = \mathcal{A}(E(a,b), E(b,a))$ is a similarity measure.*

Proof. We will begin by demonstrating that $\mathcal{S}_{\mathcal{A}}^{E}$ is well-defined, meaning it takes values in the interval $[0,1]$. Consider any $a, b \in \mathcal{L}([0,1])$. We have that $E(a,b), E(b,a) \in [0,1]$, so when we apply an aggregation function, it results in a value in the interval $[0,1]$.

Now, let us prove that it satisfies the three axioms that define a similarity measure:

A1: $\mathcal{S}_{\mathcal{A}}^{E}$ is symmetric by construction since the aggregation function $\mathcal{A}$ is symmetric.

A2: $\mathcal{S}_{\mathcal{A}}^{E}(a,b) = \mathcal{A}(E(a,b), E(b,a)) = 1 \Leftrightarrow E(a,b) = E(b,a) = 1$ since $\mathcal{A}$ is a two-variable one-strict aggregation function. This is equivalent to stating that $a = b$ because E satisfies property *I1*.

A3: For $a, b, c \in \mathcal{L}([0,1])$, if $a \subseteq b \subseteq c$, then:

- By applying *I1*, we have $E(a,c) = E(a,b) = E(b,c) = 1$.
- By applying *I2*, we have $E(c,a) \leq E(c,b)$.
- By applying *I3*, we have $E(c,a) \leq E(b,a)$.

Applying the monotonicity of the aggregation function, as well as properties *I1* and *I3*, we have $\mathcal{S}_{\mathcal{A}}^{E}(a,c) = \mathcal{A}(E(a,c), E(c,a)) \leq \mathcal{A}(E(a,b), E(b,a)) = \mathcal{S}_{\mathcal{A}}^{E}(a,b)$.

Applying the monotonicity of $\mathcal{A}$, again as well as properties *I1* and *I2*, we have $\mathcal{S}_{\mathcal{A}}^{E}(a,c) = \mathcal{A}(E(a,c), E(c,a)) \leq \mathcal{A}(E(b,c), E(c,b)) = \mathcal{S}_{\mathcal{A}}^{E}(b,c)$.

□

In particular, we can apply the previous result to the ratio model, which is a pre-inclusion by Proposition 3.3.

Corollary 4.2. *If $\alpha \neq 0$ and $\beta = 0$, the operator $\mathcal{S}_{\mathcal{A}}^{RM}(a,b) = \mathcal{A}(S_{RM}^{\alpha,0}(a,b), S_{RM}^{\alpha,0}(b,a))$ is a similarity measure for any $\mathcal{A}$ symmetric one-strict aggregation function.*

However, the aggregation of pre-inclusions E as defined in Proposition 4.1 does not necessarily result in a robust similarity measure. When using a t-norm T, which is a particular case of a symmetric one-strict aggregation function, it is not guaranteed that $T(S_{RM}^{\alpha,\beta}(a,b), S_{RM}^{\alpha,\beta}(b,a))$ will always exhibit robust similarity properties. For instance, if we take T to be the minimum operator ($T = \min$) and choose $S_{RM}^{\alpha,\beta}$ to be equal to S_J, then $\min(S_J(a,b), S_J(b,a)) = S_J(a,b)$. This demonstrates that the intervals discussed in Example 2.4 can serve as valid counterexamples, highlighting the non-fulfillment of Axiom 5. The same issue arises when considering $S_{RM}^{0.5,0.5} = S_D$. As we mentioned earlier, this is the main reason why we will not be considering the Jaccard and Dice similarities in the study of their behaviour over the properties.

4.2 Robust similarity measures

To obtain a similarity measure, it seems reasonable to introduce specific properties for both the pre-inclusions E and the aggregation function $\mathcal{A}$.

Proposition 4.3. *Let E be a robust pre-inclusion and let $\mathcal{A}$ be a symmetric, one-strict and zero-strict aggregation function. The operator $\mathcal{S}_{\mathcal{A}}^{E} : \mathcal{L}([0,1]) \times \mathcal{L}([0,1]) \to [0,1]$ defined by $\mathcal{S}_{\mathcal{A}}^{E}(a,b) = \mathcal{A}(E(a,b), E(b,a))$ is a robust similarity measure.*

Proof. We have already seen in Proposition 4.1 that $\mathcal{S}_{\mathcal{A}}^{E}$ is a measure of similarity; therefore, we only need to verify the additional axioms of robust similarity measure in Definition 2.2. Obviously, due to the properties of the pre-inclusion E, axioms *A5*, *A7*, and *A8* are trivial. We only need to prove *A4* and *A6*:

A4: $\mathcal{S}_{\mathcal{A}}^{E}(a,b) = \mathcal{A}(E(a,b), E(b,a)) = 0$ iff $E(a,b) = 0$ and $E(b,a) = 0$ since $\mathcal{A}$ is zero-strict, iff $a \cap b = \emptyset$ by axiom *I4*, iff $w(a \cap b) = 0$.

A6: Let $a \subset b$, then $E(a,b) = 1$ and $E(b,a) < 1$. So $\mathcal{S}^E_{\mathcal{A}}(a,b) = \mathcal{A}(1, E(b,a)) < 1$ since $\mathcal{A}$ is one-strict.

Therefore, $\mathcal{S}^E_{\mathcal{A}}(a,b) = \mathcal{A}(E(a,b), E(b,a))$ is a robust similarity measure. □

The converse of Proposition 4.3 does not hold in general. In other words, the fact that $\mathcal{S}^E_{\mathcal{A}}$ is a robust similarity measure does not necessarily imply that the underlying pre-inclusion E is robust or that the aggregation function $\mathcal{A}$ fulfills all the required properties. As will be seen in Corollary 4.7, the subsethood geometric mean $S_{g\text{-mean}}$ constitutes a robust similarity measure, even though its associated aggregation function (the geometric mean) is not zero-strict. This illustrates that, in some cases, the robustness of $\mathcal{S}^E_{\mathcal{A}}$ may arise from a compensatory interaction between E and A, rather than from each component independently satisfying all robustness axioms.

After examining the theoretical characterization of robust similarities, we now turn our attention to the specific case of similarities based on the ratio model. Note that, as highlighted in Example 2.4, Jaccard and Dice similarities are not robust similarity measures. It is also worth noting that Jaccard and Dice similarities satisfy $\alpha = \beta$ ($\alpha = \beta = 1$ in the case of Jaccard, and $\alpha = \beta = 0.5$ in the case of Dice). This relationship between α and β is precisely what determines whether a similarity based on the ratio model is robust, as shown in the following result.

Corollary 4.4. *Let $S^{\alpha,\beta}_{RM}$ be a ratio model and $\mathcal{A}$ an aggregation function that satisfies symmetric, one-strict and zero-strict properties. If $\alpha \neq 0$ and $\beta = 0$, then $\mathcal{S}^{RM}_{\mathcal{A}}(a,b) = \mathcal{A}(S^{\alpha,0}_{RM}(a,b), S^{\alpha,0}_{RM}(b,a))$ is a robust similarity measure.*

Proof. If $\alpha \neq 0$ and $\beta = 0$, it is immediate as $S^{\alpha,0}_{RM}$ is a robust pre-inclusion by Proposition 3.5. □

Corollary 4.4 introduces a general method to generate robust similarity measures based on the ratio model.

In addition, from Proposition 4.3, it is straightforward to check that S_{min} and S_{prod} are both robust similarity measures, although Kabir et al. proved that in [13], considering S_{min} and S_{prod} two cases from which $T(S_h(a,b), S_h(b,a))$ is a robust similarity measure, being $a, b \in \mathcal{L}([0,1])$ and T a t-norm. This leads us to consider what happens if we take another t-norm. Let us consider, for instance, the Lukasievic t-norm, which gives rise to Lukasievic subsethood (see Definition 2.6). However, S_{LK} is not a robust similarity measure, as the following example shows:

Example 4.5. *Consider S_{LK} and the intervals $[0.1, 0.6]$ and $[0.5, 1]$. Then axiom A4 is not fulfilled:*

$$S_{LK}([0.1,0.6],[0.5,1]) = \max\left\{0, \frac{0.1}{0.5} + \frac{0.1}{0.5} - 1\right\} = \max\{0, 0.2 + 0.2 - 1\} = 0,$$

while $[0.1, 0.6] \cap [0.5, 1] \neq \emptyset$.

Also, S_{LK} does not fulfill Axiom A5 showing aliasing: $S_{LK}([0,0.6],[0.3,0.4]) = S_{LK}([0,0.6],[0.25,0.85]) = 1/6$.

On the other hand, this is not the case for S_{mean} and S_{g-mean}.

Corollary 4.6. *The subsethood mean S_{mean} is a robust similarity measure in the sense of Definition 2.2.*

Proof. The subsethood mean uses a symmetric, one-strict and zero-strict aggregation function, and $\alpha = 1$ and $\beta = 0$. Hence, by Corollary 4.4, S_{mean} is a robust similarity measure. □

On the other hand, the subsethood S_{g-mean} is based on the geometric mean, which is not a zero-strict aggregation function.

Corollary 4.7. *The subsethood geometric mean S_{g-mean} is a robust similarity measure in the sense of Definition 2.2.*

Proof. The aggregation function in S_{g-mean} is symmetric and one-strict but not zero-strict. However, note that for the case $E(a,b) = E(b,a) = 0$ (i.e., when the intersection $a \cap b$ is a zero-measure set, the geometric mean also gives $\mathcal{A}(0,0) = 0$). Thus axiom A4 is still satisfied. For all the other axioms, the argument is analogous to the case of S_{mean}. Therefore S_{g-mean} is also a robust similarity measure. □

4.3 Strong robust similarity measures

In this subsection, we introduce the notion of strong robust similarity measures as follows:

Definition 4.8. *A robust similarity measure for intervals $RS : \mathcal{L}([0,1]) \times \mathcal{L}([0,1]) \to [0,1]$ is called a strong robust similarity measure if it satisfies properties* I9 *and* I10 *in Definition 3.6 and it also satisfies the property:*

- *Symmetric with respect to the width-symmetric pre-inclusion E:*
  A9: If $w(a) = w(b)$, then $RS(a,b) = E(a,b)$.

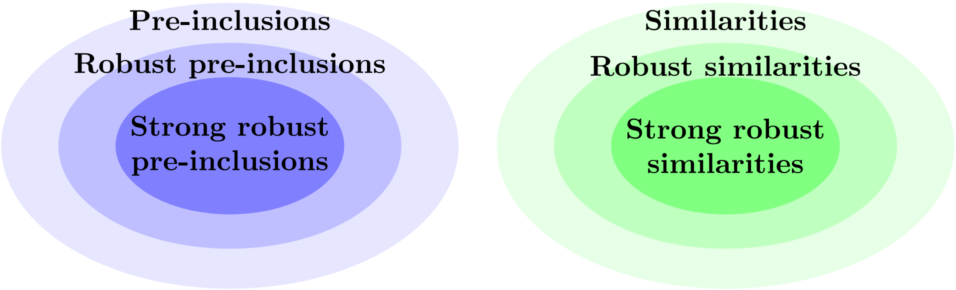


Figure 1: Hierarchy among pre-inclusions (left) and similarities (right).

Every strong robust pre-inclusion is also a robust pre-inclusion, and every robust pre-inclusion is in turn, a pre-inclusion. Analogously, every strong robust similarity measure is also a robust similarity measure, and every robust similarity measure is in turn a similarity measure. These hierarchical relationships are summarized in Figure 1, which shows the inclusion relations among the different classes of functions.

The strong robust similarity measures can be characterized based on pre-inclusions and aggregation functions, just like robust similarities. Next, we introduce the properties that pre-inclusions and aggregation functions must satisfy to fulfill the previous definition.

Proposition 4.9. *Let E be a strong robust pre-inclusion and let $\mathcal{A}$ be an idempotent, homogeneous, symmetric, one-strict and zero-strict aggregation function. Hence the operator $\mathcal{S}_{\mathcal{A}}^{E} : \mathcal{L}([0,1]) \times \mathcal{L}([0,1]) \to [0,1]$ defined by $\mathcal{S}_{\mathcal{A}}^{E}(a,b) = \mathcal{A}(E(a,b), E(b,a))$ is a strong robust similarity measure.*

Proof. $\mathcal{S}_{\mathcal{A}}^{E}$ is a robust similarity measure by Proposition 4.3. Let us check the rest of the axioms:

A9 If $w(a) = w(b)$ as E is a pre-inclusion that fulfills *I8* and $\mathcal{A}$ is idempotent:

$$\mathcal{S}_{\mathcal{A}}^{E}(a,b) = \mathcal{A}(E(a,b), E(b,a)) = E(a,b) = E(b,a).$$

Therefore, $\mathcal{S}_{\mathcal{A}}^{E}$ is symmetric with respect to the pre-inclusion E.

I9 It follows straightforwardly as E is a pre-inclusion that fulfills *I9* and $\mathcal{A}$ is a homogeneous aggregation function.

I10 It is immediate as E is a width-ordered pre-inclusion and $\mathcal{A}$ is an increasing function.

□

The converse of Proposition 4.9 also fails in general, a strong robust similarity measure does not always arise from a strong robust pre-inclusion and an aggregation with all the required properties. In fact, as will be shown later for the subsethood geometric mean, it is possible to obtain a strong robust similarity measure even when the aggregation function is not zero-strict. This illustrates that strong robustness may emerge from the interaction between the pre-inclusion and the aggregation, rather than from the strict fulfillment of every individual property.

This proposition to the ratio model as it is not a strong robust pre-inclusion, however, we obtain interesting results using the subsethood degree, which is a strong robust pre-inclusion by Proposition 3.9.

Corollary 4.10. *The operator $\mathcal{S}_{\mathcal{A}}^{h}(a,b) = \mathcal{A}(S_h(a,b), S_h(b,a))$ is a strong robust similarity measure for any aggregation function $\mathcal{A}$ that fulfills idempotent, homogeneous, symmetric, one-strict and zero-strict properties.*

Proof. By Proposition 3.9, the subsethood degree S_h is a strong robust pre-inclusion. Then, $\mathcal{S}_{\mathcal{A}}^{h}$ is a strong robust similarity measure by Proposition 4.9. □

In general, it is difficult for a similarity measure to satisfy all the requirements to be considered strong robust. Nevertheless, the subsethood minimum S_{min}, the subsethood mean S_{mean} and the subsethood geometric mean S_{g-mean} fulfill all the necessary conditions, and can therefore be identified as strong robust similarity measures.

Corollary 4.11. *The subsethood minimum S_{min}, the subsethood mean S_{mean}, and the subsethood g-mean S_{g-mean} are strong robust similarity measures.*

Proof. It is straightforward as min and *mean* are symmetric, one-strict, zero-strict, idempotent and homogeneous aggregation functions.

On the other hand, the subsethood *g-mean* is a robust similarity measure by Corollary 4.7. As the geometric mean is idempotent, homogeneous and increasing, the proof of Proposition 4.9 is still valid. ☐

Although the subsethood product is not strong robust similarity measure, it fulfills one of the properties given in Definition 4.8.

Corollary 4.12. *The subsethood product* S_{prod} *is width-ordered.*

Proof. Since the product is an aggregation function and S_h is a strong robust pre-inclusion, the conclusion follows immediately from the reasoning outlined in the proof of Proposition 4.9. ☐

4.4 Analysis of similarity measures with respect to the strong robustness axioms

In what follows, we analyze more in depth how the properties of the aggregation and/or the preinclusion used to construct the similarities affect the type of similarity. We illustrate the effects using the similarity measures derived from the ratio model (Definition 4.8).

From Figure 2, it can be observed that each example displays intervals of the same size. We consider the following subsethood degree percentages to use easy-reading intervals: 75%, 62.5%, 50%, 37.5%, 25%, 12.5%, and 0%. Given that the size of the intervals in each example is the same, we get that $S_h(a,b) = S_h(b,a)$. As a result, since the subsethood degrees are equal, the similarity should also be the same. This property is fulfilled by S_{min}, S_{mean} and S_{g-mean}, but not by S_{LK} and S_{prod}. In the case of S_{LK} and S_{prod}, this occurs as the aggregation functions used in their definition are not idempotent.

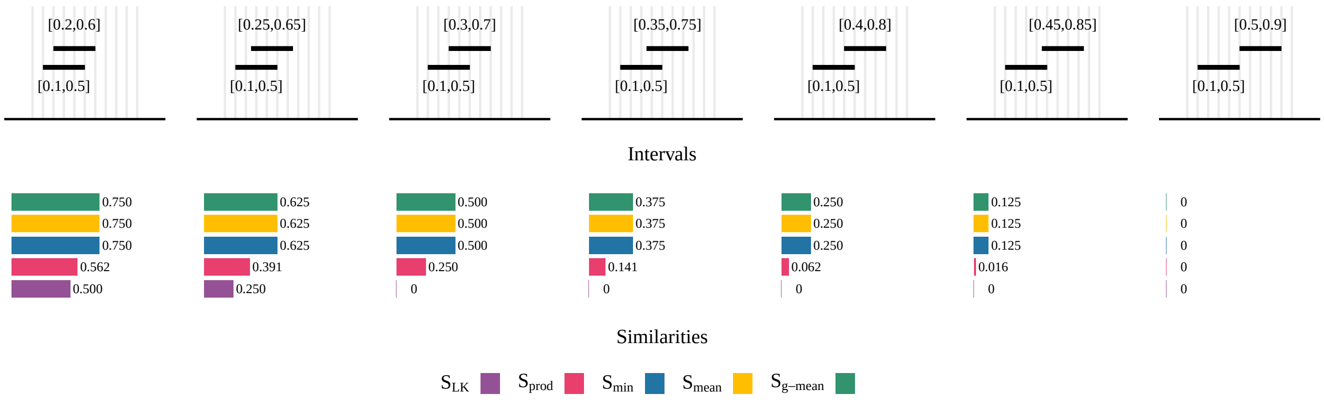


Figure 2: Analysis of Axiom *A9*.

Linearity is a relevant property of any strong robust similarity, it ensures that the similarity increases proportionally to the increase in overlap [23]. As we showed in Example 3.8, $S_{RM}^{\alpha,0}$ is not a strong robust pre-inclusion as there is no linearity. However, the subsethood degree S_h is a linear pre-inclusion. Figure 3 displays the linear property in relation to the overlapping degree for which the similarity is measured. Then, linearity is not fulfilled by the similarity measures S_{LK} and S_{prod}.

The behaviour of the measure is analysed in order to study its performance under different conditions. Thus, it will be studied when $\mathcal{S}_{\mathcal{A}}^{E}$ fulfills the width-ordered property.

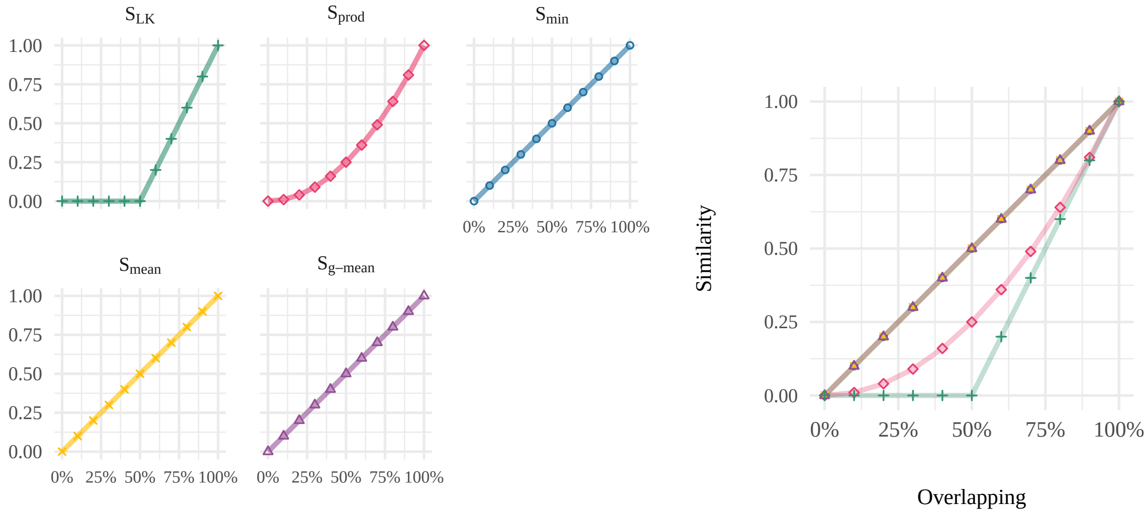


Figure 3: Analysis of Axiom *I9*: linear property in relation to the overlapping degree.

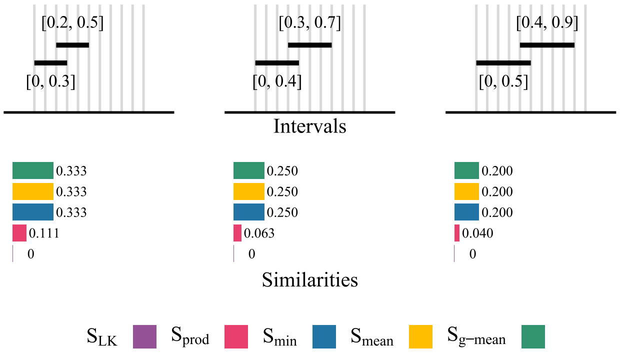


Figure 4: Analysis of Axiom *I10*.

Figure 4 illustrates that S_{prod}, S_{min}, S_{mean} and S_{g-mean} satisfy the width-ordered property required by Axiom *I10*. In contrast, S_{LK} does not comply with this property, since its values do not always decrease when the overlapping degree decreases. This axiom evaluates the behaviour of similarity measures with respect to inclusion relations, rewarding with higher values those pairs of intervals in which one is significantly contained within the other. Unlike Axiom *A4*, this property does not require strict containment of one interval in the other, but only equal widths of intersection and union.

5 Analysis of the behaviour of the similarities

Table 1 summarizes which properties, from those introduced in Definitions 2.2 and 4.8, are satisfied by the similarity measures introduced in Definition 2.6. Note that S_{LK} is the one that satisfies the fewest properties. On the other hand, S_{min}, S_{mean} and S_{g-mean} are the only three that fulfill all the axioms as they are strong robust similarity measures.

	A1	A2	A3	A4	A5	A6	A7	A8	A9	I9	I10	
S_{LK}	✓	✓	✓	✗	✗	✗	✗	✗	✗	✗	✗	}Similarity
S_{prod}	✓	✓	✓	✓	✓	✓	✓	✓	✗	✗	✓	}Robust Similarity
S_{min}	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	Strong Robust Similarity
S_{mean}	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	
S_{g-mean}	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	

Table 1: Summary of fulfillment of the properties by S_{LK} S_{prod}, S_{min}, S_{mean} and S_{g-mean}.

This comparative table not only summarizes the axiomatic compliance of each similarity, but also reveals the structural role of the aggregation function in determining robustness. In particular, it helps identify how progressively adding properties such as homogeneity, symmetry or strictness transforms a simple similarity into a robust or even strong one.

From this analysis, we can identify three progressive levels of compliance with the axioms: basic similarities (S_{LK}), robust similarities (S_{prod}), and strong robust similarities (S_{min}, S_{mean} and S_{g-mean}). This layered view not only clarifies the structural relationships among them, but also guides the practical selection of a similarity measure depending on the desired balance between sensitivity and robustness.

Once we know that S_{min}, S_{mean} and S_{g-mean} are robust similarity measures, we will investigate whether it is possible to establish an ordering between them and the other similarities generated by the aggregation of the subsethood degree.

Proposition 5.1. *The similarity measures generated by the aggregation of subsethood degrees are ordered as follows:*

$$S_{LK} \leq S_{prod} \leq S_{min} \leq S_{g-mean} \leq S_{mean}.$$

Proof. It is well-known that the Lukasievic t-norm T_{LK}, the product t-norm T_P and the minimum t-norm T_{min} satisfy $T_{LK} \leq T_P \leq T_{min}$. Moreover, the geometric mean GM dominates the minimum (for nonnegative inputs), i.e. fulfils that $T_{min} \leq GM$, since the geometric mean is an average and in particular is never smaller than the minimum of its (nonnegative) arguments. Because $S_{min}(a,b) = T_{min}(S_h(a,b), S_h(b,a))$, $S_{prod}(a,b) = T_P(S_h(a,b), S_h(b,a))$, $S_{g-mean}(a,b) = GM(S_h(a,b), S_h(b,a))$ and $S_{LK}(a,b) = T_{LK}(S_h(a,b), S_h(b,a))$, it only remains to prove the last inequality.

Let us suppose that $S_{g-mean}(a,b) > S_{mean}(a,b)$, i.e.,

$$\frac{w(a \cap b)}{\sqrt{w(a)\, w(b)}} > \frac{w(a \cap b)(w(a) + w(b))}{2w(a)\, w(b)}.$$

Thus, $4\, w(a)\, w(b) > (w(a) + w(b))^2$, which is equivalent to say that $0 > (w(a) - w(b))^2$ and this is a contradiction. □

The order given by the preceding proposition shows that first appear similarity measures, followed by robust similarity measures, and finally, strong robust similarity measures.

In particular cases, some of these similarities are equal, as we can prove in the following result:

Corollary 5.2. *Let $a, b \in \mathcal{L}([0,1])$ such that $a \subseteq b$, then $S_{LK}(a,b) = S_{prod}(a,b) = S_{min}(a,b)$.*

Proof. If $a \subseteq b$, then

$$S_{LK}(a,b) = \max\left\{0, \frac{w(a \cap b)}{w(a)} + \frac{w(a \cap b)}{w(b)} - 1\right\} = \max\left\{0, \frac{w(a)}{w(b)}\right\} = \frac{w(a)}{w(b)} \text{ and } S_{min}(a,b) = \min\left\{\frac{w(a)}{w(a)}, \frac{w(a)}{w(b)}\right\} = \frac{w(a)}{w(b)}.$$

□

It should be noticed that when an interval is included in another one and, we use S_{mean}, the values obtained are larger than 0.5. This happens because $S_h(a,b) = 1$ if a is completely contained in the other interval b. Thus, it is evident that the mean between $S_h(a,b)$ and $S_h(b,a)$ is bigger than 0.5.

Although S_{mean} has the advantage of emphasizing when one interval is fully contained within another, guaranteeing that in this case, values are always above 0.5, this can also be seen as a disadvantage. This is due to the loss of sensitivity to subtle differences between intervals because of its restricted range of values, from 0.5 to 1, which may not capture small changes in the intervals leading to close values of similarity. This limited sensitivity may pose a limitation in scenarios where subtle differences between intervals need to be detected and considered.

This can be shown in the following example. Let us define the following four intervals $a_1 = [0.1, 0.2]$, $b_1 = [0.1, 0.8]$, $a_2 = [0.1, 0.5]$ and $b_2 = [0.4, 0.8]$, where $w(a_1 \cap b_1) = w(a_2 \cap b_2)$. In addition, a_1 is completely included in b_1. while a_2 is not in b_2. These intervals are shown in Figure 5. If we compute the similarities, we obtain the following pairs of values:

$$S_{mean}(a_1, b_1) = 0.571, \qquad S_{g-mean}(a_1, b_1) = 0.378,$$
$$S_{mean}(a_2, b_2) = 0.25, \qquad S_{g-mean}(a_2, b_2) = 0.25.$$

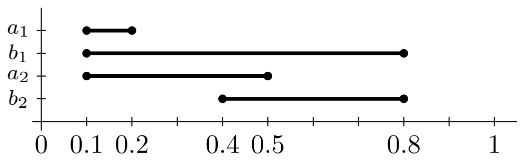


Figure 5: Pairs of intervals with same overlapping degree.

So, it is clear that the similarity between a_1 and b_1 is higher than the similarity between a_2 and b_2 because a_1 is completely included in b_1. Moreover, it should be noticed that $S_{mean}(a_1, b_1) = 0.571$ is too high compared to $S_{mean}(a_2, b_2) = 0.25$, where the number of points in common is the same, while S_{g-mean} shows values more flexible and not as drastic as S_{mean}.

The previous example illustrates the practical implications of these theoretical properties. While S_{mean} assigns higher similarity values in inclusion cases,

S_{g-mean} offers a smoother transition that better reflects partial overlaps. Interestingly, this balance emerges even though the geometric mean is not zero-strict, confirming that strong robustness can arise from the interplay between aggregation and pre-inclusion properties. In general, these results support the suitability of the proposed framework to classify and compare similarity measures in terms of their adherence to the axiomatic principles, highlighting how the choice of aggregation directly affects the robustness and interpretability of the resulting similarity.

6 An example with real data

In this section, we illustrate how the theoretical similarity measures introduced in this work can be applied to real-world data. The dataset used for this analysis was obtained from the AEMET[1], the State Meteorological Agency of Spain, which is responsible for monitoring and disseminating meteorological information across the country. Specifically, AEMET collects data on the atmospheric conditions in Spain. For our study, we retrieved the monthly maximum and minimum temperature intervals for each province during the year 2021. These intervals are displayed in Figure 6.

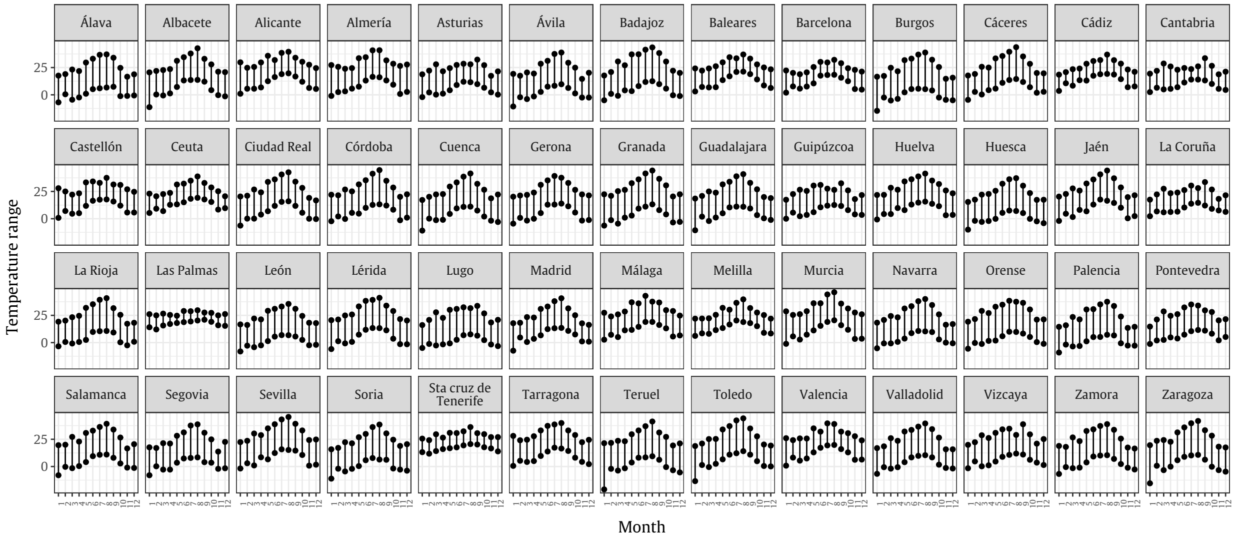


Figure 6: Intervals of temperatures of Spain in 2021 using AEMET data.

We would like to point out some parts of these data. In the following, examples will be presented using two provinces, Badajoz and Valencia. With them, we will indicate some drawbacks of certain similarity measures shown previously. In particular, we will concentrate on S_J and S_D suffering from aliasing and on S_J, S_{LK} and S_{prod} not verifying the symmetry with respect to the pre-inclusion (Axiom *A9*).

For our first analysis, let us focus on the province of Badajoz. It is located in southwestern Spain, and it has a semi-arid climate characterized by hot and dry summers and mild winters. The temperature range in this region can vary greatly depending on the time of year, with the highest temperatures typically occurring in July and August and the lowest temperatures occurring in January. The intervals of temperatures in Badajoz have been further detailed in Figure 7. Moreover, these intervals have been normalized using the minimum and maximum temperatures registered for Spain in the year 2021, which are -21°C and 46.2°C, respectively. The resulting intervals after the normalization for this province are presented in Table 2.

Our first analysis focuses on computing the similarities in the temperature intervals across different months in Badajoz and observing their behaviour. We use the theoretical framework presented to identify the characteristics described for the similarities when these are applied in real data, and we compare the results obtained with the similarity

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[1] `https://www.aemet.es/`

measures proposed (S_{mean} and S_{g-mean}) to those obtained through the non-strong robust similarity measures (S_J, S_D, S_{LK}, S_{min}, S_{prod}).

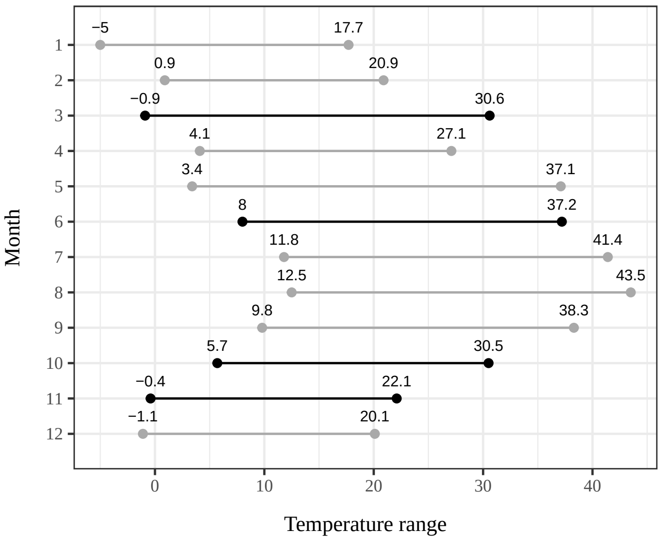

Figure 7: Interval of temperatures (in °C) in Badajoz (Spain) each month. Intervals of different color are the focus of the analysis.

Month	Normalized interval
1	[0.2380952, 0.5758929]
2	[0.3258929, 0.6235119]
3	[0.2991071, 0.7678571]
4	[0.3735119, 0.7157738]
5	[0.3630952, 0.8645833]
6	[0.4315476, 0.8660714]
7	[0.4880952, 0.9285714]
8	[0.4985119, 0.9598214]
9	[0.4583333, 0.8824405]
10	[0.3973214, 0.7663690]
11	[0.3065476, 0.6413690]
12	[0.2961310, 0.6116071]

Table 2: Interval of temperatures in Badajoz (Spain) each month normalized.

Consider the pairs of intervals of the months $\{3, 11\}$ and $\{6, 10\}$. Table 3 shows that S_J and S_D assign the same similarity values to both pairs of intervals, even though the situations are different: in $\{3, 11\}$ one interval is completely contained within the other, while in $\{6, 10\}$ the intervals only partially overlap. This confirms that these classical measures suffer from aliasing, as they are unable to distinguish between structurally different cases. By contrast, the strong robust similarities S_{mean} and S_{g-mean} provide higher values when full containment occurs ($\{3, 11\}$) and slightly lower values for partial overlap ($\{6, 10\}$), thus capturing the expected behaviour.

	S_J	S_D	S_{LK}	S_{prod}	S_{min}	S_{mean}	S_{g-mean}
$\{3, 11\}$	**0.71142857**	**0.8333333**	0.7142857	0.7142857	0.7142857	0.8571429	0.8451543
$\{6, 10\}$	**0.71142857**	**0.8333333**	0.6778059	0.6990858	0.7705479	0.8389030	0.8361135

Table 3: Similarity values for months $\{3, 11\}$ and $\{6, 10\}$ in Badajoz.

This happens as both of these similarities suffer from aliasing, so the value of similarities does not reflect that the intervals are completely different. In the presented example, S_{LK} appears to be immune to aliasing effects. However, upon comparing different months within the same province, aliasing becomes evident. For instance, let us consider the intervals **1**, **7**, and **12**, which represent the intervals for each month, respectively. The pairs of intervals **1** with **7** , and **7** with **12** , have the same width of union, width of intersection and different withs between themselves. Under this situation, it can be observed that $S_{LK}(\mathbf{1}, \mathbf{7}) = S_{LK}(\mathbf{7}, \mathbf{12}) = 0$ indicating instances of aliasing.

Also, Figure 8 displays the similarity values between all the months in a heat map. For the cases of S_D, S_{min}, S_{mean}, S_{g-mean} we can see a gray level scale, meaning that the similarities detect smaller changes between intervals. However, the grayscale scale shown for the proposed similarities reveals that they have more sensitivity, and as the month moves away and has progressively different temperature intervals, its similarity also decreases progressively. On the other hand, S_J, S_{LK} and S_{prod} have a similar behaviour. They are capable of capturing the differences between the temperature intervals, as we can see in the lower colours in the heat map.

Let us focus on another province, in this case we choose Valencia. Its Mediterranean climate is characterized by mild winters and hot summers, making the city experience long periods of sunny and warm weather. This description can be seen in Figure 9 together with the normalized intervals in Table 4.

Taking into account the intervals of the months $\{2, 9\}$ and $\{10, 12\}$, the similarity values can be observed in Table 5. The width of the interval **2** is equal to the width of the interval **9**. Also, intervals **10** and **12** have the same width

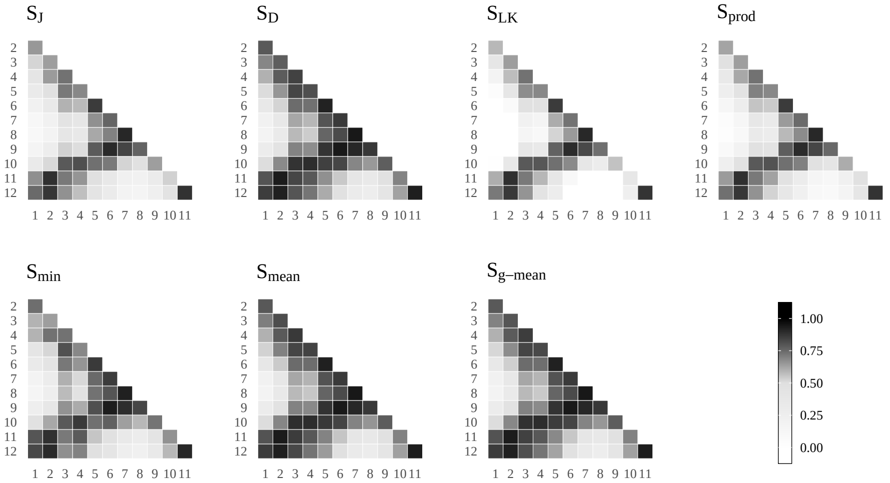

Figure 8: Matrices of comparison of the intervals obtained for the temperatures each month in Badajoz.

between them. We can observe that in this case, how Axiom *A9* is fulfilled by all similarity measures except S_J, S_{LK} and S_{prod} as we have proved theoretically, being $S_h(\mathbf{2},\mathbf{9}) = S_h(\mathbf{9},\mathbf{2}) = 0.5125$ and $S_h(\mathbf{10},\mathbf{12}) = S_h(\mathbf{12},\mathbf{10}) = 0.625$.

Table 5 confirms the expected behaviour with respect to Axiom A9. For the pairs $\{2, 9\}$ and $\{10, 12\}$, the interval widths are equal, so the similarity values should also be symmetric. Indeed, the strong robust measures S_{min}, S_{mean}, and S_{g-mean} return identical values in both directions, consistently reflecting the subsethood values $S_h(2, 9) = S_h(9, 2) = 0.5125$ and $S_h(10, 12) = S_h(12, 10) = 0.625$. By contrast, S_J, S_{LK}, and S_{prod} fail to satisfy this condition, as their values deviate from the expected symmetric results.

By applying the similarity measures to this data, we have highlighted some problems with the classical similarities available in the literature and shown how the proposed measures can overcome these issues.

Overall, the analysis provides new insights into the behaviour of similarities using real data from different provinces of Spain and demonstrates the potential of the theoretical similarities introduced in this work and how they can be applied to data analysis.

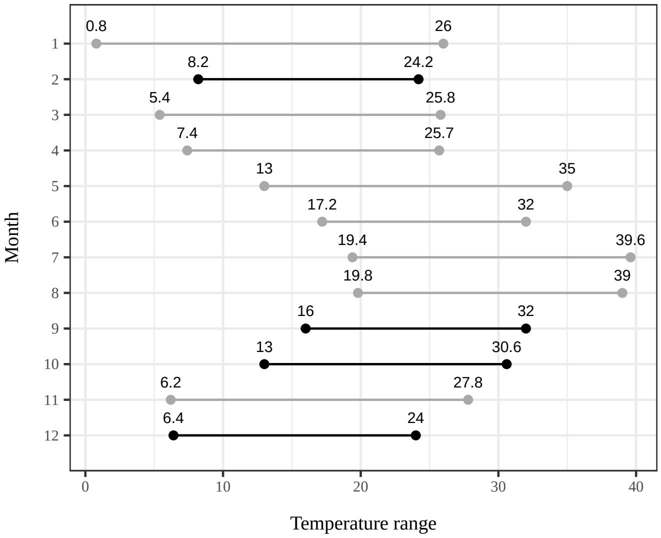

Figure 9: Interval of temperatures (in °C) in Valencia (Spain) each month.

Month	Normalized interval
1	[0.3244048, 0.6994048]
2	[0.4345238, 0.6726190]
3	[0.3928571, 0.6964286]
4	[0.4226190, 0.6949405]
5	[0.5059524, 0.8333333]
6	[0.5684524, 0.7886905]
7	[0.6011905, 0.9017857]
8	[0.6071429, 0.8928571]
9	[0.5505952, 0.7886905]
10	[0.5059524, 0.7678571]
11	[0.4047619, 0.7261905]
12	[0.4077381, 0.6696429]

Table 4: Interval of temperatures in Valencia (Spain) each month normalized.

	S_J	S_D	S_{LK}	S_{prod}	S_{min}	S_{mean}	S_{g-mean}
$\{2,9\}$	**0.3445378**	0.5125000	**0.0249999**	**0.2626563**	0.5125000	0.5125000	0.5125000
$\{10,12\}$	**0.4545455**	0.6250000	**0.2500002**	**0.3906250**	0.6250000	0.6250000	0.6250000

Table 5: Similarity values for months $\{2,9\}$ and $\{10,12\}$ in Valencia.

Beyond the numerical comparison, these results also have practical implications. In climate data analysis, the ability to capture differences between temperature intervals in a robust way is essential for tasks such as clustering provinces with similar seasonal behaviours, detecting anomalous years, or supporting decision-making in environmental management. For example, the fact that strong robust similarities can distinguish between pairs of months with the same intersection but different inclusion relationships provides more reliable information for grouping or ranking regions according to their climatic patterns. Thus, the proposed measures not only advance the theoretical framework but also provide useful tools for real applications in areas where decisions must be made under uncertainty and imprecision, such as climate monitoring, regional planning, or energy demand forecasting.

7 Conclusions

In this paper, we introduced a new family of functions, termed pre-inclusions, designed to facilitate the characterization of both strong and robust similarity measures. This concept provides a unified and flexible framework that generalizes classical inclusion measures and supports the axiomatic study of interval similarities. Building on pre-inclusions, we proposed a general methodology for constructing strong robust similarity measures through their combination with suitable aggregation functions, and we demonstrated its applicability with several illustrative examples. From a theoretical standpoint, we established precise conditions under which similarity measures ensure robustness and strong robustness, showing that only a small set of operators fulfill all the required axioms.

Notably, the subsethood minimum, subsethood mean, and subsethood geometric mean were identified as strong robust similarity measures, offering principled solutions to the aliasing problem and related limitations. From a practical perspective, we assessed the proposed measures on a real dataset of temperature intervals. The experiments confirmed the superior performance of the subsethood mean and geometric mean, which distinguished interval pairs more effectively than classical approaches. These results highlight the potential of strong robust similarity measures to enhance both the accuracy and interpretability of interval-based data analysis across diverse application domains.

As future work, further exploration of operators derived from the ratio model, such as the harmonic mean, appears promising. Additionally, applications in clustering, decision-making, and environmental data analysis merit investigation, as well as a deeper examination of the links between pre-inclusions and traditional inclusion measures. Taken together, these directions point toward more robust and versatile tools for addressing similarity under uncertainty and imprecision. In this sense, the proposed framework serves as a bridge between axiomatic foundations and practical implementations, providing a consistent basis for the systematic development of new similarity measures. This integration of theory and application is expected to stimulate further advances in fuzzy modeling and approximate reasoning.

Acknowledgement

This work has been partially supported by the projects PID2022-139886NB-I00 and SEK-25-GRU-GIC-24-018; and the European project SCIMIN-CRM (Grant Agreement No. 101177746).

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