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Minimum-Time Control of Constrained Systems via Phase-Plane Technique: Application in Robotic Manipulators

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ABSTRACT

The pursuit of time-optimal performance is a fundamental objective for high-speed robotic and mechatronic systems, where minimizing settling-time directly enhances throughput and operational efficiency. Despite its importance, deriving exact solutions for practical systems remains a formidable challenge. Existing methodologies, which predominantly rely on numerical optimization or simplified plant models, often yield suboptimal results; they fail to provide guarantees of global optimality and frequently prove inadequate when confronting realistic system constraints such as actuator saturation. To address this critical gap, this paper introduces an analytical framework for synthesizing time-optimal control laws for a class of second-order systems. The proposed method is based on phase portrait analysis, a powerful geometric approach that facilitates the direct derivation of the exact switching curve. This curve is the critical element that defines the globally optimal, bang-bang controller. The final control law is presented in a closed-form expression, thereby enabling computationally efficient and straightforward implementation. Simulation results validate the theoretical framework, demonstrating that the proposed controller consistently achieves the theoretical performance by tracking the optimal trajectory perfectly.

I. Introduction

The pursuit of optimal performance is a fundamental objective in control engineering, particularly in applications where speed and precision are critical. Among various optimality criteria, the minimum-time control problem holds significant practical importance. In these problems, it is of interest to move the system's states in the shortest possible time duration from an initial value to a desired point. This problem is especially relevant in various engineering fields such as robotics, aerospace, and manufacturing, where minimizing cycle time directly enhances productivity and efficiency [1-3].

However, a fundamental challenge in realizing time-optimal control arises from the inherent physical limitations of actuators. All real-world systems, such as the motors in a robotic manipulator, are subject to constraints on their maximum achievable force, torque, or voltage. These saturation constraints preclude the use of controllers that demand infinite control effort and transform the control design problem into a complex, constrained optimization task. For second-order systems, which aptly model a vast array of mechanical systems, including simple robotic arms, mass-spring-damper systems, and galvanometers, the solution is known to be a bang-bang controller. This controller structure operates by applying either the maximum positive or negative control effort, switching between them at precisely calculated instants [4-6].

The time-optimal control problem has been a central topic in control theory since its modern inception, largely propelled by the pioneering work of Pontryagin and his colleagues on the maximum principle [7]. This principle provides a necessary condition for optimality, establishing that for a system with bounded controls, the optimal control must often maximize a certain Hamiltonian function, leading to a bang-bang control structure [8]. For the specific case of double-integrator (Newtonian system) and second-order linear systems, the analytical solution has been long-established in classical texts, where the optimal switching curve in the phase plane is derived [1, 9].

While the concept of bang-bang control is well-established [4], the analytical derivation of the exact switching law remains a topic of interest. The phase-plane analysis methodology offers a powerful and intuitive geometric framework for tackling this problem [10]. By examining the trajectories of system's states (position and velocity), the optimal switching curve that defines the control law can be visualized and derived exactly. This approach provides not only a control solution but also deep insight into the system's behavior under optimal control [11].

Recently, some works have discussed the minimum-time control solution in linear time-invariant (LTI) systems with constrained input. Using the state transition matrix, the minimum-time control solution is exactly computed in the LTI models with constrained control effort [12]. Then, an analytical framework is presented to find a discrete-time minimum-time controller in the LTI models with constrained control input [13].

The application of these time-optimal principles to robotic systems, particularly for point-to-point motion, has been extensively explored. Considering torque constraints, to compute time-optimal trajectories, some foundational methods were presented for robotic manipulators along pre-defined paths [14, 15]. This work, often referred to as the bang-bang switching principle, has been implemented in various motion planning algorithms.

However, a significant portion of the literature focuses on numerical approaches to solve the resulting two-point boundary value problem for higher-order or nonlinear control systems [16, 17]. These control methods, while powerful, often rely on iterative optimization techniques that can be computationally intensive and may not provide the same level of intuitive insight as a closed-form analytical solution.

Furthermore, many modern control strategies for constrained systems, such as model predictive control (MPC), handle constraints effectively but provide sub-optimal solutions for the minimum-time problem due to their discrete-time nature and finite prediction horizons. The need for simple, exact, and intuitively precise control laws for fundamental system classes remains relevant for educational purposes, direct implementation on lightweight computational platforms, and as a performance benchmark for more complex algorithms.

A significant trend in contemporary literature involves solving optimally the path tracking in complex, higher-degree of freedom robotic manipulators. Highlighting the computational challenges, a control method is presented to approximate time-optimal control for the control systems with switching dynamics [18]. Similarly, a computationally efficient algorithm is suggested for time-optimal control of robotic systems with torque limitations, focusing on real-time implementability [19]. The work [16] reformulated the issue into a convex optimization, enabling efficient numerical solutions for robots following predefined paths.

Another active branch of research integrates time-optimal principles with predictive control. To achieve near-optimal performance, the use of MPC is explored while explicitly handling state and input constraints, though acknowledging the sub-optimality introduced by discretization [20]. To address this, recent work on real-time MPC [21] seeks to bridge the gap between computational tractability and time-optimal performance.

The problem is also being revisited for systems with specific nonlinearities and uncertainties. The time-optimal control for nonlinear systems using a reinforcement learning framework is investigated in the work [22], learning near-optimal policies where analytical solutions are intractable. For uncertain systems, a robust time-optimal control strategy is proposed that guarantees performance [23, 24]. Furthermore, the focus on performance has expanded beyond pure time minimization. For instance, a multi-objective optimization problem is addressed in the studies [25, 26], balancing time-optimality with energy consumption. Most recently, summarizing the current state of this nuanced field, a review of energy-time trade-offs in optimal control of the robotic manipulators was provided [2, 5, 27, 28].

This work distinguishes itself by returning to the analytical roots of the problem. Rather than presenting a new numerical algorithm, it provides a complete and self-contained derivation of the exact minimum-time control law for second-order LTI systems using phase-plane analysis. This approach offers a geometric and highly intuitive interpretation of the optimal switching logic, from first principles to final control law. The primary contribution lies in its clarity and comprehensiveness, serving to bridge classical theory with practical application in robotic regulation, and providing a clear benchmark for optimal performance.

Inspired by the minimum-time control of the Newtonian system [9], the key contribution of this research is the exact derivation of a minimum-time control law for a class of second-order LTI systems with bounded control efforts, utilizing a phase-plane idea. The issue is formulated in the context of a constrained robotic manipulator, initially modeled by its transfer function. The paper first addresses the stabilization problem, detailing how to move the system's state to the origin under predefined actuator constraints. The exact solution to the stabilization issue is then conclusively employed to solve the output regulation in the minimum possible time.

The remainder of this report is arranged as follows. Section 2 introduces the model mathematics and formally states the minimum-time control in the robot arm. Section 3 presents the core phase-plane analysis and derives the exact structure of the time-optimal control law. Then, its application is discussed in the output regulation case. Section 4 provides a numerical example to check the applicability and the effectiveness of the designed control methodology in the robotic arm. Finally, some concluding points are summarized in Section 5.

II. Problem Formulation

This section establishes the mathematical foundation for the minimum-time control problem. In numerous robotic and mechanical applications, ignoring the spring effects, a Cartesian robot manipulator with n-links can be imagined as follows [29]:

$$M\ddot{y} + B\dot{y} = u, \tag{1}$$

where $y \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$ are some vectors containing the robot positions and the input forces, respectively. In Eq. (1), M denotes the mass matrix and B denotes the damping matrix. To specify exactly the minimum-time control solution, the above-mentioned matrices are assumed to be known. In a simplified form, in Fig. 1, a robotic manipulator with three perpendicular links is depicted.

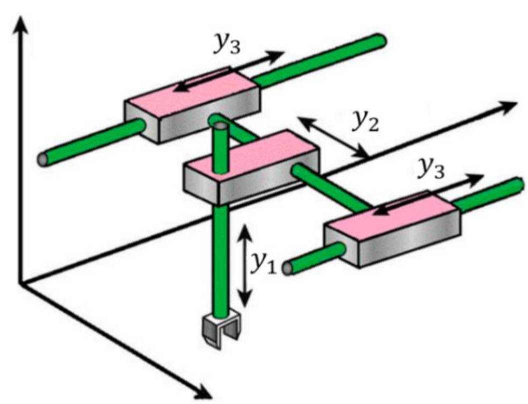

Fig. 1. A schematic of a 3-link manipulator

Here, the specific control objective is to synthesize a law $u(t)$ subjected to $|u(t)| \le u_{\max}$ that drives the system output $y(t)$, as well as its derivative (namely, the translational velocity $\dot{y}(t)$), from an primary state to a desired constant set-point in the minimum possible time, while strictly adhering to the control constraint.

In the condition which the terms M and B are some diagonal matrices, for instance like the model presented in [29], it can be decomposed as n independent subsystems. Each one may be a second-order system like

$$\ddot{y} - a\dot{y} = bu. \tag{2}$$

The control law is synthesized under the assumption that dynamic coupling between joints is negligible. This holds for operational scenarios such as slow movements or sequential joint control, where coupling effects are minimal. However, this assumption breaks down during high-speed, coordinated motions, where significant interaction forces arise and can lead to substantial tracking errors and performance loss.

As a consequence, the corresponding transfer function would be obtained as

$$G(s) = \frac{b}{s(s-a)} \tag{3}$$

where $y(t)$ is the system's output while the control effort $u(t)$ is constrained by $u_{\max}$. Moreover, the constants a and b are some known parameters for the designer. So, the control system (3) involves an integral term, while the coefficient a may be negative (stable) or positive (unstable). In the state space, the dynamic model (3) would be represented as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = ax_2 + bu \end{cases} \tag{4}$$

where $x_1 = y$ and $x_2 = \dot{y}$.

Consequently, the specific control aim is to generate a constrained controller $u(t) = \phi(x_1, x_2)$ with $|u(t)| \le u_{\max}$, that minimizes the time required to drive the system from any initial values $(x_1(t_0), x_2(t_0))$ to the desired terminal state, satisfying the output regulation $\lim_{t\to t_f} x_1(t) = y_{\text{ref}}$ and $\lim_{t\to t_f} x_2(t) = 0$. This formally defines the time-optimal control problem that will be solved in the subsequent section through phase-plane analysis. Subsequently, the closed-loop stabilization in the minimum-time is presented for the system (4). Then the results are used in the regulation problem.

III. Main Results

Having formally stated the problem, this part discusses the geometric derivation of the exact time-optimal control law using the phase-plane methodology. The phase portrait offers a powerful tool for visualizing the system's dynamics and synthesizing controllers for nonlinear systems, such as those with bang-bang control structures. The analysis begins by transforming the second-order dynamics into a system of first-order equations in the state space, defining the phase variables $x_1 = y$ (the system's output) and $x_2 = \dot{y}$ (the derivative of the output). The core of this section will be dedicated to constructing the optimal switching curve, a critical locus in the phase plane that partitions the state space into zones where the optimal controller is either $u = u_{\max}$ or $u = -u_{\max}$. The properties of system trajectories under these extreme control efforts are analyzed to prove that this switching curve indeed yields the minimum-time solution to the origin, which is then extended to the regulation problem.

Stabilization Problem. It is desired to design a command signal $u(t)$ with $|u(t)| \le u_{\max}$ so that the system's states are moved in the minimum-time from $x(t_0) = x_0$ to $x(t_f) = 0$. Thus, the cost function is

$$J = \int_{t_0}^{t_f} 1 \mathrm{d}t. \tag{5}$$

Let us construct the Hamiltonian function in the following form:

$$\mathcal{H}(x, \lambda, u) = 1 + \lambda_1 x_2 + \lambda_2 (ax_2 + bu). \tag{6}$$

The costate equations are found as follows:

$$\begin{cases} \dot{\lambda}_1 = -\frac{\partial \mathcal{H}}{\partial x_1} = 0 \\ \dot{\lambda}_2 = -\frac{\partial \mathcal{H}}{\partial x_2} = -\lambda_1 - a\lambda_2 \end{cases} \tag{7}$$

Employing the recognized Pontryagin principle with $|u(t)| \le u_{\max}$, we have:

$$\mathcal{H}\big(x^*(t), \lambda^*(t), u^*(t)\big) \le \mathcal{H}\big(x^*(t), \lambda^*(t), u(t)\big). \tag{8}$$

Consequently, the optimal control would have a solution if $|u^*(t)| = u_{\max}$ holds. Then the switching curve would be obtained as follows:

Case 1) If $u = +u_{\max}$ holds, then the system's solution is

$$\begin{cases} x_1(t) = x_1(t_0) + \frac{x_2(t_0)}{a}(e^{at}-1) \\ \qquad +u_{\max}\frac{b}{a^2}(e^{at}-1) - u_{\max}\frac{b}{a}t \quad , \quad t \geq t_0 \\ x_2(t) = x_2(t_0)e^{at} + u_{\max}\frac{b}{a}(e^{at}-1) \end{cases} \tag{9}$$

The trajectory crossing the origin is

$$\begin{cases} x_1 = u_{\max}\frac{b}{a^2}(e^{at}-1) - u_{\max}\frac{b}{a}t \\ x_2 = u_{\max}\frac{b}{a}(e^{at}-1) \end{cases} , \forall t \in \mathbb{R} \tag{10}$$

In the phase-plane, the corresponding nonlinear curve is established as follows:

$$x_1 = \frac{1}{a}x_2 - u_{\max}\frac{b}{a^2}\mathrm{Ln}\left(1 + \frac{a}{bu_{\max}}x_2\right). \tag{11}$$

Case 2) If $u = -u_{\max}$ holds, then the system's solutions are

$$\begin{cases} x_1 = x_1(t_0) + \frac{x_2(t_0)}{a}(e^{at}-1) \\ \qquad +u_{\max}\frac{b}{a^2}(1-e^{at}) + u_{\max}\frac{b}{a}t \quad , t \geq t_0 \\ x_2 = x_2(t_0)e^{at} + u_{\max}\frac{b}{a}(-e^{at}+1) \end{cases} \tag{12}$$

Similarly, the trajectory crossing the origin is

$$\begin{cases} x_1 = u_{\max}\frac{b}{a^2}(1-e^{at}) + u_{\max}\frac{b}{a}t \\ x_2 = u_{\max}\frac{b}{a}(-e^{at}+1) \end{cases} , \forall t \in \mathbb{R} \tag{13}$$

Accordingly, the following switching curve would be obtained:

$$x_1 = \frac{1}{a}x_2 + u_{\max}\frac{b}{a^2}\mathrm{Ln}\left(1 - \frac{a}{bu_{\max}}x_2\right). \tag{14}$$

For different selections of the value a and b, the phase-plane of the control system with $u_{\max} = 1$ are plotted in Fig. 2. Then the corresponding stable switching curves are shown in Fig. 3.

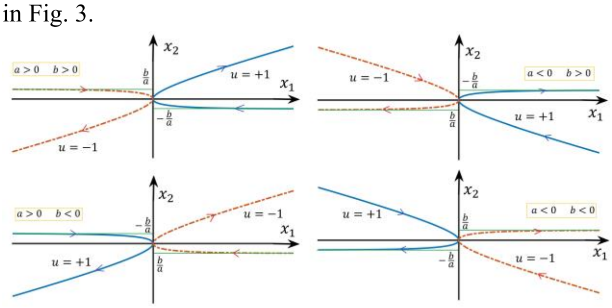

Fig. 2. The phase-planes for different signs of a and b

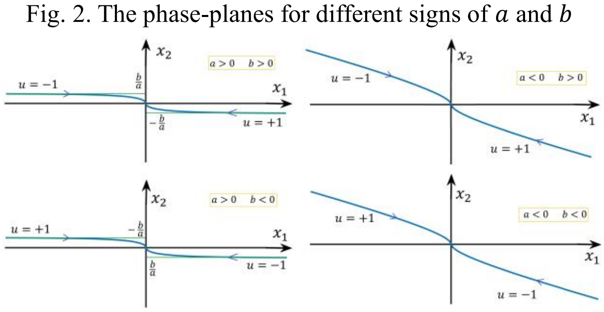

Fig. 3. Stable switching curves for different signs of a and b

According to Fig. 3, depending on a and b, a stable switching curve could be found by the following formula:

$$x_1 = \frac{1}{a}x_2 + u_{\max}\frac{|b|}{a^2}\mathrm{sgn}(x_2)\mathrm{Ln}\left(1 - \frac{a}{u_{\max}}\left|\frac{x_2}{b}\right|\right). \tag{15}$$

Then, aided by the phase plane diagram, the control signal $u(t)$ is optimally computed as follows:

$$u(t) = \begin{cases} -u_{\max}\mathrm{sgn}(bs(t)) & s(t) \neq 0 \\ -u_{\max}\mathrm{sgn}(bx_2(t)) & s(t) = 0 \end{cases} \tag{16}$$

where the switching curve $s(x_1, x_2)$ is

$$s = x_1 - \frac{1}{a}x_2 - u_{\max}\frac{|b|}{a^2}\mathrm{sgn}(x_2)\mathrm{Ln}\left(1 - \frac{a}{u_{\max}}\left|\frac{x_2}{b}\right|\right).$$

It states that the case $s(t) \neq 0$ drives the system towards the switching manifold, while the case $s(t) = 0$ with $x_2(t) \neq 0$ maintains the system on the switching surface. The control aims $(x_1(t) = x_2(t) = 0)$ would be favorably accomplished when $s(t) = 0$ and $x_2(t) = 0$ hold.

In the phase plane, it can be checked that the set $s(x_1, x_2) = 0$ or Eq. (15) constructs an invariant set. Thus the system's solutions remain on Eq. (15) for the initial conditions that start from $s(x_1, x_2) = 0$.

Unstable Region. The argument of the logarithmic function would always be positive when $a < 0$. However, for positive a, the minimum-time control problem would not have a solution when $|x_2(t_0)| > u_{\max}\frac{|b|}{a}$. For example, as far as the coefficient a is positive, either $u(t) = +u_{\max}$ combined with $x_2(t_0) = -u_{\max}\frac{b}{a}$ or $u(t) = -u_{\max}$ along with $x_2(t_0) = u_{\max}\frac{b}{a}$ is taken, applying Eq. (9) and Eq. (12), the system's solution is

$$\begin{cases} x_1(t) = x_1(t_0) + x_2(t_0)t \\ x_2(t) = x_2(t_0) \end{cases} , \quad t \geq t_0 \tag{17}$$

As a consequence, the solutions would be unbounded if both control effort $u(t)$ and initial condition $x_2(t_0)$ remain on a specified boundary set. Utilizing Eq. (9) and Eq. (12), it can be observed that the system's solutions would tend to infinity if the conditions $|x_2(t_0)| > u_{\max}\frac{|b|}{a}$ and $a > 0$ hold. In other words, the control authority is insufficient to counteract the divergent dynamics, making the time-optimal problem infeasible from such initial states.

Next, the results are discussed for the small a. To this aim, the logarithmic function $\mathrm{Ln}(1+z)$ can be approximated as the following:

$$\mathrm{Ln}(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \cdots, |z| < 1. \tag{18}$$

Fig. 3 shows, for small a (i.e., $|a| \ll 1$) and $b > 0$, if $u = +u_{\max}$, then $x_2 < 0$ and the following condition would hold:

$$x_1 = \frac{1}{a}x_2 - \frac{1}{a}x_2 + \frac{1}{2}\frac{x_2^2}{bu_{\max}} - \frac{1}{3}\frac{ax_2^3}{b^2u_{\max}^2} + \cdots = \frac{1}{2bu_{\max}}x_2^2. \tag{19}$$

Similarly, if $u = -u_{\max}$, then $x_2 > 0$ and the subsequent approximation could be written:

$$x_1 = \frac{1}{a}x_2 - \frac{1}{a}x_2 - \frac{1}{2}\frac{x_2^2}{bu_{\max}} - \frac{1}{3}\frac{ax_2^3}{b^2u_{\max}^2} + \cdots = -\frac{1}{2bu_{\max}}x_2^2. \tag{20}$$

Thus the final switching surface is reduced to the following second-order function:

$$x_1 = \frac{-1}{2bu_{\max}}|x_2|x_2, |a| \ll 1, b > 0 . \tag{21}$$

Accordingly, the same results were obtained in the study [9] for $a = 0$. The exact and approximated switching curves clearly illustrate the error introduced by the series truncation. This analysis provides the necessary context for practitioners to assess the trade-off between simplicity and optimality when deploying the simplified controller. In conclusion, the phase-plane analysis has yielded a complete solution to the minimum-time stabilization problem. The optimal switching curve has been rigorously derived, partitioning the phase plane into two distinct regions. The corresponding control law, defined by the bang-bang policy $u = +u_{\max}$ for states below the switching curve and $u = -u_{\max}$ for states above it, has been proven to move the initial states to the origin in the minimum achievable time while strictly adhering to the control constraint. Eq. (15) is a switching curve in the optimal control sense, guaranteeing that once the system trajectory reaches it, no further switching is needed. Thus, this law requires at most one switching event and provides the foundational result for addressing the output regulation in the following subsection.

Regulation Problem. The output regulation would be achieved at an invariant set-point y_{ref} with the equilibrium point $\bar{x} = [y_{\text{ref}} \quad 0]^T$ and $\bar{u} = 0$. So, satisfying the constraint $|u(t)| \le u_{\max}$, it is desired to determine $u(t)$ so that the states are moved from $x(t_0) = x_0$ to $x(t_f) = \bar{x}$ in the minimum-time. Defining $\xi(t) = x(t) - \bar{x}$, we have:

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = a\xi_2 + bu \end{cases} \tag{22}$$

Without loss of generality, it is equivalent to moving the state from $\xi_0 = x_0 - \bar{x}$ to the origin. Thus, substituting $x_1 = \xi_1 + y_{\text{ref}}$ and $x_2 = \xi_2$, a stable switching manifold is

$$\begin{aligned} x_1 &= y_{\text{ref}} + \frac{1}{a} x_2 \\ &\quad + u_{\max} \frac{|b|}{a^2} \operatorname{sgn}(x_2) \operatorname{Ln}\left(1 - \frac{a}{u_{\max}} \left|\frac{x_2}{b}\right|\right). \end{aligned} \tag{23}$$

Then the optimal control signal $u(t)$ would be found as

$$u(t) = \begin{cases} -u_{\max} \operatorname{sgn}\big(bs(t)\big) & s(t) \neq 0 \\ -u_{\max} \operatorname{sgn}\big(bx_2(t)\big) & s(t) = 0 \end{cases} \tag{24}$$

where

$$s = x_1 - y_{\text{ref}} - \frac{1}{a} x_2 - u_{\max} \frac{|b|}{a^2} \operatorname{sgn}(x_2) \operatorname{Ln}\left(1 - \frac{a}{u_{\max}} \left|\frac{x_2}{b}\right|\right).$$

Remark 1. The control signal contains instantaneous fluctuations around the equilibrium point. In the control law (24), to avoid chattering phenomena when $s(t) = 0$, the sign function can be approximated with a smoothed one like $\operatorname{sat}(.)$, $\tanh(.)$, $\tan^{-1}(.)$ and the other ones. However, smoothing the sign function would fundamentally lead to sub-optimality.

Remark 2. The proposed bang-bang controller acts as a finite-time convergent control law, ensuring robustness to infinitesimal disturbances during the reaching phase. Accordingly, the system's states are driven to the origin in a pre-computable finite settling-time.

Remark 3. In the proposed control law, the computational burden associated with the real-time calculation of the logarithmic function may be a challenging issue. Although this imposes a greater load than a linear controller, it remains feasible for standard hardware, a fact demonstrated by its prevalence in other sophisticated control algorithms. To further mitigate this cost in resource-limited environments, the option of pre-computing the switching curve into a lookup table is presented, trading a marginal amount of memory for a deterministic and fast execution time.

Remark 4. The proposed method develops a framework to design an optimal control law for the second-order LTI systems with no uncertain part and a hard limitation on the control input. The solving technique is the phase plane which is helpful for the second-order models. Thus, the problem formulation is dedicated to the second-order systems. However, it may be generalized to the high-order LTI control systems. Moreover, deriving an exact (closed form) solution would have some extra complexities in the nonlinear second-order systems.

The phase-plane analysis has yielded a complete and exact solution to the minimum-time control for the constrained second-order model. The optimal control law, defined by the switching curve, has been rigorously derived and proven to possess a bang-bang structure, requiring at most one switching event to direct the initial states to the origin in minimum time. Furthermore, by defining the error state, this stabilization law is directly applicable to the output regulation problem, enabling time-optimal set-point tracking. The following section will demonstrate the implementation and efficacy of this control law through a numerical simulation.

IV. Numerical Simulation

To confirm the theoretical derivations and demonstrate the practical performance of the designed time-optimal control law, this section presents a comprehensive numerical example. A specific second-order system, representative of a typical robotic joint, is simulated under the derived bang-bang control policy. The response of the system's states in the phase portrait, the time evolution of the control effort, and the output regulation performance is analyzed in detail. The results serve to visually confirm the existence and properties of the optimal switching curve and, most importantly, to verify that the system achieves exact regulation in the minimum achievable time while strictly adhering to the control constraint. A robot manipulator with a single arm is discussed later. To this purpose, consider a robotic model described by Eq. (3) with $u_{\max} = 3$, $a = 1$ and $b = 2$. The initial conditions of the position and velocity are $x_1(0) = -3$ and $x_2(0) = 5$.

The integration time step is one millisecond, the solver is the Runge–Kutta method, and the duration of the simulation is 20 seconds. In this example, the reference is the following piece-wise constant signal:

$$y_{\text{ref}}(t) = \begin{cases} 2 & 0 \le t < 8 \\ 3 & 8 \le t < 12 \\ 0 & 12 \le t < 20 \end{cases} \tag{25}$$

It is desired to track reference $y_{\text{ref}}(t)$ as shown in Fig. 4. To avoid the chattering phenomena when $s(t) = 0$, the function $\operatorname{sign}(bx_2)$ is approximated by $\tanh(100x_2)$. Applying the suggested procedure, the optimal control law (24) is realized with smoothed ones as follows:

$$u(t) = \begin{cases} -u_{\max}\mathrm{sgn}(s(t)) & s(t) \neq 0 \\ -u_{\max}\tanh(100x_2(t)) & s(t) = 0 \end{cases} \tag{26}$$

where $s = x_1 - y_{\mathrm{ref}} - x_2 - 6\mathrm{sgn}(x_2)\mathrm{Ln}\left(1 - \frac{|x_2|}{6}\right)$.

To show the effectiveness of the proposed technique, the system behavior is evaluated with the LQR method. Such a state feedback optimal controller would be realized as

$$u(t) = -(x_1(t) - y_{\mathrm{ref}}) - 1.5x_2(t). \tag{27}$$

The system's output is shown in Fig. 4, while the second state is illustrated in Fig. 5. It is verified that zero steady-state error is achieved via both controllers. Moreover, corresponding to $y_{\mathrm{ref}}(t) = 2$, the minimum time is 2.5545 seconds. The state trajectories of the robot in the phase-plane are seen in Fig. 6. The control effort $u(t)$ and the switching curve $s(t)$ are plotted in Fig. 7. Additionally, to show the effectiveness of the suggested control method, during interval $0 < t < 8$, some quantitative metrics like settling-time, peak output and peak control efforts are calculated and provided in Table 1.

Table 1. Comparison of some control indexes

Index	Proposed Method	LQR
Settling-Time [sec]	2.5545	5.6280
Peak Output [m]	2.7559	3.1055
Peak Control Effort [m/sec^2]	3	0.5019

Applying the proposed minimum-time controller, the output $y(t)$ follows the set-point $y_{\mathrm{ref}}(t)$ during the start and the brake phases. Compared with similar control techniques, the simulation results conclusively validate the theoretical framework developed in Section 3.

The state trajectory in the phase plane was observed to seamlessly follow the predicted optimal switching curve, switching control effort at the precise theoretical point to guide the system directly to the origin. The control input maintained its bang-bang structure, operating strictly within the defined saturation limits of $\pm u_{\max}$ throughout the process. Most significantly, the system achieved perfect regulation at a time that aligns with the theoretical minimum, as no other admissible control signal could have completed the transition faster. This numerical example effectively demonstrates the practical applicability and optimal performance of the phase-plane derived control law for the constrained second-order systems.

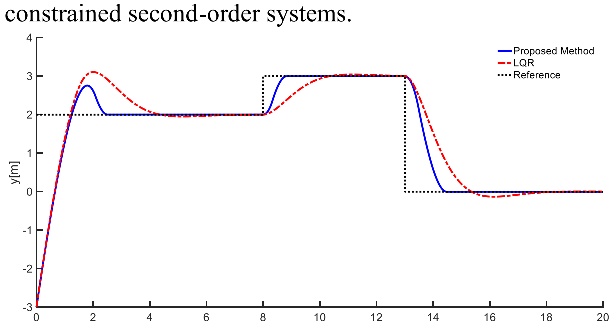

Fig. 4. The system's output and reference signal

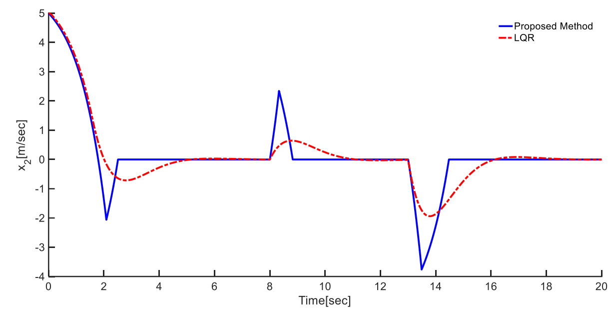

Fig. 5. The second state of the system

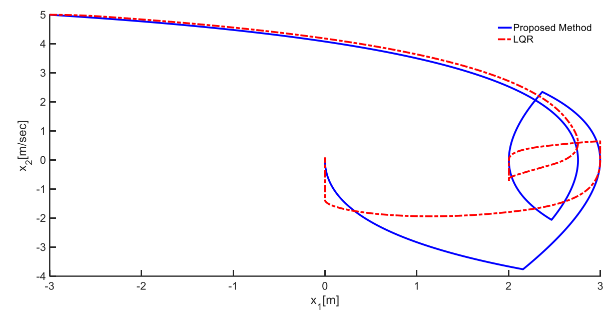

Fig. 6. The state trajectory in the phase-plane

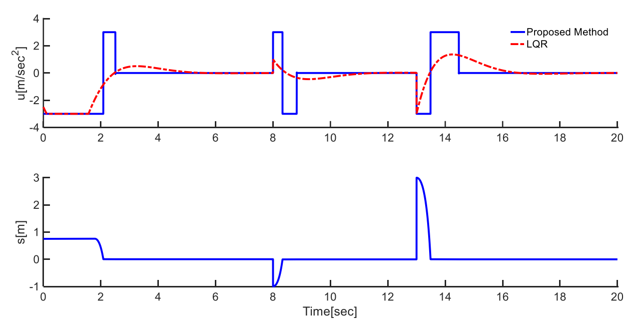

Fig. 7. The control effort $u(t)$ and the switching curve $s(t)$

To analyze the robustness against perturbations, new simulations explicitly test the controller's performance under both external disturbances and parametric model uncertainties. To this purpose, a torque disturbance with an amplitude of 2 is applied during $4 < t < 5$. Moreover, in the control law, $+20\%$ increment has been made in the parameters a and b. The output and input of the system are presented in Figs. 8 and 9. Additionally, to check the controller's robustness under external disturbance and model uncertainty, some quantitative control metrics during time interval $0 < t < 8$ are numerically calculated and reported in Table 2.

Table 2. Robustness of the proposed controller

Index	Nominal Case	Uncertain Case
Settling-Time [sec]	2.5545	2.8780
Peak Output [m]	2.7559	3.1111

Compared with the nominal case, the quantitative and figurative results demonstrate the controller's inherent robustness properties and quantify the resulting performance degradation such as the increase in settling-time.

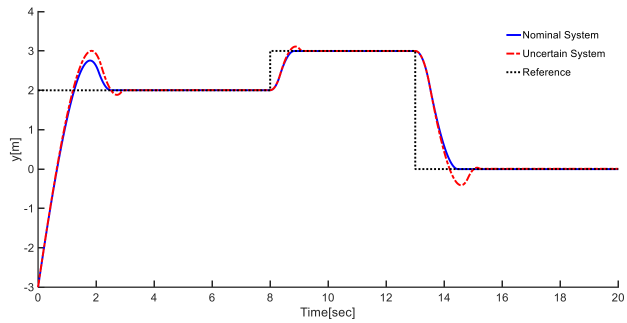

Fig. 8. The output and reference under uncertain terms

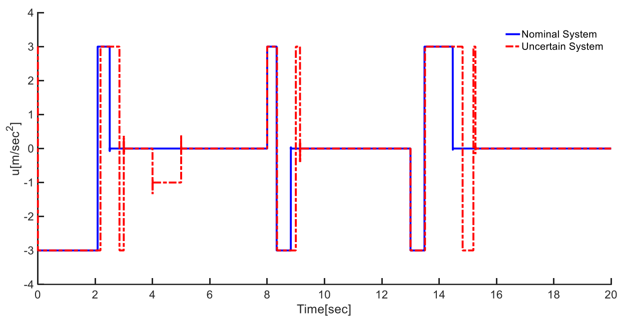

Fig. 9. The control effort $u(t)$ under uncertain terms

V. Conclusion

This paper has successfully derived and demonstrated an exact minimum-time control law for a class of second-order LTI systems subjected to control effort constraints. By employing a phase-plane analysis approach, the time-optimal control was translated into a geometric problem of finding the optimal switching curve that drives the states of the system to the origin in the shortest time. The core of the strategy was first established by solving the stabilization problem, proving that the optimal control policy is a bang-bang control law that switches between the maximum allowed control efforts. This policy was then rigorously extended to solve the output regulation, guaranteeing that the system reaches its desired set-point in the theoretical minimum time. The practical applicability of the theoretical findings was conclusively validated through a numerical example. The simulation results clearly illustrated the controller's performance, showcasing its ability to achieve precise regulation exactly as predicted by the phase-plane analysis, with no overshoot and in the fastest achievable time given the system's constraints. In summary, this work provides an exact and practically verifiable solution to the classic minimum-time control problem. The clarity of the phase-plane method offers significant intuitive insight into the structure of the optimal control law. Future work will focus on extending this approach to higher-order systems and exploring its robustness in the presence of model uncertainties and external disturbances.

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