A FUZZY VERSION OF HAHN-BANACH EXTENSION THEOREM

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Abstract. In this paper, a fuzzy version of the analytic form of Hahn-Banach extension theorem is given. As application, the Hahn-Banach theorem for r-fuzzy bounded linear functionals on r-fuzzy normed linear spaces is obtained.

1. Introduction

Hahn-Banach theorem is one of the most famous and useful result in functional analysis. Ramakrishnan [15] established the norm-preserving fuzzy completion of a fuzzy normed algebra and gave a fuzzy extension of Hahn-Banach theorem. In the same year Rhie and Hwang [16] investigated the relation between fuzzy seminorms and crisp seminorms on a linear space X and extended the analytic form of the Hahn-Banach theorem with the notion of fuzzy seminorm. In recent years, a fuzzy version of Hahn-Banach theorem on a vector space over the set of fuzzy real numbers and some related applications were proved by Binimol and Sunny Kuriakose [6, 7]. There are also many other fuzzy versions of Hahn-Banach theorem for fuzzy bounded linear operators on fuzzy normed spaces (see e.g. [2, 9, 12, 19] etc...).

In this paper, using the definition of fuzzy order due to L. A. Zadeh (see [21]), we assume that the set of real numbers $\mathbb{R}$ endowed with a fuzzy order r instead of the natural order $\leq$ and prove a new fuzzy version of the analytic form of Hahn-Banach theorem. As application, the Hahn-Banach theorem for r–fuzzy bounded linear functionals on r–fuzzy normed linear spaces is obtained.

2. Preliminaries

We begin with a number of definitions related to fuzzy orders. We follow the notation and vocabulary of Zadeh [21] closely, and refer the reader to Amroune and Davvaz [1], Beg [3], Bernadette [4], Billot [5], Bodenhofer and et.al. [8], Kundu [10], Li and Yen [11], Ovchinnikov [13, 14], Stouti and Zedam [17], Venugopalan [18], Zadeh [21] and Zimmermann [22] for elementary definitions and facts about fuzzy order relations.

The concept of a fuzzy set in a non-empty set was introduced by Zadeh [20] in 1965.

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Let X be a nonempty set, a fuzzy subset A of X is characterized by its membership function $A : X \to [0,1]$ and $A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for each $x \in X$.

In [21], Zadeh gave the following definition of fuzzy order.

Definition 2.1. [21] Let X be a nonempty set. A Zadeh's binary fuzzy partial order (briefly, fuzzy order) on X is a fuzzy subset r on $X \times X$ in which the following conditions are satisfied:
(i) for all $x \in X$, $r(x,x) = 1$, (fuzzy reflexivity);
(ii) for all $x, y \in X$, $(r(x,y) > 0$ and $x \neq y)$ implies $(r(y,x) = 0)$, (fuzzy antisymmetry);
(iii) for all $x, y, z \in X$, $r(x,z) \geq \max_{y\in X} [\min\{r(x,y), r(y,z)\}]$, (fuzzy transitivity).

Note that each crisp order $\leq$ on X can be considered a fuzzy order defined by $r(x,y) = 1$ if $x \leq y$ and $r(x,y) = 0$ if x and y are incomparable elements.

A nonempty set X with a fuzzy order r defined on it is called fuzzy ordered set (for short, foset) and we denote it by (X, r).

If Y is a subset of a foset (X, r), then the restriction of r to Y is a fuzzy order in Y and is called induced fuzzy order.

A fuzzy order r is linear (or total) on X if for every $x, y \in X$, we have $r(x,y) > 0$ or $r(y,x) > 0$. If $x \neq y$, by the fuzzy antisymmetry of r, clearly only one of these conditions can be satisfied. A fuzzy ordered set (X, r) in which r is total is called a r-fuzzy chain. Conversely, if for any $x, y \in X$, $r(x,y) > 0$ if and only if $x = y$, then (X, r) is called r-fuzzy antichain.

Next, we give some examples of fuzzy order.

Example 2.2. Let $X = \{a, b, c, d, e, f, g\}$. Then the fuzzy subset r defined on $X \times X$ by the following table:

	a	b	c	d	e	f	g
a	1	0	0	0.55	0.40	0.45	0.60
b	0	1	0	0.60	0.50	0.35	0.75
c	0.15	0	1	0.30	0.70	0.80	0.90
d	0	0	0	1	0	0.15	0
e	0	0	0	0	1	0.30	0.25
f	0	0	0	0	0	1	0
g	0	0	0	0	0	0.20	1

is a fuzzy order on X.

Example 2.3. Let $x, y \in \mathbb{R}$. Then the fuzzy subset r_λ defined for all $x, y \in \mathbb{R}$ by:

$$r_\lambda(x,y) = \begin{cases} 1\ ,\ if\ x = y; \\ \min(1, \frac{y-x}{\lambda}),\ if\ x < y\ , \\ 0,\ if\ x > y; \end{cases}$$

is a total fuzzy order on $\mathbb{R}$.

Clearly, $0 \leq r_\lambda(x,y) \leq 1$ for all $x, y \in \mathbb{R}$. Thus r_λ is well defined. Now let us show that r_λ is a fuzzy order on $\mathbb{R}$.

1) For all $x \in \mathbb{R}$, $r_\lambda(x,x) = 1$. Thus r_λ is fuzzy reflexive.

2) Let $x, y \in \mathbb{R}$ with $x \neq y$. Then, $r_\lambda(x,y) > 0$ is true only in the case $x < y$. So, r_λ is fuzzy antisymmetric.

3) Let $x, y, z \in \mathbb{R}$. Then, we have three cases to study.

3.i) If $r_\lambda(x,z) = 1$, then $r_\lambda(x,z) \geq \min\{r_\lambda(x,y), r_\lambda(y,z)\}$, for all $y \in \mathbb{R}$.

3.ii) If $r_\lambda(x,z) = \frac{z-x}{\lambda} > 0$, then $x < z$. Hence, for $y \in \mathbb{R}$ we have three cases to consider:

(a) if $x < z < y$, then $r_\lambda(y,z) = 0$.
(b) If $x \leq y \leq z$, so $\frac{z-x}{\lambda} \geq \frac{z-y}{\lambda}$. Hence, we get $r_\lambda(x,z) \geq r_\lambda(y,z)$.
(c) If $y < x < z$, then $r_\lambda(x,y) = 0$. Thus $r_\lambda(x,z) \geq \min\{r_\lambda(x,y), r_\lambda(y,z)\}$, for all $y \in \mathbb{R}$.

3.iii) If $r_\lambda(x,z) = 0$, then $x > z$. So, for every $y \in \mathbb{R}$ we have three cases:

(a) if $x > z \geq y$, then $r_\lambda(x,y) = 0$.
(b) If $x \geq y > z$, so $r_\lambda(y,z) = 0$.
(c) If $y > x > z$, hence $r_\lambda(y,z) = 0$.

Hence, $r(x,z) \geq \min\{r_\lambda(x,y), r_\lambda(y,z)\}$, for all $y \in \mathbb{R}$. Thus, r_λ is fuzzy transitive. Therefore, r_λ is a fuzzy order on $\mathbb{R}$.

Since for all $x, y \in \mathbb{R}$, such that $x \neq y$ we have either $x < y$ or $y < x$. Then, we get either $\min(1, \frac{y-x}{\lambda}) > 0$ or $\min(1, \frac{x-y}{\lambda}) > 0$ Thus, r_λ is a total fuzzy order.

Example 2.4. Let $X = \mathbb{R}$. Then, the fuzzy relation r defined for all $x, y \in \mathbb{R}$ by:

$$r(x,y) = \begin{cases} 1\ ,\ if\ x = y; \\ 0\ ,\ if\ x > y; \\ 1 - \frac{x}{y}\ ,\ if\ 0 \leq x < y; \\ 1 - \frac{y}{x}\ ,\ if\ x < y \leq 0; \\ 1\ ,\ if\ x < 0\ and\ y > 0; \end{cases} ,$$

is a total fuzzy order on $\mathbb{R}$.

Clearly, $0 \leq r(x,y) \leq 1$ for all $x, y \in \mathbb{R}$. Thus r is well defined. Now let us show that r is a fuzzy order on $\mathbb{R}$.

1) For all $x \in \mathbb{R}$, $r(x,x) = 1$. Thus r is fuzzy reflexive.

2) Let $x, y \in \mathbb{R}$ such that $x \neq y$. Then, we have $r(x,y)r(y,x) = 0$. So, r is fuzzy antisymmetric.

3) Let $x, y, z \in \mathbb{R}$. Then, we have four cases to study.

3.i) If $r(x,z) = 1$, then $r(x,z) \geq \min\{r(x,y), r(y,z)\}$, for all $y \in \mathbb{R}$.

3.ii) If $r(x,z) = 0$, then $x > z$. Hence, for every $y \in \mathbb{R}$ we distinguish the following subcases.

(a) If $x > z \geq y$, then it holds that $r(x,y) = 0$.
(b) If $x \geq y > z$, then it holds that $r(y,z) = 0$.
(c) If $y > x > z$, then it holds that $r(y,z) = 0$.

Thus, $r(x,z) \geq \min\{r(x,y), r(y,z)\}$, for all $y \in \mathbb{R}$.

3.iii) If $r(x,z) = 1 - \frac{x}{z}$, then $0 \leq x < z$. Hence, for $y \in \mathbb{R}$ we have four cases to consider:

(a) If $0 \leq x < z < y$, then $r(y,z) = 0$.
(b) If $0 \leq x < y < z$, so $1 - \frac{x}{z} \geq 1 - \frac{y}{z}$. Hence, we get $r(x,z) \geq r(y,z)$.
(c) If $0 \leq y < x < z$, then $r(x,y) = 0$.
(d) If $y < 0 \leq x < z$, so $r(x,y) = 0$.

Thus $r(x,z) \geq \min\{r(x,y), r(y,z)\}$, for all $y \in \mathbb{R}$.

3.iv) If $r(x,z) = 1 - \frac{z}{x}$, then by using a similar argument as in the case (3.iii) we can see that $r(x,z) \geq \min\{r(x,y), r(y,z)\}$, for all $y \in \mathbb{R}$.

Hence, r is fuzzy transitive. Thus, r is a fuzzy order on $\mathbb{R}$.

As for all $x, y \in \mathbb{R}$, such that $x \neq y$ we have either $x < y$ or $y < x$, then we get either $r(x,y) = 1 - \frac{x}{y} > 0$ or $r(y,x) = 1 - \frac{y}{x} > 0$. Thus, r is a total fuzzy order.

Definition 2.5. Let (X, r) be a fuzzy ordered set and A be a subset of X.

(a) An element $u \in X$ is an r-upper bound of A if $r(x,u) > 0$ for all $x \in A$. The set of all r-upper bounds of A is denoted by A^u. If u is the r-upper bound of A and $u \in A$, then u is called a greatest element of A. The r-lower bound and least element are defined analogously and the set of all r-lower bounds of A is denoted by A^ℓ.

(b) An element $m \in A$ is called a maximal element of A if there is no $x \neq m$ in A for which $r(m,x) > 0$. $x = m$. Minimal elements are defined similarly.

(c) As usual, the r-supremum of A is defined by $\sup_r(A) =$ the least element of r-upper bounds of A (if it exists). Similarly, the r-infimum of A defined by $\inf_r(A) =$ the greatest element of r-lower bounds of A (if it exists).

We write $x \vee_r y$ the r-supremum and $x \wedge_r y$ the r-infimum of the set $\{x, y\}$. For linear fuzzy order, $x \vee_r y = \max_r\{x, y\}$ and $x \wedge_r y = \min_r\{x, y\}$.

Definition 2.6. Let r be a fuzzy order on $\mathbb{R}$ and $x \in \mathbb{R}$. If $r(0,x) > 0$, then x is called an r-positive real number. The set of them all is denoted by $\mathbb{R}_r^+$. Similarly, if $r(x,0) > 0$ then x is called an r-negative real number, and the set of them all is denoted by $\mathbb{R}_r^-$.

Definition 2.7. 1) Let r be a fuzzy order on $\mathbb{R}$. We say that r is compatible with the addition if for all (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2$, we have

$$(r(x_1, y_1) > 0 \text{ and } r(x_2, y_2) > 0) \Longrightarrow (r(x_1 + x_2, y_1 + y_2) > 0).$$

2) The fuzzy order r is said to be compatible with the multiplication by scalars if for all $(x, y) \in \mathbb{R}^2$ and $\lambda > 0$, we have

$$(r(x,y) > 0) \Longrightarrow (r(\lambda x, \lambda y) > 0).$$

Example 2.8. The fuzzy order relation given in Example 2.4 is compatible with the addition and multiplication by scalars on $\mathbb{R}$.

(i) r is compatible with the addition. Indeed, let (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2$ such that $r(x_1, y_1) > 0$ and $r(x_2, y_2) > 0$. By the definition of r we get that $x_1 \leq y_1$ and

$x_2 \leq y_2$. Then, $x_1 + x_2 \leq y_1 + y_2$. Hence, $r(x_1 + x_2, y_1 + y_2) > 0$. Thus, r is compatible with the addition.
(ii) r is compatible with the multiplication by scalars. Indeed, let $(x, y) \in \mathbb{R}^2$ such that $r(x, y) > 0$ and $\lambda > 0$. By the definition of r we get that $x \leq y$. Then, $\lambda x \leq \lambda y$. Hence, $r(\lambda x, \lambda y) > 0$. Thus, r is compatible with the multiplication by scalars.

Therefore, r is compatible with the addition and multiplication by scalar on $\mathbb{R}$.

Next, we show the following two propositions which we shall need for proving a fuzzy version of Hahn-Banach theorem.

Proposition 2.9. *Let $\mathbb{R}_r = (\mathbb{R}, r)$ be the set of all real numbers endowed with a fuzzy order r compatible with the addition and the multiplication by scalar, and $x, y \in \mathbb{R}$. Then we have the following:*

i) If $r(0, x) > 0$ then $r(-x, 0) > 0$.
ii) If $r(0, x) > 0$ then $r(-x, x) > 0$.

Proof. Let $x, y \in \mathbb{R}_r$. i) Since $r(0, x) > 0$ and by the fuzzy reflexivity $r(-x, -x) =$

$1 > 0$, then from the compatibility of r with the addition we have that $r(0 + (-x), x + (-x)) > 0$. Hence, $r(-x, 0) > 0$.

ii) We assume that $r(0, x) > 0$. It is clear from (i) that $r(-x, 0) > 0$. Then, from the compatibility of r with the addition we have that $r(-x, x) > 0$. □

Proposition 2.10. *Let $\mathbb{R}_r = (\mathbb{R}, r)$ be the set of all real numbers endowed with a fuzzy order r compatible with the addition and multiplication by scalar, and let $x, y \in \mathbb{R}$ such that $x \neq y$. Then the following are equivalent.*

(i) $r(x, y) > 0$;
(ii) There exists $\tau \in \mathbb{R}$ such that $r(x, \tau) > 0$ and $r(\tau, y) > 0$, (r-fuzzy density).

Proof. Let $x, y \in \mathbb{R}_r$ such that $x \neq y$ and $r(x, y) > 0$. For the one direction, let $\tau = \frac{x+y}{2}$. Since $r(x, x) = 1 > 0$ and $r(x, y) > 0$, from the compatibility of r with the addition we get that

$$r(x + x, x + y) > 0.$$

Now, by the compatibility of r with the multiplication we obtain that

$$r(x, \frac{x + y}{2}) > 0.$$

Thus, $r(x, \tau) > 0$.

In the same way we get that $r(\tau, y) > 0$.
The other direction follows directly from the fuzzy transitivity. □

3. Results

In this section we assume that $\mathbb{R}_r$ is the set of real numbers $\mathbb{R}$ endowed with a fuzzy order r compatible with the addition and multiplication by scalar instead of the natural order $\leq$ and we shall prove a fuzzy version of Hahn-Banach extension theorem. The prove of this fuzzy version will follow the same steps as the crisp case. As application, we define the notion of r-fuzzy normed space with the help of r-fuzzy norm as a generalization of crisp normed space, we introduce the notion of r-fuzzy bounded linear functional and we prove the Hahn-Banach theorem for r-fuzzy bounded linear functionals on r-fuzzy normed linear spaces.

Definition 3.1. Let X be an real linear space, and T a mapping of X into $\mathbb{R}_r$. We say that T is a r-fuzzy sublinear functional on X if

i) $r(T(x+y), T(x)+T(y)) > 0$ for all $x, y \in X$, (r–subadditivity);
ii) $T(\lambda x) = \lambda T(x)$ for all $x \in X$ and $\lambda \in \mathbb{R}_r^+$, ($r$–positively homogeneous).

Example 3.2. The mapping $T : \mathbb{R}_r \longrightarrow \mathbb{R}_r$ defined by $T(x) = |x|_r = \max_r\{x, -x\}$ is an r-fuzzy sublinear functional on $\mathbb{R}_r$.

The following is a useful fact for r-fuzzy sublinear functionals.

Proposition 3.3. *If T is an r-fuzzy sublinear functional on a real linear space X then $r(\lambda T(x), T(\lambda x)) > 0$, for all $x \in X$ and $\lambda \in \mathbb{R}_r$.*

Proof. Let $x \in X$ and $\lambda \in \mathbb{R}_r$. If $\lambda \in \mathbb{R}_r^+$ we have $T(\lambda x) = \lambda T(x)$. Hence,

$$r(\lambda T(x), T(\lambda x)) = 1 > 0. \tag{1}$$

If $\lambda \in \mathbb{R}_r^-$, then from Proposition 2.9(i) we get that $-\lambda \in \mathbb{R}_r^+$. As $\lambda T(x) = -(-\lambda T(x))$ so by the r–positively homogeneous of T we have $\lambda T(x) = -(-\lambda T(x)) = -T(-\lambda x)$. On the other hand, since $T(\lambda x - \lambda x) = T(0) = 0$, by the r–subadditivity of T we have $r(T(\lambda x + (-\lambda x)), T(\lambda x) + T(-\lambda x)) > 0$. Hence, $r(0, T(\lambda x) + T(-\lambda x)) > 0$. Now, from the compatibility of r with the addition we have $r(-T(-\lambda x), T(\lambda x)) > 0$. Thus,

$$r(\lambda T(x), T(\lambda x)) > 0. \tag{2}$$

Therefore, (1) and (2) implies that $r(\lambda T(x), T(\lambda x)) > 0$, for all $x \in X$ and $\lambda \in \mathbb{R}_r$. □

Theorem 3.4 (Fuzzy version of Hahn-Banach theorem)**.** *Let X_0 be a subspace of a real linear space X, T a r-fuzzy sublinear functional on X, and u_0 be an linear functional on X_0 such that $r(u_0(x), T(x)) > 0$ for all $x \in X_0$. Then there exists a linear functional u on X extends u_0 to X and satisfies $r(u(x), T(x)) > 0$, for all $x \in X$.*

Proof. Let $y \in X$ such that $y \notin X_0$ and denote by Y the vector subspace generated by $X_0 \cup \{y\}$, so

$$Y = \{x_0 + \lambda y \ / \ x_0 \in X_0 \ and \ \lambda \in \mathbb{R}_r - \{0\}\}$$

Let $\tau \in \mathbb{R}_r$, and provisionally define

$$u(x_0 + \lambda y) = u_0(x_0) + \lambda\tau.$$

It is easy to show that u is a linear extension of u_0 to Y; hence it remains to choose $\tau \in \mathbb{R}_r$ such that for all $x_0 \in X_0$, and $\lambda \in \mathbb{R}_r - \{0\}$,

$$r(u_0(x_0) + \lambda\tau, T(x_0 + \lambda y)) > 0. \tag{3}$$

For all $\lambda \in \mathbb{R}_r^+ - \{0\}$, replacing x_0 by λx_0, using the $r-$positive homogeneity of T, and from the compatibility of r with the multiplication, it suffices to see that

$$r(u_0(x_0) + \tau, T(x_0 + y)) > 0. \tag{4}$$

$$r(u_0(x_0) + \tau, T(x_0 + y)) > 0.$$

Therefore, from the r-fuzzy compatibility of r with the addition, it suffices to see that

$$r(\tau, T(x_0 + y) - u_0(x_0)) > 0. \tag{5}$$

For all $\lambda \in \mathbb{R}_r^- - \{0\}$, replacing x_0 by λx_0, using the $r-$positive homogeneity of T, and from the compatibility of r with the multiplication, we observe that it suffices to see that

$$r(-u_0(x_0) - \tau, T(-x_0 - y)) > 0 \tag{6}$$

Therefore, from the r-fuzzy compatibility of r with the addition, it suffices to see that

$$r(-u_0(x_0) - T(-x_0 - y), \tau) > 0. \tag{7}$$

To see the existence of $\tau \in \mathbb{R}_r$ satisfying (5) and (6), start by observing that

$$\begin{aligned} r(-u_0(x_0) - T(-x_0 - y), T(x_0 + y) - u_0(x_0)) &\geq \min\{r(-u_0(x_0), -u_0(x_0)), \\ r(-T(-x_0 - y), T(x_0 + y))\} &\geq \min\{1, r(-T(-x_0 - y), T(x_0 + y))\}, \end{aligned}$$

In addition, from Proposition 3.3 we have $r(-T(-x_0 - y), T(x_0 + y)) > 0$.

Then $r(-u_0(x_0) - T(-x_0 - y), T(x_0 + y) - u_0(x_0)) > 0$,
and therefore by Proposition 2.10 there exists $\tau \in \mathbb{R}_r$ satisfies (5) and (6). Hence, there exists $\tau \in \mathbb{R}_r$ satisfies (3).
Now, an application of Zorn's Lemma complete the proof. □

Next, we shall give an application of r-fuzzy Hahn-Banach theorem, but in this subsection, we assume that r is linear order on $\mathbb{R}$ compatible with the addition and multiplication.

Definition 3.5. Let X be a real linear space. An r-fuzzy norm on X is a mapping $x \mapsto \|x\|_r$ from X into $\mathbb{R}_r^+$ such that for all $x, y \in X$ and $\lambda \in \mathbb{R}_r^+$, the following properties hold:

i) $\|x\|_r = 0$ if and only if $x = 0$.
ii) $\|\lambda x\|_r = |\lambda|_r \|x\|_r$.
iii) $r(\|x + y\|_r, \|x\|_r + \|y\|_r) > 0$.

A linear space X equipped with an r-fuzzy norm $\|.\|_r$ is called an r-fuzzy normed linear space. We denote it by $(X, \|.\|_r)$.

Example 3.6. The r-fuzzy absolute value $|x|_r = x \vee_r (-x)$ is an r-fuzzy norm on $\mathbb{R}_r$.

i) Let $x \in \mathbb{R}_r$, since r is a total order we have either $r(0, x) > 0$ or $r(0, -x) > 0$. Then by Proposition 2.9(ii) we have either $r(-x, x) > 0$ or $r(x, -x) > 0$.

Hence, $r(0, |x|_r) > 0$.

ii) Obvious.

iii) $\|\lambda x\|_r = \lambda x \vee_r (-\lambda x) = |\lambda|_r x \vee_r (-|\lambda|_r x) = |\lambda|_r (x \vee_r (-x)) = |\lambda|_r \|x\|_r$.

iv) Let $x, y \in \mathbb{R}_r$.To prove that $r(\|x + y\|_r, \|x\|_r + \|y\|_r) > 0$ six cases are considered.

a) If $r(0, x) > 0$ and $r(0, y) > 0$ then

$$r(|x + y|_r, |x|_r + |y|_r) = r(x + y, x + y) = r > 0.$$

b) If $r(x,0)>0$ and $r(y,0)>0$ then
$$r(|x+y|_r,|x|_r+|y|_r)=r(-x-y,-x-y)=r>0.$$

c) If $r(0,x)>0$, $r(y,0)>0$ and $r(x,-y)>0$ then
$$\begin{aligned} r(|x+y|_r,|x|_r+|y|_r) &= r(-x-y,x-y) \\ &\geq \min\{r(-x,x),r(-y,-y)\}>0. \end{aligned}$$

d) If $r(0,x)>0$, $r(y,0)>0$ and $r(-y,x)>0$ then
$$\begin{aligned} r(|x+y|_r,|x|_r+|y|_r) &= r(x+y,x-y) \\ &\geq \min\{r(x,x),r(y,-y)\}>0. \end{aligned}$$

e) If $r(x,0)>0$, $r(0,y)>0$ and $r(y,-x)>0$ then
$$\begin{aligned} r(|x+y|_r,|x|_r+|y|_r) &= r(-x-y,-x+y) \\ &\geq \min\{r(-x,-x),r(-y,y)\}>0. \end{aligned}$$

f) If $r(x,0)>0$, $r(0,y)>0$ and $r(-x,y)>0$ then
$$\begin{aligned} r(|x+y|_r,|x|_r+|y|_r) &= r(x+y,-x+y) \\ &\geq \min\{r(x,-x),r(y,y)\}>0. \end{aligned}$$

Definition 3.7. Let $(X,\|.\|_r)$ and $(Y,\|.\|_r)$ be r-fuzzy normed linear spaces. A linear operator u from X into Y is called an r-fuzzy bounded operator if there exists $K\in \mathbb{R}_r^+$ such that
$$r(\|u(x)\|_r,K\|x\|_r)>0, \qquad \text{for all } x\in X.$$

Remark 3.8. The r-fuzzy norms in X and Y are different. But we use same notation $\|.\|_r$, because there is no confusion.

Example 3.9. Let $(X,\|.\|_r)$ be an r-fuzzy normed linear space, we define an operator $u:(X,\|.\|_r)\longrightarrow (X,\|.\|_r)$ by $u(x)=\lambda x$ where $\lambda(\neq 0)\in \mathbb{R}$ is fixed. Clearly u is an r-fuzzy bounded linear operator.

In the following Lemma we describe the r-fuzzy boundedness of a linear operator between r-fuzzy normed linear spaces by means of an r-fuzzy norm of it.

Lemma 3.10. *Let u be an r-fuzzy bounded linear operator from $(X,\|.\|_r)$ into $((Y,\|.\|_r)$. Then there exists an r-fuzzy norm of u, denoted by $\|u\|_r$ such that:*
$$r(\|u(x)\|_r,\|u\|_r\|x\|_r)>0, \qquad \textit{for all } x\in X.$$

Proof. Since u is an r-fuzzy bounded linear operator, there exists $K\in \mathbb{R}_r^+$ such that
$$r(\|u(x)\|_r,K\|x\|_r)>0, \qquad \text{for all } x\in X.$$
From the compatibility of r with the multiplication we obtain
$$r(\frac{\|u(x)\|_r}{\|x\|_r},K)>0, \qquad \text{for all } x\in X.$$
Hence,
$$r(\sup_r\{\frac{\|u(x)\|_r}{\|x\|_r}\ :\ x\in X\},K)>0, \qquad \text{for all } x\in X.$$
This means that $\sup_r(\frac{\|u(x)\|_r}{\|x\|_r})$ is finite.

Now we put $\|u\|_r = \sup_r\{\frac{\|u(x)\|_r}{\|x\|_r} \ : \ x \in X\}$. It is clear that $\|u\|_r = 0$ if and only if $u = 0$, and that $\|\lambda u\|_r = |\lambda|_r \|u\|_r$. Since

$$r(\|u + v(x)\|_r, \|u(x)\|_r + \|v(x)\|_r) > 0, \qquad \text{for all } x \in X,$$

it follows from the compatibility of r with the multiplication that

$$r(\frac{\|u + v(x)\|_r}{\|x\|_r}, \frac{\|u(x)\|_r}{\|x\|_r} + \frac{\|v(x)\|_r}{\|x\|_r}) > 0, \qquad \text{for all } x \in X.$$

Then we obtain

$$r(\|u + v\|_r, \|u\|_r + \|v\|_r) > 0.$$

Hence, $\|u\|_r$ is a r-fuzzy norm of u.

In addition, as $\|u\|_r = \sup_r\{\frac{\|u(x)\|_r}{\|x\|_r} \ : \ x \in X\}$, we get

$$r(\frac{\|u(x)\|_r}{\|x\|_r}, \|u\|_r) > 0, \qquad \text{for all } x \in X,$$

which implies

$$r(\|u(x)\|_r, \|u\|_r \|x\|_r) > 0, \qquad \text{for all } x \in X.$$

□

Theorem 3.11. *Let X_0 be a subspace of an r-fuzzy normed linear space X, and u_0 be an r-fuzzy bounded linear functional on X_0. Then there exists an r-fuzzy bounded linear functional u on X such that $u(x) = u_0(x)$ for all $x \in X_0$ and $\|u\|_r = \|u_0\|_r$.*

Proof. $T(x) = \|u_0\|_r \|x\|_r$. It is easy to see that $T(x)$ is an r-fuzzy sublinear functional on X. Since u_0 is an r-fuzzy bounded linear functional on X_0, we obtain for all $x \in X_0$, that

$$r(u_0(x), T(x)) = r(u_0(x), \|u_0\|_r \|x\|_r) > 0.$$

Then from the r-fuzzy Hahn-Banach theorem there exists a linear functional u on X extends u_0 to X and satisfies $r(u(x), \|u_0\|_r \|x\|_r) > 0$, for all $x \in X$. Moreover, for all $x \in X$ we have

$$r(u(-x), \|u_0\|_r \| - x\|_r) > 0.$$

This shows that

$$r(-u(x), \|u_0\|_r \|x\|_r) > 0.$$

Hence,

$$r(|u(x)|_r, \|u_0\|_r \|x\|_r) > 0.$$

Therefore, u is an r-fuzzy bounded linear functional on X and satisfies

$$r(\|u\|_r, \|u_0\|_r) > 0.$$

But u extends u_0, so $r(\|u_0\|_r, \|u\|_r) > 0$ and therefore $\|u\|_r = \|u_0\|_r$. □

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