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FUZZY h-IDEAL OF MATRIX HEMIRING $S_2 = \begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$

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Abstract. The purpose of this paper is to study matrix hemiring S_2 via fuzzy subsets and fuzzy h-ideals.

1. Introduction

There are many concepts of universal algebras generalizing an associative ring $(R, +, \cdot)$. Some of them have been found to be very useful for solving problems in different areas of applied mathematics and information sciences. Semiring is one such concept. The structure of a semiring provides an algebraic framework for modeling and studying the key factors in these applied areas. Ideals of semiring play a central role in the structure theory and is useful for many purposes. However they do not in general coincide with the usual ring ideals and for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semiring. To solve this problem, Henriksen [7], defined a more restricted class of ideals, which are called k-ideals. A more restricted class of ideals in hemirings, which are called h-ideals is given by Iizuka [8]. LaTorre [11], investigated h-ideals and k-ideals in hemirings in an effort to obtain analogues of ring theorems for hemiring and to amend the gap between ring ideals and semiring ideals. The theory of Γ-semiring was introduced by Rao [13]. The notion of $\Gamma-$semiring theory has been enriched by the introduction of operator semirings of a $\Gamma-$semiring by Dutta and Sardar [5]. To make operator semirings effective in the study of $\Gamma-$semirings Dutta et al [5] established a correspondence between the ideals of a $\Gamma-$semiring S and the ideals of the operator semirings of S. Saha and Sardar [14] generalized the notion of Γ-semiring to Nobusawa Γ-semiring [15]. A Nobusawa Γ-semiring S is simply Γ-semiring where Γ is also an S-semiring. To each Nobusawa Γ-semiring Saha and Sardar associated a semiring called matrix semiring and it was denoted by S_2 or $\begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$ where R, L are respectively the right and the left operator semirings of the Γ-semiring S. They investigated many properties of S_2. In this paper we study S_2 via fuzzy subsets. In particular we study fuzzy h-ideals of S_2. Among other results we obtain an inclusion preserving bijection between the fuzzy h-ideals of a Nobusawa Γ-hemiring S and that of S_2. We also obtain a characterization of h-hemiregular matrix hemiring S_2. It is relevant to make clear as regards the

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terminology used here. To keep consistency with the terminology of fuzzification of hemirings [4, 10, 19, 20, 21], we call Γ-semirings to be Γ-hemirings.

2. **Preliminaries**

We recall the following definition from [6].

A *hemiring* (respectively *semiring*) is a nonempty set S on which operations addition and multiplication have been defined such that $(S,+)$ is a commutative monoid with identity 0, $(S,\cdot)$ is a semigroup (respectively monoid with identity 1_S), multiplication distributes over addition from either side, $1_S \neq 0$ and $0s = 0 = s0$ for all $s \in S$.

Let S and Γ be two additive commutative semigroups with zero. According to [16], S is called a Γ-*hemiring* if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ by $(a,\alpha,b) \mapsto a\alpha b$ satisfying the following conditions:

(1) $(a+b)\alpha c = a\alpha c + b\alpha c$,

(2) $a\alpha(b+c) = a\alpha b + a\alpha c$,

(3) $a(\alpha+\beta)b = a\alpha b + a\beta b$,

(4) $a\alpha(b\beta c) = (a\alpha b)\beta c$,

(5) $0_S\alpha a = 0_S = a\alpha 0_S$,

(6) $a0_\Gamma b = 0_S = b0_\Gamma a$,

for all $a,b,c \in S$ and for all $\alpha,\beta \in \Gamma$. In addition, if there exists a mapping $\Gamma \times S \times \Gamma \longrightarrow \Gamma$ by $(\alpha,s,\beta) \mapsto \alpha s\beta$ satisfying the above conditions, then S is called a *Nobusawa* Γ-*semiring* [15].

For simplification we write 0 instead of 0_S and 0_Γ.

Let S be the set of all $m\times n$ matrices over $\mathbb{Z}_{\mathbf{0}}^-$ (the set of all non-positive integers) and Γ be the set of all $n \times m$ matrices over $\mathbb{Z}_{\mathbf{0}}^-$. Then S forms a Γ-hemiring with usual addition and multiplication of matrices.

Now, we recall the following definitions from [5].

Let S be a Γ-hemiring and F be the free additive commutative semigroup generated by $S \times \Gamma$. We define a relation ρ on F as follows:

$$\sum_{i=1}^{m}(x_i,\alpha_i)\ \rho \sum_{j=1}^{n}(y_j,\beta_j) \text{ if and only if } \sum_{i=1}^{m}x_i\alpha_i a = \sum_{j=1}^{n}y_j\beta_j a,$$

for all $a \in S$ $(m,n \in \mathbb{Z}^+)$. Then ρ is a congruence relation on F. We denote the congruence class containing $\sum_{i=1}^{m}(x_i,\alpha_i)$ by $\sum_{i=1}^{m}[x_i,\alpha_i]$. Then F/ρ is an additive commutative semigroup. Now, F/ρ forms a hemiring with the multiplication defined by

$$\left(\sum_{i=1}^{m}[x_i,\alpha_i]\right)\left(\sum_{j=1}^{n}[y_j,\beta_j]\right) = \sum_{i,j}[x_i\alpha_i y_j,\beta_j].$$

We denote this hemiring by L and call it the *left operator hemiring* of the Γ-hemiring S. Dually we define the *right operator hemiring* R of the Γ-hemiring S.

Let S be a Γ-hemiring, L be the left operator hemiring and R be the right one. If there exists an element $\sum_{i=1}^{m}[e_i, \delta_i] \in L$(resp. $\sum_{j=1}^{n}[\gamma_j, f_j] \in R$) such that $\sum_{i=1}^{m} e_i\delta_i a = a$ (respectively, $\sum_{j=1}^{n} a\gamma_j f_j = a$) for all $a \in S$, then S is said to have the *left unity* $\sum_{i=1}^{m}[e_i, \delta_i]$ (respectively, the *right unity* $\sum_{j=1}^{n}[\gamma_j, f_j]$).

Throughout this paper unless otherwise mentioned for different elements of L (respectively, R) we take the same index say $'i'$ whose range is finite, that is, from 1 to n, for some positive integer n.

Let S be a Γ-hemiring, L be the left operator hemiring and R be the right one. If there exists an element $[e, \delta] \in L$ (respectively, $[\gamma, f] \in R$) such that $e\delta a = a$ (respectively, $a\gamma f = a$) for all $a \in S$, then S is said to have the *strong left unity* $[e, \delta]$ (respectively, strong right unity $[\gamma, f]$) [13].

Let S be a Γ-hemiring, L be the left operator hemiring and R be the right one. Let $P \subseteq L$ $(\subseteq R)$. According to [5], we define $P^+ = \{a \in S : [a, \Gamma] \subseteq P\}$ (respectively, $P^* = \{a \in S : [\Gamma, a] \subseteq P\}$) and for $Q \subseteq S$,

$$Q^{+'} = \left\{ \sum_{i=1}^{m}[x_i, \alpha_i] \in L : \left(\sum_{i=1}^{m}([x_i, \alpha_i] \right) S \subseteq Q \right\},$$

where $\left(\sum_{i=1}^{m}[x_i, \alpha_i] \right) S$ denotes the set of all finite sums $\sum_{i,k} x_i\alpha_i s_k,\ s_k \in S$ and

$$Q^{*'} = \left\{ \sum_{i=1}^{m}[\alpha_i, x_i] \in R : S \left(\sum_{i=1}^{m}([\alpha_i, x_i] \right) \subseteq Q \right\},$$

where $S \left(\sum_{i=1}^{m}[x_i, \alpha_i] \right)$ denotes the set of all finite sums $\sum_{i,k} s_k\alpha_i x_i,\ s_k \in S$.

A *fuzzy subset* μ of a non-empty set S is a function $\mu : S \longrightarrow [0, 1]$.

Let μ be a non-empty fuzzy subset of a Γ-hemiring S (i.e., $\mu(x) \neq 0$ for some $x \in S$). Then μ is called a *fuzzy left ideal* (respectively, *fuzzy right ideal*) of S [16] if

(1) $\mu(x + y) \geq \min[\mu(x), \mu(y)]$,

(2) $\mu(x\gamma y) \geq \mu(y)$ (respectively, $\mu(x\gamma y) \geq \mu(x)$),

for all $x, y \in S$ and $\gamma \in \Gamma$. A *fuzzy ideal* of a Γ-hemiring S is a non-empty fuzzy subset of S which is a fuzzy left ideal as well as a fuzzy right ideal of S. Recently, many authors studied fuzzy ideals of various kinds of algebraic structures, for example see [1, 2, 3, 9, 22]. Note that if μ is a fuzzy left or right ideal of a Γ-hemiring S, then $\mu(0) \geq \mu(x)$ for all $x \in S$.

A left ideal A of a Γ-hemiring S is called a *left h-ideal* if for any $x, z \in S$ and $a, b \in A$,

$$x + a + z = b + z \Longrightarrow x \in A.$$

A *right h-ideal* is defined analogously. A fuzzy left ideal μ of a Γ-hemiring S is called a *fuzzy left h-ideal* if for all $a, b, x, z \in$ S,

$$x + a + z = b + z \Longrightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}.$$

A *fuzzy right h-ideal* is defined similarly. By a *fuzzy h-ideal* μ, we mean that μ is both fuzzy left and fuzzy right h-ideal.

For example, let S be the additive commutative semigroup of all non-positive integers and Γ be the additive commutative semigroup of all non-positive even integers. Then S is a Γ-hemiring if $a\gamma b$ denotes the usual multiplication of integers a, γ, b where $a, b \in S$ and $\gamma \in \Gamma$. Let μ be a fuzzy subset of S, defined as follows

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.7 & \text{if } x \text{ is even} \\ 0.1 & \text{if } x \text{is odd} \end{cases}$$

The fuzzy subset μ of S is both a fuzzy ideal and a fuzzy h-ideal of S.

Let S be a Γ-hemiring and μ_1, μ_2 be two fuzzy subsets of S . Then the sum $\mu_1 \oplus \mu_2$ is defined as follows:

$$(\mu_1 \oplus \mu_2)(x) = \sup_{x=u+v} \{\min\{\mu_1(u), \mu_2(v)\} \ : \ u, v \in S\}$$

Let μ and θ be two fuzzy subsets of a Γ-hemiring S. We define *generalized h-product* of μ and θ by
$\mu \circ_h \theta(x)$

$$= \begin{cases} \sup\left\{\min_i\{\min\{\mu(a_i), \mu(c_i), \theta(b_i), \theta(d_i)\}\} : x + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z\right\} \\ 0 \ \text{ if } x \text{ can not be expressed as above,} \end{cases}$$

where $x, z, a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, for $i = 1, \ldots, n$.

For more preliminaries of semirings (hemirings) and Γ-semirings we refer to [6] and [5], respectively. Also, for more results on fuzzy h-ideals in Γ-hemirings we refer to [16]. Throughout this paper unless otherwise mentioned S denotes a Γ-hemiring with left unity and right unity and $FLh-I(S)$, $FRh-I(S)$ and $Fh-I(S)$ denote respectively the set of all fuzzy left h-ideals, the set of all fuzzy right h-ideals and the set of all fuzzy h-ideals of the Γ-hemiring S. Similar is the meaning of $FLh-I(L)$, $FLh-I(R)$, $FRh-I(L)$, $FRh-I(R)$, $Fh-I(L)$, $Fh-I(R)$, where L and R are respectively the left operator and right operator hemirings of the Γ-hemiring S. Also, in this section we assume that $\mu(0) = 1$ for a fuzzy left h-ideal (respectively, fuzzy right h-ideal, fuzzy h-ideal) μ of a Γ-hemiring S. Similarly, we assume that $\mu(0_L) = 1$ (respectively, $\mu(0_R) = 1$) for a fuzzy left h-ideal (respectively, fuzzy right h-ideal, fuzzy h-ideal) μ of the left operator hemiring (respectively, right operator hemiring R) of a Γ-hemiring S.

Throughout this section S denotes a Nobusawa Γ-hemiring with unities, R denotes the right operator hemiring and L denotes the left operator hemiring of the Nobusawa Γ-hemiring S.

We recall the following definitions and results from [17].

Let μ be a fuzzy subset of L, we define a fuzzy subset μ^+ of S by

$$\mu^+(x) = \inf_{\gamma\in\Gamma} \mu\left([x,\gamma]\right),$$

where $x \in S$. If σ is a fuzzy subset of S, we define a fuzzy subset $\sigma^{+'}$ of L by

$$\sigma^{+'}\left(\sum_i [x_i, \alpha_i]\right) = \inf_{s\in S} \sigma\left(\sum_i x_i\alpha_i s\right),$$

where $\sum_i [x_i, \alpha_i] \in L$. If δ is a fuzzy subset of R, we define a fuzzy subset δ^* of S by

$$\delta^*(x) = \inf_{\gamma\in\Gamma} \delta\left([\gamma, x]\right),$$

where $x \in S$. If η is a fuzzy subset of S, we define a fuzzy subset $\eta*'$ of R by

$$\eta^{*'}\left(\sum_i [\alpha_i, x_i]\right) = \inf_{s\in S} \eta\left(\sum_i s\alpha_i x_i\right),$$

where $\sum_i [\alpha_i, x_i] \in R$. We have:

(1) If $\mu \in Fh - I(L)$, then $\mu^+ \in Fh - I(S)$.

(2) If $\sigma \in Fh - I(S)$ (respectively, $FRh - I(S)$, $FLh - I(S)$), then $\sigma^{+'} \in Fh - I(L)$ (respectively, $FRh - I(L)$, $FLh - I(L)$).

(3) If $\delta \in Fh - I(R)$ (respectively, $FRh - I(R)$, $FLh - I(R)$), then $\delta^* \in Fh - I(S)$ (respectively, $FRh - I(S)$, $FLh - I(S)$).

(4) If $\eta \in Fh - I(S)$ (respectively, $FRh - I(S)$, $FLh - I(S)$), then $\eta^{*'} \in Fh - I(R)$ (respectively, $FRh - I(R)$, $FLh - I(R)$).

(5) The lattices of all fuzzy h-ideals of S and L are isomorphic via the inclusion preserving bijection $\sigma \mapsto \sigma^{+'}$, where $\sigma \in Fh - I(S)$ and $\sigma^{+'} \in Fh - I(L)$.

(6) The lattices of all fuzzy h-ideals of S and R are isomorphic via the inclusion preserving bijection $\sigma \mapsto \sigma^{*'}$, where $\sigma \in Fh - I(S)$ and $\sigma^{*'} \in Fh - I(R)$.

(7) The lattices of all h-ideals of S and R are isomorphic via the mapping $I \mapsto I^{*'}$, where I denotes an h-ideal of S.

(8) For any two fuzzy h-ideals μ and ν of S, $(\mu o_h \nu)^{+'} = ((\mu)^{+'} o_h (\nu)^{+'})$.

3. Fuzzy h-ideal in Matrix Hemiring $\begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$

Definition 3.1. Let S be a Nobusawa Γ-hemiring and μ be a fuzzy subset of S. Then we define a fuzzy subset $\Gamma(\mu)$ of Γ by

$$\Gamma(\mu)(\gamma) = \inf_{s,s'\in S} \mu(s\gamma s'),$$

for $\gamma \in \Gamma$.

Dually corresponding to a fuzzy subset ν of Γ, we can find a fuzzy subset $S(\nu)$ of S.

Throughout this section, unless otherwise mentioned, we consider S to be Nobusawa Γ-hemiring with unities.

Theorem 3.2. *If μ is a fuzzy h-ideal of S, then $\Gamma(\mu)$ is also a fuzzy h-ideal of Γ.*

Proof. Suppose that μ is a fuzzy h-ideal of S and $\gamma_1, \gamma_2 \in \Gamma$. Then

$$\begin{aligned}\Gamma(\mu)(\gamma_1+\gamma_2) &= \inf_{s,s^{'}\in S}\mu(s(\gamma_1+\gamma_2)s^{'})\\ &= \inf_{s,s^{'}\in S}\mu((s\gamma_1 s^{'})+(s\gamma_2 s^{'}))\\ &\geq \inf_{s,s^{'}\in S}\min\{\mu(s\gamma_1 s^{'}),\mu(s\gamma_2 s^{'})\}\\ &= \min\left\{\inf_{s,s^{'}\in S}\mu(s\gamma_1 s^{'}),\ \inf_{s,s^{'}\in S}\mu(s\gamma_2 s^{'})\right\}\\ &= \min\{\Gamma(\mu)(\gamma_1),\Gamma(\mu)(\gamma_2)\}\end{aligned}$$

and

$$\Gamma(\mu)(\gamma_1 s\gamma_2) = \inf_{s^{'},s^{''}\in S}\mu(s^{'}(\gamma_1 s\gamma_2)s^{''}) \geq \inf_{s^{'},s\in S}\mu(s^{'}\gamma_1 s) = \Gamma(\mu)(\gamma_1).$$

Similarly, we have

$$\Gamma(\mu)(\gamma_1 s\gamma_2) = \inf_{s^{'},s^{''}\in S}\mu(s^{'}(\gamma_1 s\gamma_2)s^{''}) \geq \inf_{s,s^{''}\in S}\mu(s\gamma_2 s^{''}) = \Gamma(\mu)(\gamma_2).$$

Hence, $\Gamma(\mu)$ is a fuzzy ideal of Γ. Now, suppose that $\alpha+\gamma_1+\delta = \gamma_2+\delta$, for $\alpha, \gamma_1, \gamma_2, \delta \in \Gamma$. So,

$$\begin{aligned}\Gamma(\mu)(\alpha) &= \inf_{s,s^{'}\in S}\mu(s\alpha s^{'})\\ &\geq \inf_{s,s^{'}\in S}\min\{\mu(s\gamma_1 s^{'}),\mu(s\gamma_2 s^{'})\}\\ &= \min\left\{\inf_{s,s^{'}\in S}\mu(s\gamma_1 s^{'}),\ \inf_{s,s^{'}\in S}\mu(s\gamma_2 s^{'})\right\}\\ &= \min\{\Gamma(\mu)(\gamma_1),\Gamma(\mu)(\gamma_2)\}.\end{aligned}$$

Therefore, $\Gamma(\mu)$ is a fuzzy h-ideal of Γ. □

Similarly, we can prove that corresponding to a fuzzy h-ideal ν of Γ, $S(\nu)$, defined by

$$S(\nu)(s) = \inf_{\gamma,\gamma^{'}\in S}\nu(\gamma s\gamma^{'})\ for\ s\in S,$$

is a fuzzy h-ideal of S. Now we observe that if S is a Nobusawa Γ-hemiring with strong unities, then for $\mu \in Fh-I(S)$ and $x \in S$

$$S(\Gamma(\mu))(x) = \inf_{\gamma_1,\gamma_2\in\Gamma}\Gamma(\mu)(\gamma_1 x\gamma_2) = \inf_{\gamma_1,\gamma_2\in\Gamma}\inf_{s_1,s_2\in S}\mu(s_1\gamma_1 x\gamma_2 s_2) \geq \mu(x).$$

Also we see that

$$\begin{aligned}\mu(x) &= \mu(e\delta x)\ (\textit{where } [e,\delta] \textit{ is the strong left unity of } S)\\ &= \mu(e\delta x\gamma f)\ (\textit{where } [\gamma,f] \textit{ is the strong right unity of } S)\\ &\geq \inf_{s_1,s_2\in S}\mu(s_1\delta x\gamma s_2) \geq \inf_{s_1,s_2\in S}\inf_{\gamma_1,\gamma_2\in\Gamma}\mu(s_1\gamma_1 x\gamma_2 s_2)\\ &= \inf_{\gamma_1,\gamma_2\in\Gamma}\inf_{s_1,s_2\in S}\mu(s_1\gamma_1 x\gamma_2 s_2) = S(\Gamma(\mu))(x).\end{aligned}$$

Thus $\mu = S(\Gamma(\mu))$. Similarly, for $\nu \in Fh - I(\Gamma),\ \nu = \Gamma(S(\nu))$. Consequently, we obtain the following theorem.

Theorem 3.3. *Let S be a Nobusawa Γ-hemiring with strong unities. Then there exists a bijection between the set of all fuzzy h-ideals of S and the set of all fuzzy h-ideals of Γ.*

Proposition 3.4. *Suppose that μ and ν are fuzzy h-ideals of S. Then*

(1) $\Gamma(\mu)\cap\Gamma(\nu) = \Gamma(\mu\cap\nu)$,
(2) $\Gamma(\mu)o_h\Gamma(\nu) = \Gamma(\mu o_h \nu)$,
(3) $\Gamma(\mu)\oplus\Gamma(\nu) = \Gamma(\mu\oplus\nu)$.

Proof. Let $\gamma\in\Gamma$. Then

$$\begin{aligned}(\Gamma(\mu)\cap\Gamma(\nu))(\gamma) &= \min\{\Gamma(\mu)(\gamma),\Gamma(\nu)(\gamma)\}\\ &= \min\{\inf_{s,s'\in S}\mu(s\gamma s'),\ \inf_{s,s'\in S}\nu(s\gamma s')\}\\ &= \inf_{s,s'\in S}\min\{\mu(s\gamma s'),\nu(s\gamma s')\}\\ &= \inf_{s,s'\in S}(\mu\cap\nu)(s\gamma s')\\ &= \Gamma(\mu\cap\nu)(\gamma)\end{aligned}$$

and

$$\begin{aligned}&\Gamma(\mu)o_h\Gamma(\nu)(\gamma)\\ &= \sup\Big\{\min_i\big\{\Gamma(\mu)(\gamma_i^1),\Gamma(\nu)(\gamma_i^2),\Gamma(\mu)(\gamma_i^3),\Gamma(\nu)(\gamma_i^4)\big\}\ :\\ &\qquad \gamma+\sum_i\gamma_i^1 s_i^1\gamma_i^2 + z = \sum_i \gamma_i^3 s_i^2\gamma_i^4 + z\Big\}\\ &= \sup\Big\{\min_i\Big\{\inf_{s,s'\in S}\mu(s\gamma_i^1 s'),\ \inf_{s,s'\in S}\nu(s\gamma_i^2 s'),\ \inf_{s,s'\in S}\mu(s\gamma_i^3 s'),\ \inf_{s,s'\in S}\nu(s\gamma_i^4 s')\Big\}\ :\\ &\qquad s\gamma s' + \sum_i s\gamma_i^1 s_i^1\gamma_i^2 s' + szs' = \sum_i s\gamma_i^3 s_i^2\gamma_i^4 s' + szs'\Big\}\\ &= \inf_{s,s'\in S}(\mu o_h\nu)(s\gamma s')\\ &= \Gamma(\mu o_h \nu)(\gamma).\end{aligned}$$

Also, we have

$$\begin{aligned}\Gamma(\mu\oplus\nu)(\gamma) &= \inf_{s,s^{'}\in S}(\mu\oplus\nu)(s\gamma s^{'})\\ &= \inf_{s,s^{'}\in S}\sup_{s\gamma s^{'}=p+q}\{\min\{\mu(p),\nu(q)\}\}\\ &= \inf_{s,s^{'}\in S}\sup_{\gamma=\gamma_1+\gamma_2}\{\min\{\mu(s\gamma_1 s^{'}),\nu(s\gamma_2 s^{'})\}\}\\ &= \sup_{\gamma=\gamma_1+\gamma_2}\left\{\min\left\{\inf_{s,s^{'}\in S}\mu(s\gamma_1 s^{'}),\inf_{s,s^{'}\in S}\nu(s\gamma_2 s^{'})\right\}\right\}\\ &= \sup_{\gamma=\gamma_1+\gamma_2}\{\min\{\Gamma(\mu)(\gamma_1),\Gamma(\nu)(\gamma_2)\}\}\\ &= (\Gamma(\mu)\oplus\Gamma(\nu))(\gamma).\end{aligned}$$

This completes the proof. □

Theorem 3.5. [14] *Let S be a Nobusawa Γ-hemiring; L and R be its left and right operator hemiring, respectively. Then $S_2=\begin{pmatrix}R & \Gamma\\ S & L\end{pmatrix}$ forms a hemiring with respect to the addition and multiplication defined by*

$$\begin{pmatrix}r_1 & \gamma_1\\ s_1 & l_1\end{pmatrix}+\begin{pmatrix}r_2 & \gamma_2\\ s_2 & l_2\end{pmatrix}=\begin{pmatrix}r_1+r_2 & \gamma_1+\gamma_2\\ s_1+s_2 & l_1+l_2\end{pmatrix}$$

and

$$\begin{pmatrix}r_1 & \gamma_1\\ s_1 & l_1\end{pmatrix}\cdot\begin{pmatrix}r_2 & \gamma_2\\ s_2 & l_2\end{pmatrix}=\begin{pmatrix}r_1r_2+[\gamma_1,s_2] & r_1\gamma_2+\gamma_1 l_2\\ s_1r_2+l_1s_2 & [s_1,\gamma_2]+l_1l_2\end{pmatrix}.$$

Theorem 3.6. [14] *Let S be a Nobusawa Γ-hemiring and I be an h-ideal of S. Then $I_2=\begin{pmatrix}I^{*^{'}} & \Gamma(I)\\ I & I^{+^{'}}\end{pmatrix}$ is an h-ideal of S_2.*

Let μ be a fuzzy subset of S. Then $\begin{pmatrix}\mu^{*^{'}} & \Gamma(\mu)\\ \mu & \mu^{+^{'}}\end{pmatrix}$, defined by

$$\begin{pmatrix}\mu^{*^{'}} & \Gamma(\mu)\\ \mu & \mu^{+^{'}}\end{pmatrix}\begin{pmatrix}r_1 & \gamma_1\\ s_1 & l_1\end{pmatrix}=\min\left\{\mu^{*^{'}}(r_1),\Gamma(\mu)(\gamma_1),\mu(s_1),\mu^{+^{'}}(l_1)\right\},$$

where $\begin{pmatrix}r_1 & \gamma_1\\ s_1 & l_1\end{pmatrix}\in S_2$ is a fuzzy subset of S_2.

Definition 3.7. Let μ be a fuzzy subset of S. Then the fuzzy subset $\begin{pmatrix}\mu^{*^{'}} & \Gamma(\mu)\\ \mu & \mu^{+^{'}}\end{pmatrix}$ of S_2 is called the *corresponding fuzzy subset* of S_2.

Theorem 3.8. *Let μ be a fuzzy h-ideal of S. Then $\mu_2=\begin{pmatrix}\mu^{*^{'}} & \Gamma(\mu)\\ \mu & \mu^{+^{'}}\end{pmatrix}$ is a fuzzy h-ideal of S_2. Moreover, if μ and ν are two distinct h-ideals of S, then $\mu_2\neq\nu_2$.*

Proof. Let μ be a fuzzy h-ideal of S and $\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix}, \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} \in S_2$. Then

$$\begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} \left(\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} + \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} \right)$$
$$= \begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} \begin{pmatrix} r_1 + r_2 & \gamma_1 + \gamma_2 \\ s_1 + s_2 & l_1 + l_2 \end{pmatrix}$$
$$= \min\{\mu^{*'}(r_1 + r_2), \Gamma(\mu)(\gamma_1 + \gamma_2), \mu(s_1 + s_2), \mu^{+'}(l_1 + l_2)\}$$
$$\geq \min\{\min\{\mu^{*'}(r_1), \Gamma(\mu)(\gamma_1), \mu(s_1), \mu^{+'}(l_1)\}, \min\{\mu^{*'}(r_2), \Gamma(\mu)(\gamma_2), \mu(s_2), \mu^{+'}(l_2)\}\}$$
$$= \min\left\{ \begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} \begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix}, \begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} \right\}$$

and

$$\begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} \left(\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} \cdot \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} \right)$$
$$= \begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} \begin{pmatrix} r_1 r_2 + [\gamma_1, s_2] & r_1 \gamma_2 + \gamma_1 l_2 \\ s_1 r_2 + l_1 s_2 & [s_1, \gamma_2] + l_1 l_2 \end{pmatrix}$$
$$= \min\{\min\{\mu^{*'}(r_1 r_2 + [\gamma_1, s_2]), \Gamma(\mu)(r_1 \gamma_2 + \gamma_1 l_2), \mu(s_1 r_2 + l_1 s_2), \mu^{+'}([s_1, \gamma_2] + l_1 l_2)\}$$
$$\geq \min\{\mu^{*'}(r_1 r_2), \mu^{*'}([\gamma_1, s_2]), \Gamma(\mu)(r_1 \gamma_2), \Gamma(\mu)(\gamma_1 l_2), \mu(s_1 r_2), \mu(l_1 s_2), \mu^{+'}(l_1 l_2),$$
$$\mu^{+'}([s_1, \gamma_2])\} \tag{1}$$

Now, we have the following observations:

- $\mu^{*'}([\gamma_1, s_2]) = \inf_{s \in S} \mu(s \gamma_1 s_2) \geq \mu(s_2)$,
- $\Gamma(\mu)(\gamma_1 l_2) = \inf_{s, s' \in S} \mu(s \gamma_1 l_2 s') \geq \inf_{s' \in S} \mu(l_2 s') = \mu^{+'}(l_2)$,
- $\mu^{*'}(r_2) = \inf_{s \in S} \mu(s r_2) \leq \mu(s_1 r_2)$,
- $\mu^{+'}([s_1, \gamma_2]) = \inf_{s \in S} \mu(s_1 \gamma_2 s) = \inf_{s \in S} \inf_{\gamma, \gamma' \in \Gamma} \Gamma(\mu)(\gamma s_1 \gamma_2 s \gamma') \geq \Gamma(\mu)(\gamma_2)$.

Therefore, from (1) we deduce that

$$\begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} \left(\begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} \cdot \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} \right) \geq \min\{\mu^{*'}(r_2), \Gamma(\mu)(\gamma_2), \mu(s_2), \mu^{+'}(l_2)\}$$
$$= \begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2. \end{pmatrix}$$

Hence, $\begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix}$ is a fuzzy left ideal of S_2. In a similar way we can prove that this is also a right ideal. Hence $\begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix}$ is a fuzzy ideal of S_2.

Now, suppose that

$$\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} + \begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix} + \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix} = \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} + \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}.$$

Then

$$\begin{pmatrix} x_1 + r_1 + z_1 & x_3 + \gamma_1 + z_3 \\ x_2 + s_1 + z_2 & x_4 + l_1 + z_4 \end{pmatrix} = \begin{pmatrix} r_2 + z_1 & \gamma_2 + z_3 \\ s_1 + z_2 & l_2 + z_4 \end{pmatrix}.$$

Now, we have

$$\begin{aligned}
&\begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \\
&= \min\{\mu^{*'}(x_1), \Gamma(\mu)(x_3), \mu(x_2), \mu^{+'}(x_4)\} \\
&\geq \min\{\min\{\mu^{*'}(r_1), \mu^{*'}(r_2)\}, \min\{\Gamma(\mu)(\gamma_1), \Gamma(\mu)(\gamma_2)\}, \min\{\mu(s_1), \mu(s_2)\}, \\
&\qquad \min\{\mu^{+'}(l_1), \mu^{+'}(l_2)\}\} \\
&= \min\{\min\{\mu^{*'}(r_1), \Gamma(\mu)(\gamma_1), \mu(s_1), \mu^{+'}(l_1)\}, \min\{\mu^{*'}(r_2), \Gamma(\mu)(\gamma_2), \mu(s_2), \mu^{+'}(l_2)\}\} \\
&= \min\left\{ \begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} \begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix}, \begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix} \right\}.
\end{aligned}$$

Hence, $\begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix}$ is a fuzzy h-ideal of S_2. Last part of this theorem follows from definition. □

Theorem 3.9. *Let S be a Nobusawa Γ-hemiring. Then every fuzzy ideal of S_2 is of the form $\begin{pmatrix} f^{*'} & \Gamma(f) \\ f & f^{+'} \end{pmatrix}$ for some fuzzy ideal f of S.*

Proof. Suppose that

$$\begin{pmatrix} \sum_i [\alpha_i, x_i] & \gamma_1 \\ s_1 & \sum_i [y_i, \beta_i] \end{pmatrix} \text{ and } \begin{pmatrix} \sum_i [\delta_i, p_i] & \gamma_2 \\ s_2 & \sum_i [q_i, \eta_i] \end{pmatrix}$$

be any two elements of S_2 and $\begin{pmatrix} f_2 & f_3 \\ f_1 & f_4 \end{pmatrix}$ be any fuzzy subset of S_2, where $f_1 \in FI(S)$, $f_2 \in FI(R)$, $f_3 \in FI(\Gamma)$ and $f_4 \in FI(L)$. Then

$$\begin{aligned}
&\begin{pmatrix} f_2 & f_3 \\ f_1 & f_4 \end{pmatrix}\left(\begin{pmatrix} \sum_i[\alpha_i, x_i] & \gamma_1 \\ s_1 & \sum_i[y_i, \beta_i] \end{pmatrix}\begin{pmatrix} \sum_i[\delta_i, p_i] & \gamma_2 \\ s_2 & \sum_i[q_i, \eta_i] \end{pmatrix}\right) \\
&= \begin{pmatrix} f_2 & f_3 \\ f_1 & f_4 \end{pmatrix}\left(\begin{pmatrix} \sum_i[\alpha_i, x_i]\sum_i[\delta_i, p_i] + [\gamma_1, s_2] & \sum_i \alpha_i x_i \gamma_2 + \sum_i \gamma_1 q_i \eta_i \\ \sum_i s_1\delta_i p_i + \sum_i y_i\beta_i s_2 & [s_1, \gamma_2] + \sum_i[y_i, \beta_i]\sum_i[q_i, \eta_i] \end{pmatrix}\right) \\
&= \min\Big\{ f_1(\sum_i s_1\delta_i p_i + \sum_i y_i\beta_i s_2), f_2(\sum_i[\alpha_i, x_i]\sum_i[\delta_i, p_i] + [\gamma_1, s_2]), f_3(\sum_i \alpha_i x_i \gamma_2 \\
&\qquad + \sum_i \gamma_1 q_i \eta_i), f_4([s_1, \gamma_2] + \sum_i[y_i, \beta_i]\sum_i[q_i, \eta_i]) \Big\} \\
&\geq \min\Big\{ f_1(s_1), f_1(p_i), f_1(y_i), f_1(s_2), f_2(\sum_i[\alpha_i, x_i]), f_2(\sum_i[\delta_i, p_i]), f_2([\gamma_1, s_2]), \\
&\qquad f_3(\alpha_i), f_3(\gamma_2), f_3(\gamma_1), f_3(\eta_i), f_4([s_1, \gamma_2]), f_4(\sum_i[y_i, \beta_i]), f_4(\sum_i[q_i, \eta_i]) \Big\} \qquad \text{(i)}
\end{aligned}$$

If $\begin{pmatrix} f_2 & f_3 \\ f_1 & f_4 \end{pmatrix}$ has to be an ideal of S_2, then

$$\begin{aligned}
&\begin{pmatrix} f_2 & f_3 \\ f_1 & f_4 \end{pmatrix}\left(\begin{pmatrix} \sum_i[\alpha_i, x_i] & \gamma_1 \\ s_1 & \sum_i[y_i, \beta_i] \end{pmatrix}\begin{pmatrix} \sum_i[\delta_i, p_i] & \gamma_2 \\ s_2 & \sum_i[q_i, \eta_i] \end{pmatrix}\right) \\
&\geq \min\Big\{ \min\{f_1(s_1), f_2(\sum_i[\alpha_i, x_i]), f_3(\gamma_1), f_4(\sum_i[y_i, \beta_i])\}, \\
&\qquad \min\{f_1(s_2), f_2(\sum_i[\delta_i, p_i]), f_3(\gamma_2), f_4(\sum_i[q_i, \eta_i])\} \Big\}. \qquad \text{(ii)}
\end{aligned}$$

Therefore, comparing (i) and (ii), we obtain the following:

(1) $f_1(p_i) \geq f_2(\sum_i[\delta_i, p_i])$,

(2) $f_1(y_i) \geq f_4(\sum_i[y_i, \beta_i])$,

(3a) $f_2([\gamma_1, s_2]) \geq f_3(\gamma_1)$,

(3b) $f_2([\gamma_1, s_2]) \geq f_1(s_2)$,

(4) $f_3(\alpha_i) \geq f_2(\sum_i[\alpha_i, x_i])$,

(5) $f_3(\eta_i) \geq f_4(\sum_i[q_i, \eta_i])$,

(6a) $f_4([s_1, \gamma_2]) \geq f_1(s_1)$,

(6b) $f_4([s_1, \gamma_2]) \geq f_3(\gamma_2)$.

Then, we have

(7) $f_2(\sum_i[\delta_i, p_i]) = \inf_{s\in S} f_2^*(\sum_i s\delta_i p_i) \geq f_2^*(p_i)$,

(8) $f_4(\sum_i[y_i, \beta_i]) = \inf_{s\in S} f_4^+(\sum_i y_i\beta_i s) \geq f_4^+(y_i)$,

(9) $f_2([\gamma_1, s_2]) = \inf_{s\in S} f_2^*(s\gamma_1 s_2) \geq f_2^*(s_2)$,

(10)

$$\begin{aligned} f_2([\gamma_1, s_2]) &= \inf_{s\in S} f_2^*(s\gamma_1 s_2) \\ &= \inf_{\gamma_3,\gamma_4\in\Gamma} \inf_{s\in S} \Gamma(f_2^*)(\gamma_3 s\gamma_1 s_2\gamma_4) \geq \Gamma(f_2^*)(\gamma_1), \end{aligned}$$

(11)

$$\begin{aligned} f_2(\sum_i [\alpha_i, x_i]) &= \inf_{s\in S} f_2^*(\sum_i s\alpha_i x_i) \\ &= \inf_{\gamma_3,\gamma_4\in\Gamma} \inf_{s\in S} \Gamma(f_2^*)(\sum_i \gamma_3 s\alpha_i x_i\gamma_4) \geq \Gamma(f_2^*)(\alpha_i), \end{aligned}$$

(12)

$$\begin{aligned} f_4(\sum_i [q_i, \eta_i]) &= \inf_{s\in S} f_4^+(\sum_i q_i\eta_i s) \\ &= \inf_{\gamma_3,\gamma_4\in\Gamma} \inf_{s\in S} \Gamma(f_4^+)(\sum_i \gamma_3 q_i\eta_i s\gamma_4) \geq \Gamma(f_4^+)(\eta_i), \end{aligned}$$

(13) $f_4([s_1, \gamma_2]) = \inf_{s\in S} f_4^+(s_1\gamma_2 s) = \inf_{\gamma_3,\gamma_4\in\Gamma} \inf_{s\in S} \Gamma(f_4^+)(\gamma_3 s_1\gamma_2 s\gamma_4) \geq \Gamma(f_4^+)(\gamma_2)$,

(14) $f_4([s_1, \gamma_2]) = \inf_{s\in S} f_4^+(s_1\gamma_2 s) \geq f_4^+(s_1)$.

(15) $f_1(p_i) \geq f_2^*(p_i)$ (from (1) and (7)),

(16) $f_1(y_i) \geq f_4^+(y_i)$ (from (2) and (8)),

(17) $f_2^*(s_2) \geq f_1(s_2)$ (from (3b) and (9)),

(18) $\Gamma(f_2^*)(\gamma_1) \geq f_3(\gamma_1)$ (from (3a) and (10)),

(19) $f_3(\alpha_i) \geq \Gamma(f_2^*)(\alpha_i)$ (from (4) and (11)),

(20) $f_3(\eta_i) \geq \Gamma(f_4^+)(\eta_i)$ (from (5) and (12)),

(21) $\Gamma(f_4^+)(\gamma_2) \geq f_3(\gamma_2)$ (from (6b) and(13)),

(22) $f_4^+(s_1) \geq f_1(s_1)$ (from (6a) and (14)),

Now, from (15)and (17), we obtain $f_1 = f_2^*$ which implies that $f_2 = f_1^{*'}$. From (16) and (22), we obtain $f_1 = f_4^+$ which implies that $f_4 = f_1^{+'}$. From (18) and (20) we obtain $f_3 = \Gamma(f_1)$, since $f_1 = f_2^* = f_4^+$. Therefore, if $\begin{pmatrix} f_2 & f_3 \\ f_1 & f_4 \end{pmatrix}$ has to be an ideal of S_2, then it is of the form $\begin{pmatrix} f_1^{*'} & \Gamma(f_1) \\ f_1 & f_1^{+'} \end{pmatrix}$. Hence the proof is completed. □

Note that similar result holds also for fuzzy h-ideals. Now we can easily deduce the following two theorems.

Theorem 3.10. *A fuzzy subset μ of S is a fuzzy h-ideal of S if and only if* $\begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix}$ *is a fuzzy h-ideal of S_2.*

Theorem 3.11. *Let S be a Nobusawa Γ-hemiring and R, L be its right and left operator hemiring, respectively. Then there exists an inclusion preserving bijection $\mu \to \begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix}$ between the set of all fuzzy h-ideals of S and the set of all fuzzy h-ideals of $S_2 = \begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$.*

Proof. The bijection $\mu \mapsto \begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix}$ between the set of all fuzzy h-ideals of S and the set of all fuzzy h-ideals of $S_2 = \begin{pmatrix} R & \Gamma \\ S & L \end{pmatrix}$ follows from the immediate consequence of Theorem 3.8 and Theorem 3.9. In order to show that the mapping is inclusion preserving, suppose that μ_1 and μ_2 are any two fuzzy h-ideals of S such that $\mu_1 \subseteq \mu_2$ and $\begin{pmatrix} r & \gamma \\ s & l \end{pmatrix}$ is any element of S_2. Then

$$\begin{aligned} \begin{pmatrix} \mu_1^{*'} & \Gamma(\mu_1) \\ \mu_1 & \mu_1^{+'} \end{pmatrix}\begin{pmatrix} r & \gamma \\ s & l \end{pmatrix} &= \min\{\mu_1(s), \mu_1^{*'}(r), \Gamma(\mu_1)(\gamma), \mu_1^{+'}(l)\} \\ &\leq \min\{\mu_2(s), \mu_2^{*'}(r), \Gamma(\mu_2)(\gamma), \mu_2^{+'}(l)\} \\ &= \begin{pmatrix} \mu_2^{*'} & \Gamma(\mu_2) \\ \mu_2 & \mu_2^{+'} \end{pmatrix}\begin{pmatrix} r & \gamma \\ s & l \end{pmatrix}. \end{aligned}$$

Hence, the mapping is inclusion preserving. □

Proposition 3.12. *Let μ and ν be any fuzzy h-ideals of S. Then*

(1) $\begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} o_h \begin{pmatrix} \nu^{*'} & \Gamma(\nu) \\ \nu & \nu^{+'} \end{pmatrix} = \begin{pmatrix} \mu^{*'} o_h \nu^{*'} & \Gamma(\mu) o_h \Gamma(\nu) \\ \mu o_h \nu & \mu^{+'} o_h \nu^{+'} \end{pmatrix}$

$\left(= \begin{pmatrix} (\mu o_h \nu)^{*'} & \Gamma(\mu o_h \nu) \\ \mu o_h \nu & (\mu o_h \nu)^{+'} \end{pmatrix} \right)$,

(2) $\begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} \oplus \begin{pmatrix} \nu^{*'} & \Gamma(\nu) \\ \nu & \nu^{+'} \end{pmatrix} = \begin{pmatrix} \mu^{*'} \oplus \nu^{*'} & \Gamma(\mu) \oplus \Gamma(\nu) \\ \mu \oplus \nu & \mu^{+'} \oplus \nu^{+'} \end{pmatrix}$

$\left(= \begin{pmatrix} (\mu \oplus \nu)^{*'} & \Gamma(\mu \oplus \nu) \\ \mu \oplus \nu & (\mu \oplus \nu)^{+'} \end{pmatrix} \right)$,

(3) $\begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} \cap \begin{pmatrix} \nu^{*'} & \Gamma(\nu) \\ \nu & \nu^{+'} \end{pmatrix} = \begin{pmatrix} \mu^{*'} \cap \nu^{*'} & \Gamma(\mu) \cap \Gamma(\nu) \\ \mu \cap \nu & \mu^{+'} \cap \nu^{+'} \end{pmatrix}$

$\left(= \begin{pmatrix} (\mu \cap \nu)^{*'} & \Gamma(\mu \cap \nu) \\ \mu \cap \nu & (\mu \cap \nu)^{+'} \end{pmatrix} \right)$.

Proof. (1) Let $\begin{pmatrix} r & \gamma \\ s & l \end{pmatrix}$ be an arbitrary element of S_2. If $\begin{pmatrix} r & \gamma \\ s & l \end{pmatrix}$ can be expressed as

$$\begin{pmatrix} r & \gamma \\ s & l \end{pmatrix} + \sum_i \begin{pmatrix} r_1 & \gamma_1 \\ s_1 & l_1 \end{pmatrix}_i \begin{pmatrix} r_2 & \gamma_2 \\ s_2 & l_2 \end{pmatrix}_i + \begin{pmatrix} r' & \gamma' \\ s' & l' \end{pmatrix}$$
$$= \sum_i \begin{pmatrix} r_3 & \gamma_3 \\ s_3 & l_3 \end{pmatrix}_i \begin{pmatrix} r_4 & \gamma_4 \\ s_4 & l_4 \end{pmatrix}_i + \begin{pmatrix} r' & \gamma' \\ s' & l' \end{pmatrix}$$

then every element r,γ,s,l have the representations as following:

$$r + \sum_i r_{1_i} r_{2_i} + r' = \sum_i r_{3_i} r_{4_i} + r',$$
$$s + \sum_i s_{1_i} \alpha_i s_{2_i} + s' = \sum_i s_{3_i} \beta_i s_{4_i} + s',$$
$$\gamma + \sum_i \gamma_{1_i} s_{1_i} \gamma_{2_i} + \gamma' = \sum_i \gamma_{1_i} s_{2_i} \gamma_{2_i} + \gamma',$$
$$l + \sum_i l_{1_i} l_{2_i} + l' = \sum_i l_{3_i} l_{4_i} + l'.$$

But r, γ, s, l may have some other representations. By taking sup{min{ }} over all possible representations we obtain

$$\begin{pmatrix} \mu^{*'} & \Gamma(\mu) \\ \mu & \mu^{+'} \end{pmatrix} o_h \begin{pmatrix} \nu^{*'} & \Gamma(\nu) \\ \nu & \nu^{+'} \end{pmatrix} \subseteq \begin{pmatrix} \mu^{*'} o_h \nu^{*'} & \Gamma(\mu) o_h \Gamma(\nu) \\ \mu o_h \nu & \mu^{+'} o_h \nu^{+'}. \end{pmatrix}$$

Conversely, if every element r,γ,s,l have the representations as above, then we can also find a representation of $\begin{pmatrix} r & \gamma \\ s & l \end{pmatrix}$. For example, let r, γ, s, l have the following representations, respectively,

$$r + \sum_i (\sum_j [\delta_j, z_j] \sum_k [\alpha_k, x_k]) + r' = \sum_i (\sum_m [\eta_m, y_m] \sum_n [\zeta_n, w_n]) + r',$$
$$s + \sum_k a \alpha_k x_k + s' = \sum_n b \zeta_n w_n + s',$$
$$\gamma + \sum_k \gamma_1 q_k \alpha_k + \gamma' = \sum_n \gamma_2 s_n \eta_n + \gamma',$$
$$l + \sum_i (\sum_j [p_j, \beta_j] \sum_k [q_k, \alpha_k]) + l' = \sum_i (\sum_m [r_m, \delta_m] \sum_n [s_n, \eta_n]) + l'.$$

Then $\begin{pmatrix} r & \gamma \\ s & l \end{pmatrix}$ has the expression as the following:

$$\begin{pmatrix} r & \gamma \\ s & l \end{pmatrix} + \sum_i \begin{pmatrix} \sum_j [\delta_j, z_j] & \gamma_1 \\ a & \sum_j [p_j, \beta_j] \end{pmatrix}_i \begin{pmatrix} \sum_k [\alpha_k, x_k] & 0 \\ 0 & \sum_k [q_k, \alpha_k] \end{pmatrix}_i$$
$$+ \begin{pmatrix} r' & \gamma' \\ s' & l' \end{pmatrix}$$
$$= \sum_i \begin{pmatrix} \sum_m [\eta_m, y_m] & \gamma_2 \\ b & \sum_m [r_m, \delta_m] \end{pmatrix}_i \begin{pmatrix} \sum_n [\zeta_n, w_n] & 0 \\ 0 & \sum_n [s_n, \eta_n] \end{pmatrix}_i + \begin{pmatrix} r' & \gamma' \\ s' & l' \end{pmatrix}.$$

But $\begin{pmatrix} r^{'} & \gamma^{'} \\ s^{'} & l^{'} \end{pmatrix}$ may have the other representations. So arguing in a similar way as above we deduce that

$$\begin{pmatrix} \mu^{*^{'}} & \Gamma(\mu) \\ \mu & \mu^{+^{'}} \end{pmatrix} o_h \begin{pmatrix} \nu^{*^{'}} & \Gamma(\nu) \\ \nu & \nu^{+^{'}} \end{pmatrix} \supseteq \begin{pmatrix} \mu^{*^{'}} o_h \nu^{*^{'}} & \Gamma(\mu) o_h \Gamma(\nu) \\ \mu o_h \nu & \mu^{+^{'}} o_h \nu^{+^{'}} \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} \mu^{*^{'}} & \Gamma(\mu) \\ \mu & \mu^{+^{'}} \end{pmatrix} o_h \begin{pmatrix} \nu^{*^{'}} & \Gamma(\nu) \\ \nu & \nu^{+^{'}} \end{pmatrix} = \begin{pmatrix} \mu^{*^{'}} o_h \nu^{*^{'}} & \Gamma(\mu) o_h \Gamma(\nu) \\ \mu o_h \nu & \mu^{+^{'}} o_h \nu^{+^{'}} \end{pmatrix}.$$

The rest of the proof follows by routine verification. Similarly, we can prove (2) and (3). □

Definition 3.13. [12] *A Γ-hemiring S is said to be h-hemiregular if for each $a \in S$, there exist $z, x_1, x_2 \in S$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$ such that $a + a\alpha_1 x_1 \beta_1 a + z = a\alpha_2 x_2 \beta_2 a + z$.*

In order to conclude this paper we obtain the following characterization of h-hemiregular matrix hemiring.

Theorem 3.14. *If S is an h-hemiregular, then S_2 is also an h-hemiregular.*

Proof. Suppose that $\begin{pmatrix} \mu^{*^{'}} & \Gamma(\mu) \\ \mu & \mu^{+^{'}} \end{pmatrix}$ and $\begin{pmatrix} \nu^{*^{'}} & \Gamma(\nu) \\ \nu & \nu^{+^{'}} \end{pmatrix}$ are two fuzzy h-ideals of S_2. Then

$$\begin{pmatrix} \mu^{*^{'}} & \Gamma(\mu) \\ \mu & \mu^{+^{'}} \end{pmatrix} \cap \begin{pmatrix} \nu^{*^{'}} & \Gamma(\nu) \\ \nu & \nu^{+^{'}} \end{pmatrix} = \begin{pmatrix} (\mu \cap \nu)^{*^{'}} & \Gamma(\mu \cap \nu) \\ \mu \cap \nu & (\mu \cap \nu)^{+^{'}} \end{pmatrix} = \begin{pmatrix} (\mu o_h \nu)^{*^{'}} & \Gamma(\mu o_h \nu) \\ \mu o_h \nu & (\mu o_h \nu)^{+^{'}} \end{pmatrix},$$

[since S is h-hemiregular, by Theorem 4.4 of [18], for any two fuzzy h-ideals μ and ν of S, we have $\mu \cap \nu = \mu o_h \nu$]. Now, by Proposition 3.12, we have

$$\begin{pmatrix} (\mu o_h \nu)^{*^{'}} & \Gamma(\mu o_h \nu) \\ \mu o_h \nu & (\mu o_h \nu)^{+^{'}} \end{pmatrix} = \begin{pmatrix} \mu^{*^{'}} & \Gamma(\mu) \\ \mu & \mu^{+^{'}} \end{pmatrix} o_h \begin{pmatrix} \nu^{*^{'}} & \Gamma(\nu) \\ \nu & \nu^{+^{'}} \end{pmatrix}.$$

Hence, S_2 is an h-hemiregular. □

4. **Conclusion**

As a continuation of this paper we will study various correspondence such as prime (semiprime) fuzzy h-ideal, fuzzy h-bi-ideal (h-quasi-ideal), h-hemiregularity, h-intra-hemiregularity etc between Γ-hemiring S and its matrix hemiring S_2 in details.

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