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Original paper

Self-dual pseudo-uninorms

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Abstract

Uninorms are a common generalization of t-norms and t-conorms, which are mutually dual aggregation functions. However, no uninorm is self-dual. In this paper, we show that dropping the axiom of commutativity allows a construction for self-dual pseudo-uninorms. We characterize three important classes of self-dual pseudo-uninorms, namely the representable pseudo-uninorms, pseudo-uninorms with all elements idempotent and those pseudo-uninorms that have both underlying functions continuous. Finally, it is proven that each self-dual pseudo-uninorm has continuous underlying functions. Note that such a slight change has only a little effect on the continuity and commutativity of the pseudo-uninorm.

Keywords: Pseudo-uninorm, uninorm, self-duality, semigroup, strong negation.

1 Introduction and Preliminaries

T-norms and t-conorms are important associative non-decreasing functions applied in many fields. Since there is a duality between them, results obtained for one class can be readily extended to the other. Note that t-norms have strong conjunctive behavior and dually, t-conorms have strong disjunctive behavior. To address this issue, uninorms were proposed by Yager, and Rybalov in [18] as a common ground for both t-norms and t-conorms. Uninorms exhibit some kind of bipolar behavior, which was quickly observed and applied (see a.e. [9, 13, 19]). For other applications of uninorms, see for instance [2, 8, 12, 17]. Aggregative functions [4] which are specific class of representable uninorms, which are almost self-dual were recently studied in [5, 6, 7]. Observe that, unlike for a t-conorm and a t-norm, when a duality is applied to a uninorm then another uninorm is obtained. In other words, the class of uninorms remains closed under duality. Although there is no uninorm on the unit interval that is self-dual [4]. This is mainly because each uninorm is either conjunctive ($U(0,1) = U(1,0) = 0$) and thus the dual of a conjunctive uninorm is disjunctive ($U(0,1) = U(1,0) = 1$) and vice versa.

When the axiom of commutativity is dropped from the definition of uninorms, pseudo-uninorms are obtained. Recently, a characterization of three important classes of pseudo-uninorms was done in [10, 14, 15]. It was observed that pseudo-uninorms on the unit interval might no longer be only conjunctive or disjunctive in contrast to uninorms, as was first noted in [14]. In this paper, we will show the existence and the characterization of self-dual pseudo-uninorms from those three classes. Moreover, we prove that no other self-dual pseudo-uninorm exists. Since pseudo-uninorms have strong bipolar behavior, they are suitable to model problems with negative and positive inputs. The self-duality in such a setting threats the postive and the negative inputs in a sort of symmetric manner. The obtained results are based mainly on [10]. We refer to that paper for more details. However, to keep this paper as self-contained and concise as possible, we include the necessary prerequisites in the following.

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Definition 1.1. *A function $n : [0,1] \to [0,1]$, which is non-increasing, $n(0) = 1$, $n(1) = 0$ and is involutive, i.e., $n(n(x)) = x$, is called a strong negation.*

Definition 1.2. *A bivariate function $A : [0,1]^2 \to [0,1]$ is called*

- *idempotent if $A(\xi,\xi) = \xi$ for each $\xi \in [0,1]$. If $A(\xi,\xi) = \xi$ for some $\xi \in [0,1]$ then such a point is called an idempotent point.*
- *internal if $A(\xi,\eta) \in \{\xi,\eta\}$ for each $\xi,\eta \in [0,1]$.*

Definition 1.3. [16] *We call a bivariate function $U : [0,1]^2 \to [0,1]$ a pseudo-uninorm whenever it satisfies the standard axioms: U preserves order in both arguments, is associative, and admits a neutral element $e \in [0,1]$. Furthermore,*

- *A commutative pseudo-uninorm is called a uninorm.*
- *If $e = 1$, then U is called a pseudo-t-norm.*
- *A commutative pseudo-t-norm is called a t-norm.*
- *If $e = 0$, then U is called a pseudo-t-conorm.*
- *A commutative pseudo-t-conorm is called a t-conorm.*

Each pseudo-uninorm restricted to $[0,e]^2$ ($[e,1]^2$) is a linear transformation of a pseudo-t-(co)norm T_U (S_U). T_U and S_U are called the underlying pseudo-t-norm and pseudo-t-conorm, respectively, or jointly underlying functions. If a pseudo-t-(co)norm has no idempotent points different from trivial $(0,1)$ then it is called an Archimedean pseudo-t-(co)norm. Continuous Archimedean pseudo-t-norms have strong significance in the theory of aggregation. Each continuous Archimedean pseudo-t-norm is either strict (i.e., is strictly decreasing on $]0,1[^2$) and thus is isomorphic to the product t-norm or nilpotent (i.e., $T(\xi,\eta) = 0$ for suitable $(\xi,\eta) \in]0,1[^2$) and thus is isomorphic to the Łukasiewicz t-norm. For more details on t-norms see e.g. [11].

An important class of pseudo-uninorms is formed by the so-called representable pseudo-uninorms. A pseudo-uninorm U is representable if there exists a strictly increasing bijection $f : [0,1] \to [-\infty,\infty]$ with $f(e) = 0$, such that $U(\xi,\eta) = f^{-1}(f(\xi) + f(\eta))$, with the convention $\infty + (-\infty) \in \{-\infty,\infty\}$ and $-\infty + \infty \in \{-\infty,\infty\}$. Observe that $\infty + (-\infty)$ and $-\infty + \infty$ can be chosen independently as was shown in [15].

Let $0 \le \alpha < \beta \le \gamma < \delta \le 1$, $\epsilon \in [\beta,\gamma]$ and $e \in]0,1[$ then define the function $f : [0,1] \longrightarrow [\alpha,\beta[\cup \{\epsilon\} \cup]\gamma,\delta]$, as follows.

$$f(\xi) = \begin{cases} (\beta-\alpha)\cdot\frac{\xi}{e} + \alpha & \text{if } \xi \in [0,e[\,, \\ \epsilon & \text{if } \xi = e, \\ \delta - \frac{(1-\xi)(\delta-\gamma)}{(1-e)} & \text{otherwise.} \end{cases} \tag{1}$$

Obviously, f is linear on $[0,e[$ and on $]e,1]$, altogether, f is a piece-wise linear transformation of $[0,1]$ to $([\alpha,\beta[\cup \{\epsilon\} \cup]\gamma,\delta])$. Consider a function $GU : [0,1]^2 \longrightarrow [0,1]$. Then it is possible to define the bivariate function $GU_\epsilon^{\alpha,\beta,\gamma,\delta} : ([\alpha,\beta[\cup \{\epsilon\} \cup]\gamma,\delta])^2 \longrightarrow ([\alpha,\beta[\cup \{\epsilon\} \cup]\gamma,\delta])$ given by

$$GU_\epsilon^{\alpha,\beta,\gamma,\delta}(\xi,\eta) = f(GU(f^{-1}(\xi), f^{-1}(\eta))). \tag{2}$$

Likewise, applying the inverse mapping f^{-1} allows us to transform any binary operation originally defined on $([\alpha,\beta[\cup \{\epsilon\} \cup]\gamma,\delta])^2$ to an operation acting on the unit square. Since f is an increasing bijection, the basic algebraic properties, such as commutativity, associativity, and monotonicity, are retained under transformation via f. Further, if e serves as the neutral element of GU then ϵ is the neutral element of $GU_\epsilon^{\alpha,\beta,\gamma,\delta}$.

Due to the isomorphism within the main classes of continuous Archimedean t-norms [11] we can reformulate [10][Definition 2.9] as follows.

Definition 1.4. [10] *Let $\alpha,\beta,\gamma,\delta \in [0,1]$ with $\alpha < \beta \le \gamma < \delta$, $\epsilon \in [\beta,\gamma]$. Then*

1. *A semigroup $(]\alpha,\beta[\cup \{\epsilon\} \cup]\gamma,\delta[, *)$ will be called a* representable *semigroup if it is isomorphic via (2) to a restriction of a representable uninorm to $]0,1[^2$.*
2. *A semigroup $(]\alpha,\beta[, *)$ will be called a* t-strict *semigroup if it is isomorphic to product t-norm T_P restricted to $]0,1[^2$, where $T_P(x,y) = xy$ for all $x,y \in [0,1]$.*

3. *A semigroup* $(]\gamma,\delta[\,,*)$ *will be called an* s-strict *semigroup if it is isomorphic to probabilistic sum* S_P *restricted to* $]0,1[^2$*, where* $S_P(x,y)=x+y-xy$ *for all* $x,y\in[0,1]$*.*
4. *A semigroup* $([\alpha,\beta[\,,*)$ *will be called a* t-nilpotent *semigroup if it is isomorphic to Łukasiewicz t-norm* T_L *restricted to* $[0,1[^2$*, where* $T_L(x,y)=\max(0,1-x-y)$ *for all* $x,y\in[0,1]$*.*
5. *A semigroup* $(]\gamma,\delta]\,,*)$ *will be called an* s-nilpotent *semigroup if it is isomorphic to Łukasiewicz t-conorm* S_L *restricted to* $]0,1]^2$*, where* $S_L(x,y)=\min(x+y,1)$ *for all* $x,y\in[0,1]$*.*

We denote by $\mathcal{H}$ the class of semigroups introduced above. For later use, let us also recall Clifford's ordinal sum construction [1], stated in the form given in [11]. This construction plays a central role in the forthcoming analysis of pseudo-uninorms whose underlying functions are continuous.

Theorem 1.5 (Clifford's ordinal sum [11])**.** *Let* $K\neq\emptyset$ *be a totally ordered set and* $(G_k)_{k\in K}$ *with* $G_k=(\xi_k,*_k)$ *be a family of semigroups. Assume that for all* $k,m\in K$ *with* $k\prec m$ *the sets* X_k *and* X_m *are either disjoint or that* $X_k\cap X_m=\{\xi_{k,m}\}$*, where* $\xi_{k,m}$ *is both the neutral element of* G_k *and the annihilator of* G_m *and where for each* $p\in K$ *with* $k\prec p\prec m$ *we have* $X_p=\{\xi_{k,m}\}$*. Put* $X=\bigcup\limits_{k\in K}X_k$ *and define the binary operation* $*$ *on* X *by*

$$\xi*\eta=\begin{cases}\xi *_k \eta & \text{if } (\xi,\eta)\in X_k\times X_k,\\ \xi & \text{if } (\xi,\eta)\in X_k\times X_m \text{ and } k\prec m,\\ \eta & \text{if } (\xi,\eta)\in X_k\times X_m \text{ and } m\prec k.\end{cases} \tag{3}$$

Then $G=(X,*)$ *is a semigroup. The semigroup* G *is commutative if and only if for each* $k\in K$ *the semigroup* G_k *is commutative.*

Theorem 1.6. [10] *Let* $P\colon[0,1]^2\longrightarrow[0,1]$ *be a pseudo-uninorm with continuous underlying functions and neutral element* $e\in\,]0,1[$*. Then* $([0,1],P)$ *can be expressed as an ordinal sum of a countable number of semigroups from* $\mathcal{H}$*, a possibly uncountable number of trivial semigroups and a possibly uncountable number of non-commutative, idempotent semigroups defined on two elements, where the corresponding semigroup operation is the projection to one of the coordinates.*

We use $\mathcal{H}*$ to denote the collection of semigroups specified in Theorem 1.6. Based on [10][Theorem 4.1] and [10][Remark 4.2] we formulate the following lemma.

Lemma 1.7. *Let* $(K,\preceq)$ *be a linearly indexed set and* $\{(X_k,*_k)\}_{k\in K}$ *be a family of semigroups from* $\mathcal{H}*$ *such that the ordinal sum* $([0,1],P)$ *of* $(X_k,*_k)$ *for* $k\in K$*, with respect to* $\preceq$ *is a pseudo-uninorm with the neutral element* e*. If* $e\in X_{k_0}$ *for some* $k_0\in K$ *then* $(X_{k_0},*_{k_0})$ *is either trivial a semigroup or a representable semigroup.*

2 Main results

The section is devoted to the pseudo-uninorms, which are self-dual. Firstly, we will focus on the self-duality of the most common classes of pseudo-uninorms and finally, we will show that each self-dual pseudo-uninorm has a continuous underlying function, thus completing their characterization. Let $e\in[0,1]$, we will use the notation $\mathcal{U}_e$ for the set of all pseudo-uninorms on the unit interval with the neutral element e.

Under the self-duality, we understand the following.

Definition 2.1. *Let* $F:[0,1]^2\to[0,1]$ *be a binary function. Then we say that* F *is self-dual with respect to a strong negation* n *if* $F(\xi,\eta)=n(F(n(\xi),n(\eta))$ *for whole* $(\xi,\eta)\in[0,1]^2$*.*

If U is self-dual with respect to n, then U_φ is self-dual with respect to $n\circ\varphi$, where φ is a continuous bijection from $[0,1]$ onto $[0,1]$ and $U_\varphi(x,y)=\varphi^{-1}(U(\varphi(x),\varphi(y)))$. Since each strong negation n can be represented as $n(x)=\varphi^{-1}(1-\varphi(x))$, i.e., $n(x)=\varphi^{-1}(N_C(\varphi(x)))$, where N_C is the standard negation, each suitable duality is connected with a strong negation can be obtained from the self-duality with respect to N_C. Therefore, in the following, we will restrict ourselves only on the self-dual pseudo-uninorms with respect to the standard negation. For the sake of simplicity, we will understand under the term self-dual pseudo-uninorms, those pseudo-uninorms which are self-dual with respect to the standard negation.

Since a uninorm is either conjunctive or disjunctive, no uninorm can be self-dual. However, some pseudo-uninorms are neither conjunctive nor disjunctive. This brings us to a problem: whether we can construct a self-dual pseudo-uninorm. The answer to this question is positive. In the following, we provide a characterization of all self-dual pseudo-uninorms whose underlying functions are continuous. We begin by examining the subclass of representable pseudo-uninorms.

Lemma 2.2. *Let $U \in \mathcal{U}_e$ be a self-dual then the neutral element $e = \frac{1}{2}$.*

Proof. Assume that $e \in [0,1]$ is the neutral element of a self-dual pseudo-uninorm U then

$$1 - e = U(e, 1-e) = 1 - U(1-e, e) = 1 - (1-e) = e,$$

i.e., $2e = 1$ and thus $e = \frac{1}{2}$. □

Theorem 2.3. *Let $U \in \mathcal{U}_{\frac{1}{2}}$. Then U is a self-dual and representable if and only if U has an additive generator f such that $f(1-\xi) = -f(\xi)$ with the convention $\infty + (-\infty) = \infty$ and $-\infty + \infty = -\infty$ or vice versa.*

Proof. Since U is representable from Lemma 2.2 we have $f(\frac{1}{2}) = 0$.

Now, for $\xi \in]0,1[$ the following holds.

$$f^{-1}(f(\xi) + f(1-\xi)) = U(\xi, 1-\xi) = 1 - U(1-\xi, \xi) = 1 - f^{-1}(f(\xi) + f(1-\xi)).$$

$$2f^{-1}(f(\xi) + f(1-\xi)) = 1,$$

$$f^{-1}(f(\xi) + f(1-\xi)) = \frac{1}{2},$$

$$f(\xi) + f(1-\xi) = 0,$$

$$f(1-\xi) = -f(\xi).$$

Finally,

$$U(1,0) = f^{-1}(f(1) + f(0)) = f^{-1}(\infty + (-\infty)).$$

To preserve associativity, either $\infty + (-\infty) = \infty$ or $\infty + (-\infty) = -\infty$. Thus, in the first case

$$1 = f^{-1}(\infty) = U(1,0) = 1 - U(0,1) = 1 - f^{-1}(-\infty + \infty).$$

$$0 = f^{-1}(-\infty + \infty).$$

$$-\infty = -\infty + \infty.$$

Similarly, we can show that in the second case $-\infty + \infty = \infty$.

The reverse statement can be obtained by direct calculation and is therefore omitted. □

The additive generator f from Theorem 2.3 will be called a symmetric additive generator.

Similarly, for self-dual idempotent uninorms, we can show the following.

Theorem 2.4. *Let $U \in \mathcal{U}_{\frac{1}{2}}$. Then U is a self-dual and idempotent if and only if the following constraints are fulfilled.*

1.
$$U(\xi, \eta) = \begin{cases} \min(\xi, \eta) & \text{if } \xi + \eta < 1, \\ \max(\xi, \eta) & \text{if } \xi + \eta > 1, \\ \xi \text{ or } \eta & \text{if } \xi + \eta = 1. \end{cases} \quad (4)$$

2. $U(\xi, 1-\xi) = \xi$ *if and only if* $U(1-\xi, \xi) = 1 - \xi$ *for* $\xi \in [0,1]$.

Proof. Observe that due to [3][Theorem 3] (see also [14][Proposition 1]) we have that each idempotent pseudo-uninorm on the unit interval is also internal and vice versa, i.e., for all $\xi, \eta \in [0,1]$ $U(\xi, \eta) \in \{\xi, \eta\}$. We will use this observation often in the proof.

Choose $\xi \in \left[0, \frac{1}{2}\right]$ then thanks to the internality of U we have $U(\xi, 1-\xi) \in \{\xi, 1-\xi\}$. Assume that $U(\xi, 1-\xi) = \xi$ then from self-duality of U follows $\xi = U(\xi, 1-\xi) = 1 - U(1-\xi, \xi)$, which implies $U(1-\xi, \xi) = 1-\xi$. Similarly we can show that $U(\xi, 1-\xi) = 1-\xi$ implies $U(1-\xi, \xi) = \xi$. Therefore, $U(\xi, 1-\xi) = \xi$ if and only if $U(1-\xi, \xi) = 1-\xi$ for $\xi \in [0,1]$.

Now, without loss of generality choose $\xi, \eta \in \left[0, \frac{1}{2}\right]$, $\eta < \xi$. On the one hand, since the underlying t-norm of an idempotent pseudo-uninorm U is the minimum, we have

$$U(\xi, \eta) = \min(\xi, \eta) = \eta = \min(\eta, \xi) = U(\eta, \xi),$$

and on the other hand,

$$U(1-\xi,1-\eta)=\max(1-\xi,1-\eta)=1-\eta=\max(1-\eta,1-\xi)=U(1-\eta,1-\xi).$$

Now, we can distinguish two cases if $U(\xi,1-\xi)=\xi$ and the second if $U(\xi,1-\xi)=1-\xi$.

It is enough to prove only the first case because the second one is analogous. Hence $U(\xi,1-\xi)=\xi$, thus $U(1-\xi,\xi)=1-\xi$ as was shown at the beginning of the proof. From monotonicity and internality of U follows that $U(\eta,1-\xi)=\eta=\min(\eta,1-\xi)$ since otherwise $1-\xi=U(\eta,1-\xi)>\xi=U(\xi,1-\xi)$, which is a contradiction. From the self-duality of U follows

$$U(1-\eta,\xi)=1-U(\eta,1-\xi)=1-\eta=\max(1-\eta,\xi).$$

Yet, it remains only to examine values $U(\xi,1-\eta)$ and $U(1-\xi,\eta)$. Assume for a contradiction that $U(\xi,1-\eta)=\xi$ then from associativity follows

$$1-\eta=U(1-\xi,1-\eta)=U(U(1-\xi,\xi),1-\eta)=U(1-\xi,U(\xi,1-\eta))=U(1-\xi,\xi)\in\{\xi,1-\xi\},$$

which is a contradiction. Therefore, $U(\xi,1-\eta)=1-\eta=\max(\xi,1-\eta)$ and from self-duality we have $U(1-\xi,\eta)=\eta=\min(1-\eta,\xi)$. Observe that $\eta+1-\xi<1$ and $\xi+1-\eta>1$, which finalizes the proof.

The reverse statement can be checked by a direct calculation and thus it is omitted. □

Finally, we can generalize the achieved results also for general self-dual pseudo-uninorms with continuous functions as follows.

Theorem 2.5. *Let $U\in\mathcal{U}_{\frac{1}{2}}$. Then the following items are equivalent.*

1. *U is a self-dual pseudo-uninorm with continuous underlying functions.*
2. *There is $\alpha\in\left[0,\frac{1}{2}\right]$ and U is a Clifford's ordinal sum of the following semigroups*
   - *$G_{\frac{1}{2}}=(]\alpha,1-\alpha[\,,*)$, which is a representable semigroup with $\epsilon=\frac{1}{2}$ and the additive generator of a representable uninorm on $]0,1[$ is symmetric,*
   - *$G_\xi=(\{\xi,1-\xi\},*)$ for $\xi\in[0,\alpha]$, where $*$ is the projection to one of the coordinates,*

   with the order $\preceq$, given by $\xi\preceq\eta$ if and only if $\xi\leq\eta$ for all $\xi,\eta\in[0,\alpha]\cup\{\frac{1}{2}\}$.
3. *There is $\alpha\in\left[0,\frac{1}{2}\right]$ such that*

$$U(\xi,\eta)=\begin{cases}(1-2\alpha)U_R(\frac{\xi}{1-2\alpha}+\frac{\alpha}{1-2\alpha},\frac{\eta}{1-2\alpha}+\frac{\alpha}{1-2\alpha})+\alpha & \text{if } (\xi,\eta)\in\,]\alpha,1-\alpha[^2,\\ \min(\xi,\eta) & \text{if } \xi+\eta<1 \text{ and } \min(\xi,\eta)\leq\alpha,\\ \max(\xi,\eta) & \text{if } \xi+\eta>1 \text{ and } \max(\xi,\eta)\geq1-\alpha,\\ \xi \text{ or } \eta & \text{if } \xi+\eta=1 \text{ and } (\xi,\eta)\notin\,]\alpha,1-\alpha[^2,\end{cases}\tag{5}$$

   where U_R is a representable pseudo-uninorm on $[0,1]^2$ with symmetric additive generator and $U(\xi,1-\xi)=\xi$ if and only if $U(1-\xi,\xi)=1-\xi$ for $\xi\in[0,\alpha]\cup[\alpha,1]$.

Proof. We will prove only that $1\Rightarrow2$, since $2\Rightarrow3$ and $3\Rightarrow1$ can be shown directly by a simple calculation and therefore, we decided to omit them.

Theorem 1.6 implies that any pseudo-uninorm U whose underlying functions are continuous admits a representation in the form of a Clifford ordinal sum, where the components are semigroups taken from $\mathcal{H}*$. In the next part we will show that for each $\xi\in[0,1]$ ξ either belongs to a representable semigroup $G_{\frac{1}{2}}$ or a projection semigroup G_ξ (or $G_{1-\xi}$). Choose $\xi\in[0,1]$, we will assume for simplicity that $\xi\leq\frac{1}{2}$ since otherwise we would proceed analogously. Now, we can distinguish the following cases.

1. Let $\xi=\frac{1}{2}$. Then due to Lemma 2.2 ξ is the neutral element of U. Lemma 1.7 implies that ξ belongs to a semigroup $G_{\frac{1}{2}}$, which is either a representable semigroup with the neutral element ξ, i.e., there are $\alpha\in\left[0,\frac{1}{2}\right[$, $\beta\in\left]\frac{1}{2},1\right]$ such that $G_{\frac{1}{2}}=(]\alpha,\beta[\,,*)$ or $G_{\frac{1}{2}}$ is a trivial semigroup. However, then $G_{\frac{1}{2}}=(\{\frac{1}{2}\},id)=(\{\frac{1}{2},1-\frac{1}{2}\},P_F)$, which corresponds with the statement of the theorem.

2. Let $\xi \neq \frac{1}{2}$ be an idempotent element of U. From the self-duality of U we obtain $U(1-\xi, 1-\xi) = 1-U(\xi,\xi) = 1-\xi$, i.e., $1-\xi$ is an idempotent point of U, too. Since $\xi \neq \frac{1}{2}$, hence $1-\xi \neq \xi$. Consider the Clifford's ordinal sum from Theorem 1.6. At first assume that $\xi \in X$ and $\eta \in Y$ for semigroups $G_\xi = (X, *_\xi)$, $G_\eta = (Y, *_\eta)$ such that $G_\xi \neq G_\eta$. Then due to Clifford's ordinal sum either $U(\xi, 1-\xi) = U(1-\xi, \xi) = \xi$ or $U(\xi, 1-\xi) = U(1-\xi,\xi) = 1-\xi$. In both cases U restricted to $\{\xi, 1-\xi\}^2$ commutes. However, from self-duality, we then obtain
$$U(\xi, 1-\xi) = 1 - U(1-\xi, \xi) = 1 - U(\xi, 1-\xi).$$
Hence $U(\xi, 1-\xi) = \frac{1}{2}$, which is the neutral element of U. Now, since ξ is an idempotent point of U, we have
$$\frac{1}{2} = U(\xi, 1-\xi) = U(U(\xi,\xi), 1-\xi) = U(\xi, U(\xi, 1-\xi)) = U(\xi, \frac{1}{2}) = \xi,$$
which is a contradiction with the assumption that $\xi \neq \frac{1}{2}$.

   Therefore, ξ and $1-\xi$ belong to the same semigroup G_ξ. The only semigroup with (exactly) two different idempotent points is a projection semigroup, i.e., $G_\xi = (\{\xi, 1-\xi\}, *)$ where $*$ is the projection to one of the coordinates.

3. Let $\xi \neq \frac{1}{2}$ and ξ is not an idempotent point of U then obviously $\xi < \frac{1}{2} < 1-\xi$. Observe that, similarly as in the previous item, we can show that $1-\xi$ is not an idempotent point of U.

   At first, assume that ξ belongs to a different semigroup from $1-\xi$. Then due to the ordinal sum structure we have that $U(\xi, 1-\xi) = U(1-\xi, \xi) \in \{\xi, 1-\xi\}$ and similarly as in the preceding item we can deduce that ξ and $1-\xi$ belong to the same semigroup $G_\xi = (X, U|_{X^2})$.

   Because $\xi < \frac{1}{2} < 1-\xi$ the G_ξ is a representable semigroup. Therefore, there is a unique idempotent point $\epsilon \in X$, which acts as the neutral element of U restricted to X. Since G_ξ is a representable semigroup there is $\eta \in X$ such that $U(\eta, 1-\xi) = U(1-\xi, \eta) = \epsilon$. From self-duality we have $U(1-\eta, \xi) = U(\xi, 1-\eta) = 1-\epsilon$. Since ϵ is an idempotent point of U thus $1-\epsilon$ is also an idempotent point of U. Moreover, X is closed on U and so $1-\epsilon \in X$. However, there is a unique idempotent point in a representable semigroup, which implies that $\epsilon = 1-\epsilon$, which implies $\epsilon = \frac{1}{2}$.

   We will show that $U(\xi, 1-\xi) = \frac{1}{2}$. If $\xi \leq \eta < \frac{1}{2} < 1-\eta \leq 1-\xi$, then
$$\frac{1}{2} = U(\xi, 1-\eta) \leq U(\xi, 1-\xi) \leq U(\eta, 1-\xi) = \frac{1}{2},$$
and if $\eta \leq \xi < \frac{1}{2} < 1-\xi \leq 1-\eta$, then
$$\frac{1}{2} = U(\eta, 1-\xi) \leq U(\xi, 1-\xi) \leq U(\xi, 1-\eta) = \frac{1}{2}.$$
We know that $X =]\alpha, \beta[\cup \{\frac{1}{2}\} \cup]\gamma, \delta[$, we will show that $\beta = \gamma = \frac{1}{2}$. Assume there is $\zeta \in]\xi, \frac{1}{2}[$, which belongs to a different semigroup G_ζ. Now, either $\xi \prec \zeta$ or $\xi \succ \zeta$. In the first case,
$$\frac{1}{2} = U(\xi, 1-\xi) = U(U(\zeta, \xi), 1-\xi) = U(\zeta, U(\xi, 1-\xi)) = U(\zeta, \frac{1}{2}) = \zeta,$$
hence $\zeta \in X$. In the second case, $\frac{1}{2} = U(\xi, 1-\xi) \leq U(\zeta, 1-\xi) = \zeta$, which leads to a contradiction. Consequently, $X =]\alpha, \delta[$. Moreover, since for arbitrary $\xi \in X$ also $1-\xi \in X$ and vice versa, thus $\delta = 1-\alpha$.

   By a direct calculation from (1), we compute that
$$f(\xi) = (1-2\alpha)\xi + \alpha,$$
$$f^{-1}(\xi) = \frac{\xi - \alpha}{1-2\alpha}.$$
Moreover,
$$f(1-\xi) = (1-2\alpha)(1-\xi) + \alpha = 1 - ((1-2\alpha)\xi + \alpha) = 1 - f(\xi),$$
$$f^{-1}(1-\xi) = \frac{(1-\xi)-\alpha}{1-2\alpha} = \frac{1-2\alpha-\xi+\alpha}{1-2\alpha} = 1 - \frac{\xi-\alpha}{1-2\alpha} = 1 - f^{-1}(\xi).$$
Therefore, from 2 we obtain that $GU_{\frac{1}{2}}^{\alpha,\frac{1}{2},\frac{1}{2},1-\alpha}$ is self-dual if and only if GU is self-dual. The rest follows from Theorem 2.3.

Finally, we have already shown that it is possible to decompose a self-dual pseudo-uninorm that has continuous underlying functions into the semigroups as in the statement of the theorem. We only have to check the order of these semigroups. Choose $\xi, \eta \in [0, \alpha] \cup \{\frac{1}{2}\}$. Then $U(\xi, \eta) \in \{\xi, \eta\}$ and since $U(\xi, \eta) \leq U(\xi, \frac{1}{2}) = \xi$ and $U(\xi, \eta) \leq U(\frac{1}{2}, \eta) = \eta$, we obtain $U(\xi, \eta) = \min(\xi, \eta)$, which concludes the proof. □

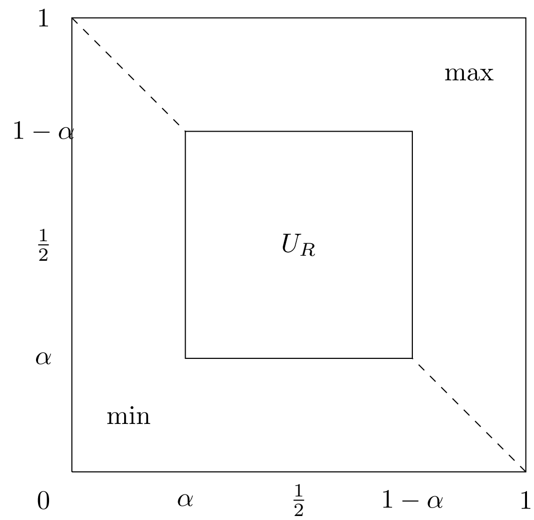

Figure 1: The structure of any self-dual pseudo-uninorm as described in Theorem 2.5. The values on the dashed lines do not commute due to the self-duality of a pseudo-uninorm. Moreover, U_R is a representable pseudo-uninorm with a symmetric additive generator.

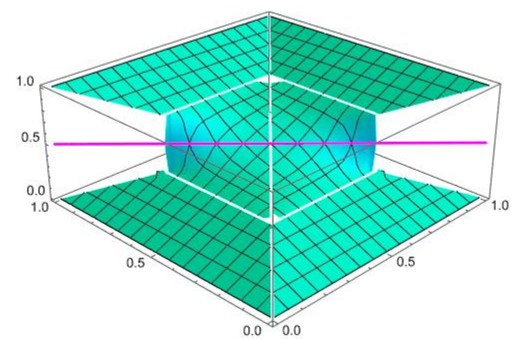

Figure 2: A self-dual pseudo-uninorm with continuous underlying functions from Example 2.7, where $\alpha = \frac{1}{4}$. The magenta line in the image is the characteristic function of U.

Remark 2.6. *In general, different dualities might be considered. Note that each suitable duality is connected with a strong negation, i.e., if n is a strong negation (involutive, non-increasing, unary function), then the self-duality of U with respect to n means $U(\xi, \eta) = n(U(n(\xi), n(\eta)))$. In this paper, we studied the case of standard negation. Since all strong negations are mutually isomorphic, achieved results can be easily extended.*

The characterizing function of a pseudo-uninorm (see [10, 14]*) is the function that divides the domain of pseudo-uninorm into the part where attained values are lower than the neutral element and the part where values are greater than the neutral element. Observe that in all cases of self-dual pseudo-uninorms with respect to a strong negation, their characterizing function coincides with the strong negation. Moreover, the neutral element is always the fixed point of a negation, as is for standard negation in $\frac{1}{2}$.*

Example 2.7. *The function U given by*

$$U(\xi, \eta) = \begin{cases} \min(\xi, \eta) & \text{if } \min(\xi, \eta) \leq \frac{1}{4} \text{ and } \xi + \eta < 1, \\ \max(\xi, \eta) & \text{if } \max(\xi, \eta) \geq \frac{3}{4} \text{ and } \xi + \eta > 1, \\ \frac{\arctan(\tan(2\pi(\xi - \frac{1}{4}) - \frac{\pi}{2}) + \tan(2\pi(\eta - \frac{1}{4})) - \frac{\pi}{2})}{2\pi} + \frac{1}{2} & \text{if } (\xi, \eta) \in \left]\frac{1}{4}, \frac{3}{4}\right[^2 \\ \xi & \text{otherwise.} \end{cases}$$

is a self-dual pseudo-uninorm by Theorem 2.5 (see also Figure 2).

Let V be given by

$$V(\xi, \eta) = \begin{cases} \min(\xi, \eta) & \text{if } \xi + \eta < 1 \text{ and } (\xi, \eta) \notin \left]0, \frac{1}{4}\right[^2 \cup \left]0, \frac{1}{4}\right[\times \left]\frac{3}{4}, 1\right[\cup \left]\frac{3}{4}, 1\right[\times \left]0, \frac{1}{4}\right[, \\ \max(\xi, \eta) & \text{if } \xi + \eta > 1 \text{ and } (\xi, \eta) \notin \left]\frac{3}{4}, 1\right[^2, \cup \left]0, \frac{1}{4}\right[\times \left]\frac{3}{4}, 1\right[\cup \left]\frac{3}{4}, 1\right[\times \left]0, \frac{1}{4}\right[, \\ U_R(\xi, \eta) & \text{if } (\xi, \eta) \in (\left]0, \frac{1}{4}\right[\cup \left]\frac{3}{4}, 1\right[)^2, \\ \frac{1}{4} & \text{if } \xi \in \left]0, \frac{1}{4}\right[\text{ and } \eta = 1 - \xi, \\ \frac{3}{4} & \text{if } \eta \in \left]0, \frac{1}{4}\right[\text{ and } \xi = 1 - \eta, \\ \xi & \text{otherwise,} \end{cases}$$

where U_R is a transformation of a representable uninorm with a symmetric additive generator via (2). Then V is a self-dual binary function, which is not a pseudo-uninorm since V is not associative. Observe that for $\xi \in \left]0, \frac{1}{4}\right[$ we have

$$\frac{1}{4} = V(\xi, 1-\xi) = V(V(\frac{3}{4}, \xi), 1-\xi) \neq V(\frac{3}{4}, V(\xi, 1-\xi)) = V(\frac{3}{4}, \frac{1}{4}) = \frac{3}{4}.$$

Now, we will show that each self-dual pseudo-uninorm can be expressed via (5). Before that, observe that for any $U \in \mathcal{U}_e$ and a pair of idempotent elements $\xi, \eta \in [0,1]$ of U, the point $U(\xi, \eta)$ is also idempotent, i.e., the set of idempotent elements is closed under U.

Lemma 2.8. *Let $U \in \mathcal{U}_{\frac{1}{2}}$ be self-dual then for all $\xi \in [0,1]$, $U(U(\xi, 1-\xi), \xi) = \xi$ holds.*

Proof. Either $U(\xi, 1-\xi) \leq e$ or $U(1-\xi, \xi) \leq e$. Let the first case hold. Then from self-duality of U follows $U(1-\xi, \xi) = 1 - U(\xi, 1-\xi) \geq e$, i.e.,

$$\xi = U(\xi, e) \leq U(\xi, U(1-\xi, \xi)) = U(U(\xi, 1-\xi), \xi) \leq U(e, \xi) = \xi.$$

□

Finally, we conclude with the complete characterization of self-dual pseudo-uninorms.

Theorem 2.9. *Let $U \in \mathcal{U}_{\frac{1}{2}}$ be self-dual, then U can be expressed via (5).*

Proof. Let $\xi \in [0, e]$, then one of the following holds. Since the proof of the statement is robust, we add an explanation to each of the following items.

1. $U(\xi, 1-\xi) = \xi$ and $U(1-\xi, \xi) = 1-\xi$. In this case we show the following.
   - ξ and $1-\xi$ are idempotent points.
   - ξ and $1-\xi$ are left annihilators on $[\xi, 1-\xi]$.
2. $U(\xi, 1-\xi) = \eta \in]\xi, e[$ and $U(1-\xi, \xi) = 1-\eta \in]e, 1-\xi[$. In this case we show the following.
   - $U(\eta, 1-\eta), U(1-\eta, \eta) \in \{\eta, 1-\eta\}$ and $\eta, 1-\eta$ are idempotent points of U.
   - The situation can be described either by Table 1 or 2.
   - Such case cannot occur due to a contradiction, which happens when we choose $\zeta \in]\xi, \alpha[$ respectively, $\zeta \in]\alpha, \xi[$
3. $U(\xi, 1-\xi) = e = U(1-\xi, \xi)$. In this case we can show the following.
   - $\xi, 1-\xi$ are not idempotent points.
   - Sections $U(\xi, \cdot), U(\cdot, \xi), U(1-\xi, \cdot), U(\cdot, 1-\xi)$ are continuous on whole unit interval.
   - There is a boundary point ξ_0 such that there is no idempotent point in $]x_0, 1-\xi_0[\setminus \{\frac{1}{2}\}$
4. $U(\xi, 1-\xi) = 1-\xi$ and $U(1-\xi, \xi) = \xi$.
5. $U(\xi, 1-\xi) = 1-\eta \in]e, 1-\xi[$ and $U(1-\xi, \xi) = \eta \in]\xi, e[$.

Since cases 1 and 4, respectively 2 and 5, are dual, we will examine only cases 1-3.

1. $U(\xi, \xi) = U(U(\xi, 1-\xi), \xi) = \xi$, i.e., ξ is an idempotent point of U and thus $1-\xi$ is an idempotent point due to self-duality of U. This implies that $\xi, 1-\xi$ are both left annihilators of U on $[\xi, 1-\xi]^2$, i.e., $U(\xi, \eta) = \xi$ and $U(1-\xi, \eta) = 1-\xi$ for all $\eta \in [\xi, 1-\xi]$. Set $\zeta = U(\eta, \xi)$ for some $\eta \in]\xi, 1-\xi[$. Observe that ζ is then left annihilator of U on $[\xi, 1-\xi]^2$ since $U(\zeta, \xi) = U(U(\eta, \xi), \xi) = U(\eta, U(\xi, \xi)) = U(\eta, \xi) = \zeta$. Without loss of generality, let $\zeta \leq e$ then $\zeta = U(\zeta, \xi) \leq \min(\zeta, \xi) = \xi$ and similarly $\zeta \geq e$ implies $\zeta = 1-\xi$. Therefore, for all $\eta \in]\xi, 1-\xi[$ we have $U(\eta, \xi) \in \{\xi, 1-\xi\} \cap [\xi, \eta]$, i.e., $U(\eta, \xi) = \xi$ and hence also $U(\eta, 1-\xi) = 1-\xi$.

	α	ξ	η	e	$1-\eta$	$1-\xi$	$1-\alpha$
$1-\alpha$	η	$1-\eta$	$1-\xi$	$1-\alpha$	$1-\alpha$		
$1-\xi$	η	$1-\eta$	$1-\xi$	$1-\xi$	$1-\alpha$		
$1-\eta$	α	ξ	η	$1-\eta$	$1-\eta$	$1-\xi$	$1-\alpha$
e	α	ξ	η	e	$1-\eta$	$1-\xi$	$1-\alpha$
η	α	ξ	η	η	$1-\eta$	$1-\xi$	$1-\alpha$
ξ			α	ξ	ξ	η	$1-\eta$
α			α	α	ξ	η	$1-\eta$

Table 1: Values of U in case when $U(\eta, 1-\eta) = 1-\eta$

2. We can easily show that η (and $1-\eta$) is an idempotent point of U. By Lemma 2.8 we have

$$U(\eta,\eta) = U(U(U(\xi,1-\xi),\xi),1-\xi) = U(\xi,1-\xi) = \eta.$$

Now, either $U(\eta,1-\eta) \in \{\eta, 1-\eta\}$ as in the previous item or $U(\eta,1-\eta) \in]\eta,e[\cup]e,1-\eta[$ or $U(\eta,1-\eta) = e$. If $U(\eta,1-\eta) = e$ then from idempotency of η follows $\eta = U(\eta,e) = U(\eta,U(\eta,1-\eta)) = U(U(\eta,\eta),1-\eta) = U(\eta,1-\eta) = e$, which is a contradiction.

Let $U(\eta,1-\eta) = \zeta \in]\eta,e[$, since $\zeta \in]e,1-\eta[$ is analogous. Then as was proven, both $\zeta,\eta \leq e$ and ζ,η are idempotent points, therefore $U(\eta,\zeta) = \min(\eta,\zeta) = \eta$ and we obtain

$$\zeta = U(\eta,1-\eta) = U(U(\eta,\eta),1-\eta) = U(\eta,U(\eta,1-\eta)) = U(\eta,\zeta) = \eta,$$

which is again a contradiction.

Thus $U(\eta,1-\eta) \in \{\eta,1-\eta\}$, which was examined in the previous item. Moreover, from Lemma 2.8 follows that $\xi = U(\eta,\xi) = U(\xi,1-\eta)$ and dually, $1-\xi = U(1-\eta,1-\xi) = U(1-\xi,\eta)$. Based on the previous item, we can distinguish two cases.

(a) If $U(1-\eta,\eta) = \eta$ and $U(\eta,1-\eta) = 1-\eta$, then

$$U(1-\eta,\xi) = U(1-\eta,U(\eta,\xi)) = U(U(1-\eta,\eta),\xi) = U(\eta,\xi) = \xi,$$

thus also $U(\eta,1-\xi) = 1-\xi$. Set $\alpha = U(\xi,\eta)$ and $1-\alpha = U(1-\xi,1-\eta)$ then

$$\begin{aligned} U(\alpha,1-\alpha) &= U(U(\xi,\eta),U(1-\xi,1-\eta)) = \\ &U(U(\xi,\eta),U(U(1-\eta,1-\xi),1-\eta)) = U(\xi,U(U(\eta,1-\eta),U(1-\xi,1-\eta))) = \\ &U(U(\xi,1-\eta),U(1-\xi,1-\eta)) = U(U(\xi,1-\xi),1-\eta) = U(\eta,1-\eta) = 1-\eta, \end{aligned}$$

and so $U(1-\alpha,\alpha) = \eta$. From associativity, we are able to obtain the values as given in Table 1. Choose $\zeta \in]\alpha,\xi[$ then $U(\zeta,1-\zeta) \in [\eta,1-\eta]$ from monotonicity of U. Then $\beta = U(\eta,\zeta) = U(U(\eta,\eta),\zeta) = U(\eta,\beta)$ and $\beta \in]\alpha,\xi[$ or $\beta = \alpha$. Therefore, we will examine directly only the case when $U(\eta,\zeta) \in \{\alpha,\zeta\}$ and similarly $U(1-\eta,\zeta) \in \{\zeta,\xi\}$. If $U(1-\eta,\zeta) = \zeta$ with $U(\eta,\zeta) = \alpha$ then

$$\zeta = U(1-\eta,\zeta) = U(U(\eta,1-\eta),\zeta) = U(\eta,U(1-\eta,\zeta)) = U(\eta,\zeta) = \alpha,$$

which is a contradiction. Therefore, either $U(\eta,\zeta) = U(1-\eta,\zeta) = \zeta$ or $U(\eta,\zeta) = \alpha$ with $U(1-\eta,\zeta) = \xi$.

i. If $U(\eta,\zeta) = U(1-\eta,\zeta) = \zeta$, then $U(\zeta,1-\zeta) = U(U(\eta,\zeta),1-\zeta) = U(\eta,U(\zeta,1-\zeta))$. Since η is the annihilator on $[\eta,1-\eta[^2$, and $U(\zeta,1-\zeta) \in [\eta,1-\eta]$ thus $U(\zeta,1-\zeta) \in \{\eta,1-\eta\}$. Therefore, either $U(1-\zeta,\zeta) = \eta$ or $U(1-\zeta,\zeta) = 1-\eta$. We will only prove the second case since the first one is analogous.

$$\xi = U(\alpha,1-\eta) = U(\alpha,U(1-\zeta,\zeta)) = U(U(\alpha,1-\zeta),\zeta) = U(\eta,\zeta) = \zeta,$$

which is a contradiction.

ii. If $U(\eta,\zeta) = \alpha$ and $U(1-\eta,\zeta) = \xi$. Due to monotonocity of U we have $U(\zeta,1-\zeta) \in [\eta,1-\eta]$ and from self-duality of U follows

$$U(\eta,U(\zeta,1-\zeta)) = U(U(\eta,\zeta),1-\zeta) = U(\alpha,1-\zeta) = U(U(\alpha,\eta),1-\zeta) = U(\alpha,U(\eta,1-\zeta)) = U(\alpha,1-\xi) = \eta,$$

	ξ	α	η	e	$1-\eta$	$1-\alpha$	$1-\xi$
$1-\xi$	$1-\eta$	$1-\eta$	$1-\xi$	$1-\xi$	$1-\xi$		
$1-\alpha$	η	η	$1-\alpha$	$1-\alpha$	$1-\alpha$		
$1-\eta$	α	α	$1-\eta$	$1-\eta$	$1-\eta$	$1-\xi$	$1-\xi$
e	ξ	α	η	e	$1-\eta$	$1-\alpha$	$1-\xi$
η	ξ	ξ	η	η	η	$1-\alpha$	$1-\alpha$
α			α	α	α	$1-\eta$	$1-\eta$
ξ			ξ	ξ	ξ	η	η

Table 2: Values of U in case when $U(\eta,1-\eta)=\eta$

$$U(1-\eta,U(\zeta,1-\zeta))=U(U(1-\eta,\zeta),1-\zeta)=U(\xi,1-\zeta)=U(U(\xi,1-\eta),1-\zeta)=U(\xi,U(1-\eta,1-\zeta))$$
$$=U(\xi,1-\alpha)=1-\eta,$$

thus $U(\zeta,1-\zeta)\in\,]\eta,1-\eta[$, because η and $1-\eta$ are right annihilators of U on $[\eta,1-\eta]^2$.

(b) If $U(\eta,1-\eta)=\eta$ and $U(1-\eta,\eta)=1-\eta$ then

$$U(\xi,\eta)=U(U(\xi,1-\eta),\eta)=U(\xi,U(1-\eta,\eta))=U(\xi,1-\eta)=\xi,$$

thus also $U(1-\xi,1-\eta)=1-\xi$. Set $\alpha=U(1-\eta,\xi)$ then $U(\xi,\alpha)=U(\xi,U(1-\eta,\xi))=U(U(\xi,1-\eta),\xi)=U(\xi,\xi)<\xi$, hence $e\geq\eta>\alpha\geq\xi$. The values that follows from associativity and monotonicity of U are summarized in Table 2. From now on, we will proceed similarly as in the previous way, choosing $\zeta\in\,]\xi,\alpha[$. Similarly, we are able to show that $U(\zeta,1-\zeta)\in\,]\eta,1-\eta[$ and $U(\zeta,\eta)=\alpha$, $U(\zeta,1-\eta)=\xi$.

Regardless of which of the previous situations occurred, we can study these two cases jointly, since they are analogous. Observe that in both cases only the values of $U(1-\alpha,\alpha)$ and $U(1-\xi,\xi)$ are always exchanged between η and $1-\eta$.

Therefore, without loss of generality, we will assume that the first case (described by Table 1) occurred, as the second one is analogous.

Note that $U(U(\xi,\xi),1-U(\xi,\xi))=U(U(\xi,U(\xi,1-\xi)),1-\xi)=U(U(\xi,\eta),1-\xi)=U(\xi,1-\xi)=\eta$, i.e., $U(\xi,\xi)\neq\alpha$. Analogously, $U(U(\alpha,\alpha),U(1-\alpha,1-\alpha))=1-\eta$. Moreover, for $\zeta\in\,]\alpha,\xi[$ we have that $U(\zeta,1-\zeta)\in\,]\eta,1-\eta[$ is an idempotent point so the situation is similar to the one described in Table 1 or Table 2 since $U(1-\zeta,\zeta)=1-U(\xi,1-\zeta)$. Now, there is an element β (which represents an element α in Tables 1 and 2, respectively) such that $U(\zeta,1-\zeta)=U(1-\beta,\beta)$ as we have discussed. Both $\beta,\zeta\in\,]\alpha,\xi[$.

If $\beta\leq\zeta$ then set $\zeta_2=U(\zeta,\zeta)$. $U(\alpha,\alpha)<U(\zeta,\zeta)<U(\xi,\xi)$ and $U(\zeta_2,1-\zeta_2)=U(\zeta,1-\zeta)$. For all elements $\gamma\in\,]U(\zeta,\zeta),\beta[$ we have that $U(\gamma,1-\gamma)\in\,]\min(U(\zeta,1-\zeta),U(1-\zeta,\zeta)),\max(U(\zeta,1-\zeta),U(1-\zeta,\zeta))[$ due to the previous items. Nevertheless, $\xi\in\,]\zeta_2,\beta[$, which is a contradiction with $U(\zeta,1-\zeta)\in\,]\eta,1-\eta[$. Similarly, we can show that neither $\beta>\zeta$.

3. $U(\xi,1-\xi)=e=U(1-\xi,\xi)$. Note that $U(\xi,\cdot)$ is a surjective mapping on $[0,1]$ since for any $\zeta\in[0,1]$, $U(\xi,U(1-\xi,\zeta))=U(U(\xi,1-\xi),\zeta)=U(e,\zeta)=\zeta$. A surjective non-decreasing unary function is continuous, thus $U(\xi,\cdot),U(\cdot,\xi),U(1-\xi,\cdot),U(\cdot,1-\xi)$ are all continuous functions. Observe that ξ is not idempotent since otherwise

$$\xi=U(\xi,e)=U(\xi,U(\xi,1-\xi))=U(U(\xi,\xi),1-\xi)=U(\xi,1-\xi)=e.$$

Denote $\xi_U^{(n)}=U(\underbrace{\xi,U(\xi,\ldots)}_{n\text{-times}})$. $\{\xi_U^{(n)}\}_{n=1}^{\infty}$ is a decreasing sequence on a compact space, i.e., there exists a limit $\xi_0=\lim_{n\to\infty}\xi_U^{(n)}$.

We will assume that there is $\zeta\in\,]\xi_0,1-\xi_0[$ such that $U(\zeta,1-\zeta)\neq e$. Obviously, then ζ is an idempotent point. If $\zeta<\xi$ (or $\zeta>1-\xi$) then $\xi_U^{(n-1)}\leq\zeta\leq\xi_U^{(n)}$ ($\xi_U^{(n-1)}\geq\zeta\geq\xi_U^{(n)}$) for some $n\in\mathbb{N}$. However, $\zeta=U(\zeta,\zeta)\geq U(\xi_U^{(n-1)},\xi)=\xi_U^{(n)}$, which is a contradiction with the non-decreasing nature of U. Set $\zeta_0=\inf(\zeta\in\,]\xi_0,1-\xi_0[\,|U(\zeta,1-\zeta)\neq e)$. Then either

(a) $U(\zeta_0,1-\zeta_0)=e$. Note that due to the continuity of $U(\zeta_0,\cdot)$ there is $1-\zeta<1-\zeta_0$ such that $U(\zeta_0,1-\zeta)>1-\zeta$ and $U(\zeta,1-\zeta)\neq e$. Then either $U(\zeta,1-\zeta)=\zeta<1-\zeta<U(\zeta_0,1-\zeta)$ or $U(\zeta,1-\zeta)=1-\zeta<U(\zeta_0,1-\zeta)$, which is a contradiction with the fact that U is non-decreasing.

(b) $U(\zeta_0, 1-\zeta_0) \neq e$. Thus $\zeta_0, 1-\zeta_0$ are idempotent points of U. Then either $U(\zeta_0, 1-\zeta_0) = \zeta_0$ and $U(1-\zeta_0, \zeta_0) = 1-\zeta_0$ or $U(\zeta_0, 1-\zeta_0) = 1-\zeta_0$ and $U(1-\zeta_0, \zeta_0) = \zeta_0$. Assume the first as the second one is analogous. Choose $\eta \in]\zeta_0, 1-\zeta_0[$ then $U(\zeta_0, \eta) = U(\eta, \zeta_0) = \zeta_0$ and $U(1-\zeta_0, \eta) = U(\eta, 1-\zeta_0) = 1-\zeta_0$ for all $\eta \in]\zeta_0, 1-\zeta_0[$ as was proven before.

Now, $U(\xi, 1-\zeta_0) \in [\xi, 1-\zeta_0]$ due to $\xi \leq e \leq 1-\zeta_0$. Moreover, $U(\xi, \eta) \leq U(\zeta, \eta) = \zeta_0$ for all $\eta \in]\zeta_0, 1-\zeta_0[$ thus from continuity of $U(\xi, \cdot)$ follows that $U(\xi, 1-\zeta_0) \leq \zeta_0$. Similarly, we can also show that $U(1-\zeta_0, \xi), U(\xi, 1-\zeta_0) \in [\xi, \zeta_0]$ and $U(\zeta_0, 1-\xi), U(1-\xi, \zeta_0) \in [1-\zeta_0, 1-\xi]$.

Let $U(\xi, 1-\zeta_0) = \alpha < \zeta_0$ then $U(\alpha, 1-\alpha) = U(1-\alpha, \alpha) = e$ and $\alpha = U(\xi, 1-\zeta_0) = U(\xi, U(1-\zeta_0, 1-\zeta_0)) = U(U(\xi, 1-\zeta_0), 1-\zeta_0) = U(\alpha, 1-\zeta_0)$.

$$1-\zeta_0 = U(e, 1-\zeta_0) = U(U(1-\alpha, \alpha), 1-\zeta_0) = U(1-\alpha, U(\alpha, 1-\zeta_0)) = U(1-\alpha, \alpha) = e,$$

which is a contradiction.

Therefore, $U(1-\zeta_0, \xi), U(\xi, 1-\zeta_0) = \zeta_0$ and $U(\zeta_0, 1-\xi), U(1-\xi, \zeta_0) = 1-\zeta_0$. However,

$$U(1-\zeta_0, \zeta_0) = U(1-\zeta_0, U(1-\zeta_0, \xi)) = U(U(1-\zeta_0, 1-\zeta_0), \xi) = U(1-\zeta_0, \xi) = \zeta_0.$$

$$U(1-\zeta_0, \zeta_0) = U(U(\xi, \zeta_0), \zeta_0) = U(\xi, U(\zeta_0, \zeta_0)) = U(\xi, \zeta_0) = 1-\zeta_0,$$

which is a contradiction. Therefore, there is no ζ_0.

Finally, we will show that U has continuous underlying functions. Now, for any $\xi \in \left[0, \frac{1}{2}\right]$, either $U(\xi, 1-\xi) \in \{\xi, 1-\xi\}$ and then ξ is an idempotent point of U or $U(\xi, 1-\xi) = e$. If $U(\xi, 1-\xi) = e$ then there exists $\xi_0 \in \left[0, \frac{1}{2}\right]$ such that $\xi \in \left]\xi_0, \frac{1}{2}\right]$. Observe that in the third item it was proven that there is no idempotent point in $\left]\xi_0, \frac{1}{2}\right]$ different from $\frac{1}{2}$. Moreover, from the third item it follows that if $\xi \in [0, \xi_0]$ then it is an idempotent point. Therefore, the underlying pseudo-t-norm of U is idempotent on $[0, \xi_0]^2$ and continuous Archimedean on $]\xi_0, e]^2$ and continuous on $[0, e]^2 \setminus [0, \xi_0]^2$ due to the third item. Therefore, U restricted to $[0, \xi_0]^2$ coincides with the minimum which is continuous. Moreover, for any $\xi \in [\xi_0, e]$ we have

$$\xi_0 = U(\xi_0, e) \geq U(\xi_0, \xi) \geq U(\xi_0, \xi_0) = \xi_0,$$

and similarly one could show that $U(\xi, \xi_0) = \xi_0$, too. For $\eta < \xi_0$ can be found that

$$U(\xi, \eta) = U(\xi, \min(\xi_0, \eta)) = U(U(\xi, \xi_0), \eta) = U(\xi_0, \eta) = \eta.$$

Since $U(\xi, \cdot)$ is a continuous function and U restricted to $[0, e]^2 \setminus]\xi_0, e[^2$ is also continuous, we found out that U is continuous on $[0, e]^2$, i.e., U has continuous underlying pseudo-t-norm and from the self-duality we obtain that both underlying functions of a self-dual pseudo-uninorm are continuous. The rest of the proof follows from Theorem 2.5. □

3 Conclusion

In this paper, we extended the results from [10], where pseudo-uninorms with continuous underlying pseudo-t-norm and pseudo-t-conorms were characterized. We have proven that, unlike for uninorms, there are pseudo-uninorms that are self-dual. Further, we have characterized those representable, idempotent and pseudo-uninorms with continuous underlying functions. Note that this slight change results in self-dual pseudo-uninorms and has only a little effect on the continuity and commutativity of the pseudo-uninorm. Moreover, we have shown that whenever a pseudo-uninorm is self-dual then both of its underlying functions must be continuous.

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