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Original paper

Several fuzzifying topologies on the fuzzy real line

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Abstract

In [13], three natural fuzzifying topologies are presented on the fuzzy real line. It is proved that the three fuzzifying topologies can be induced respectively by three fuzzifying pseudo-quasi-metrics. In this paper, we present several novel fuzzifying topologies on the fuzzy real line, and discass their relations.

Keywords: Fuzzy number, the fuzzy real line, fuzzifying topology, fuzzifying pseudo-metric.

1 Introduction

As is well known, the real line is a very important topological space, and they are the foundation of various branches of mathematics and physics. Similarly, the fuzzy real line $\mathbb{R}(M)$ is also an important foundation for various branches of fuzzy mathematics. However, the topological properties of $\mathbb{R}(M)$ have not been fully investigated.

The M-fuzzy unit interval was first presented by Hutton [6] and the M-fuzzy real line was introduced by Gantner et al.[2] and Höhle [4]. Then Goetschel and Voxman presented the other definition of fuzzy numbers [3], which is generalized to M-fuzzy set theory by Huang and Shi in [5]. A GV-fuzzy number can be decomposed into two Hutton's fuzzy numbers [7]. It was proved that the M-fuzzy real line is pointwise pseudo-metrizable and the pointwise pseudo-metric function on the M-fuzzy real line was given [10]. Moreover a natural M-topology is constructed on the set of M-fuzzy numbers in the sense of [3, 5] and it can be induced by a pointwise pseudo-metric [12].

A probabilistic metric space (or a statistical metric space) is classically defined relative to a so-called t-norm [9]. In fact, it is also called a fuzzy metric space in the sense of [8]. When $* = \min$ and $L = \mathbf{2}$, a KM fuzzy metric can be regarded as an (L, M)-fuzzy metric [11].

As we know, a KM fuzzy metric can induce a fuzzifying topology [11]. In [13], three natural fuzzifying topologies are presented on the fuzzy real line. Then the notion of fuzzifying pseudo-quasi-metrics is introduced. It is proved that the three fuzzifying topologies can be induced respectively by three fuzzifying pseudo-quasi-metrics. It is worth noting that our definition of fuzzifying pseudo-metric is slightly different from that of KM-fuzzy metric.

The fuzzifying topologies and fuzzifying pseudo-quasi-metrics on the fuzzy real line are important. Recently, based on the fuzzy pseudo-metric in [13], we have developed the four fundamental arithmetic operations for fuzzy numbers, which yield highly satisfactory results.

In this paper, we aim to present several novel fuzzifying topologies on the fuzzy real line and their relations is discussed.

2 Preliminaries

Throughout this paper, M denotes a completely distributive lattice with an order-reversing involution " $'$ " and I always denotes a unit interval $[0, 1]$. The maximal element and the minimal element in M are respectively denoted by

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1 and 0. $\mathbb{R}$ denotes the set of all real numbers.

The binary relation $\prec$ in M is defined as follows: for $a, b \in M$, $a \prec b$ if and only if for every subset $D \subseteq M$, the relation $b \leq \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [1].

Definition 2.1. [2, 4, 6] *An M-fuzzy real number is an equivalence class $[x]$ of antitone maps $x : \mathbb{R} \to M$ satisfying*

$$x(-\infty) = \bigvee_{t\in\mathbb{R}} x(t) = 1 \quad \text{and} \quad x(+\infty) = \bigwedge_{t\in\mathbb{R}} x(t) = 0,$$

where the equivalence identifies two such maps x, y if and only if $\forall t > 0$, $x(t-) = y(t-)$ or $x(t+) = y(t+)$.

The set of all M-fuzzy real numbers is denoted by $\mathbb{R}(M)$, which is called the M-fuzzy real line. When $M = I$, the I-fuzzy real line is also called the fuzzy real line.

The representative function x of an M-fuzzy real number is said to be left-continuous if $x(t) = \bigwedge_{s<t} x(s) = x(t-)$ for all $t \in \mathbb{R}$. We shall not distinguish an M-fuzzy real number $[x]$ from its representative function x being left continuous.

Definition 2.2. [15] *A map $\mathcal{T} : 2^X \to M$ is called an M-fuzzifying topology if it satisfies the following conditions:*

(FYT1) $\mathcal{T}(X) = \mathcal{T}(\emptyset) = 1$;

(FYT2) $\forall A, B \in 2^X, \mathcal{T}(A \cap B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$;

(FYT3) $\forall \{A_i \mid i \in \Delta\} \subseteq 2^X, \mathcal{T}\left(\bigcup_{i\in\Delta} A_i\right) \geq \bigwedge_{i\in\Delta} \mathcal{T}(A_i)$.

Theorem 2.3. [13] *Let $\mathbb{R}(M)$ be the M-fuzzy real line. Define three mappings $\mathcal{R}, \mathcal{L}, \mathcal{T} : 2^{\mathbb{R}(M)} \to M$ such that for all $A \subseteq \mathbb{R}(M)$,*

(1) $\mathcal{R}(A) = \bigwedge_{x\in A} \bigvee_{s\in\mathbb{R}} \left(x(s+) \wedge \bigwedge_{y\notin A} y(s+)' \right)$;

(2) $\mathcal{L}(A) = \bigwedge_{x\in A} \bigvee_{r\in\mathbb{R}} \left(x(r-)' \wedge \bigwedge_{y\notin A} y(r-) \right)$;

(3) $\mathcal{T}(A) = \bigwedge_{x\in A} \bigvee_{r,s\in\mathbb{R}} \left(x(s+) \wedge x(r-)' \wedge \bigwedge_{y\notin A} (y(s+)' \vee y(r-)) \right)$.

Then $\mathcal{R}, \mathcal{L}$ and $\mathcal{T}$ are fuzzifying topologies on $\mathbb{R}(M)$.

Lemma 2.4. [10, 12, 13] *Let $\mathbb{R}(I)$ be the fuzzy real line. Define two maps $\varepsilon, \sigma : (0, 1] \times \mathbb{R}(I) \to \mathbb{R}$ such that for all $a \in (0, 1]$ and for all $x \in \mathbb{R}(I)$,*

$$\varepsilon(a, x) = \sup \{t \mid a \leq x(t-)\}, \quad \sigma(a, x) = \inf \{t \mid a \leq x(t+)'\}.$$

Then we have the following results.

(1) $\varepsilon(a, x) = \max\{t \mid a \leq x(t-)\}$, $\sigma(a, x) = \min\{t \mid a \leq x(t+)'\}$.

(2) *For all $a \in (0, 1]$ and for all $x \in \mathbb{R}(I)$,*

$$\varepsilon(a, x) = \bigwedge_{b<a} \varepsilon(b, x), \quad \sigma(a, x) = \bigvee_{b<a} \sigma(b, x).$$

(3) *For all $a \in (0, 1]$ and for all $x \in \mathbb{R}(I)$,*

$$\varepsilon(a, x) = \bigwedge_{c>1-a} \sigma(c, x), \quad \sigma(a, x) = \bigvee_{c>1-a} \varepsilon(c, x).$$

Definition 2.5. [13] *A fuzzifying pseudo-quasi-metric on a set X is a fuzzy set $D : X^2 \times [0, \infty) \to [0, 1]$ which satisfies the following (FM1)–(FM5): $\forall x, y, z \in X$ and $\forall r, s \in (0, +\infty)$,*

(FM1) $D(x, y)(0) = 1$;

(FM2) $D(x, x)(r) \leq 0.5$;

(FM3) $\bigwedge_{r>0} D(x,y)(r) = 0$;

(FM4) $D(x,y)(r) = \bigwedge_{s<r} D(x,y)(s)$, *i.e.*, $D(x,y) : [0,\infty) \to [0,1]$ *is left continuous;*

(FM5) $D(x,z)(s+r) \leq D(x,y)(s) + D(y,z)(r)$.

A fuzzifying pseudo-quasi-metric D is said to be a fuzzifying pseudo-metric if it satisfies the following (FM6):

(FM6) $D(x,y)(r) = D(y,x)(r)$.

Theorem 2.6. [13] *Let $\mathbb{R}(I)$ be the fuzzy real line. For all $x, y \in \mathbb{R}(I)$ and for all $r \in \mathbb{R}$, define*

$$D_1(x,y)(r) = \bigvee \left\{ a \in (0,1] \;\middle|\; \max\left\{ \varepsilon(a,y) - \bigvee_{c>a'} \varepsilon(c,x), 0 \right\} \geq r \right\},$$
$$D_2(y,x)(r) = \bigvee \left\{ a \in (0,1] \;\middle|\; \max\left\{ \bigwedge_{c>a'} \sigma(c,y) - \sigma(a,x), 0 \right\} \geq r \right\},$$
$$D(x,y)(r) = D_1(x,y)(r) \vee D_2(x,y)(r).$$

Then

(1) $\forall b \in (0,1]$, $b < D_1(x,y)(r) \Rightarrow \max\left\{ \varepsilon(b,y) - \bigvee_{c>b'} \varepsilon(c,x), 0 \right\} \geq r$.

(2) $\forall b \in (0,1]$, $b < D_2(y,x)(r) \Rightarrow \max\left\{ \bigwedge_{c>b'} \sigma(c,y) - \sigma(b,x), 0 \right\} \geq r$.

(3) D_1, D_2 *are fuzzifying pseudo-quasi-metrics.*

(4) D *is a fuzzifying pseudo-metric.*

The following theorem shows three fuzzifying topologies on $\mathbb{R}(I)$ can be induced by the above fuzzifying pseudo-quasi-metrics.

Theorem 2.7. [13] *Let D_1, D_2 be respectively the above fuzzifying pseudo-quasi-metrics on $\mathbb{R}(I)$ and let D be the above fuzzifying pseudo-metric on $\mathbb{R}(I)$. Then $\forall A \subseteq \mathbb{R}(I)$,*

(1) $\mathcal{L}(A) = \bigwedge_{x\in A} \bigvee_{s>0} \bigwedge_{y\notin A} D_1(x,y)(s)$;

(2) $\mathcal{R}(A) = \bigwedge_{x\in A} \bigvee_{s>0} \bigwedge_{y\notin A} D_2(x,y)(s)$;

(3) $\mathcal{T}(A) = \bigwedge_{x\in A} \bigvee_{s>0} \bigwedge_{y\notin A} D(x,y)(s)$.

3 Several fuzzifying topologies on the M-fuzzy real line

In Theorem 2.7, three fuzzifying topologies can be induced by the fuzzifying pseudo-quasi-metrics when $M = I$. In this section, we shall present several novel fuzzifying topologies on the M-fuzzy real line, which can be regarded as generalizations of crisp topologies on $\mathbb{R}$.

Theorem 3.1. *Let $\mathbb{R}(M)$ be the M-fuzzy real line. Define three mappings $\mathcal{R}_1, \mathcal{L}_1, \mathcal{T}_1 : 2^{\mathbb{R}(M)} \to M$ such that for all $A \subseteq \mathbb{R}(M)$,*

(1) $\mathcal{R}_1(A) = \bigwedge_{x\in A} \bigvee_{s\in\mathbb{R}} \left\{ x(s+) \mid \bigvee_{y\notin A} y(s+) < 1 \right\}$;

(2) $\mathcal{L}_1(A) = \bigwedge_{x\in A} \bigvee_{r\in\mathbb{R}} \left\{ x(r-)' \mid \bigwedge_{y\notin A} y(r-) > 0 \right\}$;

(3) $\mathcal{T}_1(A) = \bigwedge_{x\in A} \bigvee_{r,s\in\mathbb{R}} \left\{ x(s+) \wedge x(r-)' \mid \bigvee_{y\notin A} (y(s+) \wedge y(r-)') < 1 \right\}$.

Then $\mathcal{R}_1, \mathcal{L}_1, \mathcal{T}_1$ are fuzzifying topologies on $\mathbb{R}(M)$.

Proof. We only prove that $\mathcal{T}_1$ is a fuzzifying topology on $\mathbb{R}(M)$. The other proofs are analogous. It is obvious that $\mathcal{T}_1(\emptyset) = \mathcal{T}_1(\mathbb{R}(M)) = 1$. Now we prove that for all $A, B \in 2^{\mathbb{R}(M)}$,

$$\mathcal{T}_1(A \cap B) \geq \mathcal{T}_1(A) \wedge \mathcal{T}_1(B).$$

Suppose that $a \in M$ and $a \prec \mathcal{T}_1(A) \wedge \mathcal{T}_1(B)$. Then $a \prec \mathcal{T}_1(A)$ and $a \prec \mathcal{T}_1(B)$. Further we have

$$a \prec \mathcal{T}_1(A) = \bigwedge_{x \in A} \bigvee_{s,r \in \mathbb{R}} \left\{ x(s+) \wedge x(r-)' \mid \bigvee_{y \notin A} (y(s+) \wedge y(r-)') < 1 \right\},$$

and

$$a \prec \mathcal{T}_1(B) = \bigwedge_{x \in B} \bigvee_{s,r \in \mathbb{R}} \left\{ x(s+) \wedge x(r-)' \mid \bigvee_{y \notin B} (y(s+) \wedge y(r-)') < 1 \right\}.$$

Hence, for all $x \in A$, there exist $r, s \in \mathbb{R}$ such that

$$a \leq x(r+) \wedge x(s-)', \quad \bigvee_{y \notin A} (y(r+) \wedge y(s-)') < 1,$$

and for all $x \in B$, there exist $u, v \in \mathbb{R}$ such that

$$a \leq x(u+) \wedge x(v-)', \quad \bigvee_{y \notin B} (y(u+) \wedge y(v-)') < 1.$$

Take $c, b \in (0,1)$ such that

$$\bigvee_{y \notin A} (y(r+) \wedge y(s-)') < c < 1, \text{ and } \bigvee_{y \notin B} (y(r+) \wedge y(s-)') < b < 1.$$

This implies for all $y \notin A \cap B$, it follows

$$y(r+) \wedge y(s-)' < c < 1, \text{ or } y(r+) \wedge y(s-)' < b < 1,$$

and

$$a \quad \leq \quad x(r+) \wedge x(u+) \wedge x(s-)' \wedge x(v-)' = x(\max\{r,u\}+) \wedge x(\min\{s,v\}-)'. \tag{1}$$

Obviously, for all $y \notin A \cap B$, it holds

$$y(r+) \wedge y(s-)' \wedge y(u+) \wedge y(v-)' = y(\max\{r,u\}+) \wedge y(\min\{s,v\}-)' < c \wedge b,$$

which implies

$$\bigvee_{y \notin A \cap B} (y(r+) \wedge y(s-)' \wedge y(u+) \wedge y(v-)') = \bigvee_{y \notin A \cap B} (y(\max\{r,u\}+) \wedge y(\min\{s,v\}-)') \leq c \wedge b < 1. \tag{2}$$

By means of formula (1) and formula (2) we obtain

$$a \quad \leq \quad \bigvee_{r,s,u,v \in \mathbb{R}} \left\{ x(\max\{r,u\}+) \wedge x(\min\{s,v\}-)' \mid \bigvee_{y \notin A \cap B} (y(\max\{r,u\}+) \wedge y(\min\{s,v\}-)') < 1 \right\}.$$

This shows

$$a \leq \bigwedge_{x \in A \cap B} \bigvee_{t,w \in \mathbb{R}} \left\{ x(t+) \wedge x(w-)' \mid \bigvee_{y \notin A \cap B} (y(t+) \wedge y(w-)') < 1 \right\} = \mathcal{T}_1(A \cap B).$$

Therefore we obtain the proof of $\mathcal{T}_1(A \cap B) \geq \mathcal{T}_1(A) \wedge \mathcal{T}_1(B)$.

Next we prove that for any family of sets $\{A_i \mid i \in \Omega\} \subseteq 2^{\mathbb{R}(M)}$,

$$\mathcal{T}_1\left(\bigcup_{i\in\Omega} A_i\right) \geq \bigwedge_{i\in\Omega} \mathcal{T}_1(A_i).$$

Suppose that $a \in M$ and $a \prec \bigwedge_{i\in\Omega} \mathcal{T}_1(A_i)$. Then for all $i \in \Omega$, it follows that

$$a \prec \mathcal{T}_1(A_i) = \bigwedge_{x\in A_i} \bigvee_{s,r\in\mathbb{R}} \left\{ x(s+) \wedge x(r-)' \mid \bigvee_{y\notin A_i} (y(r+) \wedge y(s-)') < 1 \right\}.$$

This implies that for all $i \in \Omega$ and for all $x \in A_i$,

$$a \prec \bigvee_{s,r\in\mathbb{R}} \left\{ x(r+) \wedge x(s-)' \mid \bigvee_{y\notin A_i} (y(r+) \wedge y(s-)') < 1 \right\}.$$

Hence for all $x \in \bigcup_{i\in\Omega} A_i$, there exists $k \in \Omega$ and $r_k, s_k \in \mathbb{R}$ such that $x \in A_k$ and it follows

$$a \leq x(r_k+) \wedge x(s_k-)' \ \text{ and } \bigvee_{y\notin A_k} (y(r_k+) \wedge y(s_k-)') < 1.$$

In particular, we have

$$\bigvee_{y\notin \bigcup_{i\in\Omega} A_i} (y(r_i+) \wedge y(s_i-)') \leq \bigvee_{y\notin A_k} (y(r_i+) \wedge y(s_i-)') < 1.$$

This implies

$$a \leq \bigvee_{u,v\in\mathbb{R}} \left\{ x(u+) \wedge x(v-)' \mid \bigvee_{y\notin \bigcup_{i\in\Omega} A_i} (y(u+) \wedge y(v-)') < 1 \right\}.$$

Therefore we can obtain

$$a \leq \bigwedge_{x\in \bigcup_{i\in\Omega} A_i} \bigvee_{u,v\in\mathbb{R}} \left\{ x(u+) \wedge x(v-)' \mid \bigvee_{y\notin \bigcup_{i\in\Omega} A_i} (y(u+) \wedge y(v-)') < 1 \right\} = \mathcal{T}_1\left(\bigcup_{i\in\Omega} A_i\right).$$

It is proved that $\mathcal{T}_1\left(\bigcup_{i\in\Omega} A_i\right) \geq \bigwedge_{i\in\Omega} \mathcal{T}_1(A_i)$. Therefore $\mathcal{T}_1$ is an M-fuzzifying topology on $\mathbb{R}(M)$. □

Theorem 3.2. *Let $\mathbb{R}(M)$ be the M-fuzzy real line. Define three mappings $\mathcal{R}_2, \mathcal{L}_2, \mathcal{T}_2 : 2^{\mathbb{R}(M)} \to M$ such that for all $A \subseteq \mathbb{R}(M)$,*

(1) $\mathcal{R}_2(A) = \bigwedge_{x\in A} \bigvee_{s\in\mathbb{R}} \left\{ x(s+) \mid \bigvee_{y\notin A} y(s+) = 0 \right\};$

(2) $\mathcal{L}_2(A) = \bigwedge_{x\in A} \bigvee_{r\in\mathbb{R}} \left\{ x(r-)' \mid \bigwedge_{y\notin A} y(r-) = 1 \right\};$

(3) $\mathcal{T}_2(A) = \bigwedge_{x\in A} \bigvee_{r,s\in\mathbb{R}} \left\{ x(s+) \wedge x(r-)' \mid \bigvee_{y\notin A} (y(s+) \wedge y(r-)') = 0 \right\}.$

Then $\mathcal{R}_2, \mathcal{L}_2, \mathcal{T}_2$ are fuzzifying topologies on $\mathbb{R}(M)$.

Proof. We only prove that $\mathcal{T}_2$ is a fuzzifying topology on $\mathbb{R}(M)$. The other proofs are analogous. It is obvious that $\mathcal{T}_2(\emptyset) = \mathcal{T}_2(\mathbb{R}(I)) = 1$. Now we prove that for all $A, B \in 2^{\mathbb{R}(M)}$,

$$\mathcal{T}_2(A \cap B) \geq \mathcal{T}_2(A) \wedge \mathcal{T}_2(B).$$

Suppose that $a \in M$ and $a \prec \mathcal{T}_2(A) \wedge \mathcal{T}_2(B)$. Then $a \prec \mathcal{T}_2(A)$ and $a \prec \mathcal{T}_2(B)$. Further we have

$$a \prec \mathcal{T}_2(A) = \bigwedge_{x \in A} \bigvee_{s,r \in \mathbb{R}} \left\{ x(s+) \wedge x(r-)' \mid \bigvee_{y \notin A} (y(s+) \wedge y(r-)') = 0 \right\},$$

and

$$a \prec \mathcal{T}_2(B) = \bigwedge_{x \in B} \bigvee_{s,r \in \mathbb{R}} \left\{ x(s+) \wedge x(r-)' \mid \bigvee_{y \notin B} (y(s+) \wedge y(r-)') = 0 \right\}.$$

Hence for all $x \in A$, there exist $r, s \in \mathbb{R}$ such that

$$a \leq x(r+) \wedge x(s-)', \ \bigvee_{y \notin A} (y(r+) \wedge y(s-)') = 0,$$

and for all $x \in B$, there exist $u, v \in \mathbb{R}$ such that

$$a \leq x(u+) \wedge x(v-)', \ \bigvee_{y \notin B} (y(u+) \wedge y(v-)') = 0.$$

This implies for all $y \notin A \cap B$, it follows $y(r+) \wedge y(s-)' = 0$ or $y(u+) \wedge y(v-)' = 0$. Further, for all $x \in A \cap B$, we have

$$a \quad \leq \quad x(r+) \wedge x(u+) \wedge x(s-)' \wedge x(v-)' = x(\max\{r,u\}+) \wedge x(\min\{s,v\}-)'.$$

Obviously, for all $y \notin A \cap B$, it holds

$$y(r+) \wedge y(s-)' \wedge y(u+) \wedge y(v-)' = y(\max\{r,u\}+) \wedge y(\min\{s,v\}-)' = 0.$$

Thus we have

$$a \quad \leq \quad \bigvee_{r,s \in \mathbb{R}} \left\{ x(\max\{r,u\}+) \wedge x(\min\{s,v\}-)' \mid \bigvee_{y \notin A \cap B} (y(\max\{r,u\}+) \wedge y(\min\{s,v\}-)) = 0 \right\}.$$

This shows

$$a \leq \bigwedge_{x \in A \cap B} \bigvee_{t,w \in \mathbb{R}} \left\{ x(t+) \wedge x(w-)' \mid \bigvee_{y \notin A \cap B} (y(t+) \wedge y(w-)') = 0 \right\} = \mathcal{T}_2(A \cap B).$$

Therefore, we obtain the proof of $\mathcal{T}_2(A \cap B) \geq \mathcal{T}_2(A) \wedge \mathcal{T}_2(B)$.

Next we prove that for any family of sets $\{A_i \mid i \in \Omega\} \subseteq 2^{\mathbb{R}(I)}$,

$$\mathcal{T}_2\left(\bigcup_{i \in \Omega} A_i \right) \geq \bigwedge_{i \in \Omega} \mathcal{T}_2(A_i).$$

Suppose that $a \in M$ and $a \prec \bigwedge_{i \in \Omega} \mathcal{T}_2(A_i)$. Then for all $i \in \Omega$, it follows that

$$a \prec \mathcal{T}_2(A_i) = \bigwedge_{x \in A_i} \bigvee_{s,r \in \mathbb{R}} \left\{ x(s+) \wedge x(r-)' \mid \bigvee_{y \notin A_i} (y(r+) \wedge y(s-)') = 0 \right\}.$$

This implies that for all $i \in \Omega$ and for all $x \in A_i$,

$$a \prec \bigvee_{s,r \in \mathbb{R}} \left\{ x(r+) \wedge x(s-)' \mid \bigvee_{y \notin A_i} (y(r+) \wedge y(s-)') = 0 \right\}.$$

Hence, for all $x \in \bigcup_{i\in\Omega} A_i$, there exist $k \in \Omega$ and $r_k, s_k \in \mathbb{R}$ such that $x \in A_k$ and it follows

$$a \leq x(r_k+) \wedge x(s_k-)' \text{ and } \bigvee_{y \notin A_k} (y(r_k+) \wedge y(s_k-)') = 0.$$

In particular, we have $\bigvee_{y \notin \bigcup_{i\in\Omega} A_i} (y(r_k+) \wedge y(s_k-)') \leq \bigvee_{y\notin A_k} (y(r_k+) \wedge y(s_k-)') = 0$. in this case, we have

$$a \leq \bigwedge_{x \in \bigcup_{i\in\Omega} A_i} \bigvee_{u,v\in\mathbb{R}} \left\{ x(u+) \wedge x(v-)' \mid \bigvee_{y \notin \bigcup_{i\in\Omega} A_i} (y(u+) \wedge y(v-)') = 0 \right\} = \mathcal{T}_2\left(\bigcup_{i\in\Omega} A_i\right).$$

It is proved that $\mathcal{T}_2\left(\bigcup_{i\in\Omega} A_i\right) \geq \bigwedge_{i\in\Omega} \mathcal{T}_2(A_i)$. Therefore $\mathcal{T}_2$ is an M-fuzzifying topology on $\mathbb{R}(M)$. □

4 Fuzzifying topologies on the [0, 1]-fuzzy real line

In this section, we shall discuss a special case, that is, $M = I = [0, 1]$. We shall present three fuzzifying topologies on $\mathbb{R}(I)$. They are also natural extensions of three topologies on $\mathbb{R}$.

Theorem 4.1. *Let $\mathbb{R}(I)$ be the fuzzy real line. Define three mappings $\mathcal{R}_3, \mathcal{L}_3, \mathcal{T}_3 : 2^{\mathbb{R}(I)} \to I$ such that for all $A \subseteq \mathbb{R}(I)$,*

(1) $\mathcal{R}_3(A) = \bigwedge_{x\in A} \bigvee_{s\in\mathbb{R}} \bigwedge_{y\notin A} \max\{x(s+) - y(s+), 0\}$;

(2) $\mathcal{L}_3(A) = \bigwedge_{x\in A} \bigvee_{r\in\mathbb{R}} \bigwedge_{y\notin A} \max\{y(r-) - x(r-), 0\}$;

(3) $\mathcal{T}_3(A) = \bigwedge_{x\in A} \bigvee_{r,s\in\mathbb{R}} \bigwedge_{y\notin A} (\max\{\max\{x(s+) - y(s+), 0\}, \max\{y(r-) - x(r-), 0\}\})$.

Then $\mathcal{R}_3, \mathcal{L}_3$ and $\mathcal{T}_3$ are fuzzifying topologies on $\mathbb{R}(I)$.

Proof. We only prove that $\mathcal{T}_3$ is a fuzzifying topology on $\mathbb{R}(I)$. The other proofs are analogous.

It is obvious that $\mathcal{T}_3(\emptyset) = \mathcal{T}_3(\mathbb{R}(I)) = 1$. Now we prove that for all $A, B \in 2^{\mathbb{R}(I)}$,

$$\mathcal{T}_3(A \cap B) \geq \mathcal{T}_3(A) \wedge \mathcal{T}_3(B).$$

Suppose that $a \in [0, 1)$ and $a < \mathcal{T}_3(A) \wedge \mathcal{T}_3(B)$. Then $a < \mathcal{T}_3(A)$ and $a < \mathcal{T}_3(B)$. Further we have

$$a < \mathcal{T}_3(A) = \bigwedge_{x\in A} \bigvee_{s,r\in\mathbb{R}} \bigwedge_{y\notin A} \max\{\max\{y(r-) - x(r-), 0\}, \max\{x(s+) - y(s+), 0\}\},$$

and

$$a < \mathcal{T}_3(B) = \bigwedge_{x\in B} \bigvee_{s,r\in\mathbb{R}} \bigwedge_{y\notin B} \max\{\max\{y(r-) - x(r-), 0\}, \max\{x(s+) - y(s+), 0\}\}.$$

Hence for all $x \in A$, there exists $s, r \in \mathbb{R}$ such that

$$a < \max\{\max\{y(r-) - x(r-), 0\}, \max\{x(s+) - y(s+), 0\}\}, \ \forall y \notin A,$$

and for all $x \in B$, there exists $u, v \in \mathbb{R}$ such that

$$a < \max\{\max\{y(u-) - x(u-), 0\}, \max\{x(v+) - y(v+), 0\}\}, \ \forall y \notin B.$$

This implies for all $x \in A \cap B$ and all $y \notin A \cap B$, there exists $t \in \{r, u\}$ and $w \in \{s, v\}$ such that

$$a < \max\{\max\{y(t-) - x(t-), 0\}, \max\{x(w+) - y(w+), 0\}\}, \ \forall y \notin A \cap B.$$

This shows

$$\begin{aligned} a &\leq \bigwedge_{x\in A\cap B} \bigvee_{t,w\in\mathbb{R}} \bigwedge_{y\notin A\cap B} \max\{\max\{y(t-) - x(t-), 0\}, \max\{x(w+) - y(w+), 0\}\} \\ &= \mathcal{T}_3(A \cap B). \end{aligned}$$

Therefore we have $\mathcal{T}_3(A\cap B)\geq \mathcal{T}_3(A)\wedge \mathcal{T}_3(B)$.

Next we prove that for any family of sets $\{A_i \mid i\in\Omega\}\subseteq 2^{\mathbb{R}(I)}$,

$$\mathcal{T}_3\left(\bigcup_{i\in\Omega}A_i\right)\geq \bigwedge_{i\in\Omega}\mathcal{T}_3(A_i).$$

Suppose that $a\in[0,1)$ and $a<\bigwedge_{i\in\Omega}\mathcal{T}_3(A_i)$. Then for all $i\in\Omega$, it follows that

$$a<\mathcal{T}_3(A_i)=\bigwedge_{x\in A_i}\bigvee_{s,r\in\mathbb{R}}\bigwedge_{y\notin A_i}\max\left\{\max\left\{y(r-)-x(r-),0\right\},\max\left\{x(s+)-y(s+),0\right\}\right\}.$$

This implies that for all $i\in\Omega$ and for all $x\in A_i$, there exists $s_i,r_i\in\mathbb{R}$ such that

$$a<\max\left\{\max\left\{y(r_i-)-x(r_i-),0\right\},\max\left\{x(s_i+)-y(s_i+),0\right\}\right\},\forall y\notin A_i.$$

In particular, for all $x\in\bigcup_{i\in\Omega}A_i$ and for all $y\notin\bigcup_{i\in\Omega}A_i$, there exists $k\in\Omega$ such that $x\in A_k$, and

$$a<\max\left\{\max\left\{y(r_k-)-x(r_k-),0\right\},\max\left\{x(s_k+)-y(s_k+),0\right\}\right\}.$$

Therefore we can obtain

$$a\leq\bigwedge_{x\in\bigcup_{i\in\Omega}A_i}\bigvee_{r,s\in\mathbb{R}}\bigwedge_{y\notin\bigcup_{i\in\Omega}A_i}\max\left\{\max\left\{y(r-)-x(r-),0\right\},\max\left\{x(s+)-y(s+),0\right\}\right\}=\mathcal{T}_3\left(\bigcup_{i\in\Omega}A_i\right).$$

It is proved that $\mathcal{T}_3\left(\bigcup_{i\in\Omega}A_i\right)\geq\bigwedge_{i\in\Omega}\mathcal{T}_3(A_i)$. The proof is completed. □

5 Comparison of several fuzzy topologies

In this section, we shall discuss the relations among $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{T}$.

Theorem 5.1. $\forall A\subseteq\mathbb{R}(M)$*, the following facts are true.*

(1) $\mathcal{R}(A)\leq\mathcal{R}_2(A)\leq\mathcal{R}_1(A)$,

(2) $\mathcal{L}(A)\leq\mathcal{L}_2(A)\leq\mathcal{L}_1(A)$,

(3) $\mathcal{T}(A)\leq\mathcal{T}_2(A)\leq\mathcal{T}_1(A)$.

Proof. We only prove (3). The proofs of (1) and (2) are similar to (3).

(3) In order to prove $\forall A\subseteq\mathbb{R}(I)$, $\mathcal{T}(A)\leq\mathcal{T}_2(A)$, we need to prove

$$\begin{aligned}\mathcal{T}(A)&=\bigwedge_{x\in A}\bigvee_{r,s\in\mathbb{R}}\left(x(s+)\wedge x(r-)'\wedge\bigwedge_{y\notin A}\left(y(s+)'\vee y(r-)\right)\right)\\&\leq\bigwedge_{x\in A}\bigvee_{r,s\in\mathbb{R}}\left\{x(s+)\wedge x(r-)'\mid\bigvee_{y\notin A}\left(y(s+)\wedge y(r-)'\right)=0\right\}=\mathcal{T}_2(A).\end{aligned}$$

Suppose

$$a<\mathcal{T}(A)=\bigwedge_{x\in A}\bigvee_{r,s\in\mathbb{R}}\left(x(s+)\wedge x(r-)'\wedge\bigwedge_{y\notin A}\left(y(s+)'\vee y(r-)\right)\right).$$

Then for all $x\in A$, there exists $s,r\in\mathbb{R}$ such that

$$a<x(s+)\wedge x(r-)'\wedge\bigwedge_{y\notin A}\left(y(s+)'\vee y(r-)\right).$$

This implies

$$a < x(s+) \wedge x(r-)', \text{ if } \bigwedge_{y \notin A} (y(s+)' \vee y(r-)) = 1.$$

Thus we have

$$a \leq \bigwedge_{x \in A} \bigvee_{r,s \in \mathbb{R}} \left\{ x(s+) \wedge x(r-)' \mid \bigvee_{y \notin A} (y(s+) \wedge y(r-)') = 0 \right\} = \mathcal{T}_2(A).$$

Therefore it follows $\mathcal{T}(A) \leq \mathcal{T}_2(A)$. The proof of $\mathcal{T}_2(A) \leq \mathcal{T}_1(A)$ is obvious. □

When $M = I$, we obtain the following result.

Theorem 5.2. $\forall A \subseteq \mathbb{R}(I)$, *the following facts are true.*

(1) $\mathcal{L}_3(A) \leq \mathcal{L}(A)$;

(2) $\mathcal{R}_3(A) \leq \mathcal{R}(A)$;

(3) $\mathcal{T}_3(A) \leq \mathcal{T}(A)$.

Proof. (1) In order to prove $\mathcal{L}_3(A) \leq \mathcal{L}(A)$ holds $\forall A \subseteq \mathbb{R}(I)$, we need to prove

$$\bigwedge_{x \in A} \bigvee_{r \in \mathbb{R}} \bigwedge_{y \notin A} (y(r-) - x(r-)) \leq \bigwedge_{x \in A} \bigvee_{s>0} \bigwedge_{y \notin A} D_1(x,y)(s).$$

Suppose $a < \mathcal{L}_3(A) = \bigwedge_{x \in A} \bigvee_{r \in \mathbb{R}} \bigwedge_{y \notin A} (y(r-) - x(r-))$. Then for all $x \in A$, there exists $r \in \mathbb{R}$ such that $a < y(r-) - x(r-)$, $\forall y \notin A$, i.e., $a + x(r-) < y(r-)$. Take $b \in (0,1)$ such that $a + x(r-) < b < y(r-)$. Then we have

$$\varepsilon(a,y) = \sup\{t \in \mathbb{R} \mid a \leq y(t-)\} \geq \sup\{t \in \mathbb{R} \mid b \leq y(t-)\} \geq r, \ \forall y \not\leq A.$$

From $x(r-) < b - a$ we know that there exists $t \in \mathbb{R}$ such that $t < r$ and $x(t-) < b - a \leq 1 - a$. Thus $\forall c > 1 - a$, it follows $\varepsilon(c,x) < t$, which implies $\bigvee_{c > 1-a} \varepsilon(c,x) \leq t < r$. Let $s = r - \bigvee_{c>1-a} \varepsilon(c,x) > 0$. Then

$$\varepsilon(a,y) - \bigvee_{c>1-a} \varepsilon(c,x) \geq r - \bigvee_{c>1-a} \varepsilon(c,x) = s > 0.$$

This shows

$$D_1(x,y)(s) = \bigvee \left\{ a \in (0,1] \mid \max\left\{ \varepsilon(a,y) - \bigvee_{c>1-a} \varepsilon(c,x), 0 \right\} \geq s \right\} \geq a.$$

Hence we obtain $a \leq \bigwedge_{x \in A} \bigvee_{s>0} \bigwedge_{y \notin A} D_1(x,y)(s)$. Therefore it follows

$$\mathcal{L}_3(A) = \bigwedge_{x \in A} \bigvee_{r \in \mathbb{R}} \bigwedge_{y \notin A} \{y(r-) - x(r-), 0\} \leq \bigwedge_{x \in A} \bigvee_{s>0} \bigwedge_{y \notin A} D_1(x,y)(s) = \mathcal{L}(A).$$

(2) In order to prove $\forall A \subseteq \mathbb{R}(I)$, $\mathcal{R}_3(A) \leq \mathcal{R}(A)$, we need to prove

$$\mathcal{R}_3(A) = \bigwedge_{x \in A} \bigvee_{r \in \mathbb{R}} \bigwedge_{y \notin A} \max\{x(r+) - y(r+), 0\} \leq \bigwedge_{x \in A} \bigvee_{s>0} \bigwedge_{y \notin A} D_2(x,y)(s) = \mathcal{R}(A).$$

Suppose

$$a < \mathcal{R}_3(A) = \bigwedge_{x \in A} \bigvee_{r \in \mathbb{R}} \bigwedge_{y \notin A} \max\{x(r+) - y(r+), 0\}.$$

Then for all $x \in A$, there exists $r \in \mathbb{R}$ such that

$$a < x(r+) - y(r+), \ \forall y \notin A.$$

This implies

$$a + y(r+) < x(r+) = \bigvee_{t>r} x(t+).$$

Further we know there exists $t > r$ such that

$$a + y(r+) < x(t+) \;\; \text{or} \;\; 1 - y(r+) > 1 - x(t+) + a.$$

Take $b \in (0, 1)$ such that $1 - y(r+) > b > 1 - x(t+) + a$. Then we have

$$1 - y(r+) > b \text{ and } 1 - a \geq b - a > 1 - x(t+).$$

This shows $\sigma(b, y) \leq r$ and for all $c > 1 - a$, it holds $c > 1 - x(t+)$, that is,

$$\bigwedge_{c>1-a} \sigma(c, x) = \bigwedge_{c>1-a} \min\{t \in \mathbb{R} \mid c \leq 1 - x(t+)\} \geq t > r.$$

Let $s = \bigwedge_{c>1-a} \sigma(c, x) - r > 0$. Then

$$\bigwedge_{c>1-a} \sigma(c, x) - \sigma(a, y) \geq \bigwedge_{c>1-a} \sigma(c, x) - \sigma(b, y) \geq \bigwedge_{c>1-a} \sigma(c, x) - r = s > 0.$$

This shows

$$D_2(x, y)(s) = \bigvee \left\{ a \in (0, 1] \mid \max \left\{ \bigwedge_{c>1-a} \sigma(c, x) - \sigma(a, y), 0 \right\} \geq s \right\} \geq a.$$

Hence we obtain $a \leq \bigwedge_{x\in A} \bigvee_{s>0} \bigwedge_{y\notin A} D_2(x, y)(s)$. Therefore it follows

$$\mathcal{R}_3(A) = \bigwedge_{x\in A} \bigvee_{r\in\mathbb{R}} \bigwedge_{y\notin A} \{x(r+) - y(r+), 0\} \leq \bigwedge_{x\in A} \bigvee_{s>0} \bigwedge_{y\notin A} D_2(x, y)(s) = \mathcal{R}(A).$$

(3) In order to prove $\forall A \subseteq \mathbb{R}(I)$, $\mathcal{T}_3(A) \leq \mathcal{T}(A)$, we need to prove

$$\begin{aligned} \mathcal{T}_3(A) &= \bigwedge_{x\in A} \bigvee_{s,r\in\mathbb{R}} \bigwedge_{y\notin A} \max\left\{\max\left\{x(r+) - y(r+), 0\right\}, \max\left\{y(s-) - x(s-), 0\right\}\right\} \\ &\leq \bigwedge_{x\in A} \bigvee_{t>0} \bigwedge_{y\notin A} D(x, y)(t) = \mathcal{T}(A). \end{aligned}$$

Suppose

$$a < \mathcal{T}_3(A) = \bigwedge_{x\in A} \bigvee_{s,r\in\mathbb{R}} \bigwedge_{y\notin A} \max\left\{\max\left\{x(r+) - y(r+), 0\right\}, \max\left\{y(s-) - x(s-), 0\right\}\right\}.$$

Then for all $x \in A$, there exists $r, s \in \mathbb{R}$ such that

$$a < \max\left\{\max\left\{x(r+) - y(r+), 0\right\}, \max\left\{y(s-) - x(s-), 0\right\}\right\}, \;\; \forall y \notin A.$$

This implies

$$a < \max\left\{x(r+) - y(r+), 0\right\} \text{ or } a < \max\left\{y(s-) - x(s-), 0\right\}.$$

By the proof of (1) and (2), we know there exist $r, s > 0$ such that

$$D_1(x, y)(r) \geq a \text{ or } D_2(x, y)(s) \geq a.$$

Take $t = \min\{r, s\}$. Then we obtain

$$a \leq D_1(x, y)(r) \vee D_2(x, y)(s) \leq D_1(x, y)(t) \vee D_2(x, y)(t) = D(x, y)(t).$$

Therefore it follows

$$a \leq \bigwedge_{x\in A} \bigvee_{t>0} \bigwedge_{y\notin A} D(x, y)(t) = \mathcal{T}(A).$$

The proof of $\mathcal{T}_3(A) \leq \mathcal{T}(A)$ is completed. □

It is easy to check the following result.

Corollary 5.3. *When $M = \{0, 1\}$, we have $\mathcal{T} = \mathcal{T}_1 = T_2 = \mathcal{T}_3$, in this case, $\mathcal{T}$ is equivalent to the Euclidean topology on $\mathbb{R}$.*

Example 5.4. Let $M = \{0, e, 1\}$, where $0 < e < 1$ and $e' = e$. $u, \underline{m} \in \mathbb{R}(M)$ are defined as follows.

$$u(t) = \begin{cases} 1, & t \in [-\infty, -1], \\ e, & t \in (-1, 0], \\ 0, & t \in (0, +\infty]. \end{cases} \qquad \underline{m}(t) = \begin{cases} 1, & t \in [-\infty, m], \\ 0, & t \in (m, +\infty]. \end{cases} \quad m \in (-1, 0].$$

Let $A = \{x \in \mathbb{R}(M) \mid x \not\leq u\}$ and $A^c = \{y \in \mathbb{R}(M) \mid y \leq u\}$. $\forall x \in A$, we know that there exists $t \in \mathbb{R}$ such that $x(t) > u(t)$. In this case, we have $t > -1$. Let

$$\begin{aligned} A_1 &= \{x \in A \mid \exists s \in (-1, 0],\ \text{s.t.},\ \ x(s) > u(s))\}, \\ A_2 &= \{x \in A \mid \forall s \in (-1, 0],\ x(s) \leq u(s),\ \text{but } \exists t \in (0, +\infty), \text{s.t.}, x(t) > u(t) = 0\}. \end{aligned}$$

Obviously we have $A = A_1 \cup A_2$. From the definition of A_1 and A_2, we know that $\underline{m} \in A_1$, and $x(-1+) = 1$ for all $x \in A_1$. Further, it follows $y(s) = e$ if $s \in (-1, 0]$ for all $y \in A_2$. Moreover for all $z \in A^c$, we have

$$z(s) \leq u(s) = \begin{cases} 1, & s \in (-\infty, -1], \\ e, & s \in (-1, 0], \\ 0, & s \in (0, +\infty], \end{cases} \quad \text{which implies} \quad z(s)' \geq u(s)' = \begin{cases} 0, & s \in (-\infty, -1], \\ e, & s \in (-1, 0], \\ 1, & s \in (0, +\infty]. \end{cases}$$

In this case, we have

$$\begin{aligned} \mathcal{T}_1(A) &= \bigwedge_{x \in A} \bigvee_{r, s \in \mathbb{R}} \left\{ x(s+) \wedge x(r-)' \mid \bigvee_{z \in A^c} (z(s+) \wedge z(r-)') < 1 \right\} \\ &\geq \bigwedge_{x \in A} \bigvee_{s \in \mathbb{R}, r = +\infty} \left\{ x(s+) \wedge x(r-)' \mid \bigvee_{z \in A^c} (z(s+) \wedge z(r-)') < 1 \right\} \\ &= \bigwedge_{x \in A} \bigvee_{s \in \mathbb{R}} \left\{ x(s+) \mid \bigvee_{z \in A^c} z(s+) < 1 \right\} \\ &= \bigwedge_{x \in A} \bigvee_{s \geq -1} \left\{ x(s+) \mid \bigvee_{z \in A^c} z(s+) < 1 \right\} = \bigwedge_{x \in A} x(-1+) \geq e. \\ \mathcal{T}_2(A) &= \bigwedge_{x \in A} \bigvee_{r, s \in \mathbb{R}} \left\{ x(s+) \wedge x(r-)' \mid \bigvee_{z \in A^c} (z(s+) \wedge z(r-)') = 0 \right\} \\ &= \bigwedge_{x \in A} \bigvee_{r, s \in \mathbb{R}} \{ x(s+) \wedge x(r-)' \mid \forall z \in A^c,\ z(s+) = 0 \text{ or } \ z(r-) = 1 \} \\ &\leq \bigwedge_{x \in A} \bigvee_{s \geq 0} x(s+) = \bigwedge_{x \in A} x(0+) \leq \underline{m}(0+) = 0. \end{aligned}$$

This shows $\mathcal{T}_2 \neq \mathcal{T}_1$, i.e., $\mathcal{T}_2 < \mathcal{T}_1$.

Moreover, from the following results, we know $\mathcal{T} \neq \mathcal{T}_2$, i.e., $\mathcal{T} < \mathcal{T}_2$.

$$\begin{aligned}
\mathcal{T}(A_1) &= \bigwedge_{x\in A_1} \bigvee_{r,s\in\mathbb{R}} \bigwedge_{y\in(A_1)^c} (x(s+) \wedge x(r-)' \wedge (y(s+)' \vee y(r-))) \\
&\leq \bigwedge_{m\in(-1,0)} \bigvee_{r,s\in\mathbb{R},s\geq -1} \bigwedge_{y\in(A_1)^c} (\underline{m}(s+) \wedge \underline{m}(r-)' \wedge (y(s+)' \vee y(r-))) \\
&= \bigwedge_{m\in(-1,0)} \bigvee_{-1\leq s<m<r} \bigwedge_{y\in A^c} (\underline{m}(s+) \wedge \underline{m}(r-)' \wedge (y(s+)' \vee y(r-))) \\
&= \bigwedge_{m\in(-1,0)} \bigvee_{-1\leq s<m<r\leq 0} \bigwedge_{y\in A^c} (y(s+)' \vee y(r-)) \\
&= \bigwedge_{m\in(-1,0)} \bigvee_{-1\leq s<m<r\leq 0} \bigwedge_{y\in A^c} y(s+)' \\
&= \bigwedge_{m\in(-1,0)} \bigvee_{-1\leq s<m<r\leq 0} u(s+)' = \bigwedge_{m\in(-1,0)} u(m)' = e' = e.
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}_2(A_1) &= \bigwedge_{x\in A_1} \bigvee_{r,s\in\mathbb{R}} \left\{ x(s+) \wedge x(r-)' \mid \bigvee_{y\in(A_1)^c} (y(s+) \wedge y(r-)') = 0 \right\} \\
&\geq \bigwedge_{x\in A_1} \bigvee_{s\geq -1, r=+\infty} \left\{ x(s+) \wedge x(r-)' \mid \bigvee_{y\in(A_1)^c} (y(s+) \wedge y(r-)') = 0 \right\} \\
&= \bigwedge_{x\in A_1} \bigvee_{s\geq -1} \left\{ x(s+) \mid \bigvee_{y\in(A_1)^c} y(s+) = 0 \right\} \\
&\geq \bigwedge_{x\in A_1} \bigvee_{s\geq -1} x(s+) = \bigwedge_{x\in A_1} x(-1+) = 1.
\end{aligned}$$

Example 5.5. Let $M = [0,1]$. $u \in \mathbb{R}(M)$ and $v \in \mathbb{R}(M)$ are defined as follows.

$$u(t) = \begin{cases} 1, & t\in[-\infty,-1], \\ 0.5, & t\in(-1,1], \\ 0, & t\in(1,+\infty]. \end{cases} \qquad v(t) = \begin{cases} 1, & t\in[-\infty,-1], \\ 0.6, & t\in[-1,0], \\ 0, & t\in(0,+\infty]. \end{cases}$$

Let $A = \{x \in \mathbb{R}(M) \mid x \not\leq u\}$ and $A^c = \{y \in \mathbb{R}(M) \mid y \leq u\}$. $\forall x \in A$, we know that there exists $s \in \mathbb{R}$ such that $x(s) > u(s)$. In this case, we have $s > -1$. Let

$$\begin{aligned}
A_1 &= \{x \in A \mid \exists s \in (-1,1],\ \text{s.t.},\ \ x(s) > u(s) = 0.5\}, \\
A_2 &= \{x \in A \mid \forall s \in (-1,1],\ x(s) \leq u(s) = 0.5,\ \text{but}\ \exists t \in (1,+\infty],\ \text{s.t.},\ x(t) > u(t) = 0\}.
\end{aligned}$$

Obviously we have $A = A_1 \cup A_2$, $(A_1)^c = A^c \cup A_2$ and $v \in A_1$. From the definition of A_1 and A_2, we know that

$$\begin{gathered}
x(-1+) > 0.5, \quad \forall x \in A_1; \\
0.5 \geq x(-1+) \geq x(1+) > 0, \quad \forall x \in A_2; \\
y(s+) \leq u(s+) = \begin{cases} 1, & s\in[-\infty,-1), \\ 0.5, & s\in[-1,1), \\ 0, & s\in[1,+\infty), \end{cases} \quad \forall y \in A^c; \\
y(s+)' \geq u(s+)' = \begin{cases} 0, & s\in[-\infty,-1), \\ 0.5, & s\in[-1,1), \\ 1, & s\in[1,+\infty), \end{cases} \quad \forall y \in A^c.
\end{gathered}$$

$$
\begin{aligned}
\mathcal{T}(A_1) &= \bigwedge_{x\in A_1} \bigvee_{r,s\in\mathbb{R}} \bigwedge_{y\in(A_1)^c} (x(s+) \wedge x(r-)' \wedge (y(s+)' \vee y(r-))) \\
&\geq \bigwedge_{x\in A_1} \bigvee_{s\geq -1, r=+\infty} \bigwedge_{y\in(A_1)^c} (x(s+) \wedge x(r-)' \wedge (y(s+)' \vee y(r-))) \\
&= \bigwedge_{x\in A_1} \bigvee_{s\geq -1} \bigwedge_{y\in(A_1)^c} (x(s+) \wedge y(s+)') \\
&\geq \bigwedge_{x\in A_1} \bigvee_{s\in[-1,1)} \bigwedge_{y\in(A_1)^c} (x(s+) \wedge y(s+)') \\
&\geq \bigwedge_{x\in A_1} \bigwedge_{y\in(A_1)^c} (x(-1+) \wedge y(-1+)') \geq 0.5. \\
\mathcal{T}_3(A_1) &= \bigwedge_{x\in A_1} \bigvee_{r,s\in\mathbb{R}} \bigwedge_{y\notin A_1} (\max\{\max\{x(s+) - y(s+), 0\}, \max\{y(r-) - x(r-), 0\}\}) \\
&\leq \bigvee_{r,s\in\mathbb{R}} \bigwedge_{y\notin A_1} (\max\{\max\{v(s+) - y(s+), 0\}, \max\{y(r-) - v(r-), 0\}\}) \\
&= \bigvee_{r>-1, s<0} \bigwedge_{y\notin A_1} (\max\{\max\{v(s+) - y(s+), 0\}, \max\{y(r-) - v(r-), 0\}\}) \\
&\leq \bigvee_{r>-1, s<0} \bigwedge_{y\notin A} (\max\{\max\{v(s+) - y(s+), 0\}, \max\{y(r-) - v(r-), 0\}\}) \\
&\leq \bigvee_{r>-1, s<0} (\max\{\max\{v(s+) - u(s+), 0\}, \max\{u(r-) - v(r-), 0\}\}) \leq 0.1.
\end{aligned}
$$

This shows $\mathcal{T}_3 \neq \mathcal{T}$, i.e., $\mathcal{T}_3 < \mathcal{T}$.

6 Conclusions

In this paper, some novel fuzzifying topologies are introduced on the M-fuzzy real line. The set of all distribution functions can be regarded as a special M-fuzzy real line. The M-fuzzy real line with fuzzifying topologies can be regarded as a generalization of the real line with Euclidean topology. In the future, it may become the foundation of fuzzy calculus and fuzzy optimization. We will consider the fuzzifying topological convex structure, the fuzzifying convergence, the fuzzifying completeness, and others on the M-fuzzy real line.

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