Iranian Journal of Fuzzy Systems

Volume 23, Number 1, (2026), pp. 95-109

Original paper

University of Sistan and Baluchestan

A Newton–Cotes-based iterative scheme for nonlinear fuzzy Volterra integral equation

H. Mottaghi Golshan [1] and R. Ezzati [2]

[1] *Department of Mathematics, Shahr.C., Islamic Azad University, Shahryar, Iran*
[2] *Department of Mathematics, Ka.C., Islamic Azad University, Karaj, Iran*

ha.mottaghi@iau.ac.ir, reza.ezzati@iau.ac.ir

Abstract

This paper introduces a novel iterative numerical method for solving nonlinear fuzzy Volterra integral equations using Newton–Cotes (NC) quadrature rules. The core idea is to apply auxiliary Newton–Cotes rules (ANCR) over subintervals of the domain, enabling more flexible and accurate approximations of fuzzy integrals. A detailed convergence analysis is presented to establish the method's validity and efficiency. The scheme operates within a complete fuzzy metric space and ensures convergence under Lipschitz continuity conditions in the kernel using fixed-point theory. The results show that this method can provide a significant improvement in computational accuracy and generality compared to current methods and offers a suitable opportunity for future research in the field of nonlinear fuzzy integral equations. These results demonstrate that the ANCR scheme offers both provable convergence and practical advantage in accuracy and overall computational cost for a broad class of nonlinear fuzzy Volterra problems.

Keywords: NC rules, fuzzy Volterra integral equation, auxiliary Newton-Cotes rules, iterative method.

1 Introduction and Preliminaries

Since fuzzy integral equations offer a useful tool for modeling and analyzing systems with uncertain information, their solution is crucial in many branches of science and engineering. Even though several numerical techniques have been developed to solve these equations, there is always a need for methods that are more accurate and efficient. This work's main notion is that, in many situations, such as the Volterra integral equation (1), one may compute the integral $\int_a^t, a \leq t \leq b$ by using points outside the range $[a, t]$ as a subset of $i := [a, b]$. To deal with fuzzy Volterra integral equations, this work presents and evaluates an iterative approach based on NC rules to calculate integrals in the interval $[a, t]$ using points in I.

The study of fuzzy system equations has advanced with the second-kind fuzzy integral equation, which is written as follows:

$$x(u) = \mathcal{T}(x)(u) := f(u) \oplus (\mathrm{FH}) \int_a^u \mathcal{K}(u, v, x(v))dv, \qquad u, v \in I. \tag{1}$$

In this equation, x denotes a fuzzy-valued function that is unknown, while f is a function that is known to have fuzzy values. The term $\mathcal{K}$ expresses a fuzzy-valued kernel function, which is continuous in the specified domain $I \times I \times \mathbb{R}_{\mathcal{F}}$, mapping to $\mathbb{R}_{\mathcal{F}}$. The operator $\mathcal{T}$ signifies a transformation that integrates the influence of the known fuzzy function f and the fuzzy Henstock integral, indicated as (FH) $\int$ (concerning the definition and findings, see [1, 2, 4]).

The study of fuzzy sets and integrals is useful for analyzing situations that are not immediately apparent. These uncertainties can impact any part of an integral equation, including the initial value and boundary conditions. When these models are recognized in practical contexts, interval or fuzzy techniques are used as a stand-in. Fuzzy set theory

Corresponding Author: H. Mottaghi Golshan
Received: January 2025; Revised: October 2025; Accepted: January 2026.
https://doi.org/ 10.22111/ijfs.2026.50962.9004

has been widely used in many fields, including topology, fixed-point theory, integral equalities, fractional calculus, bifurcation, image processing, consumer electronics, control theory, artificial intelligence, and operations research. The exploration of fuzzy integral equations is rapidly advancing, notably due to their newly recognized relevance to fuzzy control methodologies. Integral equations are fundamental to mathematical modeling across various disciplines, including chemistry, engineering, biology, and physics. To account for the subtleties of fuzzy analysis, the fuzzy Henstock integral expands on the traditional concept of integration, providing more adaptability and versatility in a variety of situations. This integral equation is particularly valuable in fields where uncertainty and vagueness are prevalent, such as in decision-making processes, control systems, and economic modelling. The general form of the fuzzy integral equation of the second kind not only enriches the theoretical framework of fuzzy analysis but also provides practical tools for addressing real-world problems characterized by uncertainty. The fuzzy kernel's regulation of the interaction between the known and unknown functions highlights the dynamic character of fuzzy systems and opens the way for more study and use in this emerging subject.

In the analytical and numerical solution of functional equations, such as integral equations (see [6, 8, 11, 20, 22, 23, 26]) and fixed point theory problems using Picard sequences (or successive approximation) with its application (see [17, 21]), iterative methods are undoubtedly among the most crucial. The purpose of this paper is to present a novel iterative method rooted in numerical computation FNC rules aimed specifically at addressing nonlinear fuzzy Volterra integral equations. The increasing complexity of such mathematical formulations necessitates innovative solutions, and this study builds upon prior research in the field, notably the works of [6, 8, 10, 11, 22, 23, 26, 30, 31]. Although several numerical frameworks have been proposed for solving fuzzy Volterra integral equations, such as triangular function approximations, block pulse operational matrices, and hybrid Elzaki–Adomian decomposition methods, these approaches exhibit intrinsic limitations when extended to nonlinear kernels. Triangular and block pulse bases are low-order, piecewise-constant or piecewise-linear functions that can approximate smooth fuzzy solutions only with large numbers of subintervals, resulting in slow convergence and increased computational cost. On the other hand, the Elzaki–ADM and related semi-analytical transforms rely on series decompositions whose convergence deteriorates for strongly nonlinear or rapidly varying kernels, often requiring many terms to achieve acceptable accuracy. Moreover, most of these schemes do not provide explicit convergence in fuzzy metric spaces. Therefore, there remains a need for a numerical scheme that simultaneously ensures theoretical convergence guarantees, higher-order local accuracy, and reduced computational effort for nonlinear fuzzy Volterra integral equations.

This paper aims to develop some recent works [6, 8, 10, 11, 22, 23, 26, 30, 31] by introducing and establishing an iterative method based on NC rules for solving numerically nonlinear fuzzy Volterra integral equations (1) through Picard sequences. All of these papers used points inside the interval $[a, b]$ to estimate the fuzzy integral, therefore, they usually have to solve integral equations of the Fredholm type or use complex algorithms to calculate Volterra-type integral equations. Using the points outside the interval $[a, t] \subseteq I$ to estimate the fuzzy integral and applying auxiliary FNC rules are two of the method's novelties with respect to works [6, 8, 11, 22, 23, 26] for solving Volterra-type integral equations.

In Section 3, the proposed approach is justified, establishing both its efficacy and validity through comprehensive theoretical analysis and empirical results. The iterative method introduced herein not only extends the existing body of knowledge but also demonstrates practical applicability in solving nonlinear fuzzy Volterra integral equations. The findings underscore the potential of this method to enhance computational techniques and provide researchers with a robust tool for future investigations. The results in Section 4 show the justification and validity of the method.

Let us review some essential concepts, findings, and illustrations about fuzzy number theory. The papers [1, 2, 4] are some of the fundamentals and reliable sources on the topics. $u : \mathbb{R} \to [0, 1]$ is a function that represents a fuzzy number. The condition that u is normal is $\exists x_0 \in \mathbb{R}$ such that $u(x_0) = 1$. The fuzzy convex set u is defined as follows: 1) $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}, \forall x, y \in \mathbb{R}, \lambda \in [0, 1]$, 2) u is upper semicontinuous, and 3) $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is a compact set. It is generally proved that [27], for any $0 < r \leq 1$, an arbitrary fuzzy number is represented by an ordered pair of functions, $u^r = \left[u^r_-, u^r_+\right]$, which satisfies in: u^r_- is a bounded, left-continuous, non-decreasing function over $[0, 1]$, u^r_+ is a bounded, left-continuous, non-increasing function over $[0, 1]$, and $u^r_- \leq u^r_+$. $\mathbb{R}_{\mathcal{F}} = (\mathbb{R}_{\mathcal{F}}, \oplus, \odot)$ designates the set of fuzzy numbers, where $\oplus$ and $\odot$ stand for fuzzy addition and multiplication, respectively. Every $r \in \mathbb{R}$ may be interpreted as a fuzzy point $\tilde{r} = \chi_r$, as a result, $\mathbb{R} \subseteq \mathbb{R}_{\mathcal{F}}$ and $\tilde{0}$ and $\tilde{1}$ are neutral elements for these operations, respectively. The fuzzy set $(\mathbb{R}_{\mathcal{F}}, \oplus)$ is a semi-group. If $u^r(t) = [u^r_-(t), u^r_+(t)] = [rt^n, rt^n]$, we may show it using $\tilde{t^n}$, where $n \in \mathbb{N}$.

Refer to [12, 28] for the concept of fuzzy numbers product, and [2, 14] for the set of fuzzy numbers established with Hausdorff metric:

$$D(u, v) = \sup_{r \in [0,1]} \max\{|u^r_- - v^r_-|, |u^r_+ - v^r_+|\},$$

for every $u = [u^r_-, u^r_+], v = [v^r_-, v^r_+] \in \mathbb{R}_{\mathcal{F}}$. $(\mathbb{R}_{\mathcal{F}}, D)$ is demonstrated to be complete a metric space (see [2, 5]). The set of continuous functions x from I into $\mathbb{R}_{\mathcal{F}}$ is denoted as $\mathcal{C}_{\mathcal{F}}(I)$. Apply the weighted metric $\widehat{D}_{\mu}(x,y) = \sup\{D(e^{-\mu t} \odot x(t), e^{-\mu t} \odot y(t)), t \in I\} = \sup\{e^{-\mu t} D(x(t), y(t)), t \in I\}$, it is established that $(\mathcal{C}_{\mathcal{F}}(I), \widehat{D}_{\mu})$ is a complete metric space for $\mu = 0$ (see [2, 6, 9]). Furthermore, for all $\mu > 0$ the metrics $\widehat{D}_{\mu}$ and $\widehat{D}_0 = \widehat{D}$ are equivalent, because $\widehat{D}_{\mu} \leq \widehat{D}_0 \leq e^{b\mu}\widehat{D}_{\mu}$. Thus, $(\mathcal{C}_{\mathcal{F}}(I), \widehat{D}_{\mu})$ is a complete metric space for all $\mu \geq 0$. Since $\mathbb{R} \subseteq \mathbb{R}_{\mathcal{F}}$, the set of real continuous functions $\mathcal{C}(I)$ is a subset of $\mathcal{C}_{\mathcal{F}}(I)$. Let $x \in \mathcal{C}_{\mathcal{F}}(I)$, then

$$\Omega_{\delta}(x, I) := \sup\{D(x(t_1), x(t_2)) : t_1, t_2 \in I, d(t_1, t_2) \leq \delta\},$$

is referred to as the modulus of continuity of x (see [5]), where d is Euclidean distance. You may find some essential characteristics of this concept on [3, 5]. In the sequel, we employ $\oplus \sum$ for fuzzy sum in order to eliminate ambiguity.

2 Auxiliary FNC rules

Take n, k be numbers in $\mathbb{N}$ where $n > 1$ is a multiple of $k > 1$. For the purposes of this work, with $i = 0, k, 2k, \ldots, n$, denote $i = 0 : k : n$. Equally spaced nodes must be taken into account, i.e.,

$$I(n) = \{t_i = a + ih, i = 0, \ldots, n\}, \tag{2}$$

where $h = \frac{b-a}{n}$. In the following, we are engaged in introducing auxiliary FNC rules (abbreviated AFNCR) in the interval $[a, t] \subseteq I$. Two cases exist to calculate (FH) $\int_a^{t_i} \phi(s)ds, i = 1, \ldots, n$ using FNC rules:

Case 1: The non-repeated form, that is, when $n = k$: For all $q = 1, \ldots, k$ define

$$\text{AFNCR}(\phi, k, q, [t_0, t_k]) = \oplus \sum_{i=0}^{k} \rho_{k,q}(i) \odot \phi(t_i), \tag{3}$$

where

$$\rho_q(i) := \rho_{k,q}(i) = \int_0^q \frac{\pi_i'(t)}{\pi_i'(t_i)} dt,$$

and

$$\pi_k(t) = \prod_{j=0}^{k} (t - t_j), \pi_i'(t) = \frac{\pi_k(t)}{(t - t_i)}, i = 0, \ldots, k.$$

The well-known Newton-Cotes integration coefficients on the interval I are precisely $\rho_{0,k}, \ldots, \rho_{k,k}$ in the case $q = k$; see [13, 16, 19].

Case 2: The repeated form: Each $r = 1, \ldots, n$, may be written as $r = \bar{r}k + q$, where $0 \leq q < k$ is the reminder of r in mod k. There are two possible situations:

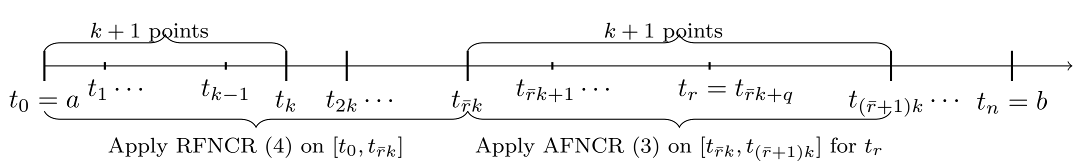
Situation 1: If $q = 0$. We have $[a, t_r] = [t_0, t_k] \cup [t_k, t_{2k}] \cup \cdots \cup [t_{(\bar{r}-1)k}, t_{\bar{r}k}]$. Define

$$\text{RFNCR}_{n,r,q}(\phi, [a, t_r]) := \oplus \sum_{j=0:k:}^{(\bar{r}-1)k} \text{NCR}_{k,k}(\phi, [t_j, t_{j+k}]) = \oplus \sum_{j=0:k:}^{(\bar{r}-1)k} \oplus \sum_{i=0}^{k} \rho_k(i) \odot \phi(t_{j+i}). \tag{4}$$

Situation 2: If $1 \leq q < k$. We have $[a, t_r] = [t_0, t_k] \cup [t_k, t_{2k}] \cup \cdots \cup [t_{(\bar{r}-1)k}, t_{\bar{r}k}] \cup [t_{\bar{r}k}, t_{\bar{r}k+q}]$. Define

$$\text{RFNCR}_{n,r,q}(\phi, [a, t_r]) := \oplus \sum_{j=0:k:}^{(\bar{r}-1)k} \oplus \sum_{i=0}^{k} \rho_k(i) \odot \phi(t_{j+i}) \oplus \sum_{i=0}^{k} \rho_q(i) \odot \phi(t_{t_{\bar{r}k+i}}). \tag{5}$$

Actually, in the above sentence, the coefficient of the first term $\rho_k(i)$ is known RFNCR on the intervals $[t_j, t_{j+k}], j = 0 : k : (\bar{r} - 1)k$, and the second one, i.e., $\rho_q(i)$, is auxiliary Newton-Cotes rules on the interval $[t_{\bar{r}k}, t_r]$. The next graphical depiction illustrates relation (5):



Theorem 2.1. The AFNCR rules (3) are exact for all $\phi = \tilde{1}, \tilde{t}, \tilde{t}^2, \ldots, \tilde{t}^k \in \mathcal{C}(I) \subseteq \mathcal{C}_{\mathcal{F}}(I)$.

Proof. Since ϕ are crisp functions, the proof is concluded from [16, Sec. 7, Theorem 1]. □

Thus, we find the following systems

$$\sum_{i=0}^{k} \rho_q(i) t_i^r = \int_a^{a+qh} t^r dt = \frac{(a+qh)^{r+1} - a^{r+1}}{r+1}, \qquad r = 0, \ldots, k, \tag{6}$$

for all $q = 1, \ldots, k$. The solutions of this k system of order $k+1$ with the unknown weights $\rho_q(i), i = 0, \ldots, k$ exist. Since t_i are distinct, thus, each of these systems is a Vandermonde matrix, which is nonsingular and has a unique solution. To calculate the coefficients $(\rho_q(i))_{k\times(k+1)}, q = 1, \ldots, k, i = 0, \ldots, k$, the Vandermonde systems (6) are used. The following example is the answer for $k = 2$ (Simpson's rule). NC rules for big k are recognized to need to be applied with caution, see [13, Subsec 2.5]. However, as you can see in Section 4, $k \leq 10$ is acceptable.

Example 2.2. Let $\phi \in \mathcal{C}_{\mathcal{F}}(I)$, the non-repeated auxiliary Simpson's rule and the known Simpson's rule are

$$\begin{cases} q = 1 \to (\text{FH}) \int_{t_0}^{t_1} \phi(t)dt & \simeq \text{AFNCR}(\phi, 2, 1, [t_0, t_1]) = \frac{h}{3}\left(\frac{5}{4}\phi_0 + 2\phi_1 - \frac{1}{4}\phi_2\right), \text{ and} \\ q = 2 \to (\text{FH}) \int_{t_0}^{t_2} \phi(t)dt & \simeq \text{AFNCR}(\phi, 2, 2, [t_0, t_2]) = \frac{h}{3}\left(\phi_0 + 4\phi_1 + \phi_2\right), \end{cases}$$

respectively, where $t_0 = a, t_1 = \frac{a+b}{2}, t_2 = b, h = \frac{b-a}{2}$.

Theorem 2.3. Let $\phi \in \mathcal{C}_{\mathcal{F}}(I)$ and $t_0 = a, t_1 = t_0 + h, \ldots, t_k = b$ be equally-spaced points. Then

$$D\left(I_q(\phi, [a, a+qh]), \text{RFNCR}(\phi, k, q, [t_0, t_q])\right) \leq \left(\sum_{i=0}^{k} |\rho_q(i)|\right) \Omega_{qh}(\phi, I), \tag{7}$$

for all $q = 1, \ldots, k$.

Proof. Put $\tilde{1}$ in (3), we get $\sum_{i=0}^{k} \rho_q(i) = qh = t_q - t_0$. According to properties of D (see [3], [14] and [27]), we have

$$\begin{aligned} &D\left((\text{FH}) \int_{t_0}^{t_q} \phi(t)dt, \text{RFNCR}(\phi, k, q, [t_0, t_q])\right) \\ &\quad = D\left(\oplus \sum_{i=0}^{k} \rho_q(i) \frac{1}{qh} (\text{FH}) \int_{t_0}^{t_q} \phi(t)dt, \frac{1}{qh} (\text{FH}) \int_{t_0}^{t_q} \oplus \sum_{i=0}^{k} \rho_q(i) \odot \phi(t_i)dt\right) \\ &\quad \leq \sum_{i=0}^{k} |\rho_q(i)| \, D\left(\frac{1}{qh} (\text{FH}) \int_{t_0}^{t_q} \phi(t)dt, \frac{1}{qh} (\text{FH}) \int_{t_0}^{t_q} \phi(t_i)\right) \\ &\quad \leq \left(\sum_{i=0}^{k} |\rho_q(i)|\right) \Omega_{qh}(\phi, [t_0, t_q]). \end{aligned} \tag{8}$$

□

3 Iterative approach based on ANCR

In sequel, let us fix $k > 1$, n be an integer and multiple of k. Consider equally spaced points (2), and $r = \bar{r}k + q$ where $0 \leq q < k$ is the remainder of $r = 1, \ldots, n$ by k. For all $x \in \mathcal{C}_{\mathcal{F}}(I)$, let us define $\mathcal{N}_n(x) : I(n) \to \mathbb{R}$ by

$$\mathcal{N}_n(x)(t_r) = \begin{cases} \oplus \sum\limits_{j=0:k:}^{(\bar{r}-1)k} \oplus \sum\limits_{i=0}^{k} \rho_k(i) \odot \mathcal{K}(t_r, t_{j+i}, x(t_{j+i})), & q = 0, \quad (9) \\ \oplus \sum\limits_{j=0:k:}^{(\bar{r}-1)k} \oplus \sum\limits_{i=0}^{k} \rho_k(i) \odot \mathcal{K}(t_r, t_{j+i}, x(t_{j+i})) \oplus \sum\limits_{i=0}^{k} \rho_q(i) \odot \mathcal{K}(t_r, t_{t_{\bar{r}k+i}}, x(t_{t_{\bar{r}k+i}})), & q = 1, \ldots, k-1, \quad (10) \end{cases}$$

where $t_r \in I(n), r = 1, \ldots, n$. For all initial values $x_0 \in X$, where X is a certain subset of $\mathcal{C}_{\mathcal{F}}(I)$, we shall show that the successive approximation sequences on equally spaced points (2), i.e., $x_i(t_0) = f(t_0)$, and

$$x_i(t_r) = f(t_r) \oplus \mathcal{N}_n(x_{i-1})(t_r), \ \forall t_r = a + rh, r = 1, \ldots, n, \tag{11}$$

where $i \in \mathbb{N} \cup \{0\}$, presents a solution for the fuzzy Volterra integral equation numerically (1) on $I(n)$.

3.1 Convergence analysis

Take into consideration the space $\mathcal{C}_{\mathcal{F}}(I)$ that has the weighted metric $\widehat{D}_\mu$ for some appropriate $\mu > 0$. Let us set

$$\lambda(x) = \max\left\{\frac{D(\mathcal{N}_n(x)(t_i), \mathcal{N}_n(x)(t_j))}{|t_i - t_j|}, \forall t_i \neq t_j \in I(n)\right\}, \tag{12}$$

For all $x \in C_{\mathcal{F}}(I)$, then $\mathcal{N}_n(x)$ are Lipschitz functions on $I(n)$ with Lipschitz constant $\lambda(x)$. Assume that $\mathcal{M}_n(x)$ represents

$$\begin{aligned}\mathcal{M}_n(x)(t) &= \inf_{y\in I(n)}[\mathcal{N}_n(x)(y) \oplus \lambda(x)|t-y|] \\ &:= \left[\inf_{y\in I(n)} \mathcal{N}_n(x)(y)^r_- + \lambda(x)|t-y|, \inf_{y\in I(n)} \mathcal{N}_n(x)(y)^r_+ + \lambda(x)|t-y|\right], \quad t \in I.\end{aligned} \tag{13}$$

Define

$$\mathcal{T}_n(x)(t) = f(t) \oplus \mathcal{M}_n(x)(t), \quad t \in I, n \in \mathbb{N}. \tag{14}$$

Then each $\mathcal{T}_n = f \oplus \mathcal{M}_n, n \in \mathbb{N}$ introduces a self-map on $C_{\mathcal{F}}(I)$, and each $\mathcal{T}_n(x) \in C_{\mathcal{F}}(I), x \in C_{\mathcal{F}}(I)$ is a Lipschitz function with Lipschitz constant $\lambda(x)$, because:

Lemma 3.1. For all $x \in \mathcal{C}_{\mathcal{F}}(I), n \in \mathbb{N}$, $\mathcal{M}_n(x), \mathcal{T}_n(x)$ are Lipschitz functions with the Lipschitz constants $\lambda(x)$ and $\mathcal{M}_n(x)(t_r) = \mathcal{N}_n(x)(t_r)$, for all $t_r \in I(n)$.

Proof. For $x, x' \in C_{\mathcal{F}}(I), t, t' \in I, r \in [0,1]$ we have

$$\begin{aligned}\mathcal{M}_n(x)(t)^r_+ &\leq \mathcal{M}_n(x)(t'')^r_+ + \lambda(x)|t-t''| \\ &\leq \mathcal{M}_n(x)(t'')^r_+ + \lambda(x)|t-t'| + \lambda(x)|t'-t''|,\end{aligned}$$

for all $t'' \in I(n)$, similarly the inequality holds for $\mathcal{M}_n(x)(t)^r_-$ instead of $\mathcal{M}_n(x)(t)^r_+$, implying

$$\begin{aligned}D(\mathcal{T}_n(x)(t), \mathcal{T}_n(x)(t')) &= D(f(t) \oplus \mathcal{M}_n(x)(t), f(t) \oplus \mathcal{M}_n(x)(t')) \\ &= D(\mathcal{M}_n(x)(t), \mathcal{M}_n(x)(t')) \\ &= \sup_{r\in[0,1]} \max\{|\mathcal{M}_n(x)(t)^r_- - \mathcal{M}_n(x)(t')^r_-|, |\mathcal{M}_n(x)(t)^r_+ - \mathcal{M}_n(x)(t')^r_+|\}, \\ &\leq \lambda(x)|t-t'|.\end{aligned}$$

Also, from (13), we get $\mathcal{M}_n(x)(t_r) = \mathcal{N}_n(x)(t_r)$, for all $t_r \in I(n)$. □

Theorem 3.2. Suppose $n \in \mathbb{N}$ is a multiples of $1 < k \in \mathbb{N}$, and

(I) Let $f \in \mathcal{C}_{\mathcal{F}}(I)$ and $\mathcal{K} \in \mathcal{C}_{\mathcal{F}}(I \times I \times \mathbb{R})$.

(II) There exist $\mu_1, \rho > 0$ such that $\mathcal{T}(\bar{B}_\rho) \subseteq \bar{B}_\rho$ and $\mathcal{T}_n(\bar{B}_\rho) \subseteq \bar{B}_\rho$, for all $n \in \mathbb{N}$, $\mu \geq \mu_1$, where $\bar{B}_\rho := \{x \in \mathcal{C}_{\mathcal{F}}(I) \mid \widehat{D}_\mu(x, f) \leq \rho\}$.

(III) Moreover, assume that there exists $L > 0$ such that

$$D(\mathcal{K}(x,y,u), \mathcal{K}(x,y,v)) \leq LD(u,v) \qquad x, y \in I, u, v \in [\mathfrak{m}, \mathfrak{M}], \tag{15}$$

where $\mathfrak{m} = \inf_{u\in\bar{B}_\rho, t\in\mathbb{R}, 0\leq r\leq 1} u(t)^r_-, \mathfrak{M} = \sup_{u\in\bar{B}_\rho, t\in\mathbb{R}, 0\leq r\leq 1} u(t)^r_+$. Then, $\mathcal{T}$ is a contraction on $(\mathcal{C}_{\mathcal{F}}(I), \hat{D}_\mu)$, for all $\mu \geq L$ and with contraction constant $\kappa = L\left(\frac{1-e^{-\mu(b-a)}}{\mu}\right) < 1$. If $u_0 \in \bar{B}_\rho$ is the unique solution of (1) on $\bar{B}_\rho$, then

$$\lim_{m\to+\infty} \lim_{n\to+\infty} \mathcal{T}_n^m x_0 = u_0,$$

for all $x_0 \in \bar{B}_\rho$.

Proof. Put $\phi = 1$ in (5), and from Theorem 2.1, we get

$$\mathrm{RFNCR}_{n,r,q}(1,[a,t_r]) = \int_a^{t_r} ds \leq b-a. \tag{16}$$

From definition of $\mathcal{T}_n, n \in \mathbb{N}$, (16) and

$$\mathcal{K}_0 = \sup_{x,y\in I, u\in[\mathfrak{m},\mathfrak{M}], 0\leq r\leq 1} |\mathcal{K}(x,y,u)|_+^r < +\infty, \tag{17}$$

we have

$$\begin{cases} \sup_{0\leq r\leq 1} \mathcal{N}_n(x)(t_r)_-^r = \mathrm{RFNCR}_{n,r,q}(\mathcal{K}(t_r,\cdot,x(\cdot))_-^r, [t_0=a, t_r=a+rh]) \leq (b-a)\mathcal{K}_0, \\ \sup_{0\leq r\leq 1} \mathcal{N}_n(x)(t_r)_+^r = \mathrm{RFNCR}_{n,r,q}(\mathcal{K}(t_r,\cdot,x(\cdot))_+^r, [t_0=a, t_r=a+rh]) \leq (b-a)\mathcal{K}_0, \end{cases}$$

and

$$\begin{cases} \mathcal{M}_n(x)(t)_-^r := \inf_{y\in I(n)}[\mathcal{N}_n(x)(y)\oplus\lambda(x)|t-y|]_-^r \leq (b-a)\mathcal{K}_0 + \inf_{y\in I(n)}\lambda(x)|t-y| \leq (b-a)\mathcal{K}_0, \\ \mathcal{M}_n(x)(t)_+^r := \inf_{y\in I(n)}[\mathcal{N}_n(x)(y)\oplus\lambda(x)|t-y|]_+^r \leq (b-a)\mathcal{K}_0 + \inf_{y\in I(n)}\lambda(x)|t-y| \leq (b-a)\mathcal{K}_0, \end{cases} \tag{18}$$

for all $r = 0,\ldots,n$. Accordingly, each $\mathcal{T}_n : \mathcal{C}_{\mathcal{F}}(I) \to \mathcal{C}_{\mathcal{F}}(I)$ is a Lipschitz function and well-defined by Lemma 3.1.

Let $x \in \bar{B}_\rho, \rho > 0$. The properties of D and [8, Lemma 5] imply

$$\begin{aligned} \hat{D}_\mu(\mathcal{T}(x), f) &= \sup_{t\in I} D(e^{-\mu t}\odot\mathcal{T}(x)(t), e^{-\mu t}\odot f(t)) \\ &= \sup_{t\in I} D\left(\mathcal{T}(x)(t), f(t)\right)e^{-\mu t} \\ &= \sup_{t\in I} D\left((\mathrm{FH})\int_a^t \mathcal{K}(t,s,x(s))ds, \tilde{0}\right)e^{-\mu t} \\ &= \sup_{t\in I} (\mathrm{L})\int_a^t D\left(\mathcal{K}(t,s,x(s)), \tilde{0}\right)e^{-\mu t} ds \\ &\leq \mathcal{K}_0 \sup_{t\in I} te^{-\mu t}. \end{aligned}$$

Therefore, $\mathcal{T}(\bar{B}_\rho) \subseteq \bar{B}_\rho$ holds for all $\mu > \mu_2$, where $\mu_2 > 0$ is a solution of the inequality

$$\mathcal{K}_0 \sup_{t\in I} te^{-\mu t} \leq \rho. \tag{19}$$

Also, from (12), (18), we get

$$\begin{aligned} \hat{D}_\mu(\mathcal{T}_n(x), f) &= \sup_{t\in I} D_\mu(e^{-\mu t}\odot\mathcal{T}_n(x), e^{-\mu t}\odot f) \\ &= \sup_{t\in I} e^{-\mu t} D_\mu(\mathcal{M}_n(x), \tilde{0}) \\ &\leq \sup_{t\in I, t_r\in I(n)} D_\mu(\mathcal{M}_n(x), \tilde{0})e^{-\mu t} \leq (b-a)\mathcal{K}_0 e^{-\mu a}. \end{aligned} \tag{20}$$

Therefore, $\mathcal{T}_n(\bar{B}_\rho) \subseteq \bar{B}_\rho$ holds for all $\mu \geq \mu_3, n \in \mathbb{N}$, where $\mu_3 > 0$ is a solution of the inequality

$$(b-a)\mathcal{K}_0 e^{-\mu a} \leq \rho, \tag{21}$$

(Without limiting generalization, we may always presume that $a > 0$). Thus, $\mathcal{T}(\bar{B}_\rho) \subseteq \bar{B}_\rho$ and $\mathcal{T}_n(\bar{B}_\rho) \subseteq \bar{B}_\rho, n \in \mathbb{N}$, for all $\mu \geq \mu_1 = \max\{\mu_2, \mu_3\}$.

We have three steps:

Step 1: We show that $\mathcal{T}$ is a contraction on $(C(I), \hat{D}_\mu)$, for all $\mu \geq L$. To see this, let $y, z \in C(I)$. From properties of D and [24, Theorem 4.3], we have

$$\hat{D}_\mu(\mathcal{T}y, \mathcal{T}z) = \sup_{t\in I} D\left(e^{-\mu t}\odot\left[f(t)\oplus(\mathrm{FH})\int_a^t \mathcal{K}(t,s,y(s)))ds\right], e^{-\mu t}\odot\left[f(t)\oplus(\mathrm{FH})\int_a^t \mathcal{K}(t,s,z(s))ds\right]\right)$$

$$
\begin{aligned}
&= \sup_{t\in I} D\left(e^{-\mu t}\odot (\mathrm{FH})\int_a^t \mathcal{K}(t,s,y(s)))ds, e^{-\mu t}\odot (\mathrm{FH})\int_a^t \mathcal{K}(t,s,z(s))ds\right)\\
&\le \sup_{t\in I} (\mathrm{L})\int_a^t e^{-\mu t} D\left(\mathcal{K}(t,s,y(s))), \mathcal{K}(t,s,z(s))\right) ds\\
&\le L\sup_{t\in I}\left\{(\mathrm{L})\int_a^t e^{-\mu t}e^{\mu s}e^{-\mu s}D(y(s),z(s))ds\right\}\\
&\le L\sup_{t\in I}\left\{(\mathrm{L})\int_a^t e^{-\mu(t-s)}ds\right\}\hat{D}_\mu(y,z)\\
&\le L\left(\frac{1-e^{-\mu(b-a)}}{\mu}\right)\hat{D}_\mu(y,z).
\end{aligned}
$$

Since for all $\mu \ge L$ we have $\kappa := L\left(\frac{1-e^{-\mu(b-a)}}{\mu}\right) < 1$, the Banach contraction principle implies that there is a unique $y \in \mathcal{C}_{\mathcal{F}}(I)$ with $y = \mathcal{T}y$; equivalently (1) has a unique solution on $\mathcal{C}_{\mathcal{F}}(I)$.

Step 2: Let us show that $\mathcal{T}_n$ covers point-wise $\mathcal{T}$ on $\mathcal{C}_{\mathcal{F}}(I)$. Consider $x \in \mathcal{C}_{\mathcal{F}}(I), t \in I$ and equally-spaced points (2). There exists $r_0 = 1, \dots, n$ such that $t_{r_0-1} \le t \le t_{r_0}$. We have

$$
\begin{aligned}
D_\mu(\mathcal{T}(x)(t), \mathcal{T}_n(x)(t)) &\le D_\mu(f(t), f(t_{r_0})) + D_\mu(\mathcal{T}(x)(t), \mathcal{T}(x)(t_{r_0}))\\
&\quad + D_\mu(\mathcal{T}(x)(t_{r_0}), \mathcal{T}_n(x)(t_{r_0})) + D_\mu(\mathcal{T}_n(x)(t_{r_0}), \mathcal{T}_n(x)(t))\\
&\le D_\mu(f(t), f(t_{r_0})) + D_\mu\left((\mathrm{FH})\int_a^t \mathcal{K}(t,s,x(s))ds, (\mathrm{FH})\int_a^{t_{r_0}} \mathcal{K}(t,s,x(s))ds\right)\\
&\quad + D_\mu\left((\mathrm{FH})\int_a^{t_{r_0}} \mathcal{K}(t,s,x(s))ds, \mathcal{T}_n(x)(t_{r_0})\right)\\
&\quad + D_\mu(f(t), f(t_{r_0})) + D_\mu(\mathcal{T}_n(x)(t_{r_0}), \mathcal{M}_n(x)(t)).
\end{aligned}
\tag{22}
$$

We demonstrate that for a sufficiently large positive integer N, each of the last three lines in (22) is small enough. Assume that $\varepsilon > 0$.

(1) From continuty of f we get $\exists\delta_f; 0 < \delta_f < h = \frac{b-a}{N_f}, \forall t, t' \in I|t-t'| < \delta_f \Rightarrow D(f(t), f(t')) < \varepsilon$.

(2) It can be choose $N_1 \in \mathbb{N}$ such that $h\mathcal{K}_0 = \frac{(b-a)\mathcal{K}_0}{N_1} < \varepsilon$, then for all $n \ge N_1$ we get

$$
\begin{aligned}
D\left((\mathrm{FH})\int_a^t \mathcal{K}(t,s,x(s))ds, (\mathrm{FH})\int_a^{t_{r_0}} \mathcal{K}(t,s,x(s))ds\right) &= D\left((\mathrm{FH})\int_t^{t_{r_0}} \mathcal{K}(t,s,x(s))ds, \tilde{0}\right)\\
&\le (\mathrm{L})\int_t^{t_{r_0}} D\left(\mathcal{K}(t,s,x(s)), \tilde{0}\right) ds\\
&\le (t_{r_0} - t)\mathcal{K}_0 \le h\mathcal{K}_0 \le \varepsilon.
\end{aligned}
$$

(3) Let $r_0 = r_0'k + q$, where $q = 0, \dots, k-1$. Since the situation $q = 0$ is handled similarly, we only take into account the cases $q = 1, \dots, k-1$. From Theorem 2.3, we derive

$$
\begin{aligned}
&D\left((\mathrm{FH})\int_a^{t_{r_0}} \mathcal{K}(t,s,x(s))ds, \mathcal{T}_n(x)(t_{r_0})\right)\\
&\quad = D\Bigg(\sum_{j=0:k:}^{(r_0'-1)k} (\mathrm{FH})\int_{t_j}^{t_{j+k}} \mathcal{K}(t,s,x(s))ds + (\mathrm{FH})\int_{t_{r_0'k}}^{t_{r_0'k+q}} \mathcal{K}(t,s,x(s))ds\ ,\\
&\quad \sum_{j=0:k:}^{(r_0'-1)k}\sum_{i=0}^{k} \rho_k(i)\mathcal{K}(t_{r_0}, t_{j+i}, x(t_{j+i})) + \sum_{i=0}^{k} \rho_q(i)\mathcal{K}(t_{r_0}, t_{t_{r_0'k+i}}, x(t_{t_{r_0'k+i}}))\Bigg)\\
&\quad \le \sum_{j=0:k:}^{(r_0'-1)k} D\left((\mathrm{FH})\int_{t_j}^{t_{j+k}} \mathcal{K}(t,s,x(s))ds, \sum_{i=0}^{k} \rho_k(i)\mathcal{K}(t_{r_0}, s, x(t_{j+i}))\right)
\end{aligned}
$$

$$+ D\left((\text{FH})\int_{t_{r_0'k}}^{t_{r_0'k+q}} \mathcal{K}(t,s,x(s))ds, \sum_{i=0}^{k}\rho_q(i)\mathcal{K}(t_{r_0},s,x(t_{t_{r_0'k+i}}))\right)$$
$$\leq \sum_{j=0:k:}^{(r_0'-1)k} L|t_{j+k}-t_j|\Omega_{\delta_2}(x,[t_{j+k},t_j]) + L|t_{r_0'k+q}-t_{r_0'k}|\Omega_{\delta_2}(x,[t_{r_0'k},t_{r_0'k+q}])$$
$$\leq L(b-a)\Omega_{\delta_2}(x,[b,a]),$$

where $\delta_2 = qh$. From [22, Sec 1 relation (7)], there exists $N_2 \in \mathbb{N}$ and $\delta_2 > 0$ such that $h = \frac{b-a}{N_2} < \delta_2$, and for all $n \geq N_2$ we have $L(b-a)\Omega_{\delta_2}(x,[b,a]) \leq \varepsilon$.

(4) For all $x \in \bar{B}_\rho$, put $\delta_3 = \frac{\varepsilon}{3\lambda(x)}$, then for all $|t-t'| < \delta_3, t, t' \in I$ we have

$$D(\mathcal{T}_n(x)(t_{r_0}), \mathcal{M}_n(x)(t)) \leq D\left(\mathcal{T}_n(x)(t_{r_0}), \inf_{y\in I(n)} \mathcal{T}_n(x)(y) \oplus \lambda(x)|t-y|\right)$$
$$\leq \inf_{y\in I(n)}[D(\mathcal{T}_n(x)(y), \mathcal{T}_n(x)(t_r)) + \lambda(x)|t-y|]$$
$$\leq \inf_{y\in I(n)}[\lambda(x)|t_r-y| + \lambda(x)|t-y|]$$
$$\leq 2\lambda(x)\delta_3 < \varepsilon.$$

Let $N_3 \in \mathbb{N}$ be an integer such that $h = \frac{b-a}{N_3} < \delta_3$, thus for all $n \geq N_3$, we have $t_{r_0} - t < \delta_3$ and $D(\mathcal{T}_n(x)(t_{r_0}),$ $\mathcal{M}_n(x)(t)) \leq \varepsilon$ holds.

So, from (1)-(4) it is concluded that $D(\mathcal{T}(x)(t), \mathcal{T}_n(x)(t)) \leq 5\varepsilon$, for all $t \in I, n \geq N = \max\{N_1, N_2, N_3, N_f\}$, this means $\mathcal{T}_n(x) \to \mathcal{T}(x)$, for all $x \in \mathcal{C}_{\mathcal{F}}(I)$.

Step 3: From Step 1, for every $\varepsilon > 0, x_0 \in X$, there exists $n_0 \in \mathbb{N}$ such that $\forall m \geq n_0, \hat{D}_\mu(\mathcal{T}^m x_0, u_0) < \frac{\varepsilon}{2}$. From Step 2, $\hat{D}_\mu(\mathcal{T}_n x, \mathcal{T}x) \to 0$, for all $x \in \bar{B}_\rho$, as a result we get $\hat{D}_\mu(\mathcal{T}_n(\mathcal{T}x), \mathcal{T}(\mathcal{T}x)) \to 0$, for all $x \in X$, and by continuity of $\mathcal{T}_n$ we get $\hat{D}_\mu(\mathcal{T}_n(\mathcal{T}_n x), \mathcal{T}_n(\mathcal{T}x)) \to 0$, thus,

$$\hat{D}_\mu(\mathcal{T}_n^2 x, \mathcal{T}^2 x) \leq \hat{D}_\mu(\mathcal{T}_n(\mathcal{T}_n x), \mathcal{T}_n(\mathcal{T}x)) + \hat{D}_\mu(\mathcal{T}_n(\mathcal{T}x), \mathcal{T}(\mathcal{T}x)) \to 0.$$

Induction may be used on m to infer

$$\hat{D}_\mu(\mathcal{T}_n^m x, \mathcal{T}^m x) \to 0, \text{ for all } x \in X, \tag{23}$$

for all $m \in \mathbb{N}$. For every $m \geq n_0$ there exists n_m such that $\forall n \geq n_m, \hat{D}_\mu(\mathcal{T}_n^m x_0, \mathcal{T}^m x_0) < \frac{\varepsilon}{2}$, thus, for all $m, n \geq n_m$ we get

$$\hat{D}_\mu(\mathcal{T}_n^m x_0, u_0) \leq \hat{D}_\mu(\mathcal{T}_n^m x_0, \mathcal{T}^m x_0) + \hat{D}_\mu(\mathcal{T}^m x_0, u_0) < \varepsilon. \tag{24}$$

This prove that $\lim_{m\to+\infty}\lim_{n\to+\infty} \mathcal{T}_n^m x_0 = u_0$. □

We do not provide the sensitivity analysis of perturbations in the fuzzy initial values, as it is somewhat similar to the proofs in [7, 15, 31] and is based on the convergence of Picard sequences in the Banach contraction principle.

4 Examples and tests

If $n \in \mathbb{N}$ is a multiple of $k \in \mathbb{N}$, where $n, k \in \mathbb{N}$. Based on [29], we used absolute errors. $e^r_{i,-}(t_i) :=| x^*(t_i)^r_- - x_m(t_i)^r_- |$ $, e^r_{i,+}(t_i) :=| x^*(t_i)^r_+ - x_m(t_i)^r_+ |$ with maximum absolute errors $e^r_{\max,-} := \max\{e^r_{i,-}(t_i), i = 0,\dots,n\}, e^r_{\max,-} :=$ $\max\{e^r_{i,-}(t_i), i = 0,\dots,n\}$. All findings are obtained using Matlab code, in which we selected $\max\{| x_{m+1}(t_i)^r_- - x_m(t_i)^r_- |\} < \varepsilon := 10^{-12}$ ($\max\{| x_{m+1}(t_i)^r_+ - x_m(t_i)^r_+ |\} < \varepsilon$) to end the iterative process, more precisely, the Algorithm 1 was used:

Example 4.1. Consider the following non-linear fuzzy Volterra integral equation:

$$x(t) = f(t) \oplus (\text{FH})\int_0^t te^{-s} \odot x^2(s)ds, \quad t \in [0,1],$$

Algorithm 1 Iterative Scheme for Solving Equation (11)

Set n and k
Set tolerance $\varepsilon > 0$
Set $i \leftarrow 0$
Set $x(t_0) \leftarrow f(t_0)$
Set $(\rho_q(i))_{k\times(k+1)}, q = 1, \dots, k, i = 0, \dots, k$ from the systems (6)
Initialize $x^0 \leftarrow$ initial guess
repeat
  for $r = 1, \dots, n$ **do**
    Set $q := r \mod k$ and $\bar{r} = \left[\frac{r}{k}\right]$
    if $q = 0$ **then**
      Compute $x^{i+1}(t_r) \leftarrow f(t_r) \oplus \mathcal{N}_n(x^i)(t_r)$ from Eq. (9)
    else if then
      Compute $x^{i+1}(t_r) \leftarrow f(t_r) \oplus \mathcal{N}_n(x^i)(t_r)$ from Eq. (10)
    end if
  end for
  Compute residual $m \leftarrow \|x^{i+1} - x^i\|_{\max}$
  $i \leftarrow i + 1$
until $m < \varepsilon$
return x^{i+1}

with

$$f^r(t) = \left[rt + r^2 t\left(e^{-t}(t^2 + 2t + 2) - 2\right), (2-r)t + t(r-2)^2\left(e^{-t}(t^2 + 2t + 2) - 2\right)\right].$$

The exact solution is $x^r = [r, (2-r)]t$. Table 1 and Figures 1 and 2 present the results.

r, n	Errors	Simpson's rule	Boole's rule	$k = 10$
$\frac{1}{4}$,20	$e^r_{\max,-}$	2.1135e-08	1.2232e-10	2.7756e-17
	$e^r_{\max,+}$	2.7605e-05	4.6020e-09	2.2204e-16
$\frac{1}{4}$,40	$e^r_{\max,-}$	1.3209e-09	1.8751e-12	2.7756e-17
	$e^r_{\max,+}$	1.6913e-06	9.6467e-09	1.3323e-15
$\frac{1}{2}$,20	$e^r_{\max,-}$	5.6146e-06	9.3600e-10	5.5511e-17
	$e^r_{\max,+}$	1.1395e-06	6.5467e-09	2.2204e-16
$\frac{1}{2}$,40	$e^r_{\max,-}$	5.5871e-09	8.1821e-12	1.1102e-16
	$e^r_{\max,+}$	7.3404e-08	1.0411e-10	4.4409e-16
$\frac{3}{4}$,20	$e^r_{\max,-}$	9.8256e-06	1.6380e-09	5.5511e-17
	$e^r_{\max,+}$	2.0119e-05	3.3540e-09	1.1102e-16
$\frac{3}{4}$,40	$e^r_{\max,-}$	1.3834e-08	2.0080e-11	2.2204e-16
	$e^r_{\max,+}$	4.6429e-08	2.6.6321e-11	4.4409e-16

Table 1: Results of errors for $n = 20, 40, k = 2, 4, 10$ and levels $r = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, respectively

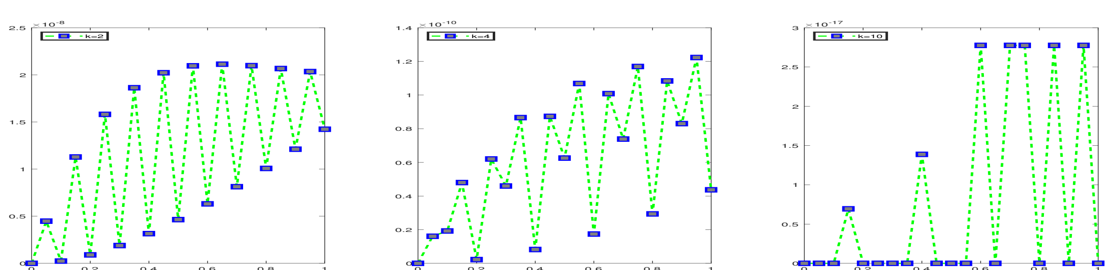


Figure 1: Plot of error functions $e^{\frac{1}{4}}_{i,-}$, where $n = 20, k = 2, 4, 10$, respectively

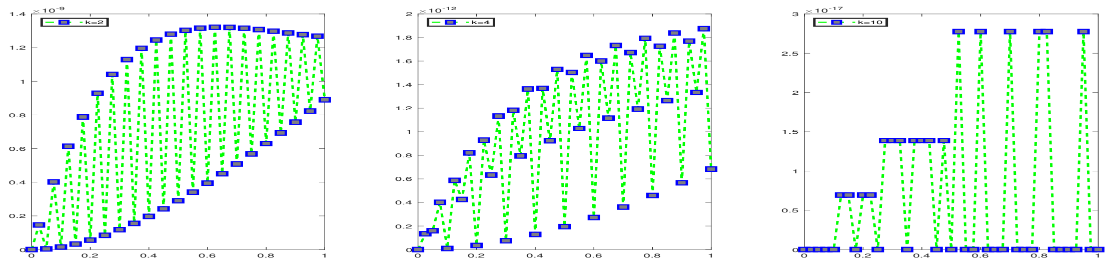


Figure 2: Plot of error functions $e_{i,-}^{\frac{1}{4}}$, where $n = 40, k = 2, 4, 10$, respectively

Example 4.2 ([18, Example 6.4] and [25, Example 6.1])**.** Consider the following linear fuzzy Volterra integral equation:

$$x^r(t) = \left[r^2 + r, 4 - r^2 - r\right] \left(\cosh(t) - \sinh^2(t)\right) \oplus (\text{FH}) \int_0^t \sinh(t) \odot x(s)ds,$$

The exact solution is $x^r = \left[r^2 + r, 4 - r^2 - r\right] \cosh(t)$. Table 2 and Figures 3 and 4 present the results.

r, n	Errors	Simpson's rule	Boole's rule	$k = 10$
$\frac{1}{4}$,10	$e^r_{\max,-}$	1.4423e-06	$4 \nmid 10$	1.3323e-15
	$e^r_{\max,+}$	1.7019e-05	$4 \nmid 10$	1.5987e-14
$\frac{1}{4}$,20	$e^r_{\max,-}$	1.0537e-07	1.2047e-10	3.3307e-16
	$e^r_{\max,+}$	1.2434e-06	1.4215e-09	1.7764e-15
$\frac{1}{2}$,10	$e^r_{\max,-}$	3.4615e-06	$4 \nmid 10$	3.1086e-15
	$e^r_{\max,+}$	1.5000e-05	$4 \nmid 10$	1.4211e-14
$\frac{1}{2}$,20	$e^r_{\max,-}$	2.5289e-07	2.8913e-10	4.4409e-16
	$e^r_{\max,+}$	1.0959e-06	1.2529e-09	1.7764e-15
$\frac{3}{4}$,10	$e^r_{\max,-}$	6.0577e-06	$4 \nmid 10$	5.7732e-15
	$e^r_{\max,+}$	2.0119e-05	$4 \nmid 10$	1.1102e-16
$\frac{3}{4}$,20	$e^r_{\max,-}$	4.4256e-07	5.0597e-10	8.8818e-16
	$e^r_{\max,+}$	9.0620e-07	1.0360e-09	1.7764e-15

Table 2: Results of errors for $n = 10, 20, k = 2, 4, 10$ and levels $r = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, respectively

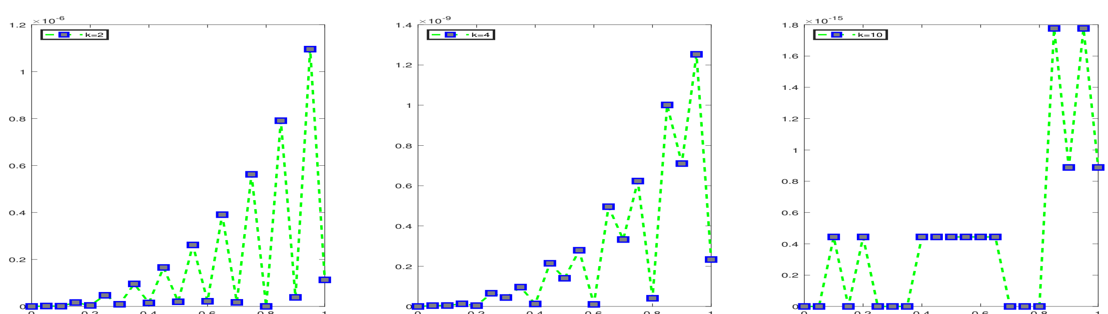


Figure 3: Plot of error functions $e_{i,+}^{\frac{1}{2}}$, where $n = 20, k = 2, 4, 10$, respectively

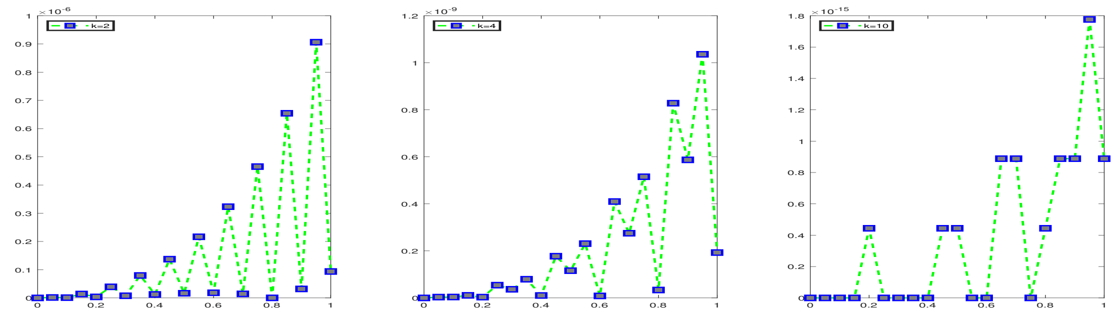
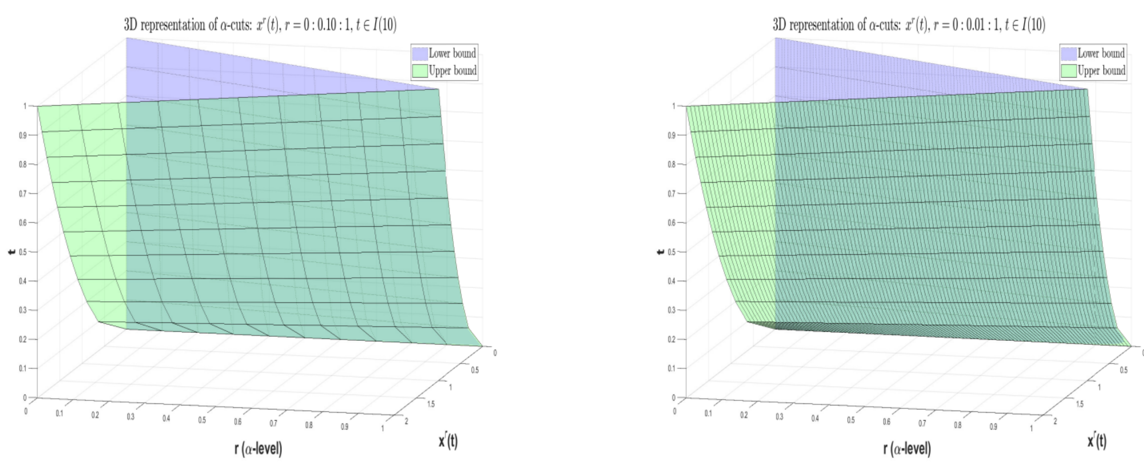


Figure 4: Plot of error functions $e^{\frac{3}{4}}_{i,+}$, where $n = 20, k = 2, 4, 10$, respectively

Example 4.3. Consider the following non-linear fuzzy Volterra integral equation:

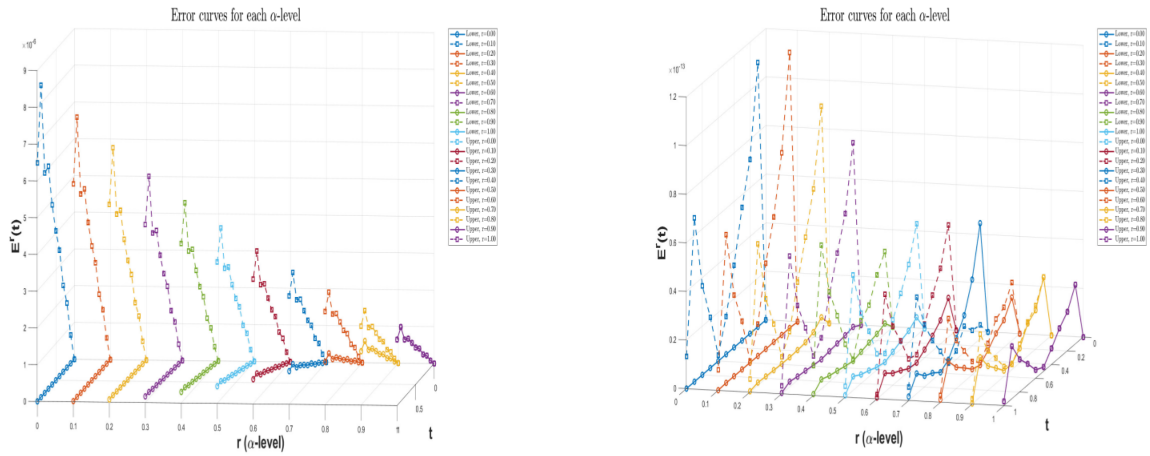
$$x(t) = \left[rt^{\frac{1}{2}} - r^2(t - \sin(t)), (2-r)^2(\cos(t) - 1) - (2-r)t^{\frac{1}{4}}\right] \oplus (\text{FH}) \int_0^x \sin(t-s) \odot x^2(s)ds.$$

The exact solution is $x^r = [r, 2-r]\sqrt{t}$. Elapsed times to calculate the numerical results and create each of the Figures 5(a) and (b) from Algorithm 1 took less than 0.5 and 2 seconds, respectively. The error functions $e^r_{i,-}$ and $e^r_{i,+}$ are shown in Figures 6(a) and (b) to support the prospect of viewing them all at once, where $n = 10, t, r = 0, 0.1, \ldots, 1$. Additionally, the computation of Algorithm 1 to produce Fig. 7 was approximately 0.5 seconds, where $n = 100, k = 10, t = 0 : 0.01 : 1, r = 0 : 0.10 : 1$.



(a) $r = 0 : 0.10 : 1, t \in I(10)$ (b) $r = 0 : 0.01 : 1, t \in I(10)$

Figure 5: 3D-Plots of $x^r(t)$, where $n = 10, k = 10$



(a) Simpson's rule (b) $k = 10$

Figure 6: 3D-Plots off error functions $e^r_{i,\pm}(t)$, where $n = 10$, $t, r = 0 : 0.10 : 1$, $k = 2$ (a) and $k = 10$ (b), respectively

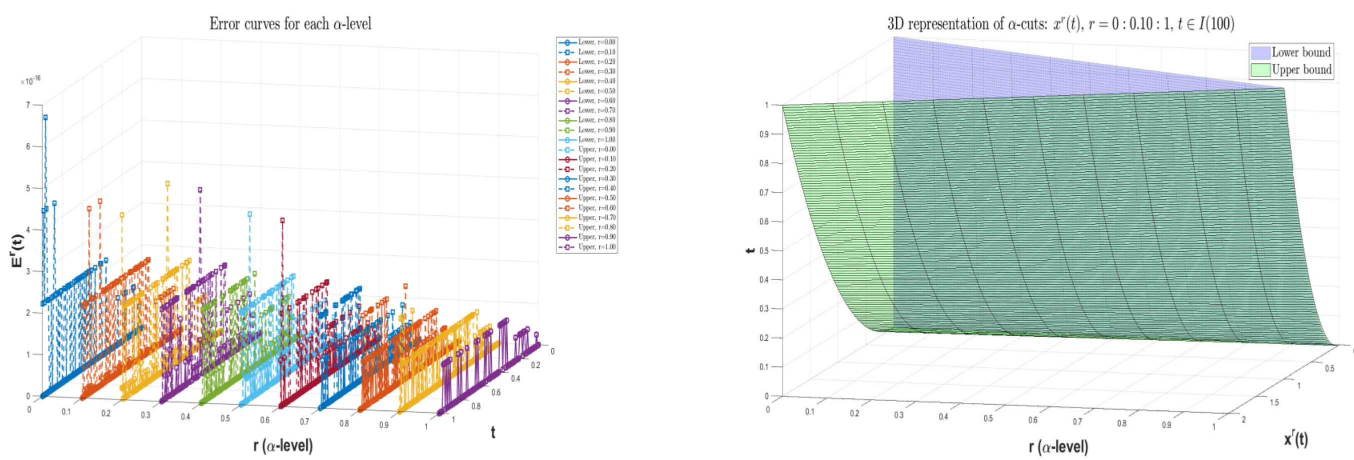


Figure 7: 3D-Plots of error functions $e^r_{i,\pm}$ and the numerical solution $x^r(t)$, where $t = 0 : 0.01 : 1, r = 0 : 0.10 : 1, k = 10$, respectively

With respect to the iterative techniques employed in [10, 11, 22, 23, 30, 31], the accuracy is comparable to [22] and superior to the others, although some of them lack Vollera-type examples.

5 Conclusions

The study's outcomes show that the suggested iterative approach can be a useful tool for resolving nonlinear fuzzy Volterra integral equations. The study's conclusions create new avenues for research and development of algorithms intended to solve challenging fuzzy integral equations. With a comprehensive analysis of convergence, this method not only provides high accuracy but can also be employed in various scientific and engineering applications requiring fuzzy integral equation solutions. Moreover, the experimental results reveal that this method offers improved computational accuracy compared to existing approaches. Theorem 3.2 guarantees that when the fuzzy kernel $\mathcal{K}$ is uniformly Lipschitz on the third column in the bounded set $[\mathfrak{m}, \mathfrak{M}]$, the iterative method converges to a unique solution of the integral equation (1). This means the approach may fail if $\mathcal{K}$ is unbounded in other columns. We examined an example that confirms this, with a singularity $\frac{1}{\sqrt{s}}, s \in [0,1]$ in its kernel. Maybe the issue can be resolved by altering FNC rules. Hence, future works can investigate further refinements and extensions of the method to improve its applicability to broader classes of less smooth kernels, such as discontinuous, (weakly) singular, or delay-type kernels.

Nomenclature

$u^r = [u^r_-, u^r_+]$	r-level representation of fuzzy number	$\oplus, \odot$	Fuzzy addition and multiplication
$(\mathbb{R}_{\mathcal{F}}, D)$	Complete metric space of fuzzy numbers	$\mathcal{C}_{\mathcal{F}}(I)$	Space of continuous fuzzy-valued functions
a, b	Endpoints of the integration interval	$I(n)$	Equally spaced partition nodes
t	Independent variable	t_i	Mesh nodes on I
n	Number of nodes or subintervals	$h = \frac{b-a}{n}$	Step size in partition
x	Unknown fuzzy-valued function	x_0	Initial guess of iteration
x^α_-, x^α_+	Lower/upper α-cuts of $x(t)$	f	Given fuzzy source function
$\mathcal{K}$	Fuzzy kernel (possibly nonlinear)	$\rho_q = \rho_{k,q}$	Newton–Cotes quadrature weights
AFNCR	Auxiliary Fuzzy Newton–Cotes Rule	RFNCR	Repeated Fuzzy Newton–Cotes Rule
π_k	Interpolation polynomial	$\mathcal{T}(x)$	Nonlinear integral operator
$\mathcal{T}_n(x)$	AFNCR numerical nonlinear operator	$\mathcal{N}_n(x)$	Discrete nonlinear operator
$\mathcal{M}_n(x)$	Lipschitz extension of $\mathcal{N}_n(x)$	x_{n+1}	$(n{+}1)$-th iterate in the AFNCR scheme
κ	Contraction constant	L	Lipschitz constant of the kernel
μ	Exponential weighting parameter: $e^{-\mu(t-s)}$	$D(\cdot,\cdot)$	Fuzzy metric induced by α-cuts
$\Omega_\delta(x, I)$	Modulus of continuity of fuzzy function x	$\hat{D}(\cdot,\cdot)$	Fuzzy metric on $\mathcal{C}_{\mathcal{F}}(I)$
ε	Convergence tolerance for iteration	$\hat{D}_\mu(\cdot,\cdot)$	Weighted fuzzy metric on $\mathcal{C}_{\mathcal{F}}(I)$
$e^r_{i,-}, e^r_{i,+}$	Lower/upper error at t_i	$e^r_{\max,-}, e^r_{\max,+}$	Lower/upper maximum errors at level r
$\lambda(x)$	Lipschitz constant of $\mathcal{N}_n(x)$		

Declarations

Funding

No funding was used in this study

Availability of data and materials

No data were used to support this study.

Conflict of interest

The authors declare that they have no competing interests.

Authors contributions

The authors' confirm sole responsibility for the following: study conception and design, data collection, analysis and interpretation of results, and manuscript preparation.

References

[1] T. Allahviranloo, S. Salahshour, *Advances in fuzzy integral and differential equations*, Studies in Fuzziness and Soft Computing, **412**, Springer, Cham, (2022), vi+228 pages. https://doi.org/10.1007/978-3-030-73711-5

[2] G. A. Anastassiou, *Fuzzy mathematics: Approximation theory*, Springer, **251** (2010). https://doi.org/10.1007/978-3-642-11220-1

[3] G. A. Anastassiou, S. G. Gal, *On a fuzzy trigonometric approximation theorem of Weierstrass-type*, The Journal of Fuzzy Mathematics, **9**(3) (2001), 701-708.

[4] B. Bede, *Mathematics of fuzzy sets and fuzzy logic*, Studies in Fuzziness and Soft Computing, Springer, Heidelberg, **295** (2013), xii+276 pages. https://doi.org/10.1007/978-3-642-35221-8

[5] B. Bede, S. G. Gal, *Quadrature rules for integrals of fuzzy-number-valued functions*, Fuzzy Sets and Systems, **145**(3) (2004), 359-380. https://doi.org/10.1016/S0165-0114(03)00182-9

[6] A. M. Bica, C. Popescu, *Numerical solutions of the nonlinear fuzzy Hammerstein-Volterra delay integral equations*, Information Sciences, **223** (2013), 236-255. https://doi.org/10.1016/j.ins.2012.10.022

[7] A. M. Bica, C. Popescu, *Approximating the solution of nonlinear Hammerstein fuzzy integral equations*, Fuzzy Sets and Systems, **245** (2014), 1-17. https://doi.org/10.1016/j.fss.2013.08.005

[8] A. M. Bica, S. Ziari, *Iterative numerical method for fuzzy Volterra linear integral equations in two dimensions*, Soft Computing, **21**(5) (2017), 1097-1108. https://doi.org/10.1007/s00500-016-2085-2

[9] A. M. Bica, S. Ziari, *Open fuzzy cubature rule with application to nonlinear fuzzy Volterra integral equations in two dimensions*, Fuzzy Sets and Systems, **358** (2019), 108-131. https://doi.org/10.1016/j.fss.2018.04.010

[10] A. M. Bica, S. Ziari, *Gauss-Legendre type quadrature iterative method for fuzzy Fredholm integral equations*, Fuzzy Sets and Systems, **477** (2024). https://doi.org/10.1016/j.fss.2023.108823

[11] A. M. Bica, S. Ziari, Z. Satmari, *An iterative method for solving linear fuzzy fractional integral equation*, Soft Computing, **26**(13) (2022), 6051-6062. https://doi.org/10.1007/s00500-022-07120-w

[12] J. J. Buckley, *Fuzzy probability and statistics*, Springer, **196** (2006). https://link.springer.com/book/10.1007/3-540-33190-5#about-this-book

[13] P. J. Davis, P. Rabinowitz, *Methods of numerical integration*, Dover Publications, Inc., Mineola, NY, (2007), xii+612 pages.

[14] S. G. Gal, *Approximation theory in fuzzy setting*, Handbook of Analytic-Computational Methods in Applied, Chapman and Hall/CRC, Boca Raton, FL Mathematics, (2000), 617-666. https://doi.org/10.1007/978-1-4612-1360-4

[15] A. M. Gholam, R. Ezzati, *Solving linear fuzzy Fredholm integral equations of the second kind via iterative method and Simpson quadrature rule: A review*, TWMS Journal of Pure and Applied Mathematics, **8**(2) (2017), 121-147.

[16] E. Isaacson, H. B. Keller, *Analysis of numerical methods*, Courier Corporation, 2012.

[17] J. Jachymski, I. Jóźwik, M. Terepeta, *The Banach fixed point theorem: Selected topics from its hundred-year history*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matematicas. RACSAM, **118**(4) (2024). https://doi.org/10.1007/s13398-024-01636-6

[18] A. F. Jameel, N. R. Anakira, A. K. Alomari, N. H. Man, *Solution and analysis of the fuzzy Volterra integral equations via homotopy analysis method*, Computer Modeling in Engineering and Sciences, **127**(3) (2021), 875-899. https://doi.org/10.32604/cmes.2021.014460

[19] P. K. Kythe, M. R. Schäferkotter, *Handbook of computational methods for integration*, Chapman and Hall/CRC, Boca Raton, FL, (2005), xxii+598 pages.

[20] H. Mottaghi Golshan, *Numerical solution of nonlinear m-dimensional Fredholm integral equations using iterative Newton-Cotes rules*, Journal of Computational and Applied Mathematics, **448** (2024). https://doi.org/10.1016/j.cam.2024.115917

[21] H. Mottaghi Golshan, *Simulation functions on metric fixed point theory*, Journal of Inequalities and Applications, **2025**(1) (2025). https://doi.org/10.1186/s13660-025-03287-6

[22] H. Mottaghi Golshan, R. Ezzati, *Successive approximation method to solve nonlinear fuzzy Fredholm integral equations using NC rules*, Soft Computing, **28**(13) (2024), 7649-7657. https://doi.org/10.1007/s00500-024-09755-3

[23] H. Mottaghi Golshan, K. Fathi Vajargah, R. Ezzati, *A new approach for solving nonlinear 3D-Volterra fuzzy integral equations*, Sahand Communications in Mathematical Analysis, **21**(4) (2024), 191-208. https://doi.org/10.22130/scma.2024.2019081.1579

[24] D. Qiu, W. Zhang, C. Lu, *On fuzzy differential equations in the quotient space of fuzzy numbers*, Fuzzy Sets and Systems, **295** (2016), 72-98. https://doi.org/10.1016/j.fss.2015.03.010

[25] F. Saberirad, S. M. Karbassi, M. Heydari, *Numerical solution of nonlinear fuzzy Volterra integral equations of the second kind for changing sign kernels*, Soft Computing, **23**(21) (2019), 11181-11197. https://doi.org/10.1007/s00500-018-3668-x

[26] S. M. Sadatrasoul, R. Ezzati, *Iterative method for numerical solution of two-dimensional nonlinear fuzzy integral equations*, Fuzzy Sets and Systems, **280** (2015), 91-106. https://doi.org/10.1016/j.fss.2014.12.008

[27] C. Wu, Z. Gong, *On Henstock integral of fuzzy-number-valued functions. I*, Fuzzy Sets and Systems, **120**(3) (2001), 523-532. https://doi.org/10.1016/S0165-0114(99)00057-3

[28] M. Zeinali, F. Maheri, *Fuzzy product rule with applications*, Iranian Journal of Fuzzy Systems, **19**(6) (2022), 75-92. https://doi.org/10.22111/ijfs.2022.7211

[29] S. Ziari, *Towards the accuracy of iterative numerical methods for fuzzy Hammerstein-Fredholm integral equations*, Fuzzy Sets and Systems, **375** (2019), 161-178. https://doi.org/10.1016/j.fss.2018.09.006

[30] S. Ziari, T. Allahviranloo, W. Pedrycz, *An improved numerical iterative method for solving nonlinear fuzzy Fredholm integral equations via Picard's method and generalized quadrature rule*, Computational and Applied Mathematics, **40**(6) (2021). https://doi.org/10.1007/s40314-021-01616-1

[31] S. Ziari, A. M. Bica, *New error estimate in the iterative numerical method for nonlinear fuzzy Hammerstein-Fredholm integral equations*, Fuzzy Sets and Systems, **295** (2016), 136-152. https://doi.org/10.1016/j.fss.2015.09.021