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Solutions to the fuzzy Pielou logistic differential equation

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Abstract

In this paper, the fuzzy Pielou logistic differential equation is studied from the perspective of the generalized Hukuhara differentiability concept. First, the uniqueness of positive or negative solutions is established. Then, the existence conditions of the solution, together with its structural representation, are obtained for two separate cases corresponding to the positivity or negativity of the fuzzy parameters of the problem. Detailed illustrative examples are also provided to clarify the results.

Keywords: Fuzzy numbers, fuzzy differential equations, logistic differential equation.

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1 Introduction

Fuzzy set theory is a powerful tool for interpreting and modeling phenomena in uncertain and ambiguous situations (for instance, [9, 15, 17]). The models extracted from uncertainty conditions are usually mathematical issues including fuzzy parameters [4, 17]. Fuzzy differential equations ($FDEs$) are one of these topics that have been interested many researchers [3, 7, 21, 23, 27]. In this context, most studies were focused on first-order linear $FDEs$ from different perspectives [2, 5, 6, 10, 12, 13, 14, 22, 25]. The investigation of more complex structures of $FDEs$ will lead to the development of theoretical results and a more precise analysis of the real behavior of phenomena. One of such equations is known as the fuzzy logistic differential equation (briefly, $FLDE$), which models population growth or similar capacity-limited processes under the uncertain conditions. The uncertainty in this equation arises from the fact that parameters such as the rate, carrying capacity, and initial population are inherently imprecise, as they are affected by measurement errors, environmental fluctuations, and inaccurate data [8]. A $FLDE$ may be represented in a general form as follows:

$$\begin{cases} x'(t) \oplus v_0 \odot x^2(t) = u_0 \odot x(t), & t > 0, \\ x(0) = x_0, \end{cases} \tag{1}$$

where x_0, u_0 and v_0 are fuzzy numbers and, the symbols $\oplus$ and $\odot$ denote the fuzzy arithmetic operators addition and multiplication. The problem (1) represents the fuzzy form of the famous Pielou $FLDE$ which appears in the modeling of many scientific issues in fields such as biological growth [16], economic [32], electrochemical dynamic [28, 34] and energy prediction [18]. An analysis of the level-wise solutions of problem (1) with positive fuzzy parameters, is given in [24]. Some $FLDEs$ have been studied in [20, 36], in their discrete form, obtained by transforming the original problem into a fuzzy difference equation. The authors in [30], have studied the $FLDE$ with an Alee effect by reformulating it as systems of ordinary differential equations under the generalized Hukuhara differentiability (gH-differentiable) concept and derived conditions for stability of equilibria, and the authors in [1], have also studied the discrete form of this equation. A power series method for the $FLDE$ of fractional order is presented in [31].

Although previous studies have addressed the $FLDE$, there remains a gap concerning the existence of analytical solution expressed as fuzzy-numerical-valued functions. Obtaining the solution of the $FLDE$ as a fuzzy-number-valued function (if exists) is mathematically significant, as it enables the direct determination of a fuzzy output for any real

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input and shows the behavior of the solution under variations of the inputs, which is the main motivation for this study. In this paper, we intend to obtain and derive analytical solutions formulas to problem (1) as fuzzy-number-valued functions. To this end, we employ the gH-differentiability concept, because it formulates derivative as a fuzzy-number-valued function and can therefore be directly applied to solve $FDEs$. Based on gH-differentiability, we study the problem (1) from two perspectives: 1. the derivative concept arising from the generalized Hukuhara difference (gH-difference) of type 1 (or (1)-differentiability) and, 2. the derivative concept arising from the gH-difference of type 2 (or (2)-differentiability). It is known that these derivatives provide two solutions to an FDE with different behavior, in the sense that with increasing time the fuzziness of the solution under (1)-differentiability is increasing and under (2)-differentiability is decreasing [4, 10, 11]. Then, it is possible to choose the desired solution from two the solutions [6]. We first study the uniqueness problem of solutions and present an interval in which the problem (1) has a unique solution. Next, we present the structure of positive solutions to problem (4) along with solving some illustrative examples in two separate cases: 1. (1)-differentiability when u_0 and v_0 are positive fuzzy numbers, and 2. (2)-differentiability when u_0 and v_0 are negative fuzzy numbers.

The structure of the paper is organized as follows: In Section 2, we review the basic concepts of fuzzy arithmetic and present some required results from the division of two trapezoidal fuzzy numbers. In the sequel, we introduce some properties of the Hausdorff metric, fuzzy-number-valued functions and fuzzy product functions and recall the (1) or (2)-differentiability concept of fuzzy-number-valued functions and their properties. In Section 3, we present the result of the uniqueness of the positive or negative solutions for problem (1), given by Theorem 3.2. In Section 4, we obtain the structure of positive solutions for some cases of problem (1) under certain conditions and explain the results by solving some examples.

2 Preliminaries

This section contains basic concepts, notations and essentially results that are required.

Definition 2.1. [4] *An ordered pair $u = (u^-, u^+)$ of functions $u^-, u^+ : [0,1] \to \mathbb{R}$, is called to be a fuzzy number defined on the real axis $\mathbb{R}$, if the following conditions hold:*
(i) u^-, called the lower branch of a fuzzy number, is a bounded non-decreasing left-continuous function on $(0,1]$, right continuous at zero,
(ii) u^+, called the upper branch fuzzy number, is a bounded non-increasing left-continuous function on $(0,1]$, right continuous at zero,
(iii) $u_1^- \le u_1^+$.
The set of all fuzzy numbers is denoted by $\mathbb{R}_F$ and for $u \in \mathbb{R}_F$, the representation $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, where $u_\alpha^* := u^*(\alpha)$, $* \in \{-,+\}$, is called the α-cut form of u.

The symbol $\mathbb{R}_F$ denotes the set of all fuzzy numbers defined on the real numbers $\mathbb{R}$. The diameter of $u \in \mathbb{R}_F$ is the function $diam(u)^{\cdot} : [0,1] \to [0, +\infty)$, defined as follows:

$$diam(u)^\alpha = u_\alpha^+ - u_\alpha^-.$$

For $u, v \in \mathbb{R}_F$ the operators addition, Hukuhara difference (H-difference), multiplication and division [17] are respectively defined in the α-cut form below:

$[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+], \quad \forall \alpha \in [0,1].$

$[u \ominus v]_\alpha = [u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+], \quad \forall \alpha \in [0,1].$

$[u \odot v]_\alpha = [\min M_\alpha, \max M_\alpha], \quad \forall \alpha \in [0,1],$

where

$$M_\alpha = \{u_\alpha^- v_\alpha^-, u_\alpha^- v_\alpha^+, u_\alpha^+ v_\alpha^-, u_\alpha^+ v_\alpha^+\}.$$

$\big[\frac{u}{v}\big]_\alpha = [\min D_\alpha, \max D_\alpha], \quad \forall \alpha \in [0,1],$

where

$$D_\alpha = \Big\{\frac{u_\alpha^-}{v_\alpha^-}, \frac{u_\alpha^-}{v_\alpha^+}, \frac{u_\alpha^+}{v_\alpha^-}, \frac{u_\alpha^+}{v_\alpha^+}\Big\},$$

provided that $v_\alpha^- > 0$ for all $\alpha \in [0,1]$ or $v_\alpha^+ < 0$ for all $\alpha \in [0,1]$. We point out that, if $\lambda \in \mathbb{R}$, then

$$[\lambda \odot u]_\alpha = \begin{cases} [\lambda u_\alpha^-, \lambda u_\alpha^+], & \lambda \geq 0, \\ [\lambda u_\alpha^+, \lambda u_\alpha^-], & \lambda < 0, \end{cases}.$$

It is noteworthy that the H-difference of two fuzzy numbers u and v may not exist and further $u \ominus v \neq u \oplus (-1) \odot v$. The following properties of H-difference are well-known
(*a*) $\lambda \odot (u \ominus v) = \lambda \odot u \ominus \lambda \odot v, \quad \forall \lambda \in \mathbb{R},\ u, v \in \mathbb{R}_F$,
(*b*) $(u \oplus v) \ominus w = (u \ominus w) \oplus v = (v \ominus w) \oplus u, \quad \forall u, v, w \in \mathbb{R}_F$,
(*c*) $(u \oplus v) \ominus (w \oplus z) = (u \ominus w) \oplus (v \ominus z), \quad \forall u, v, w, z \in \mathbb{R}_F$,
provided that, all the above H-differences exist (see [14, 22, 25], for more properties).

Definition 2.2. [35] *The Hausdorff metric (H-metric) on fuzzy numbers set is defined as the function* $D : \mathbb{R}_F \times \mathbb{R}_F \to [0, +\infty)$ *by the equation*

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} d_\alpha(u, v),$$

where

$$d_\alpha(u, v) = \max\left\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\right\},$$

is known as the H-metric between the two interval $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ *and* $[v]_\alpha = [v_\alpha^-, v_\alpha^+]$.

The following properties of H-metric are well-known [35]:
(*a*) $D(\lambda \odot u, \lambda \odot v) = |\lambda| D(u, v), \quad \forall\ \lambda \in \mathbb{R},\ u, v \in \mathbb{R}_F$,
(*b*) $D(u \oplus w, v \oplus w) = D(u, v), \quad \forall\ u, v, w \in \mathbb{R}_F$,
(*c*) $D(u \oplus v, w \oplus z) \leq D(u, w) + D(v, z), \quad \forall\ u, v, w, z \in \mathbb{R}_F$.

Lemma 2.3. [3, 21] *Let* $u, v, w, z \in \mathbb{R}_F$. *Then*
(*i*) $D(u \ominus v, u \ominus w) = D(v, w)$,
(*ii*) $D(u \ominus w, v \ominus z) \leq D(u, v) + D(w, z)$,
provided that the H-differences exist.

The space $(\mathbb{R}_F, D)$ is a complete metric space and the function $\|.\|_F : \mathbb{R}_F \to [0, +\infty)$, defined as $\|u\|_F = D(u, o)$, ($o \in \mathbb{R}_F$ is the well-known singleton zero fuzzy number whose $o_\alpha^- = o_\alpha^+ = 0$, for all $\alpha \in [0, 1]$), is a usual norm on $\mathbb{R}_F$, [4].

Definition 2.4. [13] *A fuzzy number u is said to be positive if* $u_\alpha^- > 0$, *for all* $\alpha \in [0, 1]$ *and it is negative if* $u_\alpha^+ < 0$, *for all* $\alpha \in [0, 1]$. *The notations* $\mathbb{R}_F^+$ *and* $\mathbb{R}_F^-$ *are used to show the set of positive and negative fuzzy numbers, respectively.*

According to Definitions 2.1 and 2.4, $u \in \mathbb{R}_F^+$, if $u_0^- > 0$ and $u \in \mathbb{R}_F^-$, if $u_0^+ < 0$.

Lemma 2.5. [13] *Let* $u \in \mathbb{R}_F^+ \cup \mathbb{R}_F^-$ *and* $v, w \in \mathbb{R}_F^+$ *or* $v, w \in \mathbb{R}_F^-$. *Then*

$$D(u \odot v, u \odot w) \leq \|u\|_F D(v, w).$$

Lemma 2.6. *Let* $u \in \mathbb{R}_F^+ \cup \mathbb{R}_F^-$ *and* $v, w, z \in \mathbb{R}_F^+$ *or* $v, w, z \in \mathbb{R}_F^-$. *Then*

$$D(u \odot v, w \odot z) \leq \|u\|_F D(v, w) + \|w\|_F D(u, z).$$

Proof. By using the properties of H-metric and Lemma 2.5, it simply follows that

$$\begin{aligned} D(u \odot v, w \odot z) &\leq D(u \odot v, u \odot w) + D(u \odot w, w \odot z) \\ &\leq \|u\|_F D(v, w) + \|w\|_F D(u, z). \end{aligned}$$

□

Lemma 2.7. *Let* $\{u_n\}$ *and* $\{v_n\}$ *be two sequences in* $\mathbb{R}_F$ *that respectively converge to u and v. Suppose that the H-differences* $u_n \ominus v_n$ *exist for every natural number* $n \geq 1$. *Then* $u \ominus v$ *exists and* $\lim_{n \to +\infty} u_n \ominus v_n = u \ominus v$.

Proof. See Lemma 2.6 in [22]. □

Definition 2.8. *We say that the fuzzy number* $u = (u^-, u^+)$ *is a regular fuzzy number if the functions* u^- *and* u^+ *are continuous at* $\alpha \in [0, 1]$ *and differentiable at* $\alpha \in (0, 1)$. *The set of all regular fuzzy numbers is denoted by* $\mathbb{R}_{Fr}$.

Throughout this paper, we employ the notation δf to show the derivative of a real-valued function f with respect to α.

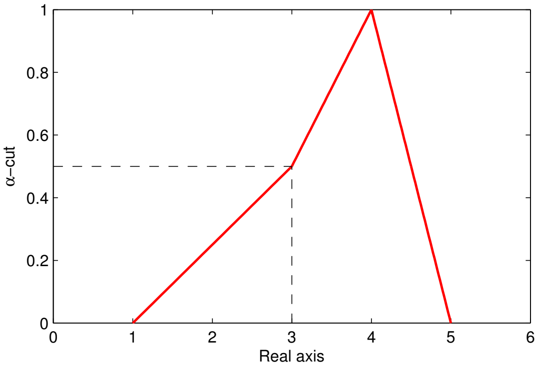

Figure 1: The fuzzy number u from Remark 2.10.

Remark 2.9. *Based on Definitions 2.1 and 2.8, the ordered pair $u = (u^-, u^+)$ is a regular fuzzy number if and only if the following conditions hold:*

- u^- *is continuous on* $[0,1]$*, differentiable on* $(0,1)$*, and* $\delta u^- \geq 0$,
- u^+ *is continuous on* $[0,1]$*, differentiable on* $(0,1)$*, and* $\delta u^+ \leq 0$,
- $diam(u)^1 \geq 0$.

In fact, if these conditions hold then, the differentiability of u^- and u^+, and that $\delta u^- \geq 0$ and $\delta u^+ \leq 0$, imply that u^- and u^+ are non-decreasing and non-increasing, respectively. Moreover, continuity on the compact interval $[0,1]$, ensures that u^- and u^+ are bounded. So that, u is a fuzzy number and it is regular. Reciprocally, if u is a regular fuzzy number, due to it is a fuzzy number, then $diam(u)^1 \geq 0$ holds. Also, u^- and u^+ are respectively non-decreasing and non-increasing which, considering the regularity of u, implies that $\delta u^- \geq 0$ and $\delta u^+ \leq 0$. Therefore, the designed conditions hold.

Remark 2.10. *Regular fuzzy numbers are used to represent many fuzzy quantities that are continuous and change gradually rather than abruptly, such as population growth, desired temperature, customer satisfaction, or appropriate light intensity. Since a logistic function exhibits smooth and gradual behavior, it is expected that its range in fuzzy modeling consists of regular fuzzy numbers. Another set of fuzzy numbers u, for which the function $\alpha \to [u]_\alpha$ is continuous under the H-metric, is introduced in [26] and denoted by $\mathbb{R}^c_F$. This set of fuzzy numbers is used to represent fuzzy quantities that are continuous. Because, the continuity of function $\alpha \to [u]_\alpha$ only requires the continuity of functions u^- and u^+. In fact, for $\alpha_0 \in [0,1]$ and sequence $\{\alpha_n\} \subseteq [0,1]$, we have*

$$d_H\big([u]_{\alpha_n}, [u]_{\alpha_0}\big) = \max\{|u^-_{\alpha_n} - u^-_{\alpha_0}|, |u^+_{\alpha_n} - u^+_{\alpha_0}|\} \to 0, \quad \textit{as} \quad \alpha_n \to \alpha_0,$$

if and only if $u^-_{\alpha_n} \to u^-_{\alpha_0}$ and $u^+_{\alpha_n} \to u^+_{\alpha_0}$, as $\alpha_n \to \alpha_0$. Consequently, $\mathbb{R}_{F_r} \subset \mathbb{R}^c_F$. But, the reverse of this inclusion relation does not hold. For example, consider the fuzzy number u with the α-cut form

$$[u]_\alpha = \begin{cases} [1+4\alpha, 5-\alpha], & 0 \leq \alpha \leq \frac{1}{2}, \\ [2+2\alpha, 5-\alpha]; & \frac{1}{2} \leq \alpha \leq 1, \end{cases}$$

as given in Figure 1. We see that $u \in \mathbb{R}^c_F$, but the function u^- is not differentiable at the point $\alpha = \frac{1}{2}$. Thus, $u \notin \mathbb{R}_{F_r}$.

A family of regular fuzzy numbers is well-known as trapezoidal fuzzy numbers, denoted by TR_F,. The fuzzy number $u = (u^-, u^+)$ is said to be trapezoidal whenever $u^-_\alpha = u^l + (u^{cl} - u^l)\alpha$ and $u^+_\alpha = u^r - (u^r - u^{cr})\alpha$, where $u^l \leq u^{cl} \leq u^{cr} \leq u^r$. Such a number is represented as $u = (u^l, u^{cl}, u^{cr}, u^r)$. In the case that $u^{cl} = u^{cr}$, then the trapezoidal fuzzy number u is called a triangular fuzzy number and represented as $u = (u^l, u^c, u^r)$, where $u^c = u^{cl}$, [4].

The following lemma is needed for some results in the paper.

Lemma 2.11. *Let $u = (u^-, u^+) \in T\mathbb{R}_F$ and $v = (v^-, v^+) \in T\mathbb{R}^-_F \cup T\mathbb{R}^+_F$ be two fuzzy numbers with the representations $u = (u^l, u^{cl}, u^{cr}, u^r)$ and $v = (v^l, v^{cl}, v^{cr}, v^r)$. Assume that the following inequalities hold*

(h_1) $u^l v^{cl} \leq v^l u^{cl}$,
(h_2) $v^r u^{cr} \leq u^r v^{cr}$,
(h_3) $u^{cl} v^{cr} \leq u^{cr} v^{cl}$.

Then $w = \left(\frac{u^-}{v^-}, \frac{u^+}{v^+}\right)$ is defined as a fuzzy number.

Proof. Let $w^-_\alpha = \frac{u^-_\alpha}{v^-_\alpha}$ and $w^+_\alpha = \frac{u^+_\alpha}{v^+_\alpha}$. Since the functions

$$u^-_\alpha = u^l + (u^{cl} - u^l)\alpha, \quad u^+_\alpha = u^r - (u^r - u^{cr})\alpha,$$

$$v_\alpha^- = v^l + (v^{cl} - v^l)\alpha, \quad v_\alpha^+ = v^r - (v^r - v^{cr})\alpha,$$

are linear on the compact interval $[0,1]$, then they are bounded and continuous on $[0,1]$. Also, the hypothesis $v \in T\mathbb{R}_F^- \cup T\mathbb{R}_F^+$, ensures that $v_\alpha^- \neq 0$ and $v_\alpha^+ \neq 0$, for all $\alpha \in [0,1]$. Therefore, the functions w^- and w^+ are bounded and continuous on $[0,1]$. Moreover, since the functions u^-, u^+, v^-, and v^+ are differentiable on $(0,1)$, we get

$$\begin{aligned} \delta w_\alpha^- &= \delta\Big(\frac{u^l + (u^{cl} - u^l)\alpha}{v^l + (v^{cl} - v^l)\alpha}\Big) \\ &= \frac{u^{cl}v^l - u^l v^{cl}}{\big(v^l + (v^{cl} - v^l)\alpha\big)^2} \geq 0, \quad \text{(by the condition } (h_1)), \end{aligned}$$

and

$$\begin{aligned} \delta w_\alpha^+ &= \delta\Big(\frac{u^r - (u^r - u^{cr})\alpha}{v^r - (v^r - v^{cr})\alpha}\Big) \\ &= \frac{u^{cr}v^r - u^r v^{cr}}{(v^r - (v^r - v^{cr})\alpha)^2} \leq 0, \quad \text{(by the condition } (h_2)). \end{aligned}$$

These inequalities ensure that w_α^- and w_α^+ are non-decreasing and non-increasing, respectively. So that, for each $\alpha \in [0,1]$, we get

$$\begin{aligned} w_\alpha^- \leq w_1^- &= \frac{u^{cl}}{v^{cl}} = \frac{u^{cl}v^{cr}}{v^{cl}v^{cr}} \\ &\leq \frac{u^{cr}v^{cl}}{v^{cl}v^{cr}} \quad \text{(by condition } (h_3)) \\ &= \frac{u^{cr}}{v^{cr}} = w_1^+ \leq w_\alpha^+. \end{aligned}$$

Consequently, the statement of the lemma is concluded. □

Remark 2.12. *If $u, v \in T\mathbb{R}_F^+$ or $u, v \in T\mathbb{R}_F^-$, then the inequalities $(h_1) - (h_3)$ from Lemma 2.11 can be ordered as follows:*

$$\frac{u^l}{v^l} \leq \frac{u^{cl}}{v^{cl}} \leq \frac{u^{cr}}{v^{cr}} \leq \frac{u^r}{v^r}. \tag{2}$$

In the case that $u = (u^l, u^c, u^r)$ and $v = (v^l, v^c, v^r)$ are two triangular fuzzy numbers, then the condition (h_3) is fulfilled. Hence, the inequalities (2) are summarized as follows:

$$\frac{u^l}{v^l} \leq \frac{u^c}{v^c} \leq \frac{u^r}{v^r}. \tag{3}$$

However, the satisfaction of the inequalities (3) does not guarantee that the fuzzy number $w = (\frac{u^-}{v^-}, \frac{u^+}{v^+})$ is a triangular fuzzy number. For example, it is easily seen that two positive triangular fuzzy numbers $u = (2, 6, 12)$ and $v = (1, 2, 3)$ satisfy the condition (3). So, we have the non-triangular fuzzy number w, given in the α-cut form below:

$$[w]_\alpha = \left[\frac{2 + 4\alpha}{1 + \alpha}, \frac{12 - 6\alpha}{3 - \alpha}\right], \ \forall \alpha \in [0,1].$$

Definition 2.13. *Let I be a subset of real numbers. A function $f : I \to \mathbb{R}_F$ is called the truly fuzzy-number-valued function on I, if there exists at least one $t \in I$, such that $f(t)$ is a non-real fuzzy number.*

The distance of two the truly fuzzy-number-valued functions $f, g : I \to \mathbb{R}_F$ is defined as follows:

$$D^*(f, g) = \sup_{t \in I} D(f(t), g(t)).$$

Definition 2.14. *Let I be a subset of real numbers. A truly fuzzy-number-valued function f is called to be the truly fuzzy product function on I, whenever $f(t)$ can be expressed as the multiplication of truly fuzzy-number-valued functions, for all $t \in I$.*

Example 2.15. *Consider the truly fuzzy-number-valued function* $f : [0, +\infty) \to \mathbb{R}_F$ *given in the* α*-cut form:*

$$[f(t)]_\alpha = \left[\frac{\alpha e^{\alpha t}}{1+e^{\alpha t}}, \frac{e^t}{1+e^t}\right], \quad \forall \alpha \in [0,1].$$

Take two functions g *and* h*, given as:*

$$[g(t)]_\alpha = \left[\frac{\alpha}{1+e^{\alpha t}}, \frac{1}{1+e^t}\right], \quad \text{and} \quad [h(t)]_\alpha = \left[e^{\alpha t}, e^t\right], \quad \forall \alpha \in [0,1].$$

It is easy to observe that g *and* h *are positive truly fuzzy-number-valued functions defined on intervals* $[0,1]$ *and* $[0,+\infty)$*, respectively. Since,* $f(t) = g(t) \odot h(t)$*, for each* $t \in [0,1]$*, then the restriction of* f *to* $[0,1]$ *is a truly fuzzy product function.*

Definition 2.16. [5, 10] *Let* f *be a fuzzy-number-valued function defined on open interval* $I \subset \mathbb{R}$*. Fix* $t_0 \in I$*. (1) The function* f *is called to be* (1)*-differentiable at* t_0*, if the H-differences* $f(t_0+h) \ominus f(t_0)$ *and* $f(t_0) \ominus f(t_0-h)$*, for all* $h > 0$*, sufficiently close to* 0 *exist, and an element* $D_1 f(t_0) \in \mathbb{R}_F$ *exists, such that*

$$\lim_{h \to 0^+} \frac{f(t_0+h) \ominus f(t_0)}{h} = \lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0-h)}{h} = D_1 f(t_0).$$

(2) *The function* f *is called to be* (2)*-differentiable at* t_0*, if the H-differences* $f(t_0-h) \ominus f(t_0)$ *and* $f(t_0) \ominus f(t_0+h)$*, for all* $h > 0$*, sufficiently close to* 0 *exist, and an element* $D_2 f(t_0) \in \mathbb{R}_F$ *exists, such that*

$$\lim_{h \to 0^+} \frac{f(t_0) \ominus f(t_0+h)}{-h} = \lim_{h \to 0^+} \frac{f(t_0-h) \ominus f(t_0)}{-h} = D_2 f(t_0).$$

Theorem 2.17. [7, 10] *Let* $I \subset \mathbb{R}$ *be an open interval and consider the function* $f : I \to \mathbb{R}_F$ *with* $[f(t)]_\alpha = [f_\alpha^-(t), f_\alpha^+(t)]$*,* $\forall \alpha \in [0,1]$*. In this case,*
(1) *if* f *is* (1)*-differentiable on* I*, then* $f_\alpha^-(t)$ *and* $f_\alpha^+(t)$ *are differentiable functions at* $t \in I$*, and*

$$[D_1 f(t)]_\alpha = \left[{f_\alpha^-}'(t), {f_\alpha^+}'(t)\right], \ \forall \alpha \in [0,1].$$

(2) *if* f *is* (2)*-differentiable on* I*, then* $f_\alpha^-(t)$ *and* $f_\alpha^+(t)$ *are differentiable functions at* $t \in I$*, and*

$$[D_2 f(t)]_\alpha = \left[{f_\alpha^+}'(t), {f_\alpha^-}'(t)\right], \ \forall \alpha \in [0,1].$$

Theorem 2.18. *Let* $I \subset \mathbb{R}$ *be an open interval and consider the function* $f : I \to \mathbb{R}_F$ *with* $[f(t)]_\alpha = [f_\alpha^-(t), f_\alpha^+(t)]$*,* $\forall \alpha \in [0,1]$*. Suppose that* $f_\alpha^-(t)$ *and* $f_\alpha^+(t)$ *are differentiable with respect to* t*, uniformly with respect to* $\alpha \in [0,1]$*. In this case:*
(1) *if for each* $t \in I$*, the functions* ${f_\alpha^-}'(t)$ *and* ${f_\alpha^+}'(t)$*, respectively are non-decreasing and non-increasing as functions of* α*, and* ${f_1^-}'(t) \leq {f_1^+}'(t)$*, then* f *is* (1)*-differentiable on* I*,*
(2) *if for each* $t \in I$*, the functions* ${f_\alpha^-}'(t)$ *and* ${f_\alpha^+}'(t)$*, respectively are non-increasing and non-decreasing as functions of* α*, and* ${f_1^+}'(t) \leq {f_1^-}'(t)$*, then* f *is* (2)*-differentiable on* I*.*

Proof. See Theorem 24 in [7]. □

3 The uniqueness of solutions

Let $k \in \{1, 2\}$ be fixed. We intend to provide the uniqueness of positive or negative solutions to the problem (1), considered below:

$$\begin{cases} D_k x(t) \oplus v_0 \odot x^2(t) = u_0 \odot x(t), & t > 0, \\ x(0) = x_0, \end{cases} \tag{4}$$

where $u_0, v_0, x_0 \in \mathbb{R}_F^+ \cup \mathbb{R}_F^-$.

Definition 3.1. *Let* $T > 0$ *and* $k \in \{1, 2\}$ *be fixed. We say that the fuzzy-number-valued function* x *is* (k)*-solution to problem* (4) *on* $[0, T]$*, if* x *is continuous on* $[0, T]$*,* (k)*-differentiable on* $(0, T)$ *and it satisfies the problem* (4)*, for all* $t \in (0, T)$*.*

Theorem 3.2. *Fix $T > 0$ and $k \in \{1,2\}$. Suppose that the problem (4) has (k)-solution $x : [0,T] \to \mathbb{R}_F^+$ (or $x : [0,T] \to \mathbb{R}_F^-$), then x is unique on $[0,\tau]$, for some $0 < \tau \leq T$.*

Proof. Let $k = 1$ and $x : [0,T] \to \mathbb{R}_F^+$ be (1)-solution to problem (4). The proof is similarly for other cases of choosing k and the sign of x. Since x is the (1)-solution to (4), so, it is continuous on $[0,T]$, which, by Proposition 2.3, (ii) in [13], implies that $M_x > 0$ exists such that $\|x(t)\|_F < M_x$, $\forall t \in [0,T]$. Also, the H-differences $u_0 \odot x(t) \ominus v_0 \odot x^2(t)$ exist and equal to $D_1x(t)$, for each $t \in (0,T)$. So that, the set

$$S_{M_x} = \big\{u \in \mathbb{R}_F^+ : \ \|u\|_F < M_x, \text{ and the } H\text{-difference } \ u_0 \odot u \ominus v_0 \odot u^2 \text{ exists}\big\}.$$

is a non-empty bounded subset of $\mathbb{R}_F^+$. Define the function $F : S_{M_x} \to \mathbb{R}_F$ by

$$F(u) = u_0 \odot u \ominus v_0 \odot u^2.$$

We show that F is a continuous function. Let the sequence $(u_n)_{n\geq 1} \subset S_{M_x}$ converge to u. Since $u_0, v_0 \in \mathbb{R}_F^+ \cup \mathbb{R}_F^-$ and $u_n \in \mathbb{R}_F^+$ so, by Theorem 3.5 in [13], the sequences $u_0 \odot u_n$ and $v_0 \odot u_n^2$ converge to $u_0 \odot u$ and $v_0 \odot u^2$, respectively. Therefore, based on Lemma 2.7, the H-difference $u_0 \odot u \ominus v_0 \odot u^2$ exists and we have

$$\lim_{n\to+\infty} F(u_n) = u_0 \odot u \ominus v_0 \odot u^2.$$

In addition, since $u_n \in \mathbb{R}_F^+$ and $\|u_n\|_F < M_x$, then

$$0 < u_{n,\alpha}^- < u_{n,\alpha}^+ < M_x,$$

uniformly with respect to $\alpha \in [0,1]$. So, passing to the limit, it follows that

$$0 < u_\alpha^- < u_\alpha^+ < M_x,$$

uniformly with respect to $\alpha \in [0,1]$, which result in $u \in \mathbb{R}_F^+$ and $\|u\|_F < M_x$. Consequently, $u \in S_{M_x}$ and

$$\lim_{n\to+\infty} F(u_n) = F(u),$$

which gives us the continuity of F. Now, assume that $y : [0,T] \to \mathbb{R}_F^+$ is another (1)-solution to problem (4). Similar to solution x, $M_y > 0$ exists such that $\|y(t)\|_F < M_y$, $\forall t \in [0,T]$. Take

$$0 < \tau < \min\Big\{T, \frac{1}{M}\Big\},$$

where $M = \|u_0\|_F + (M_x + M_y)\|v_0\|_F$. Since x is continuous on $[0,\tau]$ and $x([0,\tau]) \subset S_M$, and F is continuous on S_M, then the combination function $F(x(t))$ is continuous on $[0,\tau]$. By Proposition 2.3, (i) in [13], $F(x(t))$ is an integrable function. Therefore, under (1)-differentiability, we get

$$x(t) = x_0 \oplus \int_0^t F(x(s))ds, \quad \forall t \in [0,\tau].$$

Analogously,

$$y(t) = x_0 \oplus \int_0^t F(y(s))ds, \quad \forall t \in [0,\tau].$$

We now show that $x(t) = y(t)$, $\forall t \in [0,\tau]$. Let us suppose the contrary, i.e., $D(x(t_1), y(t_1)) > 0$, for some $t_1 \in (0,\tau]$.

This implies that $D^*(x,y)>0$. For arbitrary $t\in(0,\tau]$, we obtain

$$\begin{aligned}
D(x(t),y(t)) &= D\big(x_0\oplus\int_0^t F(x(s))ds, x_0\oplus\int_0^t F(y(s))ds\big)\\
&= D\Big(\int_0^t F(x(s))ds, \int_0^t F(y(s))ds\Big)\\
&\le \int_0^t D(F(x(s)),F(y(s)))ds\\
&= \int_0^t D(u_0\odot x(s)\ominus v_0\odot x^2(s), u_0\odot y(s)\ominus v_0\odot y^2(s))ds\\
&\le \int_0^t \big(D(u_0\odot x(s),u_0\odot y(s))+D(v_0\odot x^2(s),v_0\odot y^2(s))\big)ds\\
&\le \int_0^t \big(\|u_0\|_F+\|v_0\|_F\|x(s)\|_F+\|v_0\|_F\|y(s)\|_F\big)D(x(s),y(s))ds \text{ (by Lemmas 2.5 and 2.6)}\\
&\le M\int_0^t D(x(s),y(s))ds\\
&\le M\int_0^\tau D(x(s),y(s))ds\\
&\le \tau M D^*(x,y).
\end{aligned}$$

Therefore, $D^*(x,y)\le\tau MD^*(x,y)$. Considering $D^*(x,y)>0$, it follows that $\tau M-1\ge 0$, which is in clear contradiction with choosing τ. Consequently, the proof is completed. □

4 The structure of positive solutions

Because of the continuity of (k)-solutions ($k=1$ or 2), one has the same sign as x_0 in a neighborhood of zero. The solution of a logistic differential equation usually represents population or concentration, which naturally cannot take negative values. Accordingly, in this section, assuming that $x_0\in\mathbb{R}_F^+$, we demonstrate the structure of positive (k)-solutions to problem (4) in two separate cases: 1. (1)-solution when $u_0,v_0\in\mathbb{R}_F^+$ and 2. (2)-solution when $u_0,v_0\in\mathbb{R}_F^-$. In fuzzy modelled problems similar to crisp models, it may be expected to represent the solution in a closed form consisting of fuzzy-number-valued functions and fuzzy arithmetic operators, not just specifying the upper and lower branches of the solution. Accordingly, in each one of the above-mentioned cases, we first show that under certain conditions the solution function can be expressed as a truly fuzzy product function.

Case 1: (1)-solution when $\mathbf{u_0},\mathbf{v_0}\in\mathbb{R}_\mathbf{F}^+$

Theorem 4.1. *Suppose that $x_0,u_0,v_0\in\mathbb{R}_{Fr}^+$ are such that $c=(c^-,c^+)$, given by*

$$c_\alpha^-=\frac{x_{0,\alpha}^-}{u_{0,\alpha}^--v_{0,\alpha}^-x_{0,\alpha}^-},\quad \text{and}\quad c_\alpha^+=\frac{x_{0,\alpha}^+}{u_{0,\alpha}^+-v_{0,\alpha}^+x_{0,\alpha}^+}, \tag{5}$$

is defined as a positive fuzzy number. Suppose that there is $\tau>0$, such that $y(t)=\big(y^-(t),y^+(t)\big)$, given by

$$y_\alpha^-(t)=\frac{u_{0,\alpha}^-}{1+c_\alpha^-v_{0,\alpha}^-e^{u_{0,\alpha}^-t}},\quad \text{and}\quad y_\alpha^+(t)=\frac{u_{0,\alpha}^+}{1+c_\alpha^+v_{0,\alpha}^+e^{u_{0,\alpha}^+t}}, \tag{6}$$

is defined as a truly fuzzy-number-valued function on interval $I=[0,\tau]$. Then the problem (4) has the positive (1)-solution, given by

$$x(t)=c\odot y(t)\odot e^{t\odot u_0},\quad \forall t\in I, \tag{7}$$

where $e^{t\odot u_0}=\big(e^{u_0^-t},e^{u_0^+t}\big)$.

Proof. Since c described by (5) belongs to $\mathbb{R}_F^+$, so, by the hypothesis $x_0,u_0,v_0\in\mathbb{R}_{Fr}^+$, it has $c\in\mathbb{R}_{Fr}^+$. Let $x(t)=\big(x^-(t),x^+(t)\big)$. From (7), we have

$$x_\alpha^-(t)=\frac{c_\alpha^-u_{0,\alpha}^-e^{u_{0,\alpha}^-t}}{1+c_\alpha^-v_{0,\alpha}^-e^{u_{0,\alpha}^-t}}, \tag{8}$$

$$x_\alpha^+(t)=\frac{c_\alpha^+u_{0,\alpha}^+e^{u_{0,\alpha}^+t}}{1+c_\alpha^+v_{0,\alpha}^+e^{u_{0,\alpha}^+t}}. \tag{9}$$

By replacing (5) into (8) and (9), it easily follows that $x_\alpha^-(0)=x_{0,\alpha}^-$ and $x_\alpha^+(0)=x_{0,\alpha}^+$ which means that x fulfills the initial condition of problem (4). We now show that $x(t)$ is (1)-differentiable on the interval $(0,\tau)$. By using (6) and (8), we obtain that

$$\begin{aligned}u_{0,\alpha}^-&=\frac{u_{0,\alpha}^-}{1+c_\alpha^-v_{0,\alpha}^-e^{u_{0,\alpha}^-t}}+\frac{v_{0,\alpha}^-c_\alpha^-u_{0,\alpha}^-e^{u_{0,\alpha}^-t}}{1+c_\alpha^-v_{0,\alpha}^-e^{u_{0,\alpha}^-t}}\\&=y_\alpha^-(t)+v_{0,\alpha}^-x_\alpha^-(t)\\&=\big(y(t)\oplus v_0\odot x(t)\big)_\alpha^-,\end{aligned}$$

and similarly from (6) and (9), we get

$$u_{0,\alpha}^+=\big(y(t)\oplus v_0\odot x(t)\big)_\alpha^+.$$

So that

$$y(t)\oplus v_0\odot x(t)=u_0.$$

This means that the H-differences $u_0\ominus v_0\odot x(t)$ exist and equal to $y(t)$, for each $t\in I$. Using this and that for each $\alpha\in[0,1]$, the function $x_\alpha^-(t)$ given in (8), is the solution of the following logistic differential equation:

$$\begin{cases}(x_\alpha^-)'(t)+v_{0,\alpha}^-(x_\alpha^-(t))^2=u_{0,\alpha}^-x_\alpha^-(t), & t\in(0,\tau),\\ x_\alpha^-(0)=x_{0,\alpha}^-,\end{cases} \tag{10}$$

we obtain

$$\begin{aligned}\delta(x_\alpha^-)'(t)&=x_\alpha^-(t)\delta u_{0,\alpha}^-+u_{0,\alpha}^-\delta x_\alpha^-(t)-\big(x_\alpha^-(t)\big)^2\delta v_{0,\alpha}^--2v_{0,\alpha}^-x_\alpha^-(t)\delta x_\alpha^-(t)\\&=x_\alpha^-(t)\big(\delta u_{0,\alpha}^--x_\alpha^-(t)\delta v_{0,\alpha}^--v_{0,\alpha}^-\delta x_\alpha^-(t)\big)+\big(u_{0,\alpha}^--v_{0,\alpha}^-x_\alpha^-(t)\big)\delta x_\alpha^-(t)\\&=x_\alpha^-(t)\big(\delta(u_0\ominus v_0\odot x(t))_\alpha^-\big)+(u_0\ominus v_0\odot x(t))_\alpha^-\delta x_\alpha^-(t)\\&=x_\alpha^-(t)\delta y_\alpha^-(t)+y_\alpha^-(t)\delta x_\alpha^-(t)\\&=\delta(x\odot y)_\alpha^-(t),\end{aligned}$$

which implies $\delta(x_\alpha^-)'(t)\geq 0$. Similarly, since the function $x_\alpha^+(t)$ given in (9), is the solution of the following logistic differential equation:

$$\begin{cases}(x_\alpha^+)'(t)+v_{0,\alpha}^+(x_\alpha^+(t))^2=u_{0,\alpha}^+x_\alpha^+(t), & t\in(0,\tau),\\ x_\alpha^+(0)=x_{0,\alpha}^+,\end{cases} \tag{11}$$

we obtain

$$\delta(x_\alpha^+)'(t)=\delta(x\odot y)_\alpha^+(t)\leq 0.$$

Also, by (10) and (11), we get

$$\begin{aligned}diam(D_1x(t))^1&=(x_1^+)'(t)-(x_1^-)'(t)\\&=u_{0,1}^+x_1^+(t)-v_{0,1}^+(x_1^+(t))^2-u_{0,1}^-x_1^-(t)+v_{0,1}^-(x_1^-(t))^2\\&=x_1^+(t)(u_{0,1}^+-v_{0,1}^+x_1^+(t))-x_1^-(t)(u_{0,1}^--v_{0,1}^-x_1^-(t))\\&=x_1^+(t)y_1^+(t)-x_1^-(t)y_1^-(t)\\&=diam\big((x\odot y)(t)\big)^1\geq 0.\end{aligned}$$

Therefore, by Theorem 2.18, the function x is (1)-differentiable on $(0,\tau)$ and $D_1x(t)=\big((x^-)'(t),(x^+)'(t)\big)$. Finally, since $x_\alpha^-(t)$ and $x_\alpha^+(t)$ satisfy the equations (10) and (11), then for fixed arbitrary $t\in(0,\tau)$, we have

$$\begin{aligned}&D\big(D_1x(t)\oplus v_0\odot x^2(t),u_0\odot x(t)\big)\\&\quad=\sup_{\alpha\in[0,1]}\max\big\{|(x_\alpha^-)'(t)+v_{0,\alpha}^-(x_\alpha^-(t))^2-u_{0,\alpha}^-x_\alpha^-(t)|,|(x_\alpha^+)'(t)+v_{0,\alpha}^+(x_\alpha^+(t))^2-u_{0,\alpha}^+x_\alpha^+(t)|\big\}\\&\quad=0.\end{aligned}$$

Considering t is arbitrary, then

$$D^*\big(D_1x \oplus v_0 \odot x^2, u_0 \odot x\big) = 0.$$

Consequently, x is satisfies the equation (4), and this completes the proof. $\square$

Example 4.2. *Consider the following problem*

$$\begin{cases} D_1x(t) \oplus \frac{1}{8} \odot x^2(t) = (\frac{1}{4}, \frac{1}{2}, \frac{3}{4}) \odot x(t), & t > 0, \\ x(0) = (1, 2, 3). \end{cases} \tag{12}$$

Here $x_0 = (1, 2, 3)$, $u_0 = (\frac{1}{4}, \frac{1}{2}, \frac{3}{4})$ *and* $v_0 = \frac{1}{8}$. *From* (5), *it is easy to observe that* $c_\alpha^- = c_\alpha^+ = 8$, $\forall \alpha \in [0, 1]$. *So, by* (6), *we get*

$$y_\alpha^-(t) = \frac{1+\alpha}{4 + 4e^{(\frac{1+\alpha}{4})t}},$$

and

$$y_\alpha^+(t) = \frac{3-\alpha}{4 + 4e^{(\frac{3-\alpha}{4})t}},$$

for each $\alpha \in [0, 1]$. *We look for* $\tau > 0$ *such that* $y(t)$ *with* $[y(t)]_\alpha = \big[y_\alpha^-(t), y_\alpha^+(t)\big]$, $\forall \alpha \in [0, 1]$, *be a positive truly fuzzy-number-valued function defined on the interval* $[0, \tau]$. *For this, we check the conditions introduced by Remark 2.9. It can be observed that*

$$diam(y(t))^1 = y_1^+(t) - y_1^-(t) = 0.$$

We then should have

$$\delta y_\alpha^-(t) = \frac{e^{(\frac{1+\alpha}{4})t}}{\big(4 + 4e^{(\frac{1+\alpha}{4})t}\big)^2}\big(4 - t(1+\alpha) + 4e^{-(\frac{1+\alpha}{4})t}\big) \geq 0,$$

and

$$\delta y_\alpha^+(t) = -\frac{e^{(\frac{3-\alpha}{4})t}}{\big(4 + 4e^{(\frac{3-\alpha}{4})t}\big)^2}\big(4 - t(3-\alpha) + 4e^{-(\frac{3-\alpha}{4})t}\big) \leq 0,$$

which imply that

$$4 - t(1+\alpha) + 4e^{-(\frac{1+\alpha}{4})t} \geq 0, \tag{13}$$

and

$$4 - t(3-\alpha) + 4e^{-(\frac{3-\alpha}{4})t} \geq 0. \tag{14}$$

Since, for fixed $t > 0$, *we have*

$$\min_{\alpha \in [0,1]} \big\{4 - t(1+\alpha) + 4e^{-(\frac{1+\alpha}{4})t}\big\} = 4 - 2t + 4e^{-\frac{1}{2}t},$$

and

$$\min_{\alpha \in [0,1]} \big\{4 - t(3-\alpha) + 4e^{-(\frac{3-\alpha}{4})t}\big\} = 4 - 3t + 4e^{-\frac{3}{4}t}.$$

So, it is sufficient that inequalities $4 - 2t + 4e^{-\frac{1}{2}t} \geq 0$ *and* $4 - 3t + 4e^{-\frac{3}{4}t} \geq 0$ *hold simultaneously. By the Matlab software, it follows that* $0 \leq t \leq 1.7$. *Consequently, we set* $\tau = 1.7$, *and the problem* (12) *has* (1)*-solution on the interval* $[0, 1.7]$ *given as follows:*

$$[x(t)]_\alpha = \left[\frac{2(1+\alpha)}{1 + e^{-(\frac{1+\alpha}{4})t}}, \frac{2(3-\alpha)}{1 + e^{-(\frac{3-\alpha}{4})t}}\right], \quad \forall \alpha \in [0, 1].$$

The graphic of this solution is given in Figure 2, which confirms the behavior of a (1)*-differentiable function, because its dimension increases with respect to time.*

The following example shows that the conditions of Theorem 4.1 do not hold for a simple family of problems (4) in which only the initial value is a non-real fuzzy number.

Example 4.3. *Suppose that* $p, q > 0$ *are two real numbers and* $x_0 \in \mathbb{R}_{Fr}^+$ *is a non-real fuzzy number. Consider the following problem*

$$\begin{cases} D_1x(t) \oplus p \odot x^2(t) = q \odot x(t), & t > 0, \\ x(0) = x_0. \end{cases} \tag{15}$$

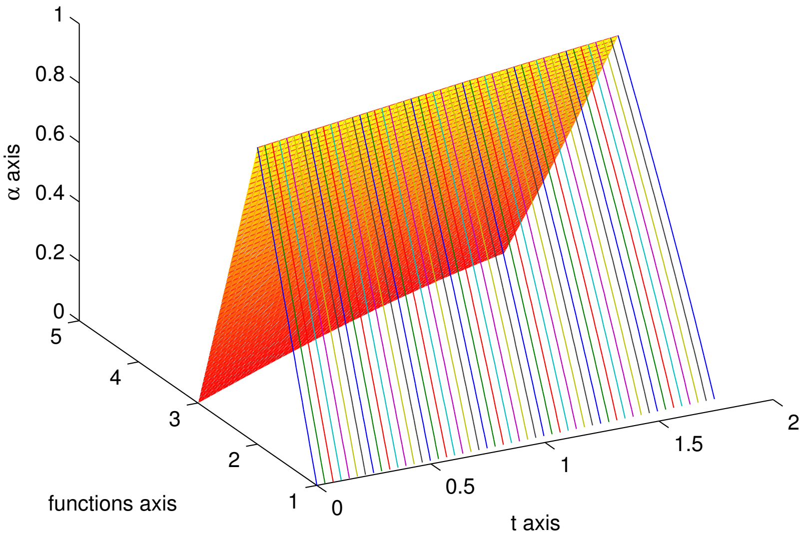

Figure 2: The solution function $x(t)$ of Example 12 where the diagonal lines represent function $x^-_\alpha(t)$ and the coloured graph represent function $x^+_\alpha(t)$.

Here $u_0 = q$ and $v_0 = p$ are positive real numbers. From (5), we get

$$c = \Big(\frac{x_0^-}{q - px_0^-}, \frac{x_0^+}{q - px_0^+}\Big).$$

So, if $q - px^+_{0,1} \le 0$, then c does not define a fuzzy number. Let us assume that $q - px^+_{0,1} > 0$. Then we have $0 < q - px^+_{0,1} \le q - px^-_{0,1}$. Using this and that $0 < x^-_{0,1} \le x^+_{0,1}$, it follows that $c^-_1 \le c^+_1$. Moreover, since $\delta c^-_\alpha = \frac{q\delta x^-_{0,\alpha}}{(q-px^-_{0,\alpha})^2} \ge 0$ and $\delta c^+_\alpha = \frac{q\delta x^+_{0,\alpha}}{(q-px^+_{0,\alpha})^2} \le 0$, then c is defined as a positive fuzzy number. However, by (6), we have

$$y^-_\alpha(t) = \frac{q}{1 + c^-_\alpha p e^{qt}}, \quad \text{and} \quad y^+_\alpha(t) = \frac{q}{1 + c^+_\alpha p e^{qt}},$$

which do not form a fuzzy-number-valued function, because $\delta y^-_\alpha(t) = -\frac{pqe^{qt}\delta c^-_\alpha}{\left(1+c^-_\alpha pe^{qt}\right)^2} \le 0$.

Remark 4.4. *The conditions of Theorem 4.1 guarantee the (1)-differentiability of the function $x(t)$ with the product representation of (7). Also, $D_1x(t)$ is the truly fuzzy product function because, from the proof of Theorem 4.1, we get*

$$D_1x(t) = x(t) \odot y(t).$$

However, according to Example 4.3, the function $y(t)$ may not exist. This shortcoming may be fixed in the next theorem, which is a direct result of Theorem 4.1.

Theorem 4.5. *Let $x_0, u_0, v_0 \in \mathbb{R}^+_{Fr}$. Suppose that $\tau > 0$ exists such that $x(t) = \big(x^-(t), x^+(t)\big)$, given by*

$$x^-_\alpha(t) = \frac{x^-_{0,\alpha} u^-_{0,\alpha} e^{u^-_{0,\alpha} t}}{u^-_{0,\alpha} + v^-_{0,\alpha} x^-_{0,\alpha}(e^{u^-_{0,\alpha} t} - 1)}, \tag{16}$$

and

$$x^+_\alpha(t) = \frac{x^+_{0,\alpha} u^+_{0,\alpha} e^{u^+_{0,\alpha} t}}{u^+_{0,\alpha} + v^+_{0,\alpha} x^+_{0,\alpha}(e^{u^+_{0,\alpha} t} - 1)}, \tag{17}$$

is a truly fuzzy-number-valued function defined on $[0, \tau]$ and is (1)-differentiable on $(0, \tau)$. Then, $x(t)$ is (1)-solution of (4).

Example 4.6. *Consider the problem* (15) *in the special case* $p = 0.1$, $q = 1$ *and* $x_0 = (0.01, 0.05, 0.09)$. *From* (16) *and* (17), *we get*

$$x^-_\alpha(t) = \frac{10(1+4\alpha)e^t}{1000 + (1+4\alpha)(e^t - 1)},$$

$$x^+_\alpha(t) = \frac{10(9-4\alpha)e^t}{1000 + (9-4\alpha)(e^t - 1)}.$$

It is easy to check that $x(t) = \big(x^-(t), x^+(t)\big)$ *is a truly fuzzy-number-valued function on* $[0, +\infty)$. *For checking* (1)*-differentiability of* x, *we use Theorem 2.18. So, we consider the following functions*

$$(x^-_\alpha)'(t) = \frac{10000(1+4\alpha)e^t - 10(1+4\alpha)^2 e^t}{(1000 + (1+4\alpha)(e^t - 1))^2},$$

$$(x^+_\alpha)'(t) = \frac{10000(9-4\alpha)e^t - 10(9-4\alpha)^2 e^t}{(1000 + (9-4\alpha)(e^t - 1))^2}.$$

It can be observed that $(x^+_1)'(t) - (x^-_1)'(t) = 0$, *for each* $t > 0$. *Also, after simple calculations, we obtain*

$$\delta(x^-_\alpha)'(t) = \frac{40000e^t}{(1000 + (1+4\alpha)(e^t - 1))^3}\big(1000 - (1+4\alpha)(e^t + 1)\big),$$

$$\delta(x^+_\alpha)'(t) = \frac{-40000e^t}{(1000 + (9-4\alpha)(e^t - 1))^3}\big(1000 - (9-4\alpha)(e^t + 1)\big).$$

Then, we should have simultaneously

$$1000 - (1+4\alpha)(e^t + 1) \geq 0,$$

and

$$1000 - (9-4\alpha)(e^t + 1) \geq 0,$$

for all $\alpha \in [0, 1]$. *Since, for* $t > 0$ *fixed, we have*

$$\min_{\alpha\in[0,1]}\{1000 - (1+4\alpha)(e^t + 1)\} = 1000 - 5(e^t + 1),$$

and

$$\min_{\alpha\in[0,1]}\{1000 - (9-4\alpha)(e^t + 1)\} = 1000 - 9(e^t + 1).$$

Therefore, it is sufficient that inequalities $1000 - 5(e^t + 1) \geq 0$ *and* $1000 - 9(e^t + 1) \geq 0$ *hold simultaneously, which gives us* $0 \leq t \leq 4.7$. *Consequently,* $x(t)$ *is* (1)*-differentiability on the interval* $(0, 4.7)$ *and by Theorem 4.5, it is* (1)*-solution of the problem. Figure 3, shows the graphic of* $x(t)$ *in the interval* $[0, 10]$. *Figure 3 clearly also shows that the obtained* (1)*-solution constitutes a part of a truly fuzzy-number-valued function exhibiting logistic behavior.*

Case 2: (2)-solution when $\mathbf{u_0}, \mathbf{v_0} \in \mathbb{R}^-_{\mathbf{Fr}}$

Theorem 4.7. *Suppose that* $x_0 \in \mathbb{R}^+_{Fr}$ *and* $u_0, v_0 \in \mathbb{R}^-_{Fr}$ *are such that* $d = (d^-, d^+)$, *given by*

$$d^-_\alpha = \frac{x^+_{0,\alpha}}{u^-_{0,\alpha} - v^-_{0,\alpha}x^+_{0,\alpha}}, \quad \text{and} \quad d^+_\alpha = \frac{x^-_{0,\alpha}}{u^+_{0,\alpha} - v^+_{0,\alpha}x^-_{0,\alpha}}, \tag{18}$$

is defined as a negative fuzzy number. Suppose that there is $\tau > 0$, *such that* $z(t) = \big(z^-(t), z^+(t)\big)$, *given by*

$$z^-_\alpha(t) = \frac{u^-_{0,\alpha}}{1 + d^-_\alpha v^-_{0,\alpha} e^{u^-_{0,\alpha}t}}, \quad \text{and} \quad z^+_\alpha(t) = \frac{u^+_{0,\alpha}}{1 + d^+_\alpha v^+_{0,\alpha} e^{u^+_{0,\alpha}t}}, \tag{19}$$

and $w(t) = \big(w^-(t), w^+(t)\big)$, *given by*

$$w^-_\alpha(t) = z^-_\alpha(t)e^{u^-_{0,\alpha}t}, \quad \text{and} \quad w^+_\alpha(t) = z^+_\alpha(t)e^{u^+_{0,\alpha}t}, \tag{20}$$

are truly fuzzy-number-valued functions defined on interval $I = [0, \tau]$. *Then the problem* (4) *has the positive* (2)*-solution, given by*

$$x(t) = d \odot w(t), \quad \forall t \in I. \tag{21}$$

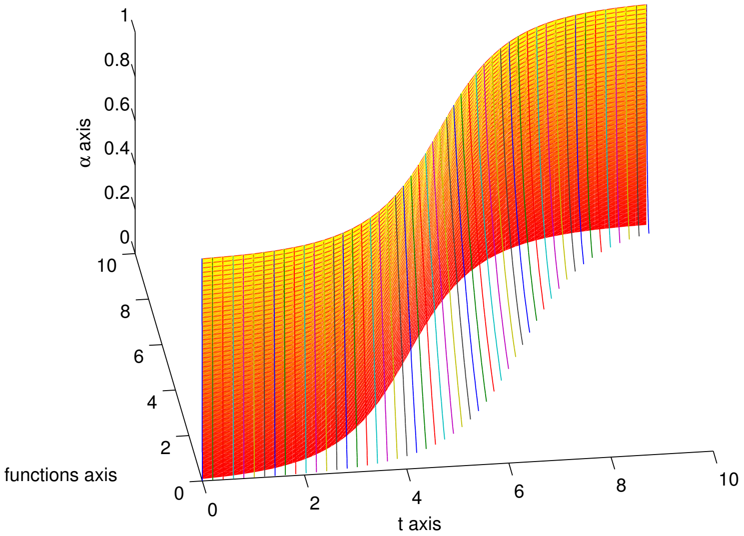

Figure 3: The function $x_\alpha^-(t)$ is denoted by diagonal lines and the function $x_\alpha^+(t)$ is denoted by colouring from Example 4.6

Proof. Similar to the proof of Theorem 4.1, we first note that, the assumptions $x_0 \in \mathbb{R}_{Fr}^+$, $u_0, v_0 \in \mathbb{R}_{Fr}^-$ and $d \in \mathbb{R}_F^-$, imply that $d \in \mathbb{R}_{Fr}^-$ and $z(t), w(t) \in \mathbb{R}_{Fr}^-$ for all $t \in I$. So, by (21), $x(t) \in \mathbb{R}_{Fr}^+$ for all $t \in I$, whose the branches are

$$x_\alpha^-(t) = \frac{d_\alpha^+ u_{0,\alpha}^+ e^{u_{0,\alpha}^+ t}}{1 + d_\alpha^+ v_{0,\alpha}^+ e^{u_{0,\alpha}^+ t}}, \tag{22}$$

$$x_\alpha^+(t) = \frac{d_\alpha^- u_{0,\alpha}^- e^{u_{0,\alpha}^- t}}{1 + d_\alpha^- v_{0,\alpha}^- e^{u_{0,\alpha}^- t}}. \tag{23}$$

It is easy to observe that $x_\alpha^-(0) = x_{0,\alpha}^-$ and $x_\alpha^+(0) = x_{0,\alpha}^+$ that means $x(0) = x_0$. We show that $x(t)$ is (2)-differentiable on the interval $(0, \tau)$. From (19) and (21), we conclude that the H-difference $u_0 \ominus v_0 \odot x(t)$ exists and equals $z(t)$, for each $t \in I$. Using this and that for each $\alpha \in [0,1]$, the function $x_\alpha^-(t)$ given in (22), is the solution of the following logistic differential equation:

$$\begin{cases} (x_\alpha^-)'(t) + v_{0,\alpha}^+ (x_\alpha^-(t))^2 = u_{0,\alpha}^+ x_\alpha^-(t), & t \in (0,\tau), \\ x_\alpha^-(0) = x_{0,\alpha}^-, \end{cases} \tag{24}$$

we obtain

$$\begin{aligned} \delta(x_\alpha^-)'(t) &= x_\alpha^-(t)\delta u_{0,\alpha}^+ + u_{0,\alpha}^+ \delta x_\alpha^-(t) - \big(x_\alpha^-(t)\big)^2 \delta v_{0,\alpha}^+ - 2v_{0,\alpha}^+ x_\alpha^-(t)\delta x_\alpha^-(t) \\ &= x_\alpha^-(t)\delta(u_{0,\alpha}^+ - v_{0,\alpha}^+ x_\alpha^-(t)) + (u_{0,\alpha}^+ - v_{0,\alpha}^+ x_\alpha^-(t))\delta x_\alpha^-(t) \\ &= x_\alpha^-(t)\delta\big((u_0 \ominus v_0 \odot x(t))_\alpha^+\big) + (u_0 \ominus v_0 \odot x(t))_\alpha^+ \delta x_\alpha^-(t) \\ &= x_\alpha^-(t)\delta z_\alpha^+(t) + z_\alpha^+(t)\delta x_\alpha^-(t) \\ &= \delta(x \odot z)_\alpha^+(t) \leq 0. \end{aligned}$$

Similarly, since the function $x_\alpha^+(t)$ in (23), is the solution of the following logistic differential equation:

$$\begin{cases} (x_\alpha^+)'(t) + v_{0,\alpha}^- (x_\alpha^+(t))^2 = u_{0,\alpha}^- x_\alpha^+(t), & t \in (0,\tau), \\ x_\alpha^+(0) = x_{0,\alpha}^+, \end{cases} \tag{25}$$

it follows that

$$\delta(x_\alpha^+)'(t) = \delta(x \odot z)_\alpha^-(t) \geq 0.$$

Also, by (24) and (25), we get

$$(x_1^-)'(t) - (x_1^+)'(t) = diam\big((x \odot z)(t)\big)^1 \geq 0, \quad \forall t \in (0,\tau).$$

Therefore, $D_2x(t) = \big((x^+)'(t), (x^-)'(t)\big)$ is defined as a truly fuzzy-number-valued function on $(0, \tau)$. Finally, similar to the proof of Theorem 4.1, we infer that

$$D^*\big(D_2x \oplus v_0 \odot x^2, u_0 \odot x\big) = 0,$$

which finishes the proof. □

Example 4.8. *Consider the following problem*

$$\begin{cases} x'(t) \oplus (-\frac{1}{8}) \odot x^2(t) = (-\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}) \odot x(t), \quad t > 0, \\ x(0) = (1, 2, 3). \end{cases} \tag{26}$$

Here $x_0 = (1, 2, 3)$*,* $u_0 = (-\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4})$ *and* $v_0 = -\frac{1}{8}$*. Then, from* (18)*, we get* $d^-_\alpha = d^+_\alpha = -8$*, for all* $\alpha \in [0, 1]$*. By replacing into* (19)*, we get*

$$z^-_\alpha(t) = \frac{-3 + \alpha}{4 + 4e^{(\frac{-3+\alpha}{4})t}},$$

and

$$z^+_\alpha(t) = \frac{-1 - \alpha}{4 + 4e^{(\frac{-1-\alpha}{4})t}},$$

which are defined as the lower and upper branches of a negative truly fuzzy-number-valued function $z(t)$*, for each* $t > 0$*. Therefore, we check the existence of function* $w(t)$*, given in* (20)*. We obtain*

$$w^+_1(t) - w^-_1(t) = z^+_1(t)e^{-\frac{1}{2}t} - z^-_1(t)e^{-\frac{1}{2}t} = 0.$$

We then should have

$$\delta w^-_\alpha(t) = \frac{e^{(\frac{-3+\alpha}{4})t}}{\big(4 + 4e^{(\frac{-3+\alpha}{4})t}\big)^2}\big(4 + t(-3 + \alpha) + 4e^{(\frac{-3+\alpha}{4})t}\big) \geq 0,$$

and

$$\delta w^+_\alpha(t) = -\frac{e^{-(\frac{1+\alpha}{4})t}}{\big(4 + 4e^{(-\frac{1+\alpha}{4})t}\big)^2}\big(4 - t(1 + \alpha) + 4e^{-(\frac{1+\alpha}{4})t}\big) \leq 0.$$

These inequalities require the same inequalities as (13) *and* (14)*. Consequently, we get* $\tau = 1.7$ *and the problem* (26) *has* (2)*-solution on interval* $[0, 1.75]$*, given as follows:*

$$[x(t)]_\alpha = \left[\frac{2(1+\alpha)}{1 + e^{(\frac{1+\alpha}{4})t}}, \frac{2(3-\alpha)}{1 + e^{(\frac{3-\alpha}{4})t}}\right], \quad \forall \alpha \in [0, 1].$$

The graphic of this solution is given in Figure 4, which confirms the behavior of a (2)*-differentiable function, because its dimension decreases with time.*

Theorem 4.9. *Let* $x_0 \in \mathbb{R}^+_{Fr}$*,* $u_0, v_0 \in \mathbb{R}^-_{Fr}$*. Suppose that* $\tau > 0$ *exists such that* $x(t) = \big(x^-(t), x^+(t)\big)$*, given by*

$$x^-_\alpha(t) = \frac{x^-_{0,\alpha} u^+_{0,\alpha} e^{u^+_{0,\alpha} t}}{u^+_{0,\alpha} - v^+_{0,\alpha} x^-_{0,\alpha}(1 - e^{u^+_{0,\alpha} t})}, \tag{27}$$

and

$$x^+_\alpha(t) = \frac{x^+_{0,\alpha} u^-_{0,\alpha} e^{u^-_{0,\alpha} t}}{u^-_{0,\alpha} - v^-_{0,\alpha} x^+_{0,\alpha}(1 - e^{u^-_{0,\alpha} t})}, \tag{28}$$

is a truly fuzzy-number-valued function defined on $[0, \tau]$ *and is* (2)*-differentiable on* $(0, \tau)$*. Then,* $x(t)$ *is* (2)*-solution of* (4)*.*

Remark 4.10. *Case* $u_0 \in \mathbb{R}^+_F$ *and* $v_0 \in \mathbb{R}^-_F$*, (or* $u_0 \in \mathbb{R}^-_F$ *and* $v_0 \in \mathbb{R}^+_F$*) excludes problem* (4) *from the real world. To illustrate, since the solution of the fuzzy case is expected to reproduce the behavior of the crisp solution, let us consider the following corresponding crisp problems:*

$$\begin{cases} x'(t) + bx^2(t) = ax(t), \\ x(0) = c > 0, \end{cases}$$

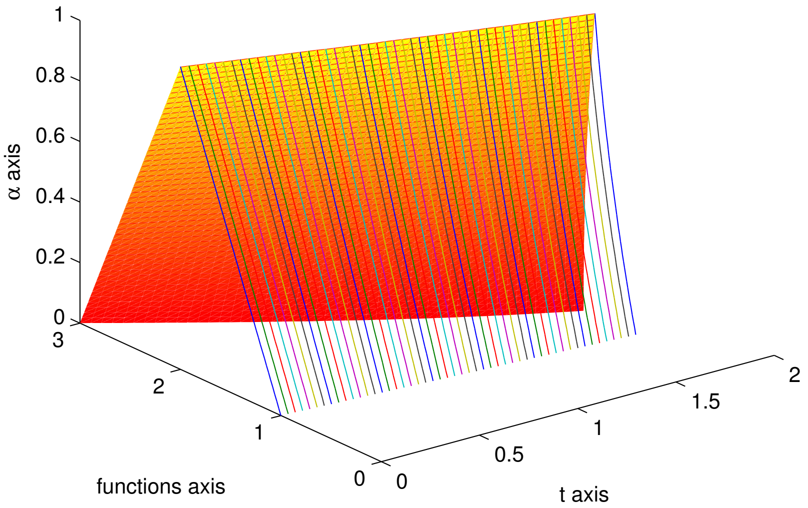

Figure 4: The solution function $x(t)$ of Example 4.8. The diagonal lines represent function $x_\alpha^-(t)$ and the coloured graph represents function $x_\alpha^+(t)$.

where $a \in [u_0]_1$, $b \in [v_0]_1$ *and* $c \in [x_0]_1$*. If, for example,* $u_0 \in \mathbb{R}_F^+$ *and* $v_0 \in \mathbb{R}_F^-$*, then* $a > 0$ *and* $b < 0$*, which implies that the positive solution* x *satisfies the inequality* $x'(t) > 0$*, for all* $t > 0$*. This means that* x *increases without bound, and since* $x'(t) = ax(t) - bx^2(t)$*, then, as* x *becomes larger, the rate of this increase is greater. This behavior is incompatible with biological reality because, typically, constraints such as limited resources, space, or environmental factors exist, which cause subjects like populations or concentrations to reach an equilibrium. Furthermore, examining the problem for these cases is challenging from a theoretical point of view. In fact, we have observed that when* u_0 *and* v_0 *have the same sign, it is possible to obtain separate equations in terms of* x_α^- *and* x_α^+*, as given* (10), (11), (24) *and* (25)*, for which analytical solutions are available. However, when* u_0 *and* v_0 *have opposite signs, the resulting crisp systems consist of coupled logistic differential equations whose the analytical solutions are generally not available (see* [19, 29, 33]*, for example).*

Remark 4.11. *In the present section, we have obtained the values of* τ *such that the* $FLDE$ (4) *has* (1)*-solution when* $u_0, v_0 \in \mathbb{R}_F^+$ *and* (2)*-solution when* $u_0, v_0 \in \mathbb{R}_F^-$*, on the interval* $[0, \tau]$*. Therefore, there is no point within this interval at which the derivative of* x *changes from* (1)*-differentiability to* (2)*-differentiability (or vice versa), meaning that no switching points exist. Accordingly, to find the solution that has a switching point at* $t = \tau$*, we need to solve the problems*

$$\begin{cases} D_2x(t) \oplus v_0 \odot x^2(t) = u_0 \odot x(t), & t > \tau, \quad u_0, v_0 \in \mathbb{R}_F^+, \\ x(\tau) = x_\tau, \end{cases}$$

where $x_\tau = \big(x^-(\tau), x^+(\tau)\big)$ *and* $x^-(t)$ *and,* $x^+(t)$ *are described by* (16) *and* (17)*, and*

$$\begin{cases} D_1x(t) \oplus v_0 \odot x^2(t) = u_0 \odot x(t), & t > \tau, \quad u_0, v_0 \in \mathbb{R}_F^-, \\ x(\tau) = x_\tau, \end{cases}$$

where $x_\tau = \big(x^-(\tau), x^+(\tau)\big)$ *and* $x^-(t)$ *and,* $x^+(t)$ *are described by* (27) *and* (28)*. If these equations have analytical solutions, then the upper and lower branches of those solutions respectively satisfy the following crisp systems:*

$$\begin{cases} (x_\alpha^+)'(t) + v_{0,\alpha}^-(x_\alpha^-(t))^2 = u_{0,\alpha}^- x_\alpha^-(t), \\ (x_\alpha^-)'(t) + v_{0,\alpha}^+(x_\alpha^+(t))^2 = u_{0,\alpha}^+ x_\alpha^+(t), \\ x_{\tau,\alpha}^- = x_\alpha^-(\tau), \\ x_{\tau,\alpha}^+ = x_\alpha^+(\tau), \end{cases}$$

and

$$\begin{cases} (x_\alpha^-)'(t) + v_{0,\alpha}^-(x_\alpha^+(t))^2 = u_{0,\alpha}^- x_\alpha^+(t), \\ (x_\alpha^+)'(t) + v_{0,\alpha}^+(x_\alpha^-(t))^2 = u_{0,\alpha}^+ x_\alpha^-(t), \\ x_{\tau,\alpha}^- = x_\alpha^-(\tau), \\ x_{\tau,\alpha}^+ = x_\alpha^+(\tau). \end{cases}$$

However, due to the nonlinear and coupled nature of these systems, the variables cannot be separated; consequently, as mentioned in Remark 4.10, the analytical solutions to these systems are not generally available.

5 Conclusion and future researche

In this work, the well-known Pielou logistic differential equation along with fuzzy initial value and fuzzy coefficients is studied from the perspective of the generalized Hukuhara differentiability. The conditions for the uniqueness of positive or negative solutions are provided, and the structure of the positive solutions is explained with illustrative examples.

For future researche, we suggest the use of numerical methods to find approximate solutions to fuzzy logistic differential equations under the gH-derivative on intervals containing the switch points. It can also be interesting to study the problem from the perspective of other concepts of derivative such as fuzzy differential inclusions. Extending analytical results to more complex structures of the problem, such as delay fuzzy logistic differential equation and fuzzy logistic differential equation with variable growth coefficient, could be other topics for research.

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