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Original paper

Shewhart control chart based on fuzzy data with ranked set sampling

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Abstract

Quality control charts with fuzzy data have been successfully used in many real-world applications in recent years. These methods have been extended to estimate the fuzzy population means based on simple random sampling techniques. In this study, a different strategy is used to develop Shewhart control charts with fuzzy means based on fuzzy data. For this purpose, the conventional rank set sampling is first extended to a well-established fuzzy random variable. Then, based on the concept of fuzzy mean and exact variance, the lower, mean, and upper fuzzy control charts are introduced. Additionally, an estimation procedure is presented that can be used to evaluate the proposed fuzzy control limits in cases where the fuzzy mean and exact variance of the population are unknown. An inclusion degree for monitoring process variability is also introduced and discussed. A real case study from photolithography is presented to demonstrate the efficiency of the proposed method for monitoring control charts with fuzzy data based on fuzzy rank set sampling.

Keywords: Fuzzy random variable, fuzzy control chart, fuzzy ranked set sampling, degree of inclusion, fuzzy control region, fuzzy mean, exact variance.

1 Introduction

Since Shewhart's first work [42], control charts have become important statistical tools used in quality management and production to monitor, control, and improve processes using statistical methods [43, 44]. A control chart provides information about changes in process parameters so that corrective action can be taken as early as possible, leading to reduced variability and improved productivity and quality. By comparing actual data with a range of expected areas, a Shewhart control chart can visually examine whether a process is under control or out of control. In particular, many applications of control charts have been developed in industry to monitor the mean value of the process [15, 33]. However, selecting suitable sampling plans plays a vital role in achieving pre-specified goals, such as stabilizing the variation in the process. Most of the methods reported in the literature are based on the assumption that samples are drawn from a process based on a simple random sampling (**SRS**) scheme. The ranked set sampling (**RSS**) method was first suggested by McIntyre [32], who noted that it is highly beneficial and superior to the standard **SRS** for estimation of the population mean [45]. Hall and Dell [16] observed that using **RSS**, the samples can be ranked more efficiently than using **SRS** when they are measured at the actual sample size in the same situation. The **RSS** scheme is more efficient than **SRS** because the method uses extra information by ranking the units, but we are measuring the same number of units for both methods [1]. It also provides a cost-effective means of obtaining a more representative sample when direct measurement of sampling units is expensive or destructive, as it is easier to classify observations. It was revealed that the **SRS** scheme is less effective in the estimation of the population mean compared to other new sampling schemes, such as the **RSS** scheme, when using the same sample size. The **RSS** method has numerous applications in various fields, including environmental studies [27] and medical applications [41], among others [6, 9, 19].

The application of the **RSS** method in the development of control charts is a relatively new field of research [34, 40]. Quality control charts with **RSS** yield higher efficiency compared to the actual measure under the same sample size

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because the control limits are narrower, leading to better quality control. The **RSS** methods are also recommended [18, 36] as these sampling approaches decrease variability and improve the efficiency of the associated control charts. Muttlak and Al-Sabah [35] showed that Shewhart control charts based on **RSS** are far better than the **SRS**-based control charts. They observed that for any sample size, if the process starts to get out of control, then the **RSS** chart reduces the average run length (**ARL**) substantially. Additionally, several studies [4, 5, 7] have shown that the use of **RSS** for analyzing performance criteria on monitoring control charts significantly outperforms traditional **SRS** on monitoring control charts.

However, in cases where there is ambiguity in the reporting of sample data, conventional Shewhart control charts cannot be accurately defined. Several studies have explored the issue of Shewhart control charts using fuzzy data. Ahmad and Cheng [2] developed a fuzzy control chart based on fuzzy process capability indices using triangular fuzzy data. Ozdemir [37] developed a fuzzy $\overline{X}$ and S control charts while dealing with unbalanced triangular fuzzy data. Kaplan et al. [14] proposed a fuzzy exponentially weighted moving average control chart based on α-cuts of triangular fuzzy data. Razali et al. [39] provided a comprehensive review of numerous fuzzy type 1 and type 2 fuzzy control charts to identify the past and current developments until 2020. Refaie et al. [8] proposed exponentially weighted moving average and cumulative sum control charts with triangular fuzzy data in a manufacturing process under the existence of mean shift, utilizing the fuzzy logic based on the α-cut approach. Aslam et al. [10] introduced fuzzy logic-based control charts, including the fuzzy moving average control chart, fuzzy weighted moving control chart, and fuzzy moving range control chart for individual measurements to detect shifts within hematocrit levels effectively based on triangular fuzzy data. Ahmad et al. [3] proposed fuzzy unbalanced control limits by using trapezoidal fuzzy data. Dilipkumar and Nanthakumar [11] suggested the fuzzy mean using a standard deviation ($\overline{X}$-S) control chart with the assistance of process capability based on trapezoidal fuzzy data. Hesamian et al. [21] developed a notion of fuzzy EWMA control chart based on fuzzy random variables in cases where fuzzy mean and/or non-fuzzy variance were unknown parameters.

All of the above methods assumed that fuzzy data are collected using an **SRS** method. As previously mentioned, the efficiency of the **RSS** method in monitoring Shewhart control charts is significantly higher compared to the **SRS** method. Therefore, it is crucial to incorporate the **RSS** sampling technique into control charts when dealing with fuzzy data instead of precise data. This research aims to introduce a new approach for fuzzy Shewhart quality control charts by utilizing an extended fuzzy **RSS** scheme that combines fuzzy means with precise variances. Furthermore, a measure of inclusion is proposed to determine if specific observed fuzzy data points fall outside the defined fuzzy control limits. The effectiveness of this approach is illustrated through a practical case study, considering scenarios where the population fuzzy mean is either known or unknown.

The remainder of this paper is organized as follows: The necessary basics of fuzzy numbers and fuzzy random variables are briefly presented in Section 2. In this section, a notion for an LR-fuzzy random variable with a fuzzy mean is first recalled. Subsequently, the concept of the exact variance of LR-fuzzy random variables is introduced and discussed. Section 3 presents the development of the conventional **RSS** for fuzzy data. In Section 4, fuzzy Shewhart quality control is extended for two cases in which the population mean is a known or unknown fuzzy quantity. The degree of inclusion is also used to monitor the process, which determines the degree to which the fuzzy data belong to or are excluded from the proposed fuzzy control region. Section 5 provides an example in which the effectiveness of the new method is investigated. Finally, the paper concludes in Section 6.

2 Preliminaries

In this section, we recall some preliminaries related to fuzzy numbers and fuzzy random variables used in the next sections.

2.1 Fuzzy numbers

A fuzzy set $\widetilde{A}$ allocates degrees of membership ranging from zero to one to each element of a universe of discourse $\mathbb{X}$, that is, $0 \leq \widetilde{A}(x) \leq 1$ for all $x \in \mathbb{X}$. A fuzzy number (**FN**) $\widetilde{A}$ represents a fuzzy set that is convex and normalized along the real number line ($\mathbb{X} = \mathbb{R}$), and $\widetilde{A}(x)$ is upper semi-continuous [29]. Within the classification of **FN**s, there exist what are known as Left-Right trapezoidal **FN**s (LR-**TZFN**s), denoted as $\widetilde{A} = (a^L, a^{c_1}, a^{c_2}, a^U)_{LR}$ with $a^L < a^{c_1} < a^{c_2} < a^U$ with the following membership function:

$$\widetilde{A}(x) = \begin{cases} L(\dfrac{a^{c_1} - x}{a^{c_1} - a^L}), & a^L \leq x \leq a^{c_1}, \\ 1, & a^{c_1} \leq x < a^{c_2}, \\ R(\dfrac{x - a^{c_2}}{a^U - a^{c_2}}), & a^{c_2} < x < a^U. \end{cases} \tag{1}$$

In Eq. (1), the continuous functions $L(.)$ and $R(.)$ characterize the left and right spreads of the LR-**TZFN**, and it holds $L(0) = R(0) = 1$, $L(1) = R(1) = 0$. The α-cuts [29] of an LR-**TZFN** $\widetilde{A} = (a^L, a^{c_1}, a^{c_2}, a^U)_{LR}$ can be evaluated as follows:

$$\widetilde{A}[\alpha] = [a^{c_1} - (a^{c_1} - a^L)L^{-1}(\alpha), a^{c_2} + (a^U - a^{c_2})R^{-1}(\alpha)]. \tag{2}$$

The set $Supp(\widetilde{A}) = \{x \in \mathbb{R} : \widetilde{A}(x) > 0\} = [\widetilde{A}_0^L, \widetilde{A}_0^U]$ is referred to as the support of $\widetilde{A}$. An LR-**TZFN** for which $a^{c_1} = a^{c_2}$ is designated as an LR-**FN**, represented as $\widetilde{A} = (a^L, a, a^U)_{LR}$. The entirety of LR-**FN**s is denoted by $\mathcal{F}_{LR}(\mathbb{R})$. Several standard operations between two LR-**FN**s, $\widetilde{A} = (a^L, a, a^U)_{LR}$ and $\widetilde{B} = (b^L, b, b^U)_{LR}$, may be established in the following manner [12, 17]:

1) (Addition):
$$\widetilde{A} \oplus \widetilde{B} = (a^L + b^L, a + b, a^U + b^U)_{LR}. \tag{3}$$

2) (Scalar multiplication):
$$\lambda \otimes \widetilde{A} = \begin{cases} (\lambda a^L, \lambda a, \lambda a^U)_{LR} & \lambda > 0, \\ (\lambda a^U, \lambda a, \lambda a^L)_{RL} & \lambda < 0. \end{cases} \tag{4}$$

3) (Difference from a constant):
$$\widetilde{A} \ominus k = (a^L - k, a - k, a^U - k)_{LR}, \ k \in \mathbb{R}. \tag{5}$$

4) (Inclusion):
$$\widetilde{A} \subseteq \widetilde{B} \text{ if } \widetilde{A}(x) \leq \widetilde{B}(x), \ x \in \mathbb{R}. \tag{6}$$

5) (Compliment):
$$\widetilde{A}^c(x) = 1 - \widetilde{A}(x), \ x \in \mathbb{R}. \tag{7}$$

Note that $\widetilde{A} \subseteq \widetilde{B}$ if and only if $\widetilde{A}[\alpha] \subseteq \widetilde{B}[\alpha]$ for every $\alpha \in [0, 1]$.

Definition 2.1. [20] *The square error distance between two LR-**FN**s $\widetilde{A} = (a^L, a, a^U)_{LR}$ and $\widetilde{B} = (b^L, b, b^U)_{LR}$ is defined as:*

$$D^2(\widetilde{A}, \widetilde{B}) = p_1(a - b)^2 + p_2\left(\frac{(a^L - b^L)^2 + (a^U - b^U)^2}{2}\right), \tag{8}$$

where

1) $p_1 = 1/(1 + c_1 + c_2)$ *and* $p_2 = (c_1 + c_2)/(1 + c_1 + c_2)$, *and*
2) $c_1 = \int_0^1 L^{-1}(\alpha)d\alpha$ *and* $c_2 = \int_0^1 R^{-1}(\alpha)d\alpha$.

Remark 2.2. D^2 *meets the following conditions for any LR-**FN** of $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$:*

1) $D^2(\widetilde{A}, \widetilde{B}) = 0$ *if and only if* $\widetilde{A} = \widetilde{B}$,
2) $D^2(\widetilde{A}, \widetilde{B}) = D^2(\widetilde{B}, \widetilde{A})$,
3) $D^2(\widetilde{A}, \widetilde{C}) \leq D^2(\widetilde{A}, \widetilde{B}) + D^2(\widetilde{B}, \widetilde{C})$.

Further, if $\widetilde{A} = (a^L, a, a^U)_{LR}$ and $\widetilde{B} = (b^L, b, b^U)_{LR}$ are reduced to the exact values a and b, meaning $a^L = a = a^U$ and $b^L = b = b^U$, then $D^2(a, b) = (a - b)^2$, which is the conventional square error distance.

A degree of inclusion is being introduced here, which quantifies the extent to which one fuzzy quantity is contained within another [46].

Definition 2.3. *Let $\widetilde{A}$ be an LR-**FN** and $\widetilde{R}$ be an LR-**TZFN**. Then, the degree of inclusion $\widetilde{A}$ to $\widetilde{R}$ is defined by* $d(\widetilde{A} \in \widetilde{R}) = \frac{\int_0^1 \int_{\widetilde{R}[\alpha]} \widetilde{A}(x)dxd\alpha}{\int_{\mathbb{R}} \widetilde{A}(x)dx}$.

In the following sections, we will use d to evaluate the degree of inclusion of a given fuzzy sample to fuzzy control limits.

Lemma 2.4. *Let $\widetilde{A}$ be an LR-**FN** and $\widetilde{R}$ be an LR-**TZFN**. Then:*

1) *$d(\widetilde{A} \in \widetilde{R}^c) = 1 - d(\widetilde{A} \in \widetilde{R})$.*

2) *$\widetilde{R}_1 \subseteq \widetilde{R}_2$ if and only if $d(\widetilde{A} \in \widetilde{R}_1) \leq d(\widetilde{A} \in \widetilde{R}_2)$.*

3) *$d(\widetilde{A} \in \widetilde{R}) = 1$ if and only if $\widetilde{A} \subseteq \widetilde{R}$.*

Proof. Since the membership function of $\widetilde{A}$ continues, according to Definition 2.3, we have

$$\begin{aligned} d(\widetilde{A} \in \widetilde{R}^c) &= \frac{\int_0^1 \int_{\widetilde{R}^c[\alpha]} \widetilde{A}(x)dxd\alpha}{\int_{\mathbb{R}} \widetilde{A}(x)dx} \\ &= \frac{\int_0^1 \int_{\{x:\widetilde{R}(x)\leq 1-\alpha\}} \widetilde{A}(x)dxd\alpha}{\int_{\mathbb{R}} \widetilde{A}(x)dx} \\ &= \frac{\int_0^1 \int_{\{x:\widetilde{R}(x)\leq \alpha\}} \widetilde{A}(x)dxd\alpha}{\int_{\mathbb{R}} \widetilde{A}(x)dx} \\ &= \frac{\int_0^1 (\int_{\mathbb{R}} \widetilde{A}(x)dx - \int_{\widetilde{R}[\alpha]} \widetilde{A}(x)dx)d\alpha}{\int_{\mathbb{R}} \widetilde{A}(x)dx} \\ &= 1 - d(\widetilde{A} \in \widetilde{R}). \end{aligned} \tag{9}$$

This verifies assertion (1). If $\widetilde{R}_1 \subseteq \widetilde{R}_2$ then $\widetilde{R}_1[\alpha] \subseteq \widetilde{R}_2[\alpha]$ for any $\alpha \in [0,1]$ and thus

$$\int_0^1 \int_{\widetilde{R}_1[\alpha]} \widetilde{A}(x)dxd\alpha \leq \int_0^1 \int_{\widetilde{R}_2[\alpha]} \widetilde{A}(x)dxd\alpha,$$

which concludes that $d(\widetilde{A} \in \widetilde{R}_1) \leq d(\widetilde{A} \in \widetilde{R}_2)$. Furthermore, $d(\widetilde{A} \in \widetilde{R}_1) \leq d(\widetilde{A} \in \widetilde{R}_2)$ concludes that

$$\int_{\widetilde{R}_1[\alpha]} \widetilde{A}(x)dx \leq \int_{\widetilde{R}_2[\alpha]} \widetilde{A}(x)dx,$$

for any $\alpha \in [0,1]$. This means $\widetilde{R}_1[\alpha] \subseteq \widetilde{R}_2[\alpha]$ for any $\alpha \in [0,1]$ that is $\widetilde{R}_1 \subseteq \widetilde{R}_2$. This verifies assertion (2). To establish assertion (3), first note that

$$\begin{aligned} d(\widetilde{A} \in \widetilde{R}) &= \frac{\int_0^1 \int_{Supp(\widetilde{A})\cap\widetilde{R}[\alpha]} \widetilde{A}(x)dxd\alpha}{\int_0^1 \int_{Supp(\widetilde{A})\cap\widetilde{R}[\alpha]\cup(Supp(\widetilde{A})-\widetilde{R}[\alpha])} \widetilde{A}(x)dxd\alpha} \\ &= \frac{\int_0^1 \int_{Supp(\widetilde{A})\cap\widetilde{R}[\alpha]} \widetilde{A}(x)dxd\alpha}{\int_0^1 \int_{Supp(\widetilde{A})\cap\widetilde{R}[\alpha]} \widetilde{A}(x)dxd\alpha + \int_0^1 \int_{Supp(\widetilde{A})-\widetilde{R}[\alpha]} \widetilde{A}(x)dxd\alpha}. \end{aligned} \tag{10}$$

Hence, $d(\widetilde{A} \in \widetilde{R}) = 1$ if and only if $\int_{Supp(\widetilde{A})-\widetilde{R}[\alpha]} \widetilde{A}(x)dx = 0$ for every $\alpha \in [0,1]$, if and only if the set $Supp(\widetilde{A}) - \widetilde{R}[\alpha]$ is empty for every $\alpha \in [0,1]$, if and only if $\widetilde{A}[\alpha] \subseteq Supp(\widetilde{A}) \subseteq \widetilde{R}[\alpha]$ for every $\alpha \in [0,1]$, implying that $\widetilde{A}$ is a subset of $\widetilde{R}$. □

Definition 2.5. *Let $\widetilde{A}$ be an LR-**FN** and $\widetilde{R}$ be an LR-**TZFN**. Then, we say that $\widetilde{A} \in_d \widetilde{R}$ if $d(\widetilde{A} \in \widetilde{R}) \geq d(\widetilde{A} \in \widetilde{R}^c)$, that is, $d(\widetilde{A} \in \widetilde{R}) \geq 0.5$.*

2.2 Fuzzy random variables

Here, a common definition of a fuzzy random variable [26, 28] is simplified within the domain of LR-**FN**s as follows.

Definition 2.6. [20, 25] *Let $(\Omega, \mathcal{A}, P)$ be a probability space. We say $\widetilde{X} = (X^L, X, X^U)_{LR}$ is an LR-fuzzy random variable (**FRV**) if X^L, X, and $X^U : \Omega \to \mathbb{R}$ are three ordinary random variables and it holds that $P(X^L < X < X^U) = 1$. In addition, two LR-**FRV**s $\widetilde{X}_1$ and $\widetilde{X}_2$ are independent and identically distributed if (X_1, X_2), (X_1^L, X_2^L) and (X_1^U, X_2^U) are independent and identically distributed random variables. Furthermore, $\widetilde{X}_1, ..., \widetilde{X}_n$ is said to be an LR-fuzzy random sample if the $\widetilde{X}_i$'s are independent and identically distributed LR-**FRV**s. An observed LR-**FRV** can be denoted by $\widetilde{x}_1, \ldots, \widetilde{x}_n$.*

Remark 2.7. *According to Kwakernaak's definition* [28], *a fuzzy quantity* $\widetilde{X}$ *is an* ***FRV*** *on* $(\Omega, \mathcal{A}, P)$ *if* $\widetilde{X}^L_\alpha$ *and* $\widetilde{X}^U_\alpha$ *are ordinary random variables for any* $\alpha \in [0,1]$*. Therefore, based on Kwakernaak's concept of* ***FRVs****, an LR-****FN*** $\widetilde{X} = (X^L, X, X^U)_{LR}$ *is an* ***FRV*** *if* $\widetilde{X}^L_\alpha = a - (a - a^L)L^{-1}(\alpha)$ *and* $\widetilde{X}^U_\alpha = a + (a^U - a)R^{-1}(\alpha)$ *are ordinary random variables for all* $\alpha \in [0,1]$*. Specifically,* $\widetilde{X}^L_0 = a^L$*,* $\widetilde{X}^U_0 = a^U$*, and* $\widetilde{X}^U_1 = a$ *are ordinary random variables for* $\alpha = 0$ *and* 1*, respectively. Therefore, Definition 2.6 is a special case of the* ***FRV*** *notion introduced by Kwakernaak* [28].

A common type of *LR*-**FRV** [24] that is often applied is the form of

$$\widetilde{X} = \begin{cases} (X - b_1^X, X, X + b_2^X)_{LR}, & 0 \in Support(X), \\ (\max\{0, X - b_1^X\}, X, X + b_2^X)_{LR}, & Support(X) \subseteq \mathbb{R}^+, \end{cases} \tag{11}$$

where $b_1^X, b_2^X > 0$ and $Support(X)$ is the set of all possible values that X can take. Specifically, a common type of the aforementioned *LR*-**FRV**, utilized in fuzzy statistics, is the normal **FRV** [38], which is defined as $\widetilde{X} = (\mu^L + \epsilon, \mu + \epsilon, \mu^U + \epsilon)_{LR}$ where $\mu^L < \mu < \mu^U$ and $\epsilon \sim N(0, \sigma^2)$ in which $N(\mu, \sigma^2)$ stands for normal distribution with mean μ and variance of σ^2.

Building on the work of Hesamian [20] and Hesamian et al. [22, 24], it is possible to simplify the concept of fuzzy expectation and precise variance for all types of **FRV**s in the context of *LR*-**FRV**s using the definitions provided.

Definition 2.8. *Let* $\widetilde{X}$ *be an LR-****FRV*** *on the probability space of* $(\Omega, \mathcal{A}, P)$*. The expectation value of* $\widetilde{X} = (X^L, X, X^U)_{LR}$ *is defined to be an LR-****FRV*** *as* $\widetilde{E}(\widetilde{X}) = (E(X^L), E(X), E(X^U))_{LR}$*.*

Definition 2.9. *Let* $\widetilde{X}$ *be an LR-****FRV*** *on the probability space of* $(\Omega, \mathcal{A}, P)$*. The variance of an LR-****FRV*** $\widetilde{X}$ *is defined by* $\sigma^2_{\widetilde{X}} = \boldsymbol{var}(\widetilde{X}) = E(D^2(\widetilde{X}, \widetilde{E}(\widetilde{X}))$ *where* D^2 *is the square error distance between two LR-****FRVs*** *introduced in Eq.* (8)*.*

Lemma 2.10. *Let* $\widetilde{X} = (X^L, X, X^U)_{LR}$ *be an LR-****FRV*** *and* $\widetilde{b} = (b^L, b, b^U)_{LR}$ *be an LR-****FN****. Then*

1) $\widetilde{E}((a \otimes \widetilde{X}) \oplus \widetilde{b}) = (a \otimes \widetilde{E}(\widetilde{X})) \oplus \widetilde{b}$ *for any* $a > 0$*.*

2) $\boldsymbol{var}((a \otimes \widetilde{X}) \oplus \widetilde{b}) = a^2 \boldsymbol{var}(\widetilde{X})$ *for any* $a \in \mathbb{R}$*.*

Proof. To validate the initial claim, one may employ the operations of addition and scalar multiplication on two *LR*-**FNs**. This results in $(a \otimes \widetilde{X}) \oplus \widetilde{b} = (aX^L + b^L, aX + b, aX^U + b^U)_{LR}$ for any $a > 0$. Thus, $\widetilde{E}((a \otimes \widetilde{X}) \oplus \widetilde{b})) = (aE(X^L) + b^L, aE(X) + b, aE(X^U) + b^U)_{LR} = (aE(X^L), aE(X), aE(X^U))_{LR} \oplus \widetilde{b} = (a \otimes \widetilde{E}(\widetilde{X})) \oplus \widetilde{b}$. To demonstrate the second assertion, let $c_1 = \int_0^1 L^{-1}(\alpha)d\alpha$ and $c_2 = \int_0^1 R^{-1}(\alpha)d\alpha$. Suppose that $a < 0$. By the arithmetic operation of *LR*-**FN**s, we can deduce that $a \otimes \widetilde{X} = (aX^U, aX, aX^L)_{RL}$, which implies

$$\mathbf{var}((a \otimes \widetilde{X})) = \frac{1}{1 + c_1 + c_2}\Bigg(var(aX + b) + (\frac{c_1 + c_2}{1 + c_1 + c_2})(\frac{var(aX^U) + var(aX^L)}{2})\Bigg) =$$

$$\frac{1}{1 + c_1 + c_2}\Bigg(a^2 var(X) + (\frac{c_1 + c_2}{1 + c_1 + c_2})(\frac{a^2 var(X^U) + a^2 var(X^L)}{2})\Bigg)$$

$$= a^2\big(p_1 var(X) + p_2(\frac{var(X^L) + var(X^U)}{2})\big) = a^2 \mathbf{var}(\widetilde{X}).$$

The same outcome holds true when $a > 0$. □

Remark 2.11. *Let* $\widetilde{X} = (X^L, X, X^U)_{LR}$ *be an LR-****FRV****. If* $var(X) = \sigma^2$*,* $var(X^L) = \sigma^{2L}$ *and* $var(X^U) = \sigma^{2U}$*, according to the above definition, we have*

$$\sigma^2_{\widetilde{X}} = p_1\sigma^2 + p_2(\frac{\sigma^{2L} + \sigma^{2U}}{2}).$$

Remark 2.12. *For a normal* ***FRV*** $\widetilde{X} = (\mu^L + \epsilon, \mu + \epsilon, \mu^U + \epsilon)_{LR}$*, it is easy to check that* $\widetilde{E}(\widetilde{X}) = \widetilde{\mu}$ *and* $\boldsymbol{var}(\widetilde{X}) = \sigma^2$*.*

Definition 2.13. [20, 22] *Let* $\widetilde{X}_1, \cdots, \widetilde{X}_n$ *be an LR-****FRS****. The* i^{th} *fuzzy LR-order statistic is defined to be an LR-****FRV*** *as* $\widetilde{X}_{(i)} = (X^L_{(i)}, X_{(i)}, X^U_{(i)})_{LR}$ *where* $X^L_{(i)}, X_{(i)}$ *and* $X^U_{(i)}$ *represent the* i^{th} *order statistics of the ordinary random samples of* X^L_1*,* X^L_2*, ...,* X^L_n*,* X_1*,* X_2*, ...,* X_n*, and* X^U_1*,* X^U_2*, ...,* X^U_n*, respectively.*

3 Fuzzy ranked set sampling (FRSS)

This section extends the conventional **RSS** [32] for fuzzy data.
Consider a fuzzy data set $\widetilde{x}_1, \widetilde{x}, ..., \widetilde{x}_N$ from a population where $\widetilde{x}_i = (x_i^L, x_i, x_i^U)_{LR}$. In the **FRSS**, the sample selection is composed of five steps to obtain a sample of size $N = mn$:
Step 1: Randomly select m^2 fuzzy data $\widetilde{x}_1, \widetilde{x}_2, ..., \widetilde{x}_{m^2}$.
Step 2: Randomly allocate the m^2 selected units into m sets, each of size m.
Step 3: Identify $\widetilde{x}_{(1)}, \widetilde{x}_{(2)}, ..., \widetilde{x}_{(m)}$ within each set.
Step 4: Select $\widetilde{x}_{(1)}$ from the first set of m units, $\widetilde{x}_{(2)}$ from the second set, $\widetilde{x}_{(3)}$ from the third set, ..., $\widetilde{x}_{(m)}$ from the last set.
Step 5: Repeat steps 1 through 4 n cycles.

Definition 3.1. *Let $\widetilde{x}_{i[i:m]j}$ be the i^{th} ordered fuzzy random sample data from the i^{th} set of size m for a given cycle j. The **FRSS** estimator of $\widetilde{\mu}$ is defined as:*

$$\widetilde{\overline{x}}_{FRSS} = \frac{1}{mn} \otimes \Big(\bigoplus_{j=1}^{n} \bigoplus_{i=1}^{m} \widetilde{x}_{i[i:m]j} \Big). \tag{12}$$

Remark 3.2. *Note that $\widetilde{\overline{x}}_{FRSS}$ is an LR-**FN** as follows:*

$$\widetilde{\overline{x}}_{FRSS} = (\overline{x}^L_{FRSS}, \overline{x}_{FRSS}, \overline{x}^U_{FRSS})_{LR}, \tag{13}$$

where

1) $\overline{x}^L_{FRSS} = \frac{1}{mn} \sum_{j=1}^{n} \sum_{i=1}^{m} x^L_{i[i:m]j}$,
2) $\overline{x}_{FRSS} = \frac{1}{mn} \sum_{j=1}^{n} \sum_{i=1}^{m} x_{i[i:m]j}$,
3) $\overline{x}^U_{FRSS} = \frac{1}{mn} \sum_{j=1}^{n} \sum_{i=1}^{m} x^U_{i[i:m]j}$,

Lemma 3.3. *Consider an **FRSS** from a population with fuzzy mean $\widetilde{\mu} = (\mu^L, \mu, \mu^U)_{LR}$. Then,*

$$\sigma^2_{\widetilde{\overline{X}}_{FRSS}} = p_1 \sigma^2_{FRSS} + p_2(\sigma^{2L}_{FRSS} + \sigma^{2U}_{FRSS}), \tag{14}$$

where

1) $\sigma^{2L}_{FRSS} = \frac{\sigma^{2L}}{m} - \frac{1}{m^2} \sum_{i=1}^{m} (\mu^L_{i[i:m]} - \mu^L)^2$ *where* $\mu^L_{i[i:m]}$ *is the mean of the* i^{th} *order statistics of a sample of size* m *that is* $\mu^L_{i[i:m]} = E(X^L_{i[i:m]})$ *and* $\sigma^{2L} = Var(X^L)$,
2) $\sigma^2_{FRSS} = \frac{\sigma^2}{m} - \frac{1}{m^2} \sum_{i=1}^{m} (\mu_{i[i:m]} - \mu)^2$ *where* $\mu^L_{i[i:m]} = E(X_{i[i:m]})$ *and* $\sigma^2 = Var(X)$,
3) $\sigma^{2U}_{FRSS} = \frac{\sigma^{2U}}{m} - \frac{1}{m^2} \sum_{i=1}^{m} (\mu^U_{i[i:m]} - \mu^L)^2$ *where* $\mu^L_{i[i:m]} = E(X^U_{i[i:m]})$ *and* $\sigma^{2U} = Var(X^U)$.

Proof. According to the definition of a variance, an LR-**FRV** (Definition 2.9), we have

$$\sigma^2_{\widetilde{\overline{X}}_{FRSS}} = \mathbf{var}(\widetilde{\overline{X}}_{FRSS}) = E(D^2(\widetilde{\overline{X}}_{FRSS}, \widetilde{E}(\widetilde{\overline{X}}_{FRSS})))$$
$$= p_1 E(\overline{X}_{FRSS} - E(\overline{X}_{FRSS}))^2 + p_2 \Big(E(\overline{X}^L_{FRSS} - E(\overline{X}^L_{FRSS}))^2 + E(\overline{X}^U_{FRSS} - E(\overline{X}^U_{FRSS}))^2 \Big).$$

According to traditional statistical inference, the outcome is as follows:

1) $\sigma^{2L}_{FRSS} = E(\overline{X}^L_{FRSS} - E(\overline{X}^L_{FRSS}))^2 = \frac{\sigma^{2L}}{m} - \frac{1}{m^2} \sum_{i=1}^{m} (\mu^L_{i[i:m]} - \mu^L)^2$ where $\mu^L_{i[i:m]}$ is the mean of the i^{th} order statistics of a sample of size m that is $\mu^L_{i[i:m]} = E(X^L_{i[i:m]})$ and $\sigma^{2L} = Var(X^L)$,
2) $\sigma^2_{FRSS} = E(\overline{X}_{FRSS} - E(\overline{X}_{FRSS}))^2 = \frac{\sigma^2}{m} - \frac{1}{m^2} \sum_{i=1}^{m} (\mu_{i[i:m]} - \mu)^2$ where $\mu^L_{i[i:m]} = E(X_{i[i:m]})$ and $\sigma^2 = Var(X)$,
3) $\sigma^{2U}_{FRSS} = E(\overline{X}^U_{FRSS} - E(\overline{X}^U_{FRSS}))^2 = \frac{\sigma^{2U}}{m} - \frac{1}{m^2} \sum_{i=1}^{m} (\mu^U_{i[i:m]} - \mu^L)^2$ where $\mu^L_{i[i:m]} = E(X^U_{i[i:m]})$ and $\sigma^{2U} = Var(X^U)$.

□

4 Fuzzy Shewhart control charts based on the FRSS

Consider a fuzzy data set from a population of size N with a fuzzy mean $\widetilde{\mu}$ and an exact variance of σ^2. The goal is to establish fuzzy control limits for the fuzzy $\widetilde{\mu}$ using a given **FRSS**. Two scenarios were examined for constructing these fuzzy control limits: (1) when $\widetilde{\mu}$ is known and (2) when $\widetilde{\mu}$ is unknown.

Definition 4.1. *Consider an **FRSS** from a population with known variance σ^2 and fuzzy population mean $\widetilde{\mu} = (\mu^L, \mu, \mu^U)_{LR}$. The following is the definition of the fuzzy lower ($\widetilde{LCL}$), center ($\widetilde{CL}$), and upper limits ($\widetilde{UCL}$) for $\widetilde{\mu} = (\mu^L, \mu, \mu^U)_{LR}$:*

$$\widetilde{LCL} = \widetilde{\mu} \ominus (3\sigma^2_{\widetilde{\overline{X}}_{FRSS}}), \tag{15}$$

$$\widetilde{CL} = \widetilde{\mu}, \tag{16}$$

$$\widetilde{UCL} = \widetilde{\mu} \oplus (3\sigma^2_{\widetilde{\overline{X}}_{FRSS}}). \tag{17}$$

Remark 4.2. *The fuzzy control limits can be substituted as follows if $\widetilde{\mu} = (\mu^L, \mu, \mu^U)_{LR}$ and σ^{2L}, σ^2, and σ^{2U} are unknown quantities in Definition 4.1:*

$$\widetilde{LCL} = \widetilde{\overline{X}}_{FRSS} \ominus (3\widehat{\sigma}_{\widetilde{\overline{X}}_{FRSS}}), \tag{18}$$

$$\widetilde{CL} = \widetilde{\overline{X}}_{FRSS}, \tag{19}$$

$$\widetilde{UCL} = \widetilde{\overline{X}}_{FRSS} \oplus (3\widehat{\sigma}_{\widetilde{\overline{X}}_{FRSS}}), \tag{20}$$

where

$$\widehat{\sigma}^2_{\widetilde{\overline{X}}_{FRSS}} = \frac{1}{m}(\widehat{\sigma}^2_{FRSS} - \frac{1}{m}\widehat{\sigma}^2_{[i]}), \tag{21}$$

in which

1) $\widehat{\sigma}^2_{FRSS} = \frac{1}{mn-1}\sum_{j=1}^{n}\sum_{i=1}^{m} D^2(\widetilde{x}_{i[i:m]}, \widetilde{\overline{x}}_{FRSS})$ *with*

$$D^2(\widetilde{x}_{i[i:m]}, \widetilde{\overline{x}}_{FRSS}) = p_1(x_{i[i:m]} - \overline{x}_{FRSS})^2 + p_2(\frac{(x^L_{i[i:m]} - \overline{x}^L_{FRSS})^2 + (x^U_{i[i:m]} - \overline{x}^U_{FRSS})^2}{2}).$$

2) $\widehat{\sigma}^2_{[i]} = \frac{1}{m}\sum_{i=1}^{m} D^2(\widetilde{\overline{x}}_{[i]}, \widetilde{\overline{x}}_{FRSS})$ *where*

$$\widetilde{\overline{x}}_{[i]} = (\frac{1}{n}\otimes)\bigoplus_{j=1}^{n}\widetilde{x}_{i[i:m]j} = (\frac{1}{n}\sum_{j=1}^{n}x^L_{i[i:m]j}, \frac{1}{n}\sum_{j=1}^{n}x_{i[i:m]j}, \frac{1}{n}\sum_{j=1}^{n}x^U_{i[i:m]j})_{LR}, \tag{22}$$

and

$$D^2(\widetilde{\overline{x}}_{[i]}, \widetilde{\overline{x}}_{FRSS}) = p_1(\overline{x}_{[i]} - \overline{x}_{FRSS})^2 + p_2(\frac{(\overline{x}^L_{[i]} - \overline{x}^L_{FRSS})^2 + (\overline{x}^U_{[i]} - \overline{x}^U_{FRSS})^2}{2}).$$

Here, we propose criteria to interpret the concepts of "under control" or "out of control" based on an **FRSS**.

Definition 4.3. *Consider an **FRSS** from a population with $\widetilde{\mu}$ and an exact variance of σ^2. Let $\widetilde{LCL} = (LCL^L, LCL, LCL^U)_{LR}$ and $\widetilde{UCL} = (UCL^L, UCL, UCL^U)_{LR}$ denote the lower and upper fuzzy control limits of $\widetilde{\mu}$. The fuzzy control region is defined by $\widetilde{R} = (LCL^L, LCL, UCL, UCL^U)_{LR}$. Furthermore, the degree to which fuzzy data $\widetilde{x}^*$ is outside of $\widetilde{R} = (LCL^L, LCL, UCL, UCL^U)_{LR}$ is defined by $d^* = d(\widetilde{x}^* \in \widetilde{R}^c)$. According to Definition 2.3, $\widetilde{x}^*$ is then considered an outlier if $d^* \geq 0.5$.*

Remark 4.4. *Faraz and Shapiro [13] proposed a fuzzy control chart based on fuzzy random variables with fuzzy means and exact variance. In their approach, they introduced a degree of inclusion for fuzzy statistical control charts based on fuzzy normal distributions. They proposed a criterion to interpret the concepts of "under control" or "out of control" based on a fuzzy **SRS** scheme with fuzzy mean and exact variance as follows:*

$$d_{FA}(\widetilde{A} \in \widetilde{R}) = \frac{\int \min\{\widetilde{A}(x), \widetilde{R}(x)\}dx}{\int \widetilde{A}(x)dx}. \tag{23}$$

*It is straightforward to confirm that d_{FA} satisfies conditions (2) and (3) in Lemma 2.4. However, the equation $d(\widetilde{A} \in \widetilde{R}^c) = 1 - d(\widetilde{A} \in \widetilde{R})$ is not true. Therefore, our proposed measure of inclusion offers a more accurate interpretation than that of Faraz and Shapiro for describing the vague concept that an LR-**FN** of $\widetilde{A}$ belongs to an LR-**TFN** of $\widetilde{R}$.*

*Furthermore, Hesamian et al. [23] proposed fuzzy control limits $\widetilde{LCL}$ and $\widetilde{UCL}$ based on normal **FRVs** with fuzzy means and fuzzy variance. By utilizing the credibility measure Cr [30], they suggested a degree to which the fuzzy sample mean $\widetilde{\overline{x}}$ of subgroup n is out of the specific fuzzy control region $[\widetilde{LCL}, \widetilde{UCL}]$ as follows:*

$$d_1 = \max\left\{Cr\{\widetilde{\overline{x}} \succ \widetilde{UCL}\}, Cr\{\widetilde{\overline{x}} \prec \widetilde{LCL}\}\right\}. \tag{24}$$

The measure d_1 has the following properties:

1. $d_1 \in [0, 1]$,
2. $1 - d_1 = \min\left\{Cr(\widetilde{\overline{x}} \preceq \widetilde{UCL}), Cr(\widetilde{\overline{x}} \succeq \widetilde{LCL})\right\}$.
3. $d_1 = 1$ *if and only if* $Cr\{\widetilde{\overline{x}} \succ \widetilde{UCL}\} = 1$ *or* $Cr\{\widetilde{\overline{x}} \prec \widetilde{LCL}\} = 1$ *if and only if* $(\widetilde{\overline{x}})_0^L > (\widetilde{UCL})_0^U$ *or* $(\widetilde{LCL})_0^L > (\widetilde{\overline{x}})_0^U$.
4. $d_1 = 0$ *if and only if* $Cr\{\widetilde{\overline{X}} \preceq \widetilde{UCL}\} = 1$ *and* $Cr\{\widetilde{\overline{x}} \succeq \widetilde{LCL}\} = 1$ *if and only if* $(\widetilde{\overline{x}})_0^U < (\widetilde{UCL})_0^L$ *and* $(\widetilde{LCL})_0^U < (\widetilde{\overline{x}})_0^L$.

However, we employ a different approach from Hesamian et al. in detecting out-of-control fuzzy control charts. It is evident that d_1 fulfills all conditions in Lemma 2.4, but the proposed criterion has a more straightforward structure for interpreting and calculating the degree of inclusion compared to the criterion of Hesamian et al.

In the following section, we offer a demonstration of the suggested fuzzy control chart using a real-world example. As this is the first presentation of the issue of fuzzy Shewhart control charts utilizing ranked set sampling, the analysis and examination of the fuzzy sample data are conducted solely according to the proposed approach.

5 Application example

In semiconductor manufacturing, the hard-bake process is a critical step that occurs after exposing the photoresist to light. It is carried out to improve the resist's characteristics, making it stronger and more resistant to subsequent processing steps such as etching and ion implantation. This additional baking process involves heating the wafer at high temperatures, usually above 120°C, to minimize internal stresses in the resist layer and enhance its adhesion to the substrate [31]. Our goal is to achieve statistical control of the resist's flow width during this process using the suggested control charts. For this purpose, twenty-five samples, each with five wafer sizes (measured in microns) were drawn using an in-control process. The complete data set is provided by Montgomery [44]. This data set can be reported by $\widetilde{x}_i = (\max\{0, x_i - 0.1x_i\}, x_i, x_i + 0.15x_i)_{LR} = (0.9x_i, x_i, 1.15x_i)_{LR}$ with $L(x) = 1 - x^2$ and $R(x) = \sqrt{1-x}$. According to Definition 2.13, note that the fuzzy order statistics are $\widetilde{x}_{(i)} = (0.9x_{(i)}, x_{(i)}, 1.15x_{(i)})_{LR}$. To generate samples via **FRSS**, the whole data set is combined to generate a population that comprises 125 observations. Then, five samples ($n = 5$), each of size five ($m = 5$), are generated via the **FRSS**, as shown in Table 1. We investigate this example in two cases.

Case I: In this example, we assume that the observed fuzzy random samples are taken from normal fuzzy random variables $\widetilde{X} = (1.35 + \epsilon, 1.50 + \epsilon, 1.63 + \epsilon)_{LR}$ where $\epsilon \sim N(0, 0.09)$. Therefore

1) $X^L \sim N(1.46, 0.09)$ concludes that X^L and $X - 0.15$ have the same distribution function, i.e. $X^L \simeq^d X - 0.15$ where ,
2) $X \sim N(1.50, 0.09)$,
3) $X^U \sim N(1.55, 0.09)$ yields $X^U \simeq^d X + 0.13$.

Therefore, from Lemma 3.3, we obtain:

1) $\mu^L_{i[i:5]} = E(X_{i[i:5]} - 0.15) = \mu_{i[i:5]} - 0.15$,
2) $\mu^U_{i[i:5]} = E(X_{i[i:5]} + 0.13) = \mu_{i[i:5]} + 0.13$.

Table 1: Center values of the **FRSS** in the hard-bake process

Cycle		Sample units
1	1st Set	$x_{111} = 1.6744, x_{112} = 1.6914, x_{113} = 1.4128, x_{114} = 1.4573, x_{115} = \mathbf{1.3235}$,
	2st Set	$x_{121} = 1.4666, x_{122} = 1.3592, x_{123} = 1.6075, x_{124} = \mathbf{1.4314}, x_{125} = 1.6109$,
	3st Set	$x_{131} = 1.4284, x_{132} = \mathbf{1.4871}, x_{133} = 1.4932, x_{134} = 1.4324, x_{135} = 1.5674$,
	4st Set	$x_{141} = 1.5028, x_{142} = 1.6352, x_{143} = 1.3841, x_{144} = 1.2831, x_{145} = \mathbf{1.5507}$,
	5st Set	$x_{151} = 1.5604, x_{152} = 1.2735, x_{153} = 1.5265, x_{154} = 1.4363, x_{155} = \mathbf{1.6441}$,
2	1st Set	$x_{211} = 1.5955, x_{212} = 1.5451, x_{213} = 1.3574, x_{214} = \mathbf{1.3281}, x_{215} = 1.4198$,
	2st Set	$x_{221} = 1.6274, x_{222} = \mathbf{1.5064}, x_{213} = 1.8366, x_{214} = 1.4177, x_{215} = 1.5144$,
	3st Set	$x_{231} = 1.6637, x_{232} = 1.4190, x_{233} = \mathbf{1.5519}, x_{234} = 1.6067, x_{235} = 1.4303$,
	4st Set	$x_{241} = 1.3688, x_{242} = 1.3884, x_{243} = 1.7277, x_{244} = \mathbf{1.5355}, x_{245} = 1.5176$,
	5st Set	$x_{251} = 1.5089, x_{252} = \mathbf{1.6697}, x_{253} = 1.4039, x_{254} = 1.4627, x_{255} = 1.5220$,
3	1st Set	$x_{311} = 1.4278, x_{312} = 1.7667, x_{313} = 1.5928, x_{314} = 1.4181, x_{315} = \mathbf{1.4158}$,
	2st Set	$x_{321} = 1.5821, x_{322} = 1.3355, x_{323} = 1.7559, x_{324} = \mathbf{1.3908}, x_{325} = 1.5777$,
	3st Set	$x_{331} = 1.4447, x_{332} = \mathbf{1.4106}, x_{333} = 1.2856, x_{334} = 1.6398, x_{335} = 1.1928$,
	4st Set	$x_{341} = 1.4036, x_{342} = \mathbf{1.5893}, x_{343} = 1.6458, x_{344} = 1.4969, x_{345} = 1.4951$,
	5st Set	$x_{351} = \mathbf{1.5996}, x_{352} = 1.2863, x_{353} = 1.2497, x_{354} = 1.3589, x_{355} = 1.5471$,
4	1st Set	$x_{411} = 1.8662, x_{412} = 1.5747, x_{413} = 1.5171, x_{414} = 1.5301, x_{415} = \mathbf{1.1839}$,
	2st Set	$x_{421} = \mathbf{1.3957}, x_{422} = 1.3680, x_{423} = 1.7269, x_{424} = 1.5014, x_{425} = 1.4449$,
	3st Set	$x_{431} = 1.5573, x_{432} = 1.3864, x_{433} = \mathbf{1.4163}, x_{434} = 1.6210, x_{435} = 1.3057$,
	4st Set	$x_{441} = \mathbf{1.6541}, x_{442} = 1.5116, x_{443} = 1.5796, x_{444} = 1.4185, x_{445} = 1.7247$,
	5st Set	$x_{451} = 1.4412, x_{452} = 1.2361, x_{453} = 1.3820, x_{454} = \mathbf{1.7601}, x_{455} = 1.7106$,
5	1st Set	$x_{511} = 1.5051, x_{512} = \mathbf{1.3485}, x_{513} = 1.4371, x_{514} = 1.5670, x_{515} = 1.4880$,
	2st Set	$x_{521} = 1.4973, x_{522} = \mathbf{1.4738}, x_{523} = 1.6583, x_{524} = 1.4720, x_{525} = 1.5936$,
	3st Set	$x_{531} = 1.5917, x_{532} = 1.4333, x_{533} = 1.5295, x_{534} = 1.6866, x_{535} = \mathbf{1.5551}$,
	4st Set	$x_{541} = 1.6399, x_{542} = \mathbf{1.5705}, x_{543} = 1.5563, x_{544} = 1.5530, x_{545} = 1.5243$,
	5st Set	$x_{551} = \mathbf{1.6887}, x_{552} = 1.3663, x_{553} = 1.6240, x_{554} = 1.5797, x_{555} = 1.3732$.

The values of $\mu_{i[i:5]}$, $i = 1, 2, ..., 5$ are given in Table 2 where

$$\mu_{i[i:5]} = \int_{-\infty}^{\infty} 5x \binom{4}{i-1} (F_X(x))^{i-1}(1 - F_X(x))^{5-i} f_X(x)dx, \tag{25}$$

in which $\binom{4}{i-1} = 4!/((i-1)!(5-i)!)$ and F_X represents the normal distribution function with mean $\mu = 1.5$ and variance of $\sigma^2 = 0.09$ and f_X is its density function. These findings immediately follow that $\sum_{i=1}^{5}(\mu^L_{i[i:5]} - \mu^L)^2 = \sum_{i=1}^{5}(\mu^U_{i[i:5]} - \mu^U)^2 = \sum_{i=1}^{5}(\mu_{i[i:5]} - \mu)^2$; thus,

$$\sigma^{2L}_{FRSS} = \sigma^{2U}_{FRSS} = \sigma^2_{FRSS} = 0.00649778.$$

This simply concludes that

$$\sigma^2_{\widetilde{\overline{X}}_{FRSS}} = p_1\sigma^2_{FRSS} + p_2(\sigma^{2L}_{FRSS} + \sigma^{2U}_{FRSS})/2 = 0.00649778(p_1 + p_2) = 0.00649778. \tag{26}$$

From Definition 4.1, therefore, the fuzzy lower ($\widetilde{LCL}$), center ($\widetilde{CL}$) and upper ($\widetilde{UCL}$) limits for $\widetilde{\mu} = (\mu^L, \mu, \mu^U)_{LR}$ can be obtained as follows:

$$\widetilde{LCL} = (1.35, 1.50, 1.63)_{LR} \ominus (3 \times 0.00649778) = (1.3305, 1.4805, 1.6105)_{LR}, \tag{27}$$

$$\widetilde{CL} = (1.35, 1.50, 1.63)_{LR}, \tag{28}$$

$$\widetilde{UCL} = (1.35, 1.50, 1.63)_{LR} \oplus (3 \times 0.00649778) = (1.3695, 1.5195, 1.6495)_{LR}. \tag{29}$$

Therefore, the fuzzy control region can be evaluated as $\widetilde{R} = (1.3305, 1.4805,\ 1.5195, 1.6495)_{LR}$, whose membership function is

$$\widetilde{R}(x) = \begin{cases} 1 - (\dfrac{1.4805 - x}{0.15})^2, & 1.3305 \leq x \leq 1.4805, \\ 1, & 1.4805 \leq x < 1.5195, \\ \sqrt{1 - \dfrac{x - 1.5195}{0.13}}, & 1.5195 < x < 1.6495. \end{cases} \tag{30}$$

The plots of $\widetilde{R}$ along with two fuzzy sample data $\widetilde{x}_{415}$ and $\widetilde{x}_{444}$ are shown in Fig. 1. Therefore, the degree to which a fuzzy sample $\widetilde{x}^*$ is under control, can be evaluated as

$$d(\widetilde{x}^* \in \widetilde{R}) = \frac{\int_0^1 \int_{1.4805-0.15\sqrt{1-\alpha}}^{1.5195+0.13(1-\alpha^2)} \widetilde{x}^*(x)dxd\alpha}{\int_0^2 \widetilde{x}^*(x)dx}. \tag{31}$$

The values of $d(\widetilde{x}^* \in \widetilde{R})$ for all cycles are given in Table 3. Table 4 shows those fuzzy samples with $d(\widetilde{x}^* \in \widetilde{R}) < 0.5$ are out of fuzzy control limits.

Case II: Assume that $\widetilde{\mu}$ is an unknown quantity. According to Remark 4.2, we need to estimate $\widetilde{\overline{X}}_{FRSS}$ and $\widehat{\sigma}^2_{\widetilde{\overline{X}}_{FRSS}}$ based on the **FRSS** distinguished in Table 1. For this purpose, we obtain

Table 2: Calculation of $\mu_{i[i:5]}$, $i = 1, 2, ..., 5$ for the case in which $\widetilde{\mu}$ is known

i	$\mu_{i[i:5]}$
1	1.15111
2	1.35149
3	1.5
4	1.64851
5	1.84889

Table 3: $d(\widetilde{x}^* \in \widetilde{R})$ for the underlying **FRSS** when $\widetilde{\mu}$ is known

Cycle 1	Cycle 2	Cycle 3	Cycle 4	Cycle 5
$d(\widetilde{x}_{115} \in \widetilde{R}) = \mathbf{0.369}$	$d(\widetilde{x}_{214} \in \widetilde{R}) = \mathbf{0.387}$	$d(\widetilde{x}_{315} \in \widetilde{R}) = 0.620$	$d(\widetilde{x}_{415} \in \widetilde{R}) = \mathbf{0.006}$	$d(\widetilde{x}_{512} \in \widetilde{R}) = \mathbf{0.460}$
$d(\widetilde{x}_{124} \in \widetilde{R}) = 0.635$	$d(\widetilde{x}_{222} \in \widetilde{R}) = 0.562$	$d(\widetilde{x}_{324} \in \widetilde{R}) = 0.578$	$d(\widetilde{x}_{421} \in \widetilde{R}) = 0.588$	$d(\widetilde{x}_{522} \in \widetilde{R}) = 0.622$
$d(\widetilde{x}_{132} \in \widetilde{R}) = 0.602$	$d(\widetilde{x}_{233} \in \widetilde{R}) = \mathbf{0.428}$	$d(\widetilde{x}_{332} \in \widetilde{R}) = 0.614$	$d(\widetilde{x}_{433} \in \widetilde{R}) = 0.621$	$d(\widetilde{x}_{535} \in \widetilde{R}) = \mathbf{0.418}$
$d(\widetilde{x}_{145} \in \widetilde{R}) = \mathbf{0.432}$	$d(\widetilde{x}_{244} \in \widetilde{R}) = \mathbf{0.481}$	$d(\widetilde{x}_{342} \in \widetilde{R}) = \mathbf{0.303}$	$d(\widetilde{x}_{441} \in \widetilde{R}) = \mathbf{0.119}$	$d(\widetilde{x}_{542} \in \widetilde{R}) = \mathbf{0.366}$
$d(\widetilde{x}_{155} \in \widetilde{R}) = \mathbf{0.142}$	$d(\widetilde{x}_{252} \in \widetilde{R}) = \mathbf{0.087}$	$d(\widetilde{x}_{351} \in \widetilde{R}) = \mathbf{0.269}$	$d(\widetilde{x}_{454} \in \widetilde{R}) = \mathbf{0.006}$	$d(\widetilde{x}_{551} \in \widetilde{R}) = \mathbf{0.057}$

Table 4: Out-of-control fuzzy samples for the case in which $\widetilde{\mu}$ is known

Cycle	fuzzy sample
Cycle 1	$\widetilde{x}_{124}, \widetilde{x}_{132}$
Cycle 2	$\widetilde{x}_{222}$
Cycle 3	$\widetilde{x}_{315}, \widetilde{x}_{324}, \widetilde{x}_{332}$
Cycle 4	$\widetilde{x}_{421}, \widetilde{x}_{433}$
Cycle 4	$\widetilde{x}_{522}$

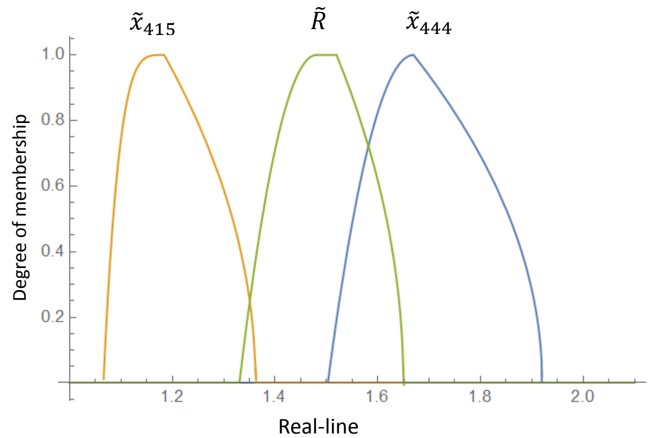

Figure 1: Membership functions of fuzzy control rejoin of $\widetilde{R}$ along with $\widetilde{x}_{415}$ and $\widetilde{x}_{444}$ for the case in which $\widetilde{\mu}$ is known

1) $\overline{x}^L_{FRSS} = \frac{1}{25}\sum_{j=1}^5\sum_{i=1}^5 x^L_{i[i:5]j} = 1.34932$ where $x^L_{i[i:5]j} = 0.9x_{i[i:5]j}$,

2) $\overline{x}_{FRSS} = \frac{1}{25}\sum_{j=1}^5\sum_{i=1}^5 x_{i[i:5]j} = 1.49925$,

3) $\overline{x}^U_{FRSS} = \frac{1}{25}\sum_{j=1}^5\sum_{i=1}^5 x^U_{i[i:5]j} = 1.72414$ where $x^U_{i[i:5]j} = 1.15x_{i[i:5]j}$,

and thus

$$\widetilde{\overline{X}}_{FRSS} = (1.34932, 1.49925, 1.72414)_{LR}. \tag{32}$$

To compute $\widehat{\sigma}^2_{\widetilde{\overline{X}}_{FRSS}}$, we first need to evaluate $\widehat{\sigma}^2_{FRSS}$ and $\widehat{\sigma}^2_{[i]}$. For this purpose, first note that $c_1 = \int_0^1 \sqrt{1-\alpha}d\alpha = c_2 = \int_0^1(1-\alpha^2)d\alpha = 2/3$. Therefore $p_1 = 1/(1+c_1+c_2) = 3/7$ and $p_2 = (c_1+c_2)/(1+c_1+c_2) = 4/7$. Further

$$\widehat{\sigma}^2_{FRSS} = \frac{1}{24}\sum_{j=1}^5\sum_{i=1}^5 D^2(\widetilde{X}_{i[i:5]j}, \widetilde{\overline{X}}_{FRSS}), \tag{33}$$

where

$$D^2(\widetilde{x}_{i[i:5]}, \widetilde{\overline{x}}_{FRSS}) = \frac{3}{7}(x_{i[i:5]} - 1.49925)^2 + \frac{4}{7}(\frac{(x^L_{i[i:5]} - 1.34932)^2 + (x^U_{i[i:5]} - 1.72414)^2}{2}). \tag{34}$$

The calculation shows that $\widehat{\sigma}^2_{FRSS} = 0.0254289$. In addition the values of $\widetilde{\overline{X}}_{[i]} = \frac{1}{5}\otimes\bigoplus_{j=1}^5 \widetilde{\overline{X}}_{i[i:5]j}$, $i = 1, 2, ..., 5$ are summarized in Table 5. Therefore, $\widehat{\sigma}^2_{[i]} = \frac{1}{5}\sum_{i=1}^5 D^2(\widetilde{\overline{x}}_{[i]}, \widetilde{\overline{x}}_{FRSS})$ where

$$D^2(\widetilde{\overline{x}}_{[i]}, \widetilde{\overline{x}}_{FRSS}) = \frac{3}{7}(\overline{x}_{[i]} - \overline{x}_{FRSS})^2 + \frac{4}{7}(\frac{(\overline{x}^L_{[i]} - 1.34932)^2 + (\overline{x}^U_{[i]} - 1.72414)^2}{2}). \tag{35}$$

A simple straightforward calculation reveals that $\widehat{\sigma}^2_{[i]} = 0.0026378$. Therefore,

$$\widehat{\sigma}^2_{\widetilde{\overline{X}}_{FRSS}} = \frac{1}{5}(0.0254289 - \frac{0.0026378}{5}) = 0.00506468. \tag{36}$$

Therefore, the fuzzy lower ($\widetilde{LCL}$), center ($\widetilde{CL}$) and upper ($\widetilde{UCL}$) limits for population $\widetilde{\mu} = (\mu^L, \mu, \mu^U)_{LR}$ can be obtained as follows:

$$\begin{aligned}\widetilde{LCL} &= (1.34932, 1.49925, 1.72414)_{LR} \ominus (3 \times 0.0711666)\\ &= (1.13582, 1.28575, 1.51064)_{LR},\end{aligned} \tag{37}$$

$$\widetilde{CL} = (1.34932, 1.49925, 1.72414)_{LR}, \tag{38}$$

Table 5: Values of $\widetilde{\overline{x}}_{[i]}$, $i = 1, 2, ..., 5$ for the case in which $\widetilde{\mu}$ is unknown

i	$\widetilde{\overline{x}}_{[i]}$
1	$(1.3386, 1.4873, 1.7104)_{LR}$
2	$(1.3664, 1.5183, 1.7460)_{LR}$
3	$(1.3331, 1.4812, 1.7034)_{LR}$
4	$(1.3338, 1.4820, 1.7043)_{LR}$
5	$(1.3745, 1.5273, 1.7564)_{LR}$

$$\begin{aligned}\widetilde{UCL} &= (1.34932, 1.49925, 1.72414)_{LR} \oplus (3 \times 0.0711666) \\ &= (1.56282, 1.71275, 1.93763)_{LR}.\end{aligned} \tag{39}$$

Therefore the fuzzy control region can be evaluated as $\widetilde{R}_s = (1.13582, 1.28575, 1.71275, 1.93763)_{LR}$ with α-cuts of $\widetilde{R}_s[\alpha] = [1.28575 - 0.14993\sqrt{1-\alpha}, 1.71275 + 0.22488(1-\alpha^2)$. The membership function of $\widetilde{R}_s$ is also

$$\widetilde{R}_s(x) = \begin{cases} 1 - (\dfrac{1.28575 - x}{0.14993})^2, & 1.13582 \le x \le 1.28575, \\ 1, & 1.28575 \le x < 1.71275, \\ \sqrt{1 - \dfrac{x - 1.71275}{0.22488}}, & 1.71275 < x < 1.93763. \end{cases} \tag{40}$$

Except $\widetilde{x}_{411}$ and $\widetilde{x}_{452}$, the other fuzzy data are within under fuzzy control limits since their supports belong to $Supp(\widetilde{R}_s)$. The membership functions of $\widetilde{x}_{411}$, $\widetilde{x}_{452}$ and $\widetilde{R}_s$ are shown in Fig. 2. Computing

$$d(\widetilde{x}^* \in \widetilde{R}_s) = \frac{\int_0^1 \int_{1.28575-0.14993\sqrt{1-\alpha}}^{1.71275+0.22488(1-\alpha^2)} \widetilde{x}^*(x)dxd\alpha}{\int_0^2 \widetilde{x}^*(x)dx}, \tag{41}$$

for $\widetilde{x}_{411}$ and $\widetilde{x}_{452}$ yields $d(\widetilde{x}_{411} \in \widetilde{R}_s) = 0.81$ and $d(\widetilde{x}_{452} \in \widetilde{R}_s) = 0.19$. Since $d(\widetilde{x}_{452} \in \widetilde{R}_s) < 0.5$, $\widetilde{x}_{452}$ is the only out-of-control fuzzy data in cases where $\widetilde{\mu}$ is unknown.

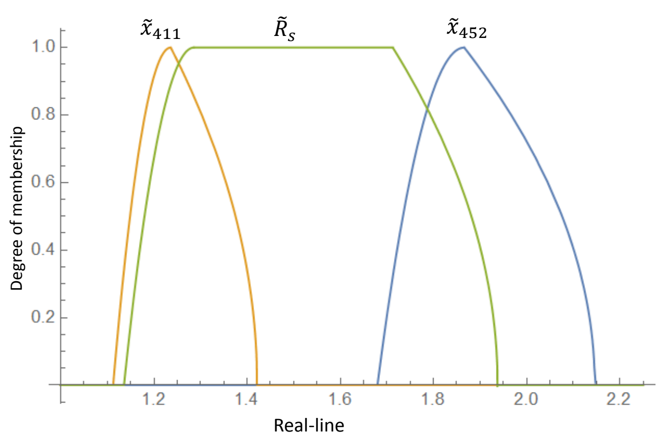

Figure 2: Membership functions of fuzzy control rejoin of $\widetilde{R}_s$ along with $\widetilde{x}_{411}$ and $\widetilde{x}_{452}$ for the case in which $\widetilde{\mu}$ is unknown

6 Conclusion

Several studies have explored control charts using fuzzy data, which are obtained from simple random samples. By incorporating additional information from ranked set sampling schemes, the effectiveness of traditional control charts can be significantly enhanced. This paper introduces a novel approach for constructing the Shewhart control chart based on LR-fuzzy data obtained through a well-established sampling method. The traditional ranked sampling framework was adapted for fuzzy data, and the fuzzy Shewhart quality control chart was extended to include an exact variance for LR-fuzzy random variables under two scenarios: when the population fuzzy mean is known and when it is unknown. A method for determining the inclusion of a fuzzy sample within the control limits using α-cuts within the fuzzy control range was proposed. The potential benefits of this approach were discussed in comparison to previous fuzzy quality control charts. The practicality and validity of the proposed fuzzy control charts were demonstrated through a real-world example. As there is currently no comparable method based on fuzzy ranked set sampling, this study focuses solely on evaluating the presented approach. Numerical assessments indicate that the proposed fuzzy ranked set sampling control charts can be effectively utilized in various industries, including manufacturing, chemical processing, healthcare, and environmental monitoring.

The primary benefit of the suggested approach is its ability to be applied to any type of LR-fuzzy numbers. However, this technique is restricted to fuzzy normal random variables with a precise variance. Therefore, exploring this methodology further for fuzzy variance is a potential area for future research. Additionally, expanding the proposed framework to include other types of ranked set sampling, such as folded ranked set sampling for fuzzy data, is a promising direction for further investigation. The expansion of non-parametric fuzzy control charts using fuzzy ranked set sampling methods could also be explored in future studies. Furthermore, developing the concept of average run length based on fuzzy ranked set sampling is another area for future research.

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