

ON LACUNARY STATISTICAL LIMIT AND CLUSTER POINTS OF SEQUENCES OF FUZZY NUMBERS

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ABSTRACT. For any lacunary sequence $\theta = (k_r)$, we define the concepts of S_θ -limit point and S_θ -cluster point of a sequence of fuzzy numbers $X = (X_k)$. We introduce the new sets $\Lambda_{S_\theta}^F(X)$, $\Gamma_{S_\theta}^F(X)$ and prove some inclusion relations between these and the sets $\Lambda_S^F(X)$, $\Gamma_S^F(X)$ introduced in [3] by Aytar [S. Aytar, Statistical limit points of sequences of fuzzy numbers, Inform. Sci. 165 (2004) 129-138]. Later, we find restriction on the lacunary sequence $\theta = (k_r)$ for which the sets $\Lambda_{S_\theta}^F(X)$ and $\Gamma_{S_\theta}^F(X)$ respectively coincides with the sets $\Lambda_S^F(X)$ and $\Gamma_S^F(X)$.

1. Introduction

The concept of statistical convergence of sequence of numbers was introduced by Fast [9] and Schoenberg [27] independently in 1951. In the recent years, since 1980 ([25] and [11]) this idea has grown a little fast and many concepts parallel to the classical concepts in mathematical analysis have been introduced for statistical convergence. Fridy [12] introduced the set Γ_x of all statistical cluster points; the set Λ_x of all statistical limit points of a sequence $x = (x_k)$ of real numbers and discussed their definitions and properties, as well as the specific relations between them. These issues have been further explored on finite dimensional spaces in [10], [14], [16], [21] and [23]. In 1993, Fridy and Orhan [14] presented an interesting generalization of statistical convergence with the help of a lacunary sequence $\theta = (k_r)$ and called it as lacunary statistical convergence or S_θ -convergence. Demirci [8] defined S_θ -limit and cluster points of a number sequence and presented some analogue of the results of Fridy [12]. On the other side, Matloka [17] introduced bounded and convergent sequences of fuzzy numbers and studied some of their properties. Nanda [18] proved the set of all convergent sequences of fuzzy numbers forms a complete metric space. In last few years, statistical convergence has been also extended to the sequences of fuzzy numbers. Nuray and Savas [19] defined statistical convergence of a sequence of fuzzy numbers and obtained the Cauchy convergence criteria. Subsequently, Nuray [20] and Kwon et. al. [15] studied S_θ -convergence of sequences of fuzzy numbers and proved some inclusion relations. Recently, Aytar [3] defined statistical limit and cluster points of a sequence of fuzzy numbers and gave some fuzzy-analogues of properties of [12] for the sets $L(X)$, $\Lambda_S^F(X)$ and $\Gamma_S^F(X)$ which respectively denote

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the sets of all ordinary limit points, statistical limit points and statistical cluster points of a sequence $X = (X_k)$ of fuzzy numbers. These issues have been further extended for sequences of fuzzy numbers in [1, 2, 4, 5, 6, 7, 22, 26] and [28].

In present paper, we define the concepts of S_θ -limit point and S_θ -cluster point of a sequence $X = (X_k)$ of fuzzy numbers and obtained some inclusion relations between the sets $L(X)$, $\Lambda_{S_\theta}^F(X)$ and $\Gamma_{S_\theta}^F(X)$ which respectively denote the sets of all ordinary limit points, S_θ -limit points and S_θ -cluster points of X . We also find some specific conditions for which the sets $\Lambda_{S_\theta}^F(X)$, $\Gamma_{S_\theta}^F(X)$ coincides, respectively, with the sets $\Lambda_S^F(X)$, $\Gamma_S^F(X)$ introduced by Aytar in [3].

2. Preliminaries

Let \mathbb{N} and \mathbb{R} , denote the set of positive integers and the set of real numbers respectively. Given any interval A , we shall denote its end points by \underline{A}, \bar{A} and by D , we denote the set of all closed bounded intervals on the real line \mathbb{R} , i.e., $D = \{A \subset \mathbb{R} : A = [\underline{A}, \bar{A}]\}$. For $A, B \in D$, we define $A \leq B$ if and only if $\underline{A} \leq \underline{B}$ and $\bar{A} \leq \bar{B}$. Moreover, the distance function d defined by $d(A, B) = \max\{|\underline{A} - \underline{B}|, |\bar{A} - \bar{B}|\}$ is a Hausdorff metric on D and (D, d) is a complete metric space. Also \leq is a partial order on D .

A fuzzy number is a function X from \mathbb{R} to $[0, 1]$ which satisfies the following conditions: (i) X is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $X(x_0) = 1$; (ii) X is fuzzy convex, i.e., for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, $X(\lambda x + (1 - \lambda)y) \geq \min\{X(x), X(y)\}$; (iii) X is upper semi-continuous and (iv) the closure of the set $\{x \in \mathbb{R} : X(x) > 0\}$ denoted by X^0 , is compact.

Properties (i)-(iv) imply that for each $\alpha \in (0, 1]$, the α -level set,

$$X^\alpha = \{x \in \mathbb{R} : X(x) \geq \alpha\} = [\underline{X}^\alpha, \bar{X}^\alpha]$$

is a non-empty compact convex subset of \mathbb{R} . Let $L(\mathbb{R})$ denotes the set of all fuzzy numbers. Linear structure of $L(\mathbb{R})$ induces an addition $X + Y$ and a scalar multiplication λX in terms of α -level sets by

$$[X + Y]^\alpha = [X]^\alpha + [Y]^\alpha \text{ and } [\lambda X]^\alpha = \lambda[X]^\alpha \quad (X, Y \in L(\mathbb{R}), \lambda \in \mathbb{R})$$

for each $\alpha \in [0, 1]$.

Define a map $\bar{d} : L(\mathbb{R}) \times L(\mathbb{R}) \rightarrow \mathbb{R}$ by $\bar{d}(X, Y) = \sup_{\alpha \in [0, 1]} d(X^\alpha, Y^\alpha)$. Puri and Ralescu [24] proved that $(L(\mathbb{R}), \bar{d})$ is a complete metric space.

Also the ordered structure on $L(\mathbb{R})$ is defined as follows:

For $X, Y \in L(\mathbb{R})$, we define $X \leq Y$ if and only if $\underline{X}^\alpha \leq \underline{Y}^\alpha$ and $\bar{X}^\alpha \leq \bar{Y}^\alpha$ for each $\alpha \in [0, 1]$. We say that $X < Y$ if $X \leq Y$ and there exists $\alpha_0 \in [0, 1]$ such that $\underline{X}^{\alpha_0} < \underline{Y}^{\alpha_0}$ or $\bar{X}^{\alpha_0} < \bar{Y}^{\alpha_0}$. The fuzzy numbers X and Y are said to be incomparable if neither $X \leq Y$ nor $Y \leq X$.

We now give the definitions of statistical convergence and lacunary statistical convergence of a sequence of fuzzy numbers.

Definition 2.1. [19] A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number X_0 , written as $S - \lim_k X_k = X_0$, if for each $\epsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : \bar{d}(X_k, X_0) \geq \epsilon\}| = 0.$$

Here vertical bars denotes the cardinality of the enclosed set.

By a lacunary sequence, we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ whereas we denote the ratio $\frac{k_r}{k_{r-1}}$ by q_r .

A lacunary sequence $\theta' = (k'_r)$ is called a lacunary refinement of the lacunary sequence $\theta = (k_r)$ if $\{k_r\} \subseteq \{k'_r\}$.

Definition 2.2. [20] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary statistically convergent to a fuzzy number X_0 , in symbol: $S_\theta - \lim_k X_k = X_0$, if for each $\epsilon > 0$,

$$\lim_r \frac{1}{h_r} |\{k \in I_r : \bar{d}(X_k, X_0) \geq \epsilon\}| = 0.$$

Example 2.3. Let $\theta = (k_r)$ be a lacunary sequence. Define a sequence $X = (X_k)$ of fuzzy numbers as follows. For $x \in \mathbb{R}$ and $k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r$, $r \in \mathbb{N}$, define

$$X_k(x) = \begin{cases} 0, & \text{if } x \in (-\infty, k) \cup (k + 2, \infty), \\ x - k & \text{if } x \in [k, k + 1], \\ -x + k + 2 & \text{if } x \in (k + 1, k + 2]. \end{cases}$$

and $X_k(x) = X_0$ if k does not satisfy $k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r$, where X_0 is given by

$$X_0(x) = \begin{cases} 0, & \text{if } x \in (-\infty, -1) \cup (1, \infty), \\ 1 + x & \text{if } x \in [-1, 0], \\ 1 - x & \text{if } x \in (0, 1]. \end{cases}$$

For $0 < \epsilon < 1$, if we denote the set $K_r(\epsilon) = \{k \in I_r : \bar{d}(X_k, X_0) \geq \epsilon\}$, then

$$\begin{aligned} K_r(\epsilon) &= \{k \in I_r : X_k \text{ with } k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r, r \in \mathbb{N}\} \\ &= \{k \in I_r : k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r, r \in \mathbb{N}\}; \end{aligned}$$

and therefore

$$\frac{1}{h_r} |K_r(\epsilon)| = \frac{1}{h_r} |\{k \in I_r : k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r, r \in \mathbb{N}\}| \leq \frac{\sqrt{h_r}}{h_r}.$$

Taking $r \rightarrow \infty$, we get

$$\lim_r \frac{1}{h_r} |K_r(\epsilon)| \leq \lim_r \frac{\sqrt{h_r}}{h_r} = 0;$$

which shows (X_k) is S_θ -convergent to X_0 . Obviously, (X_k) is not convergent to X_0 in the usual sense. \square

3. S_θ -limit and Cluster Points

Definition 3.1. Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers. Let $(X_{k(j)})$ be a subsequence of $X = (X_k)$ and $K = \{k(j) : j \in \mathbb{N}\}$ then we denote $(X_{k(j)})$ by $(X)_K$. If

$$\lim_r \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| = 0,$$

then (X_K) is called a θ -thin subsequence. On the other hand, (X_K) is a θ -nonthin subsequence of X provided that

$$\limsup_r \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| > 0.$$

Definition 3.2. For any two sequences of fuzzy numbers (X_k) and (Y_k) if

$$\lim_r \frac{1}{h_r} |\{k \in I_r : X_k \neq Y_k\}| = 0,$$

then we say that $X_k = Y_k$ for almost all k with respect to θ .

Definition 3.3. Let $\theta = (k_r)$ be a lacunary sequence. A fuzzy number X_0 is said to be a lacunary statistical limit point or S_θ -limit point of a sequence of fuzzy numbers $X = (X_k)$ provided that there is a θ -nonthin subsequence of X that converges to X_0 .

Let $\Lambda_{S_\theta}^F(X)$ denotes the set of all S_θ -limit points of the sequence $X = (X_k)$.

Definition 3.4. Let $\theta = (k_r)$ be a lacunary sequence. A fuzzy number Y_0 is said to be a lacunary statistical cluster point or S_θ -cluster point of a sequence of fuzzy numbers $X = (X_k)$ provided that for each $\epsilon > 0$,

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : \bar{d}(X_k, Y_0) < \epsilon\}| > 0.$$

Let $\Gamma_{S_\theta}^F(X)$ denotes the set of all S_θ -cluster points of the sequence $X = (X_k)$.

Theorem 3.5. For any lacunary sequence $\theta = (k_r)$ and any sequence of fuzzy numbers $X = (X_k)$, $\Lambda_{S_\theta}^F(X) \subseteq \Gamma_{S_\theta}^F(X)$.

Proof. Suppose $X_0 \in \Lambda_{S_\theta}^F(X)$, there is a subsequence $(X_{k(j)})$ with $\lim_j X_{k(j)} = X_0$ and

$$\limsup_r \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| = d > 0. \quad (1)$$

For each $\epsilon > 0$, $\{k \in I_r : \bar{d}(X_k, X_0) < \epsilon\} \supset \{k(j) \in I_r : \bar{d}(X_{k(j)}, X_0) < \epsilon\}$ gives $\{k \in I_r : \bar{d}(X_k, X_0) < \epsilon\} \supset \{k(j) \in I_r : j \in \mathbb{N}\} - \{k(j) \in I_r : \bar{d}(X_{k(j)}, X_0) \geq \epsilon\}$; and therefore we have

$$\begin{aligned} \limsup_r \frac{1}{h_r} |\{k \in I_r : \bar{d}(X_k, X_0) < \epsilon\}| &\geq \limsup_r \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| - \\ &\limsup_r \frac{1}{h_r} |\{k(j) \in I_r : \bar{d}(X_{k(j)}, X_0) \geq \epsilon\}|. \end{aligned}$$

Since, $\{k(j) \in I_r : \bar{d}(X_{k(j)}, X_0) \geq \epsilon\}$ is a finite set so $\limsup_r \frac{1}{h_r} |\{k(j) \in I_r : \bar{d}(X_{k(j)}, X_0) \geq \epsilon\}| = 0$. Hence the result follows by (1). \square

Theorem 3.6. For any lacunary sequence $\theta = (k_r)$ and any sequence of fuzzy numbers $X = (X_k)$, $\Gamma_{S_\theta}^F(X) \subseteq L(X)$, where $L(X)$ denotes the set of all ordinary limit points of $X = (X_k)$.

Proof. Suppose $Y_0 \in \Gamma_{S_\theta}^F(X)$, then for each $\epsilon > 0$

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : \bar{d}(X_k, Y_0) < \epsilon\}| > 0.$$

We set $(X)_K$ a θ -nonthin subsequence of X such that $K = \{k(j) \in I_r : \bar{d}(X_{k(j)}, Y_0) < \epsilon\}$. Since $\limsup_r \frac{1}{h_r} |\{K\}| > 0$ and there are infinitely many elements in K , so we have a θ -nonthin subsequence $(X)_K$ that converges to Y_0 . This shows that $Y_0 \in L(X)$. \square

The converse of the above theorem does not hold in general as can be seen in the following example.

Example 3.7. Let $\theta = (k_r)$ be a lacunary sequence. We define a sequence of fuzzy numbers $X = (X_k)$ as follows. For $x \in \mathbb{R}$, define

$$X_k(x) = \begin{cases} X_0, & \text{if } k_r - [\sqrt{h_r}] + 1 \leq k \leq k_r; r \in \mathbb{N}, \\ Y_0, & \text{otherwise.} \end{cases}$$

where the fuzzy numbers X_0 and Y_0 are given by

$$X_0 = \begin{cases} 0, & \text{if } x \in (-\infty, 0) \cup (2, \infty), \\ x & \text{if } x \in [0, 1], \\ 2 - x & \text{if } x \in (1, 2]. \end{cases}$$

and

$$Y_0 = \begin{cases} 0, & \text{if } x \in (-\infty, 3) \cup (5, \infty), \\ x - 3 & \text{if } x \in [3, 4], \\ 5 - x & \text{if } x \in (4, 5]. \end{cases}$$

Clearly the sequence $X = (X_k)$ has two different subsequences (X_0, X_0, \dots) and (Y_0, Y_0, \dots) which converges to X_0 and Y_0 respectively. Hence $L(X) = \{X_0, Y_0\}$, however, $\Gamma_{S_\theta}^F(X) = \{Y_0\}$.

Theorem 3.8. Let $\theta = (k_r)$ be a lacunary sequence. If $X = (X_k)$ and $Y = (Y_k)$ are two sequences of fuzzy numbers such that $X_k = Y_k$ for all most all k with respect to θ , then $\Lambda_{S_\theta}^F(X) = \Lambda_{S_\theta}^F(Y)$ and $\Gamma_{S_\theta}^F(X) = \Gamma_{S_\theta}^F(Y)$.

Proof. Assume $\lim_r \frac{1}{h_r} |\{k \in I_r : X_k \neq Y_k\}| = 0$ and let $X_0 \in \Lambda_{S_\theta}^F(Y)$, there is a θ -nonthin subsequence $(Y)_K$ of (Y_k) that converges to X_0 . Since $\lim_r \frac{1}{h_r} |\{k \in I_r : k \in K \text{ and } X_k \neq Y_k\}| = 0$, it follows that $\limsup_r \frac{1}{h_r} |\{k \in I_r : k \in K \text{ and } X_k = Y_k\}| > 0$. Therefore, the latter set yields a θ -nonthin subsequence $(X)_{K'}$ of (X_k) that converges to X_0 . Hence, $X_0 \in \Lambda_{S_\theta}^F(X)$ and $\Lambda_{S_\theta}^F(Y) \subset \Lambda_{S_\theta}^F(X)$. By symmetry we see that $\Lambda_{S_\theta}^F(X) \subset \Lambda_{S_\theta}^F(Y)$, whence $\Lambda_{S_\theta}^F(X) = \Lambda_{S_\theta}^F(Y)$. The assertion that $\Gamma_{S_\theta}^F(X) = \Gamma_{S_\theta}^F(Y)$ can be proved by a similar argument. \square

Theorem 3.9. Let $\theta = (k_r)$ be a lacunary sequence. If $X = (X_k)$ be a sequence of fuzzy numbers such that $S_\theta - \lim_k X_k = X_0$, then $\Lambda_{S_\theta}^F(X) = \Gamma_{S_\theta}^F(X) = \{X_0\}$.

Proof. We first prove that $\Lambda_{S_\theta}^F(X) = \{X_0\}$. Suppose $\Lambda_{S_\theta}^F(X) = \{X_0, Y_0\}$ where $X_0 \neq Y_0$. Let $0 < \epsilon < \frac{\bar{d}(X_0, Y_0)}{2}$, there are two θ -nonthin subsequences $(X_{k(j)})$ and $(X_{l(i)})$ of (X_k) which respectively converges to X_0 and Y_0 . Since $(X_{l(i)})$ converges

to Y_0 so for each $\epsilon > 0$, $\{l(i) \in I_r : \bar{d}(X_{l(i)}, Y_0) \geq \epsilon\}$ is a finite set and therefore we have $\limsup_r \frac{1}{h_r} |\{l(i) \in I_r : \bar{d}(X_{l(i)}, Y_0) \geq \epsilon\}| = 0$. Further, for $\epsilon > 0$ we can write $\{l(i) \in I_r : i \in \mathbb{N}\} = \{l(i) \in I_r : \bar{d}(X_{l(i)}, Y_0) < \epsilon\} \cup \{l(i) \in I_r : \bar{d}(X_{l(i)}, Y_0) \geq \epsilon\}$, which implies

$$\begin{aligned}
 (\star) \quad 0 < \limsup_r \frac{1}{h_r} |\{l(i) \in I_r : i \in \mathbb{N}\}| &\leq \limsup_r \frac{1}{h_r} |\{l(i) \in I_r : \bar{d}(X_{l(i)}, Y_0) < \epsilon\}| + \\
 &\quad \limsup_r \frac{1}{h_r} |\{l(i) \in I_r : \bar{d}(X_{l(i)}, Y_0) \geq \epsilon\}| \\
 &= \limsup_r \frac{1}{h_r} |\{l(i) \in I_r : \bar{d}(X_{l(i)}, Y_0) < \epsilon\}|
 \end{aligned}$$

Since $S_\theta - \lim_k X_k = X_0$ so for $\epsilon > 0$ $\lim_r \frac{1}{h_r} |\{k \in I_r : \bar{d}(X_k, X_0) \geq \epsilon\}| = 0$, which implies $\limsup_r \frac{1}{h_r} |\{k \in I_r : \bar{d}(X_k, X_0) < \epsilon\}| > 0$. Also for $0 < 2\epsilon < \bar{d}(X_0, Y_0)$, $\{l(i) \in I_r : \bar{d}(X_{l(i)}, Y_0) < \epsilon\} \cap \{k \in I_r : \bar{d}(X_k, X_0) < \epsilon\} = \emptyset$. But then $\{l(i) \in I_r : \bar{d}(X_{l(i)}, Y_0) < \epsilon\} \subset \{k \in I_r : \bar{d}(X_k, X_0) \geq \epsilon\}$, which immediately implies

$$\limsup_r \frac{1}{h_r} |\{l(i) \in I_r : \bar{d}(X_{l(i)}, Y_0) < \epsilon\}| \leq \limsup_r \frac{1}{h_r} |\{k \in I_r : \bar{d}(X_k, X_0) \geq \epsilon\}| = 0. \quad (2)$$

As left side of (2) can not be negative, so we must have $\limsup_r \frac{1}{h_r} |\{l(i) \in I_r : \bar{d}(X_{l(i)}, Y_0) < \epsilon\}| = 0$, which contradict (\star) . Hence, $\Lambda_{S_\theta}^F(X) = \{X_0\}$.

Let Z_0 be a S_θ -cluster point of $X = (X_k)$ different from X_0 i.e., $\Gamma_{S_\theta}^F(X) = \{X_0, Z_0\}$ where $X_0 \neq Z_0$. Choose $0 < \epsilon < \frac{\bar{d}(X_0, Z_0)}{2}$, then

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : \bar{d}(X_k, Z_0) < \epsilon\}| > 0. \quad (3)$$

Since $\{k \in I_r : \bar{d}(X_k, X_0) < \epsilon\} \cap \{k \in I_r : \bar{d}(X_k, Z_0) < \epsilon\} = \emptyset$ for every $0 < \epsilon < \frac{\bar{d}(X_0, Z_0)}{2}$, so $\{k \in I_r : \bar{d}(X_k, X_0) \geq \epsilon\} \supseteq \{k \in I_r : \bar{d}(X_k, Z_0) < \epsilon\}$, implies that

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : \bar{d}(X_k, X_0) \geq \epsilon\}| \geq \limsup_r \frac{1}{h_r} |\{k \in I_r : \bar{d}(X_k, Z_0) < \epsilon\}|. \quad (4)$$

Since $S_\theta - \lim_k X_k = X_0$ so left side of (4) is zero, however, by (3), the right side of (4) is strictly greater than zero. This contradiction shows that $\Gamma_{S_\theta}^F(X) = \{X_0\}$. \square

We next give some inclusion theorems on the sets $\Lambda_{S_\theta}^F(X)$, $\Gamma_{S_\theta}^F(X)$ and $\Lambda_S^F(X)$, $\Gamma_S^F(X)$.

Theorem 3.10. *Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers; then*

- (i) $\liminf_r q_r > 1$ implies $\Lambda_{S_\theta}^F(X) \subseteq \Lambda_S^F(X)$;
- (ii) $\limsup_r q_r < \infty$ implies $\Lambda_S^F(X) \subseteq \Lambda_{S_\theta}^F(X)$;
- (iii) $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$ implies $\Lambda_S^F(X) = \Lambda_{S_\theta}^F(X)$.

Proof. (i) Suppose $\liminf_r q_r > 1$; there exists a $\delta > 0$ such that $q_r > 1 + \delta$ for sufficiently large r , which implies that

$$\frac{h_r}{k_r} \geq \frac{\delta}{1 + \delta}.$$

If $X_0 \in \Lambda_{S_\theta}^F(X)$, there is a subsequence $(X_{k(j)})$ of (X_k) with $\lim_j X_{k(j)} = X_0$ and

$$\limsup_r \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| = d > 0 \quad (5)$$

$$\begin{aligned} \text{Now } \frac{1}{k_r} |\{k(j) \leq k_r : j \in \mathbb{N}\}| &\geq \frac{1}{k_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| = \frac{h_r}{k_r} \cdot \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| \\ &\geq \left(\frac{\delta}{1+\delta}\right) \cdot \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}|; \end{aligned}$$

implies together with (5), $\limsup_r \frac{1}{k_r} |\{k(j) \leq k_r : j \in \mathbb{N}\}| > 0$. Since $(X_{k(j)})$ already converges to X_0 , it follows that $X_0 \in \Lambda_S^F(X)$. Hence, $\Lambda_{S_\theta}^F(X) \subseteq \Lambda_S^F(X)$.

(ii) If $\limsup q_r < \infty$, then there is an H such that $q_r < H$ for all r . We may assume $H > 1$ as $\theta = (k_r)$ is a lacunary sequence. Also for each $r \in \mathbb{N}$, we have

$$\frac{h_r}{k_{r-1}} = \frac{k_r - k_{r-1}}{k_{r-1}} = q_r - 1 < H - 1.$$

If $X_0 \in \Lambda_S^F(X)$, there is a subsequence $(X_{k(j)})$ of (X_k) with $\lim_j X_{k(j)} = X_0$ and

$$\limsup_n \frac{1}{n} |\{k(j) \leq n : j \in \mathbb{N}\}| = d > 0 \quad (6)$$

Define $N_r = |\{k(j) \in I_r : j \in \mathbb{N}\}|$ and $t_r = \frac{N_r}{h_r}$, then for any integer n satisfying $k_{r-1} < n \leq k_r$; we can write

$$\begin{aligned} \frac{1}{n} |\{k(j) \leq n : j \in \mathbb{N}\}| &\leq \frac{1}{k_{r-1}} |\{k(j) \leq k_r : j \in \mathbb{N}\}| = \frac{1}{k_{r-1}} \{N_1 + N_2 + \dots + N_r\} \\ &= \frac{1}{k_{r-1}} \{t_1 h_1 + t_2 h_2 + \dots + t_r h_r\} = \frac{1}{\sum_{i=1}^{r-1} h_i} \sum_{i=1}^{r-1} h_i t_i + \frac{h_r}{k_{r-1}} t_r \\ &< \frac{1}{\sum_{i=1}^{r-1} h_i} \sum_{i=1}^{r-1} h_i t_i + (H - 1) t_r \end{aligned}$$

Suppose $t_r \rightarrow 0$ as $r \rightarrow \infty$. Since θ is lacunary and the first part on the right side of the above expression is a regular weighted mean transform of the sequence $t = (t_r)$, and therefore it, too tends to zero as $r \rightarrow \infty$. Consequently, we get $\limsup_n \frac{1}{n} |\{k(j) \leq n : j \in \mathbb{N}\}| = 0$ which contradict (6). Thus $\lim_r t_r \neq 0$ and therefore $X_0 \in \Lambda_{S_\theta}^F(X)$ as $X_{k(j)} \rightarrow X_0$ already. Hence $\Lambda_S^F(X) \subseteq \Lambda_{S_\theta}^F(X)$.

(iii) This is an immediate consequence of (i) and (ii). □

Theorem 3.11. *Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ be a sequence of fuzzy numbers; then*

- (i) $\liminf q_r > 1$ implies $\Gamma_{S_\theta}^F(X) \subset \Gamma_S^F(X)$;
- (ii) $\limsup q_r < \infty$ implies $\Gamma_S^F(X) \subset \Gamma_{S_\theta}^F(X)$;
- (iii) $1 < \liminf q_r \leq \limsup q_r < \infty$ implies $\Gamma_S^F(X) = \Gamma_{S_\theta}^F(X)$.

Proof. The proof of the theorem can be obtained in similar way as that of theorem 3.10 and therefore is omitted. □

Finally, we consider the inclusions $\Gamma_{S_\theta}^F(X) \subseteq \Gamma_{S_{\theta'}}^F(X)$ and $\Lambda_{S_\theta}^F(X) \subseteq \Lambda_{S_{\theta'}}^F(X)$ where θ' is a lacunary refinement of $\theta = (k_r)$.

Theorem 3.12. *For any lacunary refinement θ' of a lacunary sequence $\theta = (k_r)$, $\Gamma_{S_\theta}^F(X) \subseteq \Gamma_{S_{\theta'}}^F(X)$ and $\Lambda_{S_\theta}^F(X) \subseteq \Lambda_{S_{\theta'}}^F(X)$.*

Proof. Suppose each I_r of θ contains the points $\{k'_{r,i}\}_{i=1}^{v(r)}$ of θ' so that $k_{r-1} < k'_{r,1} < k'_{r,2} < \dots < k'_{r,v(r)} = k_r$, where $I'_{r,i} = (k'_{r,i-1}, k'_{r,i}]$. Note that for all r , $v(r) \geq 1$. Let $(I_j^*)_{j=1}^\infty$ be the sequence of abutting intervals $I'_{r,i}$ ordered by increasing right end points. Suppose $Y_0 \in \Gamma_{S_\theta}^F(X)$, then for each $\epsilon > 0$,

$$\limsup_r \frac{1}{h_r} |\{k \in I_r : \bar{d}(X_k, Y_0) < \epsilon\}| > 0. \quad (7)$$

As before, write $h'_{r,i} = k'_{r,i} - k'_{r,i-1}$ and $h'_{r,1} = k'_{r,1} - k_{r-1}$. For each $\epsilon > 0$, we have

$$\begin{aligned} \frac{1}{h_r} |\{k \in I_r : \bar{d}(X_k, Y_0) < \epsilon\}| &= \frac{1}{h_r} \sum_{I_j^* \subseteq I_r} h_j^* \frac{1}{h_j^*} |\{k \in I_j^* : \bar{d}(X_k, Y_0) < \epsilon\}| \\ &= \frac{1}{h_r} \sum_{I_j^* \subseteq I_r} h_j^* (C_{\theta'} \chi_K)_j \end{aligned}$$

where χ_K is the characteristics function of the set $K = \{k \in \mathbb{N} : \bar{d}(X_k, Y_0) < \epsilon\}$ and $(C_{\theta'} \chi_K)_j = \frac{|K \cap I_j^*|}{h_j^*}$. Suppose $\lim_j (C_{\theta'} \chi_K)_j = 0$, then the right side of the above expression is a regular weighted mean transform of $(C_{\theta'} \chi_K)_j$ and therefore tends to zero as $j \rightarrow \infty$. But then we obtain a contradiction to (7) and hence $\lim_j (C_{\theta'} \chi_K)_j \neq 0$. This shows that $Y_0 \in \Gamma_{S_{\theta'}}^F(X)$. The assertion that $\Lambda_{S_\theta}^F(X) \subseteq \Lambda_{S_{\theta'}}^F(X)$ can be proved by a similar argument. \square

4. Conclusion

In the present work, we introduced more general type of limit and cluster points for sequences of fuzzy numbers by use of lacunary sequences and investigated some of their properties. As in the case of real sequences, these notions may be used to predict the behavior of sequences of fuzzy number which are not lacunary statistically convergent. We also observe that if the lacunary sequence $\theta = (k_r)$ satisfy $1 < \liminf q_r \leq \limsup q_r < \infty$, then Definition 3.3 and Definition 3.4 respectively coincides with Definition 3.1 and Definition 3.2 of [3].

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