ROUGHNESS IN MODULES BY USING THE NOTION OF REFERENCE POINTS

B. DAVVAZ AND A. MALEKZADEH

Abstract. A module over a ring is a general mathematical concept for many examples of mathematical objects that can be added to each other and multiplied by scalar numbers. In this paper, we consider a module over a ring as a universe and by using the notion of reference points, we provide local approximations for subsets of the universe.

1. Introduction

The theory of rough sets was originally proposed by Pawlak [25, 26] as a formal tool for modeling and processing intelligent systems characterized by insufficient and incomplete information, also see [23]. In [24], Pawlak and Skowron discussed methods based on the combination of rough sets and Boolean reasoning with applications in pattern recognition, machine learning, data mining and conflict analysis. The basic structure of rough set theory is an approximation space consisting of a universe of discourse and a binary relation imposed on it. By introducing the concepts of lower and upper approximations of all decision classes with respect to an approximation space induced from the conditional attribute set, knowledge hidden in information tables may be unraveled and expressed in the form of decision rules. We have witnessed a rapid development of a fast growing interest in rough set theory recently and many models and methods have been proposed and studied.

Rough set theory has found practical applications in many areas such as knowledge discovery, machine learning, data analysis, approximate classification, conflict analysis, and so on. A key concept in Pawlak rough set model is an equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation is the union of all the equivalence classes which have a nonempty intersection with the set. In some probabilistic problems, complete information about the probability model may not exist. In [32, 33], Torabi et al. obtained a lower and upper probability for an arbitrary event by using rough set theory and then a measurement for inclusiveness of events is introduced. In [17], Kedukodi et al. provided local approximations for a subset of the universe by introducing the notion of a reference point. In [30, 31] a new extension of the rough set theory by means of integrating the classical Pawlak rough set theory with the interval-valued

Received: January 2012; Revised: October 2012 and January 2013; Accepted: April 2013
Key words and phrases: Rough set, Fuzzy set, Ring, Module, Submodule, Reference point.

fuzzy set theory is given, and the relationships with this model and the others rough set models are also examined. Rough approximations in a general approximation space and their fundamental properties are investigated in [11].

Biswas and Nanda [3] introduced the rough subgroup notion. Kuroki [19] defined the rough ideal in a semigroup. Then, Kuroki and Wang [20] studied the lower and upper approximations with respect to normal subgroups. In [6, 7], Davvaz studied the relationship between rough sets and ring theory; he considered the ring as a universal set and introduced the notion of rough ideals and rough subrings with respect to the ideal of a ring. In [14], the notions of rough prime ideals and rough fuzzy prime ideals in a semigroup were introduced. In [8], Davvaz studied the concept of probabilistic T-rough set is defined. In [34], the concepts of set-valued homomorphism and strong set-valued homomorphism of a ring are introduced. In [27], Rasouli and Davvaz considered a relationship between rough sets and MV-algebra theory. Rough modules have been investigated by Davvaz and Mahdavipour [10]. Dubois and Prade [12] introduced the problem of fuzzification of a rough set. Many authors analyzed the concept of a fuzzy rough set. In [15], Kazancı et al. interpreted the lower and upper approximations as subsets of the quotient hypermodule. Also, using the concept of fuzzy sets, they introduced and discussed the concept of fuzzy rough hypermodules. In [21], the notion of rough sets within the context of the commutative n-ary hypergroups is studied, also see [9]. In [28], the notion of rough approximations of Cayley graphs are studied, and rough edge Cayley graphs are introduced. Furthermore, a new algebraic definition for pseudo-Cayley graphs containing Cayley graphs is proposed, and a rough approximation is expanded to pseudo-Cayley graphs.

Now, in this paper, we consider a module over a ring as a universe and by using the notion of reference points, we provide local approximations for subsets of the universe.

2. Definitions and Preliminaries

In applied mathematics we encounter many examples of mathematical objects that can be added to each other and multiplied by scalar numbers. First of all, the real numbers themselves are such objects. Other examples are real-valued functions, the complex numbers, infinite series, vectors in n-dimensional space, and vector-valued functions. In this paper, we discuss a general mathematical concept, called a module over a ring, which includes all those examples and many others as special cases. One of the standard sources for the algebraic theory of modules is [2], also see [13, 18]. In [4], Booth and Groenewald considered nearring modules, as equiprimeness originated from nearrings. The definition given in this paper holds in particular for rings.

Throughout this paper R is a commutative ring with identity and, M is a unital R-module. For a submodule N of an R-module M, the set $\{r \in R \mid rM \subseteq N\}$ is denoted by $(N : M)$ and is called *colon* of N. A proper submodule P of an R-module M is called *prime* if $rm \in P$ for $r \in R$ and $m \in M$ implies either $m \in P$ or $r \in (P : M)$. A general theme in the studying of prime submodules is to extend results concerning prime ideals to prime submodules. Any prime ideal of a ring R is a prime submodule of the R-module R and any maximal submodule of an R-module M is a prime submodule of M. As an example, $3\mathbb{Z}$ is a prime submodule of $\mathbb{Z}$. A submodule P of an R-module M is called *equiprime* if $a \in R$ with $a \notin (P : M)$ and $x, y \in M$, for all $r \in R$ from $arx - ary \in P$ it follows $x - y \in P$ [4]. Let E be an equivalence relation on an R-module M. Then, E is called a *full congruence relation* if for all $a, b, c, d \in M$ and $r \in R$ from aEb and cEd it follows: $(a+c)E(b+d)$ and $(ra)E(rb)$. If E satisfies only the first condition, then E is called a *partial congruence relation*.

As it is well known in the fuzzy theory established by Zadeh [35], a fuzzy subset μ of a non-empty set X is defined as a map from X to the unit interval $[0, 1]$. Let μ and λ be two fuzzy subsets of X. The inclusion $\mu \subseteq \lambda$ is denoted by $\mu(x) \leq \lambda(x)$, for all $x \in X$. Let μ and λ be two fuzzy subsets of a non-empty set X. Then, $\mu \cap \lambda$ and $\mu \cup \lambda$ are defined by $(\mu \cap \lambda)(x) = \min\{\mu(x),\ \lambda(x)\}$ and $(\mu \cup \lambda)(x) = \max\{\mu(x), \lambda(x)\}$. For any fuzzy set μ of a non-empty set X and any $t \in [0, 1]$, we define the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$, which is called a *t-level cut* of μ.

The notion of fuzzy submodule of a module was introduced by Negoita and Ralescu in [22], also see [1, 29].

Definition 2.1. A fuzzy subset μ of an R-module M is called a *fuzzy submodule* of M if it satisfies the following properties:

(1) $\mu(0_M) = 1$;
(2) $\mu(rx) \geq \mu(x)$, for all $r \in R$, $x \in M$;
(3) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in M$.

Let μ be a fuzzy submodule of M. Then, for all $x \in M$ we have

(1) $\mu(0_M) \geq \mu(x)$;
(2) $\mu(-x) = \mu(x)$.

Theorem 2.2. *Let μ be a fuzzy subset of an R-module M. Then, μ is a fuzzy submodule of M if and only if the t-level cut μ_t $(\neq \emptyset)$ is a submodule of M, for all $t \in [0, 1]$.*

Example 2.3. Consider $R = \mathbb{Z}$ and $M = \mathbb{Z}_6$. Then, M is an R-module. We define

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.7 & \text{if } x \in \{2, 4\}, \\ 0.2 & \text{if } x \in \{1, 3, 5\}. \end{cases}$$

Then, μ is a fuzzy submodule of M.

If λ is a fuzzy subset of R and ν is a fuzzy subset of M, then the expression $\lambda \cdot \nu$ is defined by

$$(\lambda \cdot \nu)(x) = \sup_{x=rz} \min\{\lambda(r), \nu(z)\},$$

where as usual the supremum of an empty set is taken to be 0.

By using [16], we give the definition of equiprime fuzzy submodule as follows:

Definition 2.4. A fuzzy submodule μ of an R-module M is called *equiprime* if for all $a \in R$ and for all $x, y \in M$,

$$\inf_{r \in R} \mu(arx - ary) \leq \max\{\mu(a), \mu(x - y)\}.$$

Example 2.5. Let $M = \{0, a, b, c\}$ be a set with the binary operation $+$ defined as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then, $(M, +)$ is an abelian group. If $R = \mathbb{Z}_2$, then M is an R-module. We define

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \{0, a\}, \\ 0.5 & \text{if } x \in \{b, c\}. \end{cases}$$

Then, μ is an equiprime fuzzy submodule of M.

Let E be an equivalence relation on a universe U. For an element $x \in U$, the equivalence class containing x is given by $[x] = \{y \in U \mid xEy\}$. The set $\{[x] \mid x \in U\}$ is denoted by U/E. For a subset $A \subseteq U$, the *lower approximation* and the *upper approximation* of A are defined as follows: $\underline{U}(A) = \{x \in U \mid [x] \subseteq A\}$ and $\overline{U}(A) = \{x \in U | [x] \cap A \neq \emptyset\}$. The pair (U, E) is called the *approximation space.* The *positive region*, *negative region*, and the *boundary region* of A are defined as follows:

$POS(A) = \underline{U}(A)$,
$NEG(A) = (\overline{U}(A))^c$,
$BND(A) = \overline{U}(A) \setminus \underline{U}(A)$.

A is called a *rough set* if it has non-empty boundary region, otherwise it is precise.

Definition 2.6. Let μ be a fuzzy submodule of an R-module M and $t \in [0, 1]$. Then,

$$U(\mu, t) = \{(x, y) \in M \times M \mid \mu(x - y) \geq t\}$$

is called a *t-level relation* of μ.

Lemma 2.7. *Let $t \in [0, 1]$ and μ be a fuzzy submodule of an R-module M. Then, $U(\mu, t)$ is a full congruence relation on M.*

Proof. The proof is similar to the proof of Lemma 3.2 in [7]. □

Let A be a non-empty subset of an R-module M. For an element $x \in M$, the equivalence class containing x is given by $[x]_{(\mu,t)} = \{y \in M \mid \mu(x - y) \geq t\}$. Then, the sets

$$\underline{U}(\mu, t, A) = \{x \in M \mid [x]_{(\mu,t)} \subseteq A\} \text{ and } \overline{U}(\mu, t, A) = \{x \in M \mid [x]_{(\mu,t)} \cap A \neq \emptyset\}$$

are respectively called the *lower approximation* and the *upper approximation* of the set X with respect to $U(\mu, t)$.

3. Reference Points Applied in Modules

Definition 3.1. Let μ be a fuzzy submodule of an R-module M, $a \in R$ and $t \in [0, 1]$. Then,

$$U_e(\mu, t, a) = \{(x, y) \in M \times M \mid \mu(rax - ray) \geq t, \ \forall r \in R\}$$

is called a t-level relation with respect to the reference point a.

Let μ be a fuzzy submodule of an R-module M, $a \in R$ and $t \in [0,1]$. Then, $U_e(\mu, t, a)$ is a partial congruence relation on M. For an element $x \in M$, the equivalence class containing x is denoted by $[x]_{(\mu,t,a)}$. The following can be handled analogously as in [7].

- Let μ and λ be fuzzy submodules of an R-module M, $a \in R$ and $t \in [0,1]$. Then, $U_e(\mu \cap \lambda, t, a) = U_e(\mu, t, a) \cap U_e(\lambda, t, a)$.
- Let μ be a fuzzy submodule of an R-module M. If $a \in R$, $x, y \in M$ and $t \in [0,1]$, then $[x]_{(\mu,t,a)} + [y]_{(\mu,t,a)} = [x+y]_{(\mu,t,a)}$ and $[-x]_{(\mu,t,a)} = -[x]_{(\mu,t,a)}$.
- Let μ be a fuzzy submodule of an R-module M, $a \in R$ and $t \in [0,1]$. For any $x \in M$, we have $x + [0]_{(\mu,t,a)} = [x]_{(\mu,t,a)}$.
- Let μ and λ be fuzzy submodules of an R-module M, $a \in R$, $\lambda \subseteq \mu$ and $t \in [0,1]$. Then, for all $x \in M$, $[x]_{(\lambda,t,a)} \subseteq [x]_{(\mu,t,a)}$.

Definition 3.2. Let μ be a fuzzy submodule of an R-module M, $a \in R$ and $t \in [0,1]$. Let A be a non-empty subset of M. Then, the sets

$$\underline{U_e}(\mu, t, a, A) = \{x \in M \mid [x]_{(\mu,t,a)} \subseteq A\}$$

and

$$\overline{U_e}(\mu, t, a, A) = \{x \in M \mid [x]_{(\mu,t,a)} \cap A \neq \emptyset\},$$

are respectively called the *lower approximation* and the *upper approximation* of the set A with respect to $U_e(\mu, t, a)$. The boundary of A is given by

$$BND(A) = \overline{U_e}(\mu, t, a, A) \setminus \underline{U_e}(\mu, t, a, A).$$

A is a *rough set* if it has non-empty boundary region, otherwise it is precise.

We use the notation (M, μ, t, a) instead of approximation space $(M, U_e(\mu, t, a))$. Let $(\underline{U_e}(\mu, t, a, A), \overline{U_e}(\mu, t, a, A)$ be a rough set in the approximation space (M, μ, t, a). If $\underline{U_e}(\mu, t, a, A)$ and $\overline{U_e}(\mu, t, a, A)$ are submodules of M, then $(\underline{U_e}(\mu, t, a, A), \overline{U_e}(\mu, t, a, A)$ is called a *rough submodule* of M.

Proposition 3.3. *For every approximation space* (M, μ, t, a) *and every subsets* A, B *of an* R*-module* M*, we have*

(1) $\underline{U_e}(\mu, t, a, A) \subseteq A \subseteq \overline{U_e}(\mu, t, a, A)$;
(2) $\underline{U_e}(\mu, t, a, \emptyset) = \emptyset = \overline{U_e}(\mu, t, a, \emptyset)$;
(3) $\underline{U_e}(\mu, t, a, R) = R = \overline{U_e}(\mu, t, a, R)$;
(4) *For* $A \subseteq B$ *we have*
$\underline{U_e}(\mu, t, a, A) \subseteq \underline{U_e}(\mu, t, a, B)$ *and* $\overline{U_e}(\mu, t, a, A) \subseteq \overline{U_e}(\mu, t, a, B)$;
(5) $\underline{U_e}(\mu, t, a, \underline{U_e}(\mu, t, a, A)) = \underline{U_e}(\mu, t, a, A)$;
(6) $\overline{U_e}(\mu, t, a, \overline{U_e}(\mu, t, a, A)) = \overline{U_e}(\mu, t, a, A)$;
(7) $\overline{U_e}(\mu, t, a, \underline{U_e}(\mu, t, a, A)) = \underline{U_e}(\mu, t, a, A)$;
(8) $\underline{U_e}(\mu, t, a, \overline{U_e}(\mu, t, a, A)) = \overline{U_e}(\mu, t, a, A)$;
(9) $\underline{U_e}(\mu, t, a, A) = (\overline{U_e}(\mu, t, a, A^c))^c$;
(10) $\overline{U_e}(\mu, t, a, A) = (\underline{U_e}(\mu, t, a, A^c))^c$;
(11) $\underline{U_e}(\mu, t, a, A \cap B) = \underline{U_e}(\mu, t, a, A) \cap \underline{U_e}(\mu, t, a, B)$;
(12) $\overline{U_e}(\mu, t, a, A \cap B) \subseteq \overline{U_e}(\mu, t, a, A) \cap \overline{U_e}(\mu, t, a, B)$;

(13) $\underline{U_e}(\mu, t, a, A \cup B) \supseteq \underline{U_e}(\mu, t, a, A) \cup \underline{U_e}(\mu, t, a, B)$;
(14) $\overline{U_e}(\mu, t, a, A \cup B) = \overline{U_e}(\mu, t, a, A) \cup \overline{U_e}(\mu, t, a, B)$;
(15) $\underline{U_e}(\mu, t, a, [x]_{(\mu,t,a)}) = \overline{U_e}(\mu, t, a, [x]_{(\mu,t,a)})$, *for all* $x \in M$.

Proof. The proof is similar to the proof of Theorem 2.1 in [19]. □

Example 3.4. Suppose that $M = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{Z}_3\}$ is a module over $R = \mathbb{Z}_3$. We define fuzzy submodule $\mu : M \to [0, 1]$ as follows:

$$\mu(y) = \begin{cases} 1 & \text{if } y = 0_M, \\ 0.8 & \text{if } y \in \{1 + 2x, 2 + x\}, \\ 0.4 & \text{elsewhere.} \end{cases}$$

Then, we have

$$U_e(\mu, 1, 2) = \{(x, x) \mid x \in M\}.$$

Hence, $U_e(\mu, 1, 2)$ partitions M into 27 equivalence classes.

Also, we have $U_e(\mu, 0.4, 2) = M \times M$. Hence, $U_e(\mu, 0.4, 2)$ partitions M into one equivalence class namely M itself.

It is clear that $U_e(\mu, 0.8, 2)$ partitions M into the following 9 equivalence classes:

0	1	2	$2+2x^2$	$1+x^2$	$2+x+2x^2$	x^2	$1+2x^2$	$1+x+x^2$
$1+2x$	x	$2x$	$2x+2x^2$	$x+x^2$	$1+2x+2x^2$	$2+x+x^2$	$x+2x^2$	$2+x^2$
$2+x$	$2+2x$	$1+x$	$1+x+2x^2$	$2+2x+x^2$	$2x^2$	$1+2x+x^2$	$2+2x+2x^2$	$2x+x^2$

Take $A = \{0, 2 + 2x, 2, 2x^2, 2 + x, 1 + 2x, x\}$, and

$$B = \{1, 2 + x, 1 + 2x + x^2, 2x, 1 + x^2, x + 2x^2, 2x + 2x^2, 2 + x + 2x^2, 2 + x^2\}.$$

Clearly, $A \cap B = \{2 + x\}$ and

$$A \cup B = \{0, 1, 2, 2 + 2x, 2x^2, 2 + x, 1 + 2x, x, 1 + 2x + x^2, 2x, 1 + x^2, x + 2x^2, 2x + 2x^2, 2 + x + 2x^2, 2 + x^2\}.$$

Now, we have

$$\overline{U_e}(\mu, 0.8, 2, A) = \{0, 1+2x, 2+x, 1, x, 2+2x, 2, 2x, 1+x, 2x^2, 1+2x+2x^2, 2+x+2x^2\};$$

$$\overline{U_e}(\mu, 0.8, 2, B) = M \text{ and } \overline{U_e}(\mu, 0.8, 2, A \cap B) = \{0, 1 + 2x, 2 + x\}.$$

Hence,

$$\overline{U_e}(\mu, 0.8, 2, A \cap B) \neq \overline{U_e}(\mu, 0.8, 2, A) \cap \overline{U_e}(\mu, 0.8, 2, B).$$

Also, we have

$$\underline{U_e}(\mu, 0.8, 2, A) = \{0, 1+2x, 2+x\};\ \underline{U_e}(\mu, 0.8, 2, B) = \emptyset;$$
$$\underline{U_e}(\mu, 0.8, 2, A \cup B) = \{0, 1+2x, 2+x, 1, x, 2+2x\}.$$

Hence,

$$\underline{U_e}(\mu, 0.8, 2, A \cup B) \neq \underline{U_e}(\mu, 0.8, 2, A) \cup \underline{U_e}(\mu, 0.8, 2, B).$$

From the above computations, it is clear that the inclusions in Proposition 3.3 (1), (12) and (13) can be strict. Also, we can conclude the following:

(1) The uncertain information of $A \cap B$ may be less than the intersection of the uncertain information of A and that of B.
(2) The certain information of $A \cup B$ may be more than the union of the certain information of A and that of B.

Proposition 3.5. *Let μ and λ be fuzzy submodules of a R-module M, $a \in R$ and $t \in [0,1]$. If X is a non-empty subset of M, then*

$$\overline{U_e}(\mu \cap \lambda, t, a, X) \subseteq \overline{U_e}(\mu, t, a, X) \cap \overline{U_e}(\lambda, t, a, X).$$

Proof.

$$\begin{aligned}
x \in \overline{U_e}(\mu \cap \lambda, t, a, X) &\Longrightarrow [x]_{(\mu\cap\lambda,t,a)} \cap X \neq \emptyset \\
&\Longrightarrow \exists b \in [x]_{(\mu\cap\lambda,t,a)} \cap X \\
&\Longrightarrow (b, x) \in U_e(\mu \cap \lambda, t, a),\ b \in X \\
&\Longrightarrow (\mu \cap \lambda)(arb - arx) \geq t,\ b \in X \\
&\Longrightarrow \min\{\mu(arb - arx), \lambda(arb - arx)\} \geq t, b \in X \\
&\Longrightarrow \mu(arb - arx) \geq t,\ \lambda(arb - arx) \geq t, b \in X \\
&\Longrightarrow (b, x) \in U_e(\mu, t, a),\ (b, x) \in U_e(\lambda, t, a), b \in X \\
&\Longrightarrow (b, x) \in U_e(\mu, t, a), b \in X\ (b, x) \in U_e(\lambda, t, a), b \in X \\
&\Longrightarrow b \in [x]_{(\mu,t,a)} \cap X,\ b \in [x]_{(\lambda,t,a)} \cap X \\
&\Longrightarrow [x]_{(\mu,t,a)} \cap X \neq \emptyset,\ [x]_{(\lambda,t,a)} \cap X \neq \emptyset \\
&\Longrightarrow x \in \overline{U_e}(\mu, t, X), x \in \overline{U_e}(\lambda, t, a, X).
\end{aligned}$$

So that $\overline{U_e}(\mu \cap \lambda, t, a, X) \subseteq \overline{U_e}(\mu, t, a, X) \cap \overline{U_e}(\lambda, t, a, X)$. □

Proposition 3.6. *Let μ and λ be fuzzy submodules of an R-module M, $a \in R$ and $t \in [0,1]$. If X is a non-empty subset of M, then*

$$\underline{U_e}(\mu, t, a, X) \cap \underline{U_e}(\lambda, t, a, X) \subseteq \underline{U_e}(\mu \cap \lambda, t, a, X).$$

Proof. We have

$$\begin{aligned}
x \in \underline{U_e}(\mu, t, a, X) \cap \underline{U_e}(\lambda, t, a, X) &\Longrightarrow x \in \underline{U_e}(\mu, t, a, X),\ x \in \underline{U_e}(\lambda, t, a, X) \\
&\Longrightarrow [x]_{(\mu,t,a)} \subseteq X,\ [x]_{(\lambda,t,a)} \subseteq X \\
&\Longrightarrow [x]_{(\mu\cap\lambda,t,a)} \subseteq X \\
&\Longrightarrow x \in \underline{U_e}(\mu \cap \lambda, t, a, X).
\end{aligned}$$

So, $\underline{U_e}(\mu, t, a, X) \cap \underline{U_e}(\lambda, t, a, X) \subseteq \underline{U_e}(\mu \cap \lambda, t, a, X)$. □

Proposition 3.7. *Let μ be a fuzzy submodule of an R-module M, $a \in R$ and $t \in [0,1]$. If A is a submodule of M, then $\overline{U_e}(\mu, t, a, A)$ is a submodule of M.*

Proof. The proof is similar to the proof of Proposition 3.15 in [10]. □

Proposition 3.8. *Let μ be a fuzzy submodule of an R-module M, $a \in R$, $t \in [0,1]$ and A be a submodule of M. If $\underline{U_e}(\mu, t, a, A)$ is a non-empty set, then $[0]_{(\mu,t,a)} \subseteq A$.*

Proof. Assume that $\underline{U_e}(\mu, t, a, A) \neq \emptyset$. Then, there exists $x \in \underline{U_e}(\mu, t, a, A)$ or $[x]_{(\mu,t,a)} \subseteq A$. So, $-([x]_{(\mu,t,a)}) \subseteq -A = -\{-a|a \in A\} = A$. Now, we have

$$\begin{aligned}[0]_{(\mu,t,a)} &= [x + (-x)]_{(\mu,t,a)} = [x]_{(\mu,t,a)} + [-x]_{(\mu,t,a)} \\ &= [x]_{(\mu,t,a)} + (-[x]_{(\mu,t,a)}) \subseteq A + A = A.\end{aligned}$$

This completes the proof. □

Proposition 3.9. *Let μ be a fuzzy submodule of an R-module M, $a \in R$, $t \in [0,1]$ and A be a submodule of M. If $\underline{U_e}(\mu, t, a, A)$ is a non-empty set, then $\underline{U_e}(\mu, t, a, A) = A$.*

Proof. By Proposition 3.3(1), we have $\underline{U_e}(\mu, t, a, A) \subseteq A$. Now, we show that $A \subseteq \underline{U_e}(\mu, t, a, A)$. Assume that x is an arbitrary element of A. By Proposition 3.8, we have $[0]_{(\mu,t,a)} \subseteq A$. Since A is a submodule of R-module M, we have $x + [0]_{(\mu,t,a)} \subseteq x + A \subseteq A$. Now, we obtain $[x]_{(\mu,t,a)} \subseteq A$, which implies that $x \in \underline{U_e}(\mu, t, a, A)$. □

Proposition 3.10. *Let μ be a fuzzy submodule of an R-module M, $a \in R$ and $t \in [0,1]$. If A is a submodule of M and $(\underline{U_e}(\mu, t, a, A)$ is a non-empty set, then $(\underline{U_e}(\mu, t, a, A), \overline{U_e}(\mu, t, a, A))$ is a rough submodule of M.*

Proof. By using Propositions 3.7 and 3.9. □

Proposition 3.11. *Let μ and λ be fuzzy submodules of an R-module M, $a \in R$, $t \in [0,1]$ and X be a non-empty subset of M. If $U_e(\lambda, t, a) \subseteq U_e(\mu, t, a)$, then*

(1) $\overline{U_e}(\lambda, t, a, X) \subseteq \overline{U_e}(\mu, t, a, X)$;
(2) $\underline{U_e}(\mu, t, a, X) \subseteq \underline{U_e}(\lambda, t, a, X)$.

Proof. (1) Assume that x is an arbitrary element of $\overline{U_e}(\lambda, t, a, X)$. Then, $[x]_{(\lambda,t,a)} \cap X \neq \emptyset$. Hence, $[x]_{(\mu,t,a)} \cap X \neq \emptyset$, which implies that $x \in \overline{U_e}(\mu, t, a, X)$.

(2) Assume that $x \in \underline{U_e}(\mu, t, a, X)$. Then, $[x]_{(\mu,t,a)} \subseteq X$. Now, we obtain $[x]_{(\lambda,t,a)} \subseteq X$ which implies that $x \in \underline{U_e}(\lambda, t, a, X)$. □

Now, we provide the following example.

Example 3.12. Let $M = \{(x, y)|x, y \in \mathbb{Z}_6\}$ be a module over $R = \mathbb{Z}$. Define fuzzy submodules $\mu, \lambda : M \to [0,1]$ as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } x = (0,0), \\ 0.7 & \text{if } x \in \{(0,1), (0,2), (0,3), (0,4), (0,5)\}, \\ 0.2 & \text{elsewhere.} \end{cases}$$

and

$$\lambda(x) = \begin{cases} 1 & \text{if } x = (0,0), \\ 0.7 & \text{if } x \in \{(0,2),(0,4),(3,1),(3,3),(3,5)\}, \\ 0.5 & \text{elsewhere.} \end{cases}$$

(I) It is clear that $M/U_e(\mu, 0.7, 1) = \{\{i\} \times \mathbb{Z}_6 \mid i \in \mathbb{Z}_6\}$. Also, $U_e(\lambda, 0.7, 1)$ partition M into the following six equivalence classes:

(0, 0)	(0, 1)	(1, 0)	(1, 1)	(2, 0)	(2, 1)
(0, 2)	(0, 3)	(1, 2)	(1, 3)	(2, 2)	(2, 3)
(0, 4)	(0, 5)	(1, 4)	(1, 5)	(2, 4)	(2, 5)
(3, 1)	(3, 0)	(4, 1)	(4, 0)	(5, 1)	(5, 0)
(3, 3)	(3, 2)	(4, 3)	(4, 2)	(5, 3)	(5, 2)
(3, 5)	(3, 4)	(4, 5)	(4, 4)	(5, 5)	(5, 4)

So, $M/\underline{U_e}(\mu \cap \lambda, 0.7, 1) = \{\{i\} \times \{0,2,4\} \mid i \in \mathbb{Z}_6\} \cup \{\{i\} \times \{1,3,5\} \mid i \in \mathbb{Z}_6\}$.

Now, take $X = \{(0,0),(1,0),(2,0),(3,0),(4,0),(5,0)\}$. Then,

$$\overline{U_e}(\mu, 0.7, 1, X) = M; \quad \overline{U_e}(\lambda, 0.7, 1, X) = M;$$

and

$$\overline{U_e}(\mu \cap \lambda, 0.7, 1, X) = \{(0,0),(0,2),(0,4),(1,0),(1,2),(1,4),(2,0),(2,2),(2,4),(3,0), (3,2),(3,4),(4,0),(4,2),(4,4),(5,0),(5,2),(5,4)\}.$$

which implies that $\overline{U_e}(\mu \cap \lambda, 0.7, 1, X) \neq \overline{U_e}(\mu, 0.7, 1, X) \cap \overline{U_e}(\lambda, 0.7, 1, X)$.

Now, take $Y = \{(0,0),(0,2),(0,4),(2,0),(2,1),(2,2),(2,3),(2,4),(2,5)\}$. Then,

$$\underline{U_e}(\mu, 0.7, 1, Y) = \{(2,0),(2,1),(2,2),(2,3),(2,4),(2,5)\}, \underline{U_e}(\lambda, 0.7, 1, Y) = \emptyset,$$

$$\underline{U_e}(\mu \cap \lambda, 0.7, 1, Y) = Y,$$

which implies that $\underline{U_e}(\mu \cap \lambda, 0.7, 1, Y) \neq \underline{U_e}(\mu, 0.7, 1, Y) \cap \underline{U_e}(\lambda, 0.7, 1, Y)$.

The above computations show that the inclusions in Propositions 3.4 and 3.5 can be strict. Also we can conclude the following:

(1) The uncertain information of a set X with respect to the intersection of two fuzzy submodules μ and λ may be less than the intersection of the uncertain information of X with respect to fuzzy submodules μ and λ.

(2) The certain information of a set Y with respect to the intersection of two fuzzy submodules μ and λ may be more than the intersection of the certain information of Y with respect to the fuzzy submodules μ and λ.

(II) Now, $U_e(\mu, 1, 2)$ partition M into the following 9 equivalence classes:

(0, 0)	(0, 1)	(0, 2)	(1, 0)	(1, 1)
(0, 3)	(0, 4)	(0, 5)	(1, 3)	(1, 4)
(3, 0)	(3, 1)	(3, 2)	(4, 0)	(4, 1)
(3, 3)	(3, 4)	(3, 5)	(4, 3)	(4, 4)

(1, 2)	(2, 0)	(2, 1)	(2, 2)
(1, 5)	(2, 3)	(2, 4)	(2, 5)
(4, 2)	(5, 0)	(5, 1)	(5, 2)
(4, 5)	(5, 3)	(5, 4)	(5, 5)

Also, $\underline{U_e}(\mu, 1, 1)$ partition M into the following 36 equivalence classes:

(0, 0)	(0, 1)	(0, 2)	(0, 3)	(0, 4)	(0, 5)
(1, 0)	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
(2, 0)	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
(3, 0)	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)
(4, 0)	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)
(5, 0)	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)

Take $X = \{(0,0),(0,1),(0,2),(0,3),(3,3),(3,0),(4,1)\}$. Then,

$$\begin{aligned}\overline{U_e}(\mu, 1, 2, X) = \{&(0,0),(0,3),(0,1),(0,4),(1,1),(1,4),\\ &(3,0),(3,1),(3,3),(3,4),(4,1),(4,4)\},\end{aligned}$$

and $\overline{U_e}(\mu, 1, 1, X) = X$, which implies that $\overline{U_e}(\mu, 1, 1, X) \neq \overline{U_e}(\mu, 1, 2, X)$.

So,

$$\underline{U_e}(\mu, 1, 2, X) = \{(0,0),(0,3),(3,0),(3,3)\};\ \underline{U_e}(\mu, 1, 1, X) = X,$$

which implies that $\underline{U_e}(\mu, 1, 2, X) \neq \underline{U_e}(\mu, 1, 1, X)$. Thus, distinct reference points can yield distinct lower and upper approximations for a set.

(III) Now, take $A = \{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5),(3,0)\}$. Note that A is not a submodule of M. However,

$$\underline{U_e}(\mu, 0.7, 1, A) = \{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5)\},$$

and

$$\begin{aligned}\overline{U_e}(\mu, 0.7, 1, A) = \{&(0,0),(0,1),(0,2),(0,3),(0,4),(0,5),\\ &(3,0),(3,1),(3,2),(3,3),(3,4),(3,5)\},\end{aligned}$$

are submodules of M. Hence, $(\underline{U_e}(\mu, 0.7, 1, A), \overline{U_e}(\mu, 0.7, 1, A))$ is a rough submodule of M. This example shows that the converse of the Theorem 3.9 is not true in general.

Theorem 3.13. *Let μ be a fuzzy submodules of an R-module M. Then, the following conditions are equivalent:*

(1) *μ is an equiprime fuzzy submodule of M.*
(2) *For every $t \in [0,1]$, the level subset μ_t is an equiprime submodule of M.*

Proof. (1)⇒(2): Let μ be an equiprime fuzzy submodule of R-module M. Take $t \in [0,1]$. Then, μ_t is a submodule of an R-module M. Also, $\mu_t \neq \emptyset$, so $0 \in \mu_t$. Note that if $\mu_t = M$, then M is trivially an equiprime submodule of M. Also, if $t = 0$, then $\mu_t = M$. Hence, we assume $t \neq 0$ and $\mu_t \neq M$. Let $a \in R$ with $a \notin (\mu_t : M)$ and $x, y \in M$ such that $arx - ary \in \mu_t$ for all $r \in R$. This implies that $\mu(arx - ary) \geq t$ for all $r \in R$. Hence, $\inf_{r \in R} \mu(arx - ary) \geq t$. As μ is an equiprime fuzzy submodule of M, we get $t \leq \inf_{r\in R} \mu(arx - ary) \leq \max\{\mu(a), \mu(x-y)\}$. On the other hand, $a \notin (\mu_t : M)$ implies that $a \notin \mu_t$. So, $\mu(a) < t$. Thus, we obtain $\mu(x-y) \geq t$, This implies that $x - y \in \mu_t$.

(2)⇒(1): Let $a \in R$ and $x, y \in M$. Take $t = \inf_{r\in R} \mu(arx - ary)$. Note that $\mu(adx - ady) \geq t$ for all $d \in R$. This implies that $adx - ady \in \mu_t$ for all $d \in R$. As μ_t is an equiprime submodule of M,

$$\begin{aligned} &\mu(x-y) \geq t \text{ or } a \in (\mu_t : M) \\ &\Rightarrow \mu(x-y) \geq t \text{ or } \mu(am) \geq t, \ \forall m \in M. \end{aligned}$$

Therefore, $\max\{\mu(a), \mu(x-y)\} \geq t = \inf_{r\in R}(arx - ary)$. □

Lemma 3.14. *Let μ be an equiprime fuzzy submodule of an R-module M, $a \in R$ with $a \notin (\mu_t : M)$, $t \in (0, 1]$. Then,*

(1) $\overline{U_e}(\mu, t, a, X) \subseteq \overline{U}(\mu, t, X)$;

(2) $\underline{U}(\mu, t, X) \subseteq \underline{U_e}(\mu, t, a, X)$.

Proof. Suppose that $z \in \overline{U_e}(\mu, t, a, X)$. Then, $[z]_{(\mu,t,a)} \cap X \neq \emptyset$. Take $y \in [z]_{(\mu,t,a)} \cap X$. Then, $(z, y) \in U_e(\mu, t, a)$ and $y \in X$. This gives $\mu(arz - ary) \geq t$, for every $r \in R$. As μ is an equiprime fuzzy submodule of an R-module M, we obtain $\mu(z - y) \geq t$. This implies $(z, y) \in U(\mu, t)$. Hence, $y \in [z]_{(\mu,t)}$ and $y \in X$. This further implies that $[z]_{(\mu,t)} \cap X \neq \emptyset$. Thus, $z \in \overline{U}(\mu, t, X)$. This proves (1). In order to prove (2), take $z \in \underline{U}(\mu, t, X)$. Thus, $[z]_{(\mu,t)} \subseteq X$. We have to prove that $[z]_{(\mu,t,a)} \subseteq X$. Now,

$$\begin{aligned} y \in [z]_{(\mu,t,a)} &\Longrightarrow (z, y) \in U_e(\mu, t, a) \Longrightarrow \mu(arz - ary) \geq t \\ &\Longrightarrow \mu(z - y) \geq t \Longrightarrow (z, y) \in U(\mu, t) \Longrightarrow y \in [z]_{(\mu,t)} \subseteq X. \end{aligned}$$

Hence, $y \in X$. This completes the proof. □

Theorem 3.15. *Let μ be an equiprime fuzzy submodule of an R-module M and $a \in R$ such that $a \notin (\mu_t : M)$ and $t \in (0, 1]$. Then for any subset X of M,*

$$\underline{U}(\mu, t, X) \subseteq \underline{U_e}(\mu, t, a, X) \subseteq X \subseteq \overline{U_e}(\mu, t, a, X) \subseteq \overline{U}(\mu, t, X).$$

Proof. The proof is straightforward by using Proposition 3.3 (1) and Lemma 3.12. □

Theorem 3.16. *Let μ be a fuzzy submodule of a R-module M, $a \in R$ and $t \in (0, 1]$. Then, for any subset X of M,*

$$\underline{U_e}(\mu, t, a, X) \subseteq \underline{U}(\mu, t, X) \subseteq X \subseteq \overline{U}(\mu, t, X) \subseteq \overline{U_e}(\mu, t, a, X).$$

Proof. The proof is similar to the proof of Theorem 3.21 in [17]. □

Now, we provide an example to show that the inclusions in Theorem 3.14 can be strict.

Example 3.17. Consider the fuzzy submodule μ of the $\mathbb{Z}$-module M defined in the Example 4. Then,

$$\inf_{r\in R} \mu(2r(0, 5) - 2r(0, 2)) = \mu((0, 0)) = 1,$$

and so we have

$$\mu(x-y) = \mu((0,5)-(0,2)) = \mu((0,3)) = 0.7,$$

which implies that

$$\mu((0,5)-(0,2)) < \inf_{r\in R} \mu(2r(0,5)-2r(0,2)).$$

Hence, μ is not an equiprime fuzzy submodule of M. Now, we have

$$U(\mu,1) = \{(x,x) \mid x \in M\}.$$

Hence, $M/U(\mu,1) = \{\{x\} \mid x \in M\}$.

Take $X = \{(0,0),(0,3),(3,0),(3,3),(3,4),(4,1)\}$. Then,

$$\underline{U}(\mu,1,X) = X = \overline{U}(\mu,1,X).$$

By using Example 4, $\underline{U_e}(\mu,1,2,X) = \{(0,0),(0,3),(3,0),(3,3)\}$ and

$$\overline{U_e}(\mu,1,2,X) = \{(0,0),(0,3),(0,1),(0,4),(1,1),(1,4),$$
$$(3,0),(3,1),(3,3),(3,4),(4,1),(4,4)\}.$$

Hence, we obtain

$$\underline{U_e}(\mu,1,2,X) \subsetneq \underline{U}(\mu,1,X) = X = \overline{U}(\mu,1,X) \subsetneq \overline{U_e}(\mu,1,2,X).$$

Corollary 3.18. *Let μ be an equiprime fuzzy submodule of an R-module M and $a \in R$ such that $a \notin (\mu_t : M)$ and $t \in (0,1]$. Then, for any subset X of M,*

$$\underline{U_e}(\mu,t,a,X) = \underline{U}(\mu,t,X) \subseteq X \subseteq \overline{U}(\mu,t,X) = \overline{U_e}(\mu,t,a,X).$$

Proof. The proof follows from Theorems 3.13 and 3.14. □

Definition 3.19. [5] A *rough approximation algebra* is a structure $\langle A,\wedge,\vee,^c,l,u,0,1\rangle$ where $\langle A,\wedge,\vee,0,1\rangle$ is a bounded lattice; $\leq$ denotes the partial order; ^{c} is a De Morgan (or involutive) negation. The lower and upper approximation mappings $l,u : A \to A$ satisfy the following properties

(R1) $l(a) \leq u(a)$, for all $a \in A$,
(R2) $l(0) = u(0) = 0$ and $l(1) = u(1) = 1$,
(R3) $a \leq b$ implies $l(a) \leq l(b)$ and $u(a) \leq u(b)$.

Let $a \in A$. Then, a rough approximation of a is the pair lower-upper approximation $\langle l(a),u(a)\rangle$. A *rough approximation framework* is a collection of rough approximation algebras.

Theorem 3.20. *Let μ be a fuzzy submodule of an R-module M, $a \in R$ and $t \in (0,1]$. Then*

(1) $\langle P(M),\cap,\cup,^c,\underline{U_e}(\mu,t,a),\overline{U_e}(\mu,t,a),\emptyset,M\rangle$ *is a rough approximation algebra.*
(2) $\{\langle P(M),\cap,\cup,^c,\underline{U_e}(\mu,t,b),\overline{U_e}(\mu,t,b),\emptyset,M\rangle \mid b \in R\}$ *is a rough approximation framework.*

Proof. The proof is similar to the proof of Theorem 3.24 in [17]. □

Let $a \in R$ and E_a be an equivalence relation associated with the reference point a. Suppose E_a partitions M in equivalence classes $[x]_a = \{y \in M : xE_ay\}$. Let

$\nu, \omega \in [0,1]$ be two parameters such that $\nu > \omega$. Let $A \subseteq M$ and $a \in R$. The lower approximation of A with respect to a is defined as

$$l_{\{\nu,a\}}(A) = \Big\{x \in M | P(A \mid [x]_a) = \frac{\mid A \cap [x]_a \mid}{\mid [x]_a \mid} \geq \nu\Big\}.$$

The upper approximation of A with respect to a is defined as

$$u_{\{\omega,a\}}(A) = \Big\{x \in M | P(A \mid [x]_a) = \frac{\mid A \cap [x]_a \mid}{\mid [x]_a \mid} > \omega\Big\}.$$

If M is an R-module, then $\langle P(M), \cap, \cup, ^c, l_{\{\nu,a\}}, u_{\{\omega,a\}}, \emptyset, M\rangle$ is a rough approximation algebra and for $A, B \subseteq M$, we have

(1) $u_{\{\omega,a\}}(A) \cup u_{\{\omega,a\}}(B) \subseteq u_{\{\omega,a\}}(A \cup B)$.
(2) $u_{\{\omega,a\}}(A \cap B) \subseteq u_{\{\omega,a\}}(A) \cap u_{\{\omega,a\}}(B)$.
(3) $l_{\{\nu,a\}}(A) \cup l_{\{\nu,a\}}(B) \subseteq l_{\{\nu,a\}}(A \cup B)$.
(4) $l_{\{\nu,a\}}(A \cap B) \subseteq l_{\{\nu,a\}}(A) \cap l_{\{\nu,a\}}(B)$.
(5) $(u_{\{\omega,a\}}(A))^c \subseteq (l_{\{\nu,a\}}(A))^c$.

Example 3.21. Consider $M = \{(x, y) | x, y \in \mathbb{Z}_{10}\}$ (M is a $\mathbb{Z}$-module). Define a fuzzy submodule $\mu : M \to [0,1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = (0,0), \\ 0.6 & \text{if } x \in \{0\} \times \mathbb{Z}_{10}, \\ 0.3 & \text{elsewhere.} \end{cases}$$

Let $a \in R = \mathbb{Z}$ and $t \in [0,1]$. Then, $E_a = U_e(\mu, t, a)$ is an equivalence relation on M. Denote the equivalence classes by $[x]_a = \{y \in M : xE_a y\}$. Let $\nu = 0.5$ and $\omega = 0.15$. Take $a = 2$ and $t = 0.6$. Then,

$$M/U_e(\mu, 0.6, 2) = \{\{0,5\}\times\mathbb{Z}_{10},\ \{1,6\}\times\mathbb{Z}_{10},\ \{2,7\}\times\mathbb{Z}_{10},\ \{3,8\}\times\mathbb{Z}_{10},\ \{4,9\}\times\mathbb{Z}_{10}\}.$$

First, we illustrate that inclusions in (1), (2), (3), (4) and (5) can be strict.

Inclusion in (1): Take

$$A = \{(1,5), (2,0), (2,2), (2,4), (2,6), (2,8), (6,3), (7,2), (7,4), (7,6)\},$$

and

$$B = \{(1,2), (2,1), (2,3), (2,5), (2,7), (3,2), (3,9), (6,6), (7,0), (7,1), (7,7), (7,9), (8,1), (8,5)\}.$$

Then,

$$u_{\{0.15,2\}}(A) = \{2,7\} \times \mathbb{Z}_{10} \text{ and } u_{\{0.15,2\}}(B) = \{2,3,7,8\} \times \mathbb{Z}_{10}.$$

Note that $A \nsubseteq u_{\{0.15,2\}}(A)$ and $B \nsubseteq u_{\{0.15,2\}}(B)$. Now, we have

$$A \cup B = \{(1,2), (1,5), (2,0), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (3,2), (3,9), (6,3), (6,6), (7,0), (7,1), (7,2), (7,4), (7,6), (7,7), (7,9), (8,1), (8,5)\},$$

and

$$u_{\{0.15,2\}}(A \cup B) = \{1,2,3,6,7,8\} \times \mathbb{Z}_{10}.$$

Hence, $u_{\{0.15,2\}}(A) \cup u_{\{0.15,2\}}(B) \subsetneq u_{\{0.15,2\}}(A \cup B)$.

Inclusion in (2): Take

$$A = \{(6,5), (6,6), (6,7), (6,8)\} \text{ and } B = \{(1,0), (1,1), (1,2), (1,3)\}.$$

Then,

$$u_{\{0.15,2\}}(A) = \{1, 6\} \times \mathbb{Z}_{10} = u_{\{0.15,2\}}(B).$$

Now, we have $A \cap B = \emptyset$ and $u_{\{0.15,2\}}(A \cap B) = \emptyset$. Hence, $u_{\{0.15,2\}}(A \cap B) \subsetneq u_{\{0.15,2\}}(A) \cap u_{\{0.15,2\}}(B)$.

Inclusion in (3): Take

$$A = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9), (4,3), (4,7), (4,8), (5,0), (5,1), (9,0), (9,2), (9,5)\},$$

and

$$B = \{(0,0), (0,1), (4,2), (5,0), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (5,7), (5,8), (5,9), (9,1), (9,3), (9,7)\}.$$

Then,

$$l_{\{0.5,2\}}(A) = \{0, 5\} \times \mathbb{Z}_{10} = l_{\{0.5,2\}}(B).$$

Now, we have

$$A \cup B = \{(0,0), (0,1), (0,2), (0,3), (0,4), (0,5), (0,6), (0,7), (0,8), (0,9), (4,2), (4,3), (4,7), (4,8), (5,0), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (5,7), (5,8), (5,9), (9,0), (9,1), (9,2), (9,3), (9,5), (9,7)\},$$

and

$$l_{\{0.5,2\}}(A \cup B) = \{0, 4, 5, 9\} \times \mathbb{Z}_{10}.$$

Hence, $l_{\{0.5,2\}}(A) \cup l_{\{0.5,2\}}(B) \subsetneq l_{\{0.5,2\}}(A \cup B)$.

Inclusion in (4): Take

$$A = \{(3,0), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (3,7), (3,8), (3,9), (9,4), (9,9)\},$$

and

$$B = \{(8,0), (8,1), (8,2), (8,3), (8,4), (8,5), (8,6), (8,7), (8,8), (8,9), (9,4), (9,9)\}.$$

Then,

$$l_{\{0.5,2\}}(A) = \{3, 8\} \times \mathbb{Z}_{10} = l_{\{0.5,2\}}(B).$$

Note that $l_{\{0.5,2\}}(A) \nsubseteq A$ and $l_{\{0.5,2\}}(B) \nsubseteq B$.

Now, we have $A \cap B = \{(9,4), (9,9)\}$ and

$$l_{\{0.5,2\}}(A \cap B) = \left\{ x \in M | \frac{|\ (A \cap B) \cap [x]_2\ |}{|\ [x]_2\ |} \geq 0.5 \right\} = \emptyset.$$

Hence, $l_{\{0.5,2\}}(A \cap B) \subsetneq l_{\{0.5,2\}}(A) \cap l_{\{0.5,2\}}(B)$.

Inclusion in (5): Take

$$A = \{(0,0), (0,1), (1,0), (1,1), (1,2), (1,3), (1,4), (1,5), (2,0), (2,1), (3,0), (3,4), (3,5), (4,0), (5,0), (5,1), (6,6), (6,7), (6,8), (6,9), (8,5), (9,9)\}.$$

Then,

$$l_{\{0.5,2\}}(A) = \{1, 6\} \times \mathbb{Z}_{10} \text{ and } u_{\{0.15,2\}}(A) = \{0, 3, 5, 6, 8\} \times \mathbb{Z}_{10}.$$

Hence, $(u_{\{0.15,2\}}(A))^c \subsetneq (l_{\{0.5,2\}}(A))^c$.

Now, take $a = 1$ and $t = 0.6$. Then,

$$M/U_e(\mu, 0.6, 1) = \{\{i\} \times \mathbb{Z}_{10} \mid i \in \mathbb{Z}_{10}\}.$$

Now let us compute the lower and upper approximations of the set A with respect to the reference point $a = 1$. We have

$$l_{\{0.5,1\}}(A) = \{1\} \times \mathbb{Z}_{10} \text{ and } u_{\{0.15,1\}}(A) = \{0, 1, 2, 3, 5, 6\} \times \mathbb{Z}_{10}.$$

Note that distinct reference points, namely $a = 2$ and $a = 1$, yield distinct lower and upper approximations for the set A in the universe M.

4. Conclusion

Kedukodi et al. [17] studied reference points and roughness in rings. In a vector space, the set of scalars forms a field and acts on the vectors by scalar multiplication, subject to certain axioms such as distributive law. In a module the scalars need only be a ring. So, in this paper, we considered a module over a ring as a universe and the approximations given in the paper depend on the reference point. Moreover, the notion of a reference point expands the application domain of the rough set model presented in [10].

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B. Davvaz*, Department of Mathematics, Yazd University, Yazd, Iran
E-mail address: davvaz@yazd.ac.ir

A. Malekzadeh, Department of Mathematics, Yazd University, Yazd, Iran
E-mail address: malekzadeh_85@yahoo.com

*Corresponding author