

## AN ALGEBRAIC STRUCTURE FOR INTUITIONISTIC FUZZY LOGIC

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ABSTRACT. In this paper we extend the notion of degrees of membership and non-membership of intuitionistic fuzzy sets to lattices and introduce a residuated lattice with appropriate operations to serve as semantics of intuitionistic fuzzy logic. It would be a step forward to find an algebraic counterpart for intuitionistic fuzzy logic. We give the main properties of the operations defined and prove some theorems to demonstrate our goal.

### 1. Introduction

L.A. Zadeh introduced the concept of fuzzy subsets of a well-defined set in his paper [23] for modeling the vague concepts in the real world. After him, Goguen generalized this to L-fuzzy sets [10] where L is an appropriate lattice. Fuzzy logic based on the theory of fuzzy sets (TFS) is developed. This is called fuzzy logic in wide sense denoted by  $FL_w$  and contains mostly applications of TFS [24]. Another version of fuzzy logic called fuzzy logic in narrow sense denoted by  $FL_n$  which studies fuzzy logic as a many valued logic. There are many researchers to work on this sense of fuzzy logic [ for example 7, 11, 12, 20]. Ward and Dilworth introduced residuated lattices [22] and gave the main properties of these lattices, although there existed before this paper such as Boolean algebras and Heyting algebras. H. Ono considered residuated lattices as an algebraic structure of substructural logics in [14]. P. Hajek in 1998 [11] introduced the notion of a BL - algebra as a residuated lattice with two more conditions, namely divisibility and prelinearity to prove the completeness of Lukasiewicz logic as a many valued logic. He showed that these algebras are the best algebraic counterparts of fuzzy logics generated by continuous t-norms [11]. K. T. Atanassov [1] introduced the notion of an Intuitionistic Fuzzy Set (IFS in short) as a generalization of a Fuzzy Set (FS). In fact from his point of view for each element of the universe there are two degrees, one a degree of membership to a vague subset and the other is a degree of non -membership to that given subset. Many researchers have been working on the theory of this subject and developed it in interesting different branches [see for example 5, 13]. Many studied and applied it in a broad range of applications [16, 17, 21]. K. Atanassov and S. Stoeva generalized the concept of IFS to Intuitionistic L-fuzzy sets [2] where

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$L$  is an appropriate lattice. Tepavcevic, A. and Gerstenkorn, T. gave a new definition of lattice valued Intuitionistic fuzzy sets in [19]. Glad Deschrijver, et al. [9] considered the Intuitionistic operators and defined negator, t-norms, t-conorms and implicators on the lattice  $L^* = \{(x, y) \in [0, 1]^2 | x \leq y - 1\}$ . Andreja Tepavcevic et al. considered general lattice valued IFSs in [18]. Thus, BL - algebras as special residuated lattices are suitable algebraic structures for fuzzy logics and intuitionistic fuzzy logic based on intuitionistic fuzzy sets are supposed to be a generalization of fuzzy logic.

All above motivates us to construct a residuated lattice as an algebraic counterpart of intuitionistic fuzzy logic from a special BL - algebra that fulfils our logical requirements based on IFSs. This is towards algebraization and axiomatization of intuitionistic fuzzy logic. Whenever it is done we can use it in application areas such as approximate reasonings in artificial intelligence more easily.

This paper is organized as follows: in the next section we give the preliminaries including the basic definitions and theorems that are needed in the other parts. In section 3 we introduce the notion of intuitionistic fuzzy residuated lattices by defining the needed operators. The properties of these lattices are given. As a result we conclude that this lattice (algebra) can be an appropriate set of values for the semantics of the Intuitionistic Fuzzy Logic.

## 2. Preliminaries

In this section we give some definitions and theorems that we need in the sequel. Assume that  $\mathbf{U}$  is the universe. A fuzzy set  $\mathbf{A}$  in  $\mathbf{U}$  is characterized by the same symbol  $\mathbf{A}$  as a function  $\mathbf{A} : \mathbf{U} \rightarrow [0, 1]$  where  $\mathbf{A}(u) \in [0, 1]$  is the membership degree of the element  $u \in \mathbf{U}$  [23].

Let  $\mathbf{L} = (L, \wedge, \vee, 0, 1)$  be a bounded (complete) lattice. By an  $\mathbf{L}$ -fuzzy set  $\mathbf{A}$  in  $\mathbf{U}$  we mean a function  $\mathbf{A} : \mathbf{U} \rightarrow \mathbf{L}$  [10].

**Definition 2.1.** [1] Let  $\mathbf{U}$  be the universe. By an Intuitionistic Fuzzy Set (IFS) in  $\mathbf{U}$  we mean a set of ordered triples  $\mathbf{A} = \{(x, \mu_A(x), \nu_A(x)) | x \in \mathbf{U}\}$ , where  $\mu_A(x)$  is the membership degree of  $x$  to  $\mathbf{A}$  and  $\nu_A(x)$  is the non-membership degree of  $x$  to  $\mathbf{A}$  such that  $\mu : \mathbf{U} \rightarrow [0, 1]$  and  $\nu : \mathbf{U} \rightarrow [0, 1]$  satisfying  $0 \leq \mu_A(x) + \nu_A(x) \leq 1$  for all  $x \in \mathbf{U}$ . The complement of an IFS  $\mathbf{A}$  is defined by  $\mathbf{A}^c = \{(x, \nu_A(x), \mu_A(x)) | x \in \mathbf{U}\}$ . We omit the coordinate  $x$  when it is clear that where they are coming from.

We recall from [9] that  $\mathbf{L}^* = \{(x, y) \in [0, 1]^2 | 0 \leq x + y \leq 1\}$  is a complete lattice with the order defined by

$$(x_1, x_2) \preceq (y_1, y_2) \text{ if and only if } x_1 \leq y_1 \text{ and } y_2 \leq x_2 \quad (1)$$

There is an extension of the above IFSs [18,19], where the degrees of membership and non-membership are in a complete lattice  $L$ , i.e.,  $\mu_A(x), \nu_A(x) \in L$  such that  $\mu_A(x) \leq_L N(\nu_A(x))$  for all  $x \in U$ , where  $N$  is an involutive negator [8]. In this paper and [9] the authors also defined IF t-norms, t-conorms, R- and S-implicators on  $L^*$  defined above. The disadvantages of the given extension are explained and the authors proposed some other extensions to overcome the problems.

In the next section we are proposing a new extension using residuated lattices. These notions are defined as follows:

**Definition 2.2.** [11] A residuated lattice is an algebra  $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$  of type  $(2,2,2,2,0,0)$  such that:

- (i)  $(L, \wedge, \vee, 0, 1)$  is a bounded lattice,
- (ii)  $(L, *, 1)$  is a commutative monoid, and
- (iii) the operations  $*$  and  $\rightarrow$  form an adjoint pair, i.e.,

$$x * y \leq z \quad \text{if and only if} \quad x \leq y \rightarrow z \quad (2)$$

for all  $x, y, z \in L$ .

**Theorem 2.3.** [11] *In any residuated lattice  $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$  the following properties hold for all  $x, y, z \in L$ :*

- (1)  $x * (x \rightarrow y) \leq x \wedge y$ ,
- (2)  $x \leq y \rightarrow x * y$ ,
- (3)  $x \leq y$  implies  $x * z \leq y * z$ ,
- (4)  $x \leq y$  implies  $y \rightarrow z \leq x \rightarrow z$  and  $z \rightarrow x \leq z \rightarrow y$ ,
- (5)  $x \leq y$  if and only if  $x \rightarrow y = 1$ ,
- (6)  $(x \vee y) * z = (x * z) \vee (y * z)$ , and
- (7)  $1 \rightarrow x = x$ .

We may have a negator in a bounded lattice.

**Definition 2.4.** Let  $L = (L, \wedge, \vee, 0, 1)$  be a bounded lattice. A unary operator  $N : L \rightarrow L$  is a negator if it is non-increasing with respect to the usual order on  $L$ ,  $N(0) = 1$  and  $N(1) = 0$ . Moreover, if  $N$  satisfies  $N(N(x)) = x$  it is called an involutive negator or a strong negation.

**Lemma 2.5.** *Let  $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$  be a residuated lattice. Define  $\neg : L \rightarrow L$  by  $\neg x = x \rightarrow 0$ . Then  $\neg$  is a negator on  $L$ .*

*Proof.* From Theorem 2.3 (4) it follows that  $\neg$  is non-increasing. Since  $1 * 0 = 0$ ,  $1 \leq 0 \rightarrow 0$  by residuation (2.1). But this means that  $\neg 0 = 1$ . Moreover,  $\neg 1 = 1 \rightarrow 0 = 1 * (1 \rightarrow 0) \leq 1 \wedge 0 = 0$  by (1) in Theorem 2.3.  $\square$

**Definition 2.6.** [11] A BL-algebra is a residuated lattice with the following properties:

- (iv)  $x * (x \rightarrow y) = x \wedge y$ ,
- (v)  $(x \rightarrow y) \vee (y \rightarrow x) = 1$ .

An MV-algebra  $L$  is a BL-algebra  $L$  satisfying :

- (vi)  $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$  for all  $x, y \in L$ .

Now if  $L$  is an MV-algebra, then  $\neg(\neg x) = (x \rightarrow 0) \rightarrow 0 = (0 \rightarrow x) \rightarrow x = 1 \rightarrow x = x$  for all  $x \in L$ . Therefore  $\neg$  is an involutive negator. R. Cignoli and F. Esteva [6] call such a residuated lattices that is residuated lattices in which  $\neg x = x \rightarrow 0$  is strong as *involutive* residuated lattices. But generally if there exists an involutive

negator in a residuated lattice, they call it a *symmetric* residuated lattice. It is evident that every involutive residuated lattice is a symmetric residuated lattice.

### 3. Intuitionistic Fuzzy Residuated Lattices

In this section we introduce a lattice built from a residuated lattice. We give its construction and then give the basic properties that show that this is a good candidate for the semantics part of a logic so called Intuitionistic Fuzzy Logic (IFL for short) introduced by K.T. Atanassov and G. Garagov [3].

**Definition 3.1.** Let  $L = (L, \wedge, \vee, *, \rightarrow, \sim, 0, 1)$  be a symmetric residuated lattice. Let  $\tilde{L} = \{(x, y) \in L^2 | x \leq \sim y\}$ . Define

$$(x_1, y_1) \preceq (x_2, y_2) \quad \text{if and only if} \quad x_1 \leq x_2 \quad \text{and} \quad y_2 \leq y_1. \quad (3)$$

It is easily verified that the above relation on  $\tilde{L}$  is a partially order. This order induces the following lattice operators on  $\tilde{L}$ :

$$(x_1, y_1) \bigwedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \vee y_2) \quad (4)$$

$$(x_1, y_1) \bigvee (x_2, y_2) = (x_1 \vee x_2, y_1 \wedge y_2) \quad (5)$$

**Lemma 3.2.** Let  $L$  and  $\tilde{L}$  be as in Definition 3.1 with corresponding order and operators. If  $L$  is a complete lattice, then  $\tilde{L}$  is a complete lattice.

*Proof.* It is easy to see that  $\tilde{L} = (\tilde{L}, \bigwedge, \bigvee, \tilde{0}, \tilde{1})$ , where  $\tilde{0} = (0, 1)$  and  $\tilde{1} = (1, 0)$  is a bounded lattice with least element  $\tilde{0}$  and greatest element  $\tilde{1}$ . To show that it is complete, let  $A = \{(x_i, y_i) \in \tilde{L} | i \in I\}$  be an arbitrary subset of  $\tilde{L}$ . Since for each  $i \in I$ ,  $x_i$  and  $y_i$  are in  $L$  and  $L$  is complete,  $u = \sup_i x_i$  and  $v = \inf_i y_i$  exist and are in  $L$ . We see that  $(u, v) \in L^2$  and  $(v, u) \in L^2$  are  $\sup A$  and  $\inf A$  respectively. It is sufficient to show that  $(u, v) \in \tilde{L}$ , i.e.,  $u \leq \sim v$ . In fact since  $x_i \leq \sim y_i$  for every  $i \in I$  and  $\sim$  is an involutive negator, we have  $u = \sup_i x_i \leq \sup_i \sim y_i = \sim \inf_i y_i = \sim v$ . From above it follows that  $v \leq \sim u$ , i.e.,  $(v, u) = \inf A \in \tilde{L}$ .  $\square$

We are now extending the notions of t-norm and t-conorm to arbitrary bounded lattices.

**Definition 3.3.** Let  $L = (L, \wedge, \vee, 0, 1)$  be a bounded lattice. (a) A lattice triangular norm (Lt-norm for short) is a binary operator  $T : L^2 \rightarrow L$  which is commutative, associative, isotone and  $T(1, x) = x$  for all  $x \in L$ . (b) A lattice triangular conorm (Lt-conorm for short) is a binary operator  $S : L^2 \rightarrow L$  which is commutative, associative, isotone and  $S(0, x) = x$  for all  $x \in L$ .

It is easy to see that if  $\sim$  is an involutive negator on a bounded lattice  $L$  and  $T$  is an Lt-norm on  $L$ , then  $S$  defined by

$$S(x, y) = \sim T(\sim x, \sim y) \quad (6)$$

for all  $x, y \in L$  is an Lt-conorm. Conversely, if  $S$  is an Lt-conorm on  $L$ , then  $T$  defined by

$$T(x, y) = \sim S(\sim x, \sim y) \quad (7)$$

for all  $x, y \in L$  is an Lt-norm on  $L$ .

We can also extend the notion of an implicator to bounded lattices.

**Definition 3.4.** Let  $L = (L, \wedge, \vee, 0, 1)$  be a bounded lattice. A lattice implicator ( $L$ -implicator) on  $L$  is a binary operator  $I : L^2 \rightarrow L$  satisfying:

- (i)  $I(0, 0) = I(0, 1) = I(1, 1) = 1$  and  $I(1, 0) = 0$  (Boundary conditions), and
- (ii)  $I$  is decreasing w.r.t. first component and increasing w.r.t. second component (Monotonic conditions).

**Lemma 3.5.** Let  $L$  and  $\tilde{L}$  be as in Definition 3.1. Then  $T$  defined by

$$T((x, y), (u, v)) = (x * u, S(y, v)) \quad (8)$$

for all  $(x, y), (u, v) \in \tilde{L}$  and  $S(a, b) = \sim(\sim a * \sim b)$  for all  $a, b \in L$ , is an Lt-norm on  $\tilde{L}$ .

*Proof.* First we notice that  $x \leq \sim y$  and  $u \leq \sim v$  imply that  $x * u \leq \sim y * \sim v = \sim S(y, v)$ . Then  $T : \tilde{L}^2 \rightarrow \tilde{L}$  is well-defined. Since  $*$  and  $S$  are commutative and associative,  $T$  is commutative and associative. To show that  $T$  is isotone we reason as: If  $(x_1, y_1) \preceq (x_2, y_2)$ , then  $x_1 \leq x_2$  and  $y_2 \leq y_1$  by Definition 3.1. Since  $*$  and  $S$  are isotone, we have  $x_1 * u \leq x_2 * u$  and  $S(y, v) \leq S(y_1, v)$ . Thus  $T((x_1, y_1), (u, v)) \preceq T((x_2, y_2), (u, v))$  which means that  $T$  is increasing w.r.t. the first component. Similarly  $T$  is increasing w.r.t. the second component. Moreover,  $T(\tilde{1}, (u, v)) = (1 * u, S(0, v)) = (u, v)$ .  $\square$

We recall that this Lt-norm is an extension of IF t-norms that are defined on  $L^*$  in [8] and call them t-representable.

**Lemma 3.6.** Let  $L$  and  $\tilde{L}$  be as in Definition 3.1. Define  $I$  by

$$I((x, y), (u, v)) = ((x \rightarrow u) \wedge (\sim y \rightarrow \sim v), \sim(\sim y \rightarrow \sim v)) \quad (9)$$

for all  $(x, y), (u, v) \in \tilde{L}$ . Then  $I$  is an  $L$ -implicator on  $\tilde{L}$ .

*Proof.* First we show that  $I$  is a well-defined binary operator from  $\tilde{L}^2$  to  $\tilde{L}$ . Since  $\rightarrow$  is a well-defined operator on  $L$ , it remains only to show that  $\text{range}(I) \subseteq \tilde{L}$ , i.e.,  $(x \rightarrow u) \wedge (\sim y \rightarrow \sim v) \leq \sim(\sim y \rightarrow \sim v) = (\sim y \rightarrow \sim v)$  and this is obvious because  $(x \rightarrow u) \wedge (\sim y \rightarrow \sim v) \leq (\sim y \rightarrow \sim v)$ . Then we verify Boundary conditions (i) in Definition 3.4 :

$$\begin{aligned} I(\tilde{0}, \tilde{0}) &= ((0 \rightarrow 0) \wedge (\sim 1 \rightarrow \sim 1), \sim(\sim 1 \rightarrow \sim 1)) = (1, 0) = \tilde{1}, \\ I(\tilde{0}, \tilde{1}) &= ((0 \rightarrow 1) \wedge (\sim 1 \rightarrow \sim 0), \sim(\sim 1 \rightarrow \sim 0)) = (1, 0) = \tilde{1}, \\ I(\tilde{1}, \tilde{1}) &= ((1 \rightarrow 1) \wedge (\sim 0 \rightarrow \sim 0), \sim(\sim 0 \rightarrow \sim 0)) = (1, 0) = \tilde{1}, \\ I(\tilde{1}, \tilde{0}) &= ((1 \rightarrow 0) \wedge (\sim 0 \rightarrow \sim 1), \sim(\sim 0 \rightarrow \sim 1)) = (0, 1) = \tilde{0}. \end{aligned} \quad \square$$

For Monotonic conditions ((ii) in Definition 3.4.) we reason as follows:

Let  $(x_1, y_1) \preceq (x, y)$ . Then  $x_1 \leq x$  and  $y \leq y_1$ . Since  $\sim$  is involutive,  $\sim y_1 \leq \sim y$ . From Theorem 2.3.(4) it follows that  $x \rightarrow u \leq x_1 \rightarrow u$  and  $\sim y \rightarrow \sim v \leq \sim y_1 \rightarrow \sim v$ . Thus  $(x \rightarrow u) \wedge (\sim y \rightarrow \sim v) \leq (x_1 \rightarrow u) \wedge (\sim y_1 \rightarrow \sim v)$ . On the other hand we have  $\sim(\sim y_1 \rightarrow \sim v) \leq \sim(\sim y \rightarrow \sim v)$ . Therefore  $I((x, y), (u, v)) \preceq I((x_1, y_1), (u, v))$  which means  $I$  is decreasing w.r.t. the first component. Similarly using properties

of  $\sim$  as an involutive negator and properties of  $\rightarrow$  as an implication in a residuated lattice, we can deduce that  $I$  is increasing w.r.t. the second component.

Satisfying  $I$  in Boundary and Monotonic conditions, imply that  $I$  is an L-implicator on  $\tilde{L}$  by Definition 3.4.

In the next theorem we will show some properties of the above L-implicator  $I$ .

**Theorem 3.7.** *Let  $I : \tilde{L}^2 \rightarrow \tilde{L}$  be the L-implicator defined in Lemma 3.6. Then  $I$  has the following properties:*

- (a)  $I(\tilde{1}, (u, v)) = (u, v)$  for all  $(u, v) \in \tilde{L}$ ,
- (b)  $X \preceq Y$  if and only if  $I(X, Y) = \tilde{1}$ .

*Proof.* (a) Let  $(u, v)$  be an element of  $\tilde{L}$ . Then

$$u \leq \sim v \tag{10}$$

From the definition of  $I$  we have  $I((1, 0), (u, v)) = ((1 \rightarrow u) \wedge (\sim 0 \rightarrow \sim v), \sim (\sim 0 \rightarrow \sim v))$  using (7) in Theorem 2.3, (10) and involution of  $\sim$  we have  $I((1, 0), (u, v)) = (u, v)$ .

(b) Let  $(x, y) = X$  and  $Y = (u, v)$  be elements of  $\tilde{L}$  such that  $X \preceq Y$ , i.e.,  $x \leq u$  and  $v \leq y$  by (3). Then  $\sim y \leq \sim v$  and we have  $\sim y \rightarrow \sim v = 1$  as well as  $x \rightarrow u = 1$  by Theorem 2.3 (7) which in turn imply that  $(x \rightarrow u) \wedge (\sim y \rightarrow \sim v) = 1$ . On the other hand  $\sim (\sim y \rightarrow \sim v) = 0$ . Therefore  $I(X, Y) = \tilde{1}$ .

Conversely, assume that  $X, Y \in \tilde{L}$  and  $I(X, Y) = \tilde{1}$ . Then by definition of  $I$  and property of lattice operation  $\wedge$  we have  $x \rightarrow u = 1$ ,  $\sim y \rightarrow \sim v = 1$  and  $\sim (\sim y \rightarrow \sim v) = 0$ . Theorem 2.3 (5) implies that  $x \leq u$  and  $\sim y \leq \sim v$ . Then  $x \leq u$  and  $v \leq y$ , i.e.,  $X \preceq Y$ .  $\square$

**Theorem 3.8.** *Let  $L, \tilde{L}, T$  and  $I$  be as above. Then  $\tilde{L} = (\tilde{L}, \wedge, \vee, T, I, \tilde{0}, \tilde{1})$  is a residuated lattice.*

*Proof.* (a) By the proof of Lemma 3.2. we see that  $(\tilde{L}, \wedge, \vee, \tilde{0}, \tilde{1})$  is a bounded lattice.

(b) From Lemma 3.5. it follows that  $(\tilde{L}, T, \tilde{1})$  is a commutative monoid.

(c) We show that  $T$  and  $I$  form an adjoint pair, i.e.,

$$T(X, Y) \preceq Z \quad \text{if and only if} \quad X \preceq I(Y, Z) \tag{11}$$

for all  $X, Y, Z \in \tilde{L}$ . Let  $X = (x, y)$ ,  $Y = (u, v)$  and  $Z = (s, t)$  be elements of  $\tilde{L}$  such that  $T(X, Y) \preceq Z$ . Then by definitions of  $T$  in (8) and  $\preceq$  in (3) we have

$$x * u \leq s \tag{12}$$

and

$$t \leq S(y, v) \tag{13}$$

Since  $*$  and  $\rightarrow$  form an adjoint pair, (12) implies that

$$x \leq u \rightarrow s. \tag{14}$$

From the definition of  $S$  as the dual of  $T$  and properties of involution  $\sim$  applied to (13) we get

$$\sim y * \sim v \leq \sim t. \quad (15)$$

Again by residuation we have

$$\sim y \leq \sim v \rightarrow \sim t \quad (16)$$

Since  $X = (x, y) \in \tilde{L}$ ,  $x \leq \sim y$ . This together with (16) implies that

$$x \leq \sim v \rightarrow \sim t. \quad (17)$$

From (14) and (17) it follows that

$$x \leq (u \rightarrow s) \wedge (\sim v \rightarrow \sim t). \quad (18)$$

On the other hand (16) implies that

$$\sim (\sim v \rightarrow \sim t) \leq y. \quad (19)$$

Inequalities in (18) and (19) together mean that

$$X = (x, y) \preceq I(Y, Z). \quad (20)$$

Conversely, let  $X \preceq I(Y, Z)$ . Then  $x \leq (u \rightarrow s) \wedge (\sim v \rightarrow \sim t) \leq (u \rightarrow s)$  which by residuation implies  $x * u \leq s$ . on the other hand  $\sim (\sim v \rightarrow \sim t) \leq y$ . Then  $\sim y \leq \sim v \rightarrow \sim t$  and by residuation  $\sim y * \sim v \leq \sim t$ . Using definition of  $S$  and  $\sim$  we get  $t \leq S(y, v)$ . Thus  $T(X, Y) \preceq Z$ . Now (a), (b) and (c) prove that  $T$  and  $I$  satisfy residuation property. Residual implications are widely studied in [4].  $\square$

**Example 3.9.** Let  $L = ([0, 1], \wedge, \vee, *, \rightarrow, \sim, 0, 1)$  be the Lukasiewicz algebra, where  $x \wedge y = \min(x, y)$ ,  $x \vee y = \max(x, y)$ ,  $x * y = \max(0, x + y - 1)$ ,  $x \rightarrow y = \min(1, 1 - x + y)$ ,  $\sim x = x \rightarrow 0 = 1 - x$ .

It is well known that this algebra is an involutive residuated lattice and therefore a symmetric residuated lattice. Define

$$\tilde{L} = L^* = \{(x, y) \in [0, 1]^2 | x \leq \sim y\} \quad (21)$$

Our L-implicator in Theorem 3.8. defined by (9) applied on  $\tilde{L} = L^*$  becomes

$$I((x, y), (u, v)) = (\min(1, 1 - x + y, 1 - v + y), \max(0, v - y)) \quad (22)$$

for all  $(x, y), (u, v) \in \tilde{L}$ . This is the same implication operator that Deschrijver, G. et al. have in [9] denoted by  $\mathcal{I}_{\mathcal{T}}$ , but comparing with the implicator  $\mathcal{I}_{\mathcal{S}_2, \mathcal{N}_s}$  in [9] we see that they are very close in the sense that they have the same first component, but generally  $I \preceq \mathcal{I}_{\mathcal{S}_2, \mathcal{N}_s}$  and by some calculations we get

$$I((x, y), (u, v)) = \mathcal{I}_{\mathcal{S}_2, \mathcal{N}_s}((x, y), (u, v)) \quad \text{if and only if} \quad v \leq y \quad \text{or} \quad y = 1 - x \quad (23)$$

It is shown in [9] that  $\mathcal{I}_{\mathcal{S}_2, \mathcal{N}_s}$  satisfies the axioms (A1-A6) of Smets and Magrez for an IF implicator [15]. If we consider the generalized versions of these axioms for L -implicators (Definition 3.4. ), our proposed L -implicator  $I$  satisfies A1,A2 and A5. Axiom A6 is about continuity which is not of our concerns now in lattices. Moreover  $\mathcal{I}_{\mathcal{S}_2, \mathcal{N}_s}$  is an S-implicator generated by  $\mathcal{S}_2$  and standard negation  $\mathcal{N}_s$  where  $\mathcal{S}_2((x_1, x_2), (y_1, y_2)) = ((\min(1, x_1 + 1 - y_2, y_1 + 1 - x_2), \max(0, x_1 + y_2 - 1))$  on  $\tilde{L}$  and  $\mathcal{N}_s(x, y) = (y, x)$  as well as an R-implicator generated by  $\mathcal{T}_2((x_1, x_2), (y_1, y_2)) = (\max(0, x_1 + y_1 - 1), \min(1, 1 + x_2 - y_1, 1 - x_1 + y_2))$ [15].

We now show that the proposed L -implicator  $I$  is an R-implicator with respect to corresponding notions in lattices..

**Lemma 3.10.** *Let  $L$  and  $\tilde{L}$  be as above. Then  $I$  is an R-implicator generated by  $T$ , i.e.,*

$$I(X, Y) = \sup\{Z \in \tilde{L} | T(X, Z) \preceq Y\} \quad (24)$$

for all  $(X, Y) \in \tilde{L}_2$ .

*Proof.* Let  $A = \{Z \in \tilde{L} | T(X, Z) \preceq Y\}$ ,  $X = (x, y)$  and  $Y = (u, v)$ . First we show that  $I(X, Y)$  is an upper bound of  $A$ . In fact, let  $(s, t) \in A$ . Then

$$x * s \leq u \quad (25)$$

and

$$v \leq S(y, t). \quad (26)$$

Based on residuation property, definition of  $S$ , involution of  $\sim$ , and the fact that  $s \leq \sim t$  from (25) and (26) we get

$$s \leq x \rightarrow u \quad (27)$$

and

$$s \leq \sim t \leq \sim y \rightarrow \sim v. \quad (28)$$

(27) and (28) imply that

$$s \leq (x \rightarrow u) \wedge (\sim y \rightarrow \sim v) \quad (29)$$

and

$$\sim (\sim y \rightarrow \sim v) \leq t \quad (30)$$

From (29) and (30) it follows that

$$Z = (s, t) \preceq I(X, Y). \quad (31)$$

On the other hand we show that  $I(X, Y) \in A$ . In fact

$$x * [(x \rightarrow u) \wedge (\sim y \rightarrow \sim v)] \leq x * (x \rightarrow u) \leq u \quad (32)$$

and

$$\sim y * (\sim y \rightarrow \sim v) \leq \sim v \quad (33)$$



which is equivalent to

$$S(y, \sim(\sim y \rightarrow \sim v)) = \sim(\sim y * (\sim y \rightarrow \sim v)) \geq \sim(\sim v) = v \quad (34)$$

(32) and (34) prove that

$$T(X, I(X, Y)) \preceq Y \quad (35)$$

That is  $I(X, Y) \in A$ . Therefore indeed we proved  $I(X, Y) = \sup A = \max A$ .  $\square$

**Example 3.11.** Let  $L = \{0, a, b, 1\}$ , where  $0 < a < b < 1$ .  $L = (L, \wedge, \vee, *, \rightarrow, \sim, 0, 1)$  is a symmetric residuated lattice with the operators defined by the following tables:

$x$	$\sim x$
0	1
$a$	$b$
$b$	$a$
1	0

$*$	0	$a$	$b$	1
0	0	0	0	0
$a$	0	0	$a$	$a$
$b$	0	$a$	$b$	$b$
1	0	$a$	$b$	1

$\rightarrow$	0	$a$	$b$	1
0	1	1	1	1
$a$	$a$	1	1	1
$b$	0	$a$	1	1
1	0	$a$	$b$	1

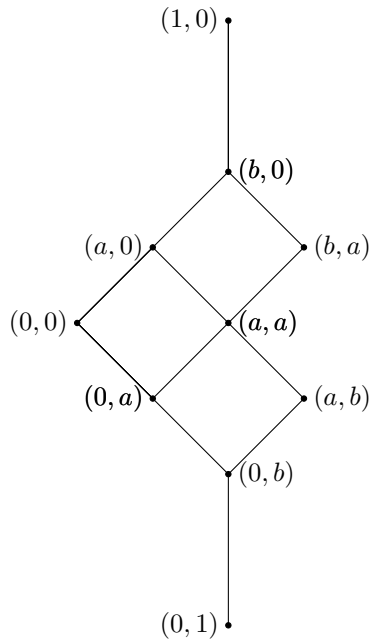
TABLE 1. Operators  $\sim$ ,  $*$  and  $\rightarrow$  on  $L$

Now  $\tilde{L} = \{(0, 0), (0, a), (0, b), (0, 1), (a, b), (a, a), (a, 0), (b, a), (b, 0), (1, 0)\}$  with the order  $\preceq$  shown by the diagram in Figure 1 and corresponding  $T$  and  $I$  defined by (8) and (9) respectively is a symmetric residuated lattice. We note that the standard involutive negation is  $N_s$  defined by  $N_s(x, y) = (y, x)$  for all  $(x, y) \in \tilde{L}$ , where as the negation generated by the implicator  $I$  is  $N_I(x, y) = I((x, y), (0, 1)) = (\neg \sim y, \sim \neg \sim y)$

Using the above residuated lattice, we may have a ten-valued intuitionistic fuzzy logic that to each propositional symbol of our language one of the nodes as an intuitionistic fuzzy value is assigned. Then we can find the intuitionistic fuzzy values of each formula via the operations  $N_s$  for negation,  $\wedge$  for "and",  $\vee$  for "or",  $T$  for bold conjunction,  $I$  for implication.

#### 4. Conclusion and Future Research

In this paper we constructed a residuated lattice called intuitionistic fuzzy residuated lattice from a usual residuated lattice suitable for the semantics of intuitionistic fuzzy logics given by Atanassov. We showed that the operations have reasonable properties to meet basic conditions. The proposed implicator  $I$  satisfies the basic properties of a logical implicator. In future works we give more properties of  $I$  and try to extract a set of axioms for this so called intuitionistic fuzzy logic. We believe that this is a type of logic different from the others but has some common features with known logics such as fuzzy logic. As a generalization of fuzzy logic can have applications in different areas such as image processing. We will show that we can have Modus Ponens as a rule of inference. However we accept the notion of standard tautology in the sense that a formula  $\varphi$  is a tautology in IFL if  $\vartheta(\varphi) = (1, 0)$  but do not agree with the definition of the tautology as a formula  $\varphi$  with IFL value  $\vartheta(\varphi) = (\mu(\varphi), \nu(\varphi)) \in \tilde{L}$  such that  $\mu(\varphi) \geq \nu(\varphi)$ , because they may not be comparable. If they are comparable and even in  $L^*$ , we must have a better

FIGURE 1. Lattice Structure of  $\tilde{L}$ 

criteria such as "  $\mu(\varphi)$  is much (very) bigger than  $\nu(\varphi)$ ". This leads us to modeling linguistic hedges in the IFL in a convincing form.

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