

## FUZZY PROJECTIVE MODULES AND TENSOR PRODUCTS IN FUZZY MODULE CATEGORIES

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ABSTRACT. Let  $R$  be a commutative ring. We write  $\text{Hom}(\mu_A, \nu_B)$  for the set of all fuzzy  $R$ -morphisms from  $\mu_A$  to  $\nu_B$ , where  $\mu_A$  and  $\nu_B$  are two fuzzy  $R$ -modules. We make  $\text{Hom}(\mu_A, \nu_B)$  into fuzzy  $R$ -module by redefining a function  $\alpha : \text{Hom}(\mu_A, \nu_B) \rightarrow [0, 1]$ . We study the properties of the functor  $\text{Hom}(\mu_A, -) : FR\text{-Mod} \rightarrow FR\text{-Mod}$  and get some unexpected results. In addition, we prove that  $\text{Hom}(\xi_P, -)$  is exact if and only if  $\xi_P$  is a fuzzy projective  $R$ -module, when  $R$  is a commutative semiperfect ring. Finally, we investigate tensor product of two fuzzy  $R$ -modules and get some related properties. Also, we study the relationships between Hom functor and tensor functor.

### 1. Introduction

Zadeh [15] introduced the concept of a fuzzy subset  $\mu$  of a nonempty set  $X$ . In 1971, Rosenfeld [12] defined the fuzzy subgroups of a group. Negoita and Ralescu [9] introduced fuzzy module. Pan [10,11] made the  $\text{Hom}(\mu_A, \nu_B)$  into a fuzzy module and investigated the properties of the functors  $\text{Hom}(\mu_A, -) : FR\text{-Mod} \rightarrow FR\text{-Mod}$  and  $\text{Hom}(-, \nu_B) : FR\text{-Mod} \rightarrow FR\text{-Mod}$ . López-Permouth [6,7] gave the definition of the tensor products and studied Mortia theory in fuzzy module categories. Isaac [5] gave an alternate definition for projective  $L$ -modules and investigated these fuzzy modules. Chen [3] studied the relation between projective S-acts and Hom functors in the category of S-acts and Liu [8] studied the Hom functors and tensor product functors in the category of fuzzy  $S$ -acts. In this paper, we study the properties of Hom functor and tensor functor in fuzzy module categories. We also obtain that a fuzzy  $R$ -module  $\xi_P$  is projective if and only if  $\text{Hom}(\xi_P, -) : FR\text{-Mod} \rightarrow FR\text{-Mod}$  is exact.

Hom functors and tensor functors play an important role in ring theory. If  $R$  is a ring and  ${}_R M_R$  is a bimodule, then  $\text{Hom}_R(M, -) : R\text{-Mod} \rightarrow R\text{-Mod}$  is left exact and  $M \otimes_R - : R\text{-Mod} \rightarrow R\text{-Mod}$  is right exact; In addition,  $(M \otimes_R -, \text{Hom}_R(M, -))$  is an adjoint pair ([13]). Also, Hom functors and tensor functors play an important role in equivalences of module categories. The concept of projective module over a ring  $R$  is a more flexible generalization of the idea of a free module. Various equivalent characterizations of these modules are available. Perhaps the most insightful and certainly the most abstract characterization of a projective  $R$ -module  $M$  is that

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the functor  $\text{Hom}_R(M, -) : R\text{-Mod} \rightarrow \text{Ab}$  is (right) exact, where  $\text{Ab}$  is the category of abelian groups ([13]).

Our aim is to develop the fuzzy technology by studying the category of fuzzy  $R$ -modules. It is natural to study the two functors and fuzzy projective modules in  $FR\text{-Mod}$ . This will help us to study equivalences of fuzzy module categories. In this paper, we investigate properties of the two functors in  $FR\text{-Mod}$  and the relationship of fuzzy projective module and Hom functor. Indeed, we get some similar results in  $FR\text{-Mod}$ . We also get some unexpected results, e.g., the functor  $\text{Hom}(\mu_A, -) : FR\text{-Mod} \rightarrow FR\text{-Mod}$  being not left exact in  $FR\text{-Mod}$ . In section 2, we give some definitions and notations. In section 3, we study Hom functor and fuzzy projective modules. Pan [10] defined fuzzy Hom set. But, when we try to study the relationship of Hom functor and fuzzy projective module, we find it hard to make  $\Gamma_{\mu_A}$  a fuzzy morphism. We resolve this problem successfully by redefining a function from  $\text{Hom}(\mu_A, \nu_B)$  to  $[0, 1]$ , which can also make  $\text{Hom}(\mu_A, \nu_B)$  into a fuzzy  $R$ -module. Also, we get some related properties of Hom functors and prove that  $\text{Hom}(\xi_P, -)$  is exact if and only if  $\xi_P$  is a fuzzy projective  $R$ -module. In section 4, we recall the definition of tensor product in the category of fuzzy  $R$ -modules and prove that the tensor product of two fuzzy  $R$ -modules exists. We also get some properties of tensor product. In addition, we study the adjointness of Hom functor and tensor functor.

## 2. Preliminaries

Throughout this paper, let  $R$  be a ring with 1. We shall denote a left  $R$ -module  $M$  by  ${}_R M$  and the set of left  $R$ -homomorphisms from  ${}_R M$  to  ${}_R N$  by  $\text{Hom}_R(M, N)$ . We denote the category of left  $R$ -modules by  $R\text{-Mod}$  and the category of right  $R$ -modules will be denoted by  $\text{Mod-}R$ .

Let  $A$  be a left  $R$ -module. A function  $\mu_A$  is called fuzzy (left)  $R$ -module, if the map  $\mu$  from the  $R$ -module  $A$  to the interval  $[0, 1]$  satisfies

- 1) for all  $x, y \in A$ ,  $\mu(x + y) \geq \bigwedge\{\mu(x), \mu(y)\}$ ,
- 2) for all  $a \in A, r \in R$ ,  $\mu(ra) \geq \mu(a)$ ,
- 3)  $\mu(0) = 1$ .

Similarly, we can define the fuzzy right  $R$ -modules. For two fuzzy  $R$ -modules  $\mu_A$  and  $\nu_B$ , a function  $\tilde{f} : \mu_A \rightarrow \nu_B$  is called fuzzy  $R$ -homomorphism, if  $f$  is an  $R$ -homomorphism and  $\nu(f(a)) \geq \mu(a)$  ( $\forall a \in A$ ).

For simplicity, denote by  $\text{Hom}(\mu_A, \nu_B)$  the set of fuzzy  $R$ -homomorphisms from  $\mu_A$  to  $\nu_B$ .

Let  $R$  be a ring. We shall denote by  $FR\text{-Mod}$  ( $\text{Mod-}FR$ ) the category of fuzzy left (right)  $R$ -modules.

A fuzzy  $R$ -homomorphism  $\tilde{f} \in \text{Hom}(\mu_A, \nu_B)$  is called fuzzy split, if there is a fuzzy  $R$ -homomorphism  $\tilde{t} \in \text{Hom}(\nu_B, \mu_A)$  such that the composition  $\tilde{t}\tilde{f} = id$ .

A fuzzy  $R$ -homomorphism  $\tilde{f} \in \text{Hom}(\mu_A, \nu_B)$  is called a fuzzy quasi-isomorphism if  $f$  is an isomorphism ([1]).

A fuzzy  $R$ -homomorphism  $\tilde{f} \in \text{Hom}(\mu_A, \nu_B)$  is called a fuzzy isomorphism, if  $f$  is an isomorphism and  $\nu\tilde{f}(a) = \mu(a)$  ( $\forall a \in A$ ).

If  $M \in R\text{-Mod}$ ,  $0_M$  represents the fuzzy  $R$ -modules  $0 : M \longrightarrow [0, 1]$  satisfying

$$0(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

While,  $1_M$  represents the fuzzy  $R$ -modules  $1 : M \longrightarrow [0, 1]$  satisfying  $1(x) = 1$ , for all  $x \in M$ .

If  $\tilde{f} : \mu_A \rightarrow \nu_B$  is a fuzzy  $R$ -homomorphism, we define that  $\text{Ker}\tilde{f} = \{a \in A \mid \nu(\tilde{f}(a)) = 1\}$  and that  $\text{Im}\tilde{f} = \{\tilde{f}(a) \mid a \in A\}$ . If  $f : A \rightarrow B$  is an  $R$ -homomorphism and  $\text{Ker}f$  is the preimage of  $\{0\}$  under  $f$ , we have  $\text{Ker}f \subseteq \text{Ker}\tilde{f}$ . Especially, if  $\nu_B = 1_B$ , then we have  $\text{Ker}\tilde{f} = A$ , for all  $\tilde{f} \in \text{Hom}_R(\mu_A, \nu_B)$ .

**Proposition 2.1.** *Let  $R$  be a ring. If  $\tilde{f} \in \text{Hom}_R(\mu_A, \nu_B)$ , where  $\mu_A$  and  $\nu_B$  are two fuzzy  $R$ -modules, then we have the following:*

- 1)  $\text{Ker}\tilde{f}$  is a submodule of  $A$ ;
- 2) Define  $\mu' : \text{Ker}\tilde{f} \rightarrow [0, 1]$  by  $\mu'(k) = \mu(k)$ , for  $k \in \text{Ker}\tilde{f}$ . Then  $\mu'_{\text{Ker}\tilde{f}}$  is a fuzzy submodule of  $\mu_A$ .

*Proof.* 1) Let  $s$  be the zero element of  $A$ . Obviously, we have  $s \in \text{Ker}\tilde{f}$ . Given  $a \in \text{Ker}\tilde{f}$  and  $r \in R$ , then

$$\nu(\tilde{f}(ra)) = \nu(r\tilde{f}(a)) \geq \nu(\tilde{f}(a)) = 1.$$

Hence, we get  $ra \in \text{Ker}\tilde{f}$ . Particularly, we have  $-a \in \text{Ker}\tilde{f}$ . If  $a, b \in \text{Ker}\tilde{f}$ , we can easily get  $a + b \in \text{Ker}\tilde{f}$ . This proves that  $\text{Ker}\tilde{f}$  is a submodule of  $A$ .

- 2) The proof is obvious. □

In  $R\text{-Mod}$ , the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a short exact sequence if  $f$  is a monomorphism,  $g$  is an epimorphism and  $\text{Im}f = \text{Ker}g$  ([13]). We know that  $\text{Ker}f = \{0\}$ .

We denote by  $\bar{s}$  the fuzzy zero module.

**Definition 2.2.** Let  $\mu_A, \nu_B$  and  $\eta_C$  be fuzzy  $R$ -modules. A short exact sequence is a sequence of the form

$$\bar{s} \longrightarrow \mu_A \xrightarrow{\tilde{f}} \nu_B \xrightarrow{\tilde{g}} \eta_C \longrightarrow \bar{s}$$

where  $\tilde{f}$  is a monomorphism,  $\tilde{g}$  is an epimorphism and  $\text{Im}\tilde{f} = \text{Ker}\tilde{g}$ .

Note that  $\text{Ker}\tilde{f}$  is usually larger than  $\{0\}$  in Definition 2.2. Hence, the crisp case of the definition is different from the well-known notion of short exact sequence in [13].

If  $\eta_C = 1_C$ , we get that  $\text{Im}\tilde{f} = \text{Ker}\tilde{g} = B$ . As  $\tilde{f}$  is monic, we can get that  $\tilde{f}$  is a quasi-isomorphism.

Let  $\{\mu_{iA_i} \mid i \in I\}$  be a family of fuzzy  $R$ -modules. It is well-known that  $\coprod_{i \in I} \mu_{iA_i}$  is the coproduct of  $\{\mu_{iA_i} \mid i \in I\}$ . The map  $\mu : \coprod_{i \in I} A_i \longrightarrow [0, 1]$  is defined by putting  $\mu((a_i)_{i \in I}) = \bigwedge \{\mu_i(a_i) \mid i \in I\}$ .

Similarly, if  $\{\mu_{iA_i} | i \in I\}$  is a family of fuzzy  $R$ -modules, the product  $\prod_{i \in I} \mu_{iA_i}$  is defined by  $\mu((a_i)_{i \in I}) = \bigwedge \{\mu_i(a_i) | i \in I\}$ .

A fuzzy  $R$ -module  $\delta_P$  is called projective if and only if for every surjective fuzzy  $R$ -homomorphism  $\tilde{f} : \mu_A \rightarrow \nu_B$  and for every fuzzy  $R$ -homomorphism  $\tilde{g} : \delta_P \rightarrow \nu_B$ , there exists a fuzzy  $R$ -homomorphism  $\tilde{h} : \delta_P \rightarrow \mu_A$  such that  $\tilde{f}\tilde{h} = \tilde{g}$ . If  $\delta_P$  is projective, then  $\delta_P = 0_P$ .

Denote by  $J(R)$  the radical of a ring  $R$ . A ring  $R$  is called semiperfect if  $R/J(R)$  is semisimple and idempotents lift modulo  $J(R)$  ([2]). Let  $R$  be a semiperfect ring and let  $P$  be a projective  $R$ -module, then  $P \cong \prod_{i \in I} Re_i$ , where  $e_i \in E(R)$  and  $E(R)$  is the set of idempotent elements of  $R$  ([2], Theorem 27.11). Hence we can obtain the following statement.

**Proposition 2.3.** *Let  $R$  be a semiperfect ring. If  $0_P$  is a fuzzy projective  $R$ -module, then  $0_P \cong \prod_{i \in I} 0_{Re_i}$ , where  $e_i \in E(R)$ .*

### 3. Hom Functors and Fuzzy Projective Modules

In the following, we will define a function from  $\text{Hom}(\mu_A, \nu_B)$  to  $[0, 1]$  and make  $\text{Hom}(\mu_A, \nu_B)$  into a fuzzy  $R$ -module.

**Theorem 3.1.** *Let  $R$  be a commutative ring and let  $\mu_A$  and  $\nu_B$  be two fuzzy  $R$ -modules. Then  $\text{Hom}(\mu_A, \nu_B)$  is a fuzzy  $R$ -module by the function  $\alpha : \text{Hom}(\mu_A, \nu_B) \rightarrow [0, 1]$  defined by*

$$\alpha(\tilde{f}) = \bigwedge \{\nu(\tilde{f}(a)) | a \in A\}.$$

*Proof.* Assume  $r \in R$  and  $\tilde{f} \in \text{Hom}(\mu_A, \nu_B)$ . Define a function  $r \cdot \tilde{f} : \mu_A \rightarrow \nu_B$  by

$$r \cdot \tilde{f}(a) = r\tilde{f}(a).$$

Then we have

$$\nu(r \cdot \tilde{f}(a)) = \nu(r\tilde{f}(a)) \geq \nu(\tilde{f}(a)) \geq \mu(a).$$

This concludes that  $r \cdot \tilde{f} \in \text{Hom}(\mu_A, \nu_B)$ . Hence we show that  $\text{Hom}(\mu_A, \nu_B)$  is an  $R$ -module.

We now have to prove that  $\text{Hom}(\mu_A, \nu_B)$  is a fuzzy  $R$ -module. Suppose  $r \in R$ ,  $\tilde{f} \in \text{Hom}(\mu_A, \nu_B)$  and  $a \in A$ . By

$$\nu(r \cdot \tilde{f}(a)) = \nu(r\tilde{f}(a)) \geq \nu(\tilde{f}(a)),$$

we have,  $\bigwedge \{\nu(r \cdot \tilde{f}(a)) | a \in A\} \geq \bigwedge \{\nu(\tilde{f}(a)) | a \in A\}$ . This concludes that  $\alpha(r \cdot \tilde{f}) \geq \alpha(\tilde{f})$ .

If  $\tilde{f}, \tilde{g} \in \text{Hom}(\mu_A, \nu_B)$ , then

$$\begin{aligned} \alpha(\tilde{f} + \tilde{g}) &= \bigwedge \{\nu(\tilde{f}(a) + \tilde{g}(a)) | a \in A\} \\ &\geq \bigwedge \{\bigwedge \{\nu(\tilde{f}(a)), \nu(\tilde{g}(a))\} | a \in A\} \\ &\geq \bigwedge \{\bigwedge \{\nu(\tilde{f}(a)) | a \in A\}, \bigwedge \{\nu(\tilde{g}(a)) | a \in A\}\} \\ &= \bigwedge \{\alpha(\tilde{f}), \alpha(\tilde{g})\}. \end{aligned}$$

We obviously have that  $\alpha(0) = 1$ . Therefore we have shown that  $\text{Hom}(\mu_A, \nu_B)$  is a fuzzy  $R$ -module.  $\square$

For  $M \in R\text{-Mod}$ , the functor  $\text{Hom}(M, -)$  is left exact in the  $R$ -module category. The situation in  $FR\text{-Mod}$  is, however, different. The functor  $\text{Hom}(\xi_M, -)$  is not left exact.

**Example 3.2.** Let  $Z$  be the integer ring and  $A = (6)$ . Define  $\nu : Z \rightarrow [0, 1]$  by

$$\nu(n) = \begin{cases} \frac{1}{10}, & \text{if } n \neq 0, \\ 1, & \text{if } n = 0. \end{cases}$$

Define  $\eta : Z_6 \rightarrow [0, 1]$  by

$$\eta(\bar{k}) = \begin{cases} \frac{1}{2}, & \text{if } \bar{k} \neq \bar{0}, \\ 1, & \text{if } \bar{k} = \bar{0}. \end{cases}$$

Then both  $\nu_A$  and  $\eta_{Z_6}$  are fuzzy  $Z$ -modules. Also, we have a short exact sequence

$$\bar{s} \longrightarrow 0_A \xrightarrow{\tilde{f}} \nu_Z \xrightarrow{\tilde{g}} \eta_{Z_6} \longrightarrow \bar{s},$$

where  $\tilde{f}$  is the inclusion homomorphism and  $\tilde{g}$  is the epimorphism. We claim that the sequence

$$\bar{s} \longrightarrow \alpha_{\text{Hom}(\nu_Z, 0_A)} \xrightarrow{F\tilde{f}} \beta_{\text{Hom}(\nu_Z, \nu_Z)} \xrightarrow{F\tilde{g}} \gamma_{\text{Hom}(\nu_Z, \eta_{Z_6})}$$

is not exact, where  $F = \text{Hom}(\nu_Z, -)$ . Define  $\tilde{h}_1 : \nu_Z \rightarrow \nu_Z$  by putting  $\tilde{h}_1(n) = 6n$  and define  $\tilde{h}_2 : \nu_Z \rightarrow \nu_Z$  by putting  $\tilde{h}_2(n) = 12n$ . We can check that both  $\tilde{h}_1$  and  $\tilde{h}_2$  are in  $\text{Ker}F\tilde{g}$ . Hence  $|\text{Ker}F\tilde{g}| \geq 2$ . Since  $\text{Hom}(\nu_Z, 0_A)$  contains only zero morphism, we have  $\text{Im}F\tilde{f} \neq \text{Ker}F\tilde{g}$ . Then  $\text{Hom}(\nu_Z, -)$  is not exact.

We can get the following statement which is similar to Pan's results [10].

**Theorem 3.3.** Let  $R$  be a commutative ring and let

$$\bar{s} \longrightarrow \mu_A \xrightarrow{\tilde{f}} \nu_B \xrightarrow{\tilde{g}} \eta_C$$

be an exact sequence, where  $\tilde{f}$  is fuzzy split. Then  $\text{Hom}(\xi_M, -)$  preserves the sequence.

*Proof.* The proof is similar to that of Theorem 2.38 in [13] or Theorem 2 in [10].

Let  $F = \text{Hom}(\xi_M, -)$ . We will show that the sequence

$$\bar{s} \longrightarrow \alpha_{\text{Hom}(\xi_M, \mu_A)} \xrightarrow{F\tilde{f}} \beta_{\text{Hom}(\xi_M, \nu_B)} \xrightarrow{F\tilde{g}} \gamma_{\text{Hom}(\xi_M, \eta_C)}$$

is exact.

1) By Theorem 2.38 [13],  $\text{Hom}(M, -)$  is left exact. So, it is clear that  $F\tilde{f}$  is monic.

2)  $\text{Im}F\tilde{f} \subseteq \text{Ker}F\tilde{g}$ . Suppose  $\tilde{h} \in \text{Im}F\tilde{f}$ , then there exists  $\tilde{k} \in \text{Hom}(\xi_M, \mu_A)$  such that  $\tilde{h} = F\tilde{f}(\tilde{k}) = \tilde{f}\tilde{k}$ . Since  $\text{Im}\tilde{f} = \text{Ker}\tilde{g}$ , we have  $\eta(\tilde{g}\tilde{f}\tilde{k}(m)) = 1$  for all  $m \in M$ . Hence,  $\gamma(F\tilde{g}(\tilde{h})) = 1$  and so  $\tilde{h} \in \text{Ker}F\tilde{g}$ .

3)  $\text{Ker}F\tilde{g} \subseteq \text{Im}F\tilde{f}$ . Suppose  $\tilde{h} \in \text{Ker}F\tilde{g}$ , so that  $\gamma(F\tilde{g}(\tilde{h})) = 1$ . If  $m \in M$ , by Theorem 3.1, we have  $\eta(\tilde{g}\tilde{h}(m)) = 1$  and  $\tilde{h}(m) \in \text{Ker}\tilde{g} = \text{Im}\tilde{f}$ . As  $\tilde{f}$  is monic, there is a unique  $a_m \in A$  with  $\tilde{f}(a_m) = \tilde{h}(m)$ . Define  $\tilde{k} : \xi_M \rightarrow \mu_A$  by  $\tilde{k}(m) = a_m$ . Then  $F\tilde{f}(\tilde{k}) = \tilde{f}\tilde{k} = \tilde{h}$ . We have to show that  $\tilde{k}$  is fuzzy. Since  $\tilde{f}$  is fuzzy split, there exists a fuzzy morphism  $\tilde{t} : \nu_B \rightarrow \mu_A$  such that  $\tilde{t}\tilde{f} = id$ . For all  $m \in M$ , we have

$$\mu(\tilde{k}(m)) = \mu(a_m) = \mu(\tilde{t}\tilde{f}(a_m)) \geq \nu(\tilde{f}(a_m)) = \nu(\tilde{h}(m)) \geq \xi(m).$$

Hence  $\tilde{k}$  is fuzzy.  $\square$

**Theorem 3.4.** *Let  $R$  be a commutative ring and let*

$$\mu_A \xrightarrow{\tilde{f}} \nu_B \xrightarrow{\tilde{g}} \eta_C \longrightarrow \bar{s}$$

*be an exact sequence, where  $\tilde{f}$  is fuzzy split. Let  $G = \text{Hom}(-, \xi_M)$ , then we have the following exact sequence*

$$\bar{s} \longrightarrow \gamma_{\text{Hom}(\eta_C, \xi_M)} \xrightarrow{G\tilde{g}} \beta_{\text{Hom}(\nu_B, \xi_M)} \xrightarrow{G\tilde{f}} \alpha_{\text{Hom}(\mu_A, \xi_M)}.$$

*Proof.* The proof is similar to that of Theorem 3.3.  $\square$

Now, we study the functor  $\text{Hom}(\mu_M, -)$ , where  $\mu_M = 0_{Re}$  and  $e \in E(R)$ .

Define a map

$$\Gamma_{\mu_A} : \begin{array}{ccc} \alpha_{\text{Hom}(0_{Re}, \mu_A)} & \longrightarrow & \mu_{eA} \\ \tilde{g} & \longmapsto & \tilde{g}(e). \end{array}$$

**Lemma 3.5.** *Let  $R$  be a commutative ring and let  $\mu_A \in FR\text{-Mod}$ . Then  $\Gamma_{\mu_A}$  is a fuzzy  $R$ -module isomorphism.*

*Proof.* Assume  $ea \in eA$ . we define a map  $\tilde{f} : 0_{Re} \longrightarrow \mu_{eA}$  by putting  $\tilde{f}(re) = rea$ . We can easily check that  $\tilde{f} \in \alpha_{\text{Hom}(0_{Re}, \mu_A)}$  and  $\Gamma_{\mu_A}(\tilde{f}) = ea$ . It follows that  $\Gamma_{\mu_A}$  is a surjective map. If  $\tilde{f} \in \alpha_{\text{Hom}(0_{Re}, \mu_A)}$ , we can see that  $\tilde{f}$  is determined by  $\tilde{f}(e)$ . This shows that  $\Gamma_{\mu_A}$  is an injective map.

Let  $\tilde{f} \in \text{Hom}(0_{Re}, \mu_A)$ . We have  $\alpha(\tilde{f}) = \mu(\tilde{f}(e))$  and  $\mu\Gamma_{\mu_A} = \alpha$ . Thus, we prove that  $\Gamma_{\mu_A}$  is a fuzzy isomorphism.  $\square$

**Proposition 3.6.** *Let  $R$  be a ring and the following diagram of fuzzy  $R$ -modules is commutative:*

$$\begin{array}{ccccccc} \bar{s} & \rightarrow & \mu_A & \xrightarrow{\tilde{f}} & \nu_B & \xrightarrow{\tilde{g}} & \eta_C \rightarrow \bar{s} \\ & & \tilde{\alpha} \downarrow & & \downarrow \tilde{\beta} & & \downarrow \tilde{\gamma} \\ \bar{s} & \rightarrow & \chi_L & \xrightarrow{\tilde{h}} & \xi_M & \xrightarrow{\tilde{p}} & \delta_N \rightarrow \bar{s}, \end{array}$$

*where  $\tilde{\alpha}, \tilde{\gamma}$  are fuzzy isomorphisms and  $\tilde{\beta}$  is a fuzzy quasi-isomorphism. The bottom row is a short exact sequence if and only if so is the top row.*

*Proof.*  $\Rightarrow$ : If the bottom row is a short exact sequence, we can easily have that  $\tilde{g}$  is an epimorphism and  $\tilde{f}$  is a monomorphism.

Given  $a \in A$ , by the commutativity, we have

$$\tilde{\gamma}\tilde{g}\tilde{f}(a) = \tilde{p}\tilde{h}\tilde{\alpha}(a).$$

As  $\delta\tilde{p}\tilde{h}\tilde{\alpha}(a) = 1$  and  $\tilde{\gamma}$  is a fuzzy isomorphism, then  $\eta\tilde{g}\tilde{f}(a) = 1$ . This shows that  $\text{Im}(\tilde{f}) \subseteq \text{Ker}(\tilde{g})$ .

Suppose  $b \in \text{Ker}(\tilde{g})$ . By the commutativity, we can get  $\delta\tilde{p}\tilde{\beta}(b) = 1$ . Since  $\text{Im}(\tilde{h}) = \text{Ker}(\tilde{p})$ , there exist  $l \in L$  such that  $\tilde{h}(l) = \tilde{\beta}(b)$ . Note that  $\tilde{\alpha}$  is a fuzzy isomorphism. Then we have  $a \in A$  satisfying  $\tilde{\alpha}(a) = l$  and so  $\tilde{\beta}\tilde{f}(a) = \tilde{h}\tilde{\alpha}(a) = \tilde{\beta}(b)$ . Hence, we get  $\tilde{f}(a) = b$ . This shows that  $\text{Ker}(\tilde{g}) \subseteq \text{Im}(\tilde{f})$  and then the top row is a short exact sequence.

$\Leftarrow$ : The proof is similar to that of " $\Rightarrow$ ".  $\square$

**Lemma 3.7.** *Let  $R$  be a commutative ring and let*

$$\bar{s} \rightarrow \mu_A \xrightarrow{\tilde{f}} \nu_B \xrightarrow{\tilde{g}} \eta_C \rightarrow \bar{s}$$

*be a short exact sequence of fuzzy  $R$ -modules. Let  $e \in E(R)$ ,  $\tilde{f}_e = \tilde{f}|_{eA}$  and  $\tilde{g}_e = \tilde{g}|_{eB}$ . The following sequence*

$$\bar{s} \rightarrow \mu_{eA} \xrightarrow{\tilde{f}_e} \nu_{eB} \xrightarrow{\tilde{g}_e} \eta_{eC} \rightarrow \bar{s}$$

*is a short exact sequence.*

*Proof.* Suppose  $ec \in eC$ . Since  $\tilde{g}$  is an epimorphism, there exists  $b \in B$  satisfying  $\tilde{g}(b) = ec$ . Since  $e$  is an idempotent, we have  $eb \in eB$  and  $\tilde{g}_e(eb) = e\tilde{g}(b) = ec$ . This proves that  $\tilde{g}_e$  is an epimorphism. We can obviously see that  $\tilde{f}_e$  is a monomorphism.

We now show that  $\text{Im}(\tilde{f}_e) = \text{Ker}(\tilde{g}_e)$ . It is obvious that  $\text{Im}(\tilde{f}_e) \subseteq \text{Ker}(\tilde{g}_e)$ . Suppose  $eb \in \text{Ker}(\tilde{g}_e)$ . We have an element  $a \in A$  satisfying  $\tilde{f}(a) = eb$ . Hence  $ea \in eA$  and  $\tilde{f}_e(ea) = \tilde{f}(ea) = e\tilde{f}(a) = eb$ , that is,  $\text{Ker}(\tilde{g}_e) \subseteq \text{Im}(\tilde{f}_e)$ . Thus we get the desired result.  $\square$

**Lemma 3.8.** *Let  $R$  be a commutative ring and let  $e \in E(R)$ . The functor  $\text{Hom}(0_{Re}, -)$  preserves the sequence*

$$\bar{s} \longrightarrow \mu_A \xrightarrow{\tilde{f}} \nu_B \xrightarrow{\tilde{g}} \eta_C \longrightarrow \bar{s} \quad (1)$$

*of fuzzy  $R$ -modules.*

*Proof.* The sequence

$$\bar{s} \longrightarrow \mu_{eA} \xrightarrow{\tilde{f}_e} \nu_{eB} \xrightarrow{\tilde{g}_e} \eta_{eC} \longrightarrow \bar{s}$$

is also a short exact sequence by Lemma 3.7. Consider the following commutative diagram of fuzzy  $R$ -modules:

$$\begin{array}{ccccccc} \bar{s} & \rightarrow & \alpha \text{Hom}(0_{Re}, \mu_A) & \rightarrow & \beta \text{Hom}(0_{Re}, \nu_B) & \rightarrow & \gamma \text{Hom}(0_{Re}, \eta_C) & \rightarrow & \bar{s} \\ & & \Gamma_{\mu_A} \downarrow & & \downarrow \Gamma_{\nu_B} & & \downarrow \Gamma_{\eta_C} & & \\ \bar{s} & \rightarrow & \mu_{eA} & \rightarrow & \nu_{eB} & \rightarrow & \eta_{eC} & \rightarrow & \bar{s}, \end{array}$$

Note that  $\Gamma_{\mu_A}, \Gamma_{\nu_B}$  and  $\Gamma_{\eta_C}$  are all fuzzy isomorphisms by Lemma 3.5. Using Proposition 3.6, we prove that the top row is a short exact sequence and so get the desired result.  $\square$

**Lemma 3.9.** *Let  $\mu_A \in FR\text{-Mod}$ ,  $e_i \in E(R)$  for any  $i \in I$ . Then we have*

$$\alpha \text{Hom}(\coprod_{i \in I} 0_{Re_i}, \mu_A) \cong \prod_{i \in I} \mu_{e_i A}.$$

*Proof.* Let

$$k : \text{Hom}(\prod_{i \in I} Re_i, A) \rightarrow \prod_{i \in I} \text{Hom}(Re_i, A)$$

given by

$$f \mapsto (f\lambda_i = f_i)_{i \in I}$$

be the isomorphism, where  $\lambda_j$  is the injection  $Re_j \rightarrow \prod_{i \in I} Re_i$ .

By the proof of Lemma 3.5, for all  $i \in I$ , we have  $\alpha_i(f_i) = \mu(f_i(e_i))$ . Note that  $\mu(\sum_{i \in I} f_i(r_i e_i)) = \bigwedge_{i \in I} \{\mu(f_i(r_i e_i))\}$  and  $\mu(f_i(r_i e_i)) = \mu(r_i f_i(e_i)) \geq \mu(f_i(e_i))$ , where  $r_i, e_i \in R, i \in I$ . For every  $f \in \text{Hom}(\prod_{i \in I} R e_i, A)$ , we have

$$\begin{aligned} \alpha(f) &= \bigwedge \{\mu \circ f((r_i e_i)_{i \in I}) | (r_i e_i)_{i \in I} \in \prod_{i \in I} R e_i\} \\ &= \bigwedge \{\mu(\sum_{i \in I} f_i(r_i e_i)) | (r_i e_i)_{i \in I} \in \prod_{i \in I} R e_i\} \\ &= \bigwedge \{\mu(f_i(e_i)) | i \in I\} \\ &\quad (\text{Note that } \{f_i(e_i) | e_i \in \prod_{i \in I} R e_i\} \subseteq \{\sum_{i \in I} f_i(r_i e_i) | r_i e_i \in \prod_{i \in I} R e_i\}) \\ &= \bigwedge \{\alpha_i(f_i) | i \in I\} \\ &= \prod_{i \in I} \alpha_i((f_i)_{i \in I}) \\ &= (\prod_{i \in I} \alpha_i) \circ k(f). \end{aligned}$$

So, we can get

$$\alpha_{\text{Hom}(\prod_{i \in I} 0_{R e_i}, \mu_A)} \cong \prod_{i \in I} \alpha_i \text{Hom}(0_{R e_i}, \mu_A). \quad \square$$

**Theorem 3.10.** *Let  $R$  be a commutative semiperfect ring. Then  $\text{Hom}(\xi_P, -)$  preserves the short exact sequence (1) of fuzzy  $R$ -modules if and only if  $\xi_P$  is a fuzzy projective  $R$ -module.*

*Proof.*  $\implies$ : Let  $\tilde{g} : \nu_B \rightarrow \eta_C$  be a fuzzy epimorphism. We denote that  $K = \text{Ker} \tilde{g}$  and  $\nu' = \nu|_K$ . By Proposition 2.1, we have that  $\nu'_K$  is a fuzzy submodule of  $\nu_B$  and so we obtain the following short exact sequence

$$\bar{s} \rightarrow \nu_K \xrightarrow{\tilde{l}} \nu_B \xrightarrow{\tilde{g}} \eta_C \rightarrow \bar{s},$$

where  $\tilde{l}$  is the inclusion map.

Since  $\text{Hom}(\xi_P, -)$  preserves the sequence,  $\text{Hom}(\xi_P, -)$  preserves the epimorphism  $\varphi$ . Hence,  $\xi_P$  is a fuzzy projective  $R$ -module.

$\impliedby$ : Since  $\xi_P$  is a fuzzy projective  $R$ -module, we have  $\xi_P \cong \prod_{i \in I} 0_{R e_i}$ , where

$e_i \in E(R)$ . Let  $\bar{s} \rightarrow \mu_A \xrightarrow{\tilde{f}} \nu_B \xrightarrow{\tilde{g}} \eta_C \rightarrow \bar{s}$  be a short exact sequence, then the sequence

$$\bar{s} \rightarrow \prod_{i \in I} \mu_{e_i A} \rightarrow \prod_{i \in I} \nu_{e_i B} \rightarrow \prod_{i \in I} \eta_{e_i C} \rightarrow \bar{s}$$

is also a short exact sequence by Lemma 3.7. Using Lemma 3.9, we have the following commutative diagram:

$$\begin{array}{ccccccc} \bar{s} \rightarrow & \alpha_{\text{Hom}(\prod_{i \in I} 0_{R e_i}, \mu_A)} & \rightarrow & \beta_{\text{Hom}(\prod_{i \in I} 0_{R e_i}, \nu_B)} & \rightarrow & \gamma_{\text{Hom}(\prod_{i \in I} 0_{R e_i}, \eta_C)} & \rightarrow \bar{s} \\ & \downarrow & & \downarrow & & \downarrow & \\ \bar{s} \rightarrow & \prod_{i \in I} \mu_{e_i A} & \rightarrow & \prod_{i \in I} \nu_{e_i B} & \rightarrow & \prod_{i \in I} \eta_{e_i C} & \rightarrow \bar{s}. \end{array}$$

Since the bottom row is a short exact sequence, the top row is also a short exact sequence by Proposition 3.6.  $\square$

#### 4. Tensor Products

In this section,  $R$  is not necessarily commutative. In [6], López-Permouth gave the definition of tensor product of two fuzzy modules. If  $A$  is an  $R$ -module,  $s(A)$ ,  $s({}_Z A)$  and  $s({}_R A)$  represent respectively the set of all grade functions on  $A$ , the set of all group grade functions on  $A$ , and the set of all  $R$ -module grade functions on  $A$ . Given two complete lattices  $L_1 \subseteq L_2$ , for  $l_0 \in L_2$ , the closure of  $l_0$  in  $L_1$  is the infimum in  $L_1$  of the set  $\{l \in L_1 | l_0 \leq l\}$ .

We denote by  $\mathcal{C}$  the category of complete lattices with all order-preserving mappings. A homomorphism  $f : X \rightarrow Y$  of  $\mathcal{C}$  preserves infima if and only if there exists a homomorphism  $g : Y \rightarrow X$  in  $\mathcal{C}$  such that  $y \leq f(x) \Leftrightarrow g(y) \leq x$ , for all  $x \in X$  and  $y \in Y$  [14]. If  $f : M \rightarrow N$  is a group homomorphism ( $R$ -homomorphism), let  $s(f) : s(N) \rightarrow s(M)$  be given by  $s(f)(\beta)(m) = \beta(f(m))$ , for  $\beta \in s(N), m \in M$ . Define  $t(A) = s(A)$ . The map  $t(f) : t(M) \rightarrow t(N)$  satisfies that for all  $\alpha \in t(M)$  and  $\beta \in t(N)$ ,  $s(f)(\beta) \geq \alpha$  if and only if  $\beta \geq t(f)(\alpha)$ . As the referee points out,  $s(p)$  and  $t(p)$  is a Galois connection between  $s(N)$  and  $s(M)$  in the sense of Definition 0-3.1 in [4], thereby yielding the closure operator  $s(p) \circ t(p)$  on  $s(M)$  in the sense of Definition 0-3.8 in [4].

Let  $M \in \text{Mod-}R$ ,  $N \in R\text{-Mod}$ . If  $p : M \times N \rightarrow M \otimes N$  is the tensor product of  $M$  and  $N$ , the fuzzy map  $\mu \otimes \nu$  on  $M \otimes N$  is defined to be the closure in  $s({}_Z M \otimes N)$  of  $t(p)(\mu \times \nu) \in s(M \otimes N)$ . Since  $s({}_Z M \otimes N) \subseteq s(M \otimes N)$ ,  $\mu \otimes \nu$  is the smallest group grade function on  ${}_Z M \otimes N$  larger than  $t(p)(\mu \times \nu)$  [7].

Let  $\mu_A$  and  $\nu_B$  be two fuzzy  $R$ -modules and  $F$  be a free abelian group with basis  $A \times B = \{(a, b) | a \in A, b \in B\}$ . We define  $\mu_A \times \nu_B : A \times B \rightarrow [0, 1]$  by  $\mu \times \nu(a, b) = \bigvee \{\mu(a), \nu(b)\}$  and  $\mu \times \nu(\sum (a_i, b_i)) = \bigwedge \{\bigvee \{\mu(a_i), \nu(b_i)\} | i \in I\}$ . Then we have defined a fuzzy membership function on  $F$ . We shall prove the tensor product of two fuzzy  $R$ -modules exists in this situation which is different from [6].

Let  $A \in \text{Mod-}R$ ,  $B \in R\text{-Mod}$  and  $C$  be an abelian group. A map  $f : A \times B \rightarrow C$  is called biadditive provided that for all  $a \in A, r \in R$  and  $b \in B$ ,

- 1)  $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$ ;
- 2)  $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$ ;
- 3)  $f(ar, b) = f(a, rb)$ .

Let  $A \in \text{Mod-}R$ ,  $B \in R\text{-Mod}$  and  $G$  be an abelian group. Let  $\varphi : A \times B \rightarrow G$  be biadditive. A pair  $(G, \varphi)$  is a tensor product of  $A$  and  $B$  if, for every biadditive  $\psi : A \times B \rightarrow H$ , where  $H$  is an abelian group, there is a unique homomorphism  $\theta \in \text{Hom}(G, H)$  satisfying  $\theta\varphi = \psi$  [13]. Define  $A \otimes B = F/S$ , where  $S$  is the subgroup generated by elements of forms  $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$ ,  $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$ , and  $(ar, b) - (a, rb)$ . Then  $A \otimes B$  is the tensor product of  $A$  and  $B$  ([13]).

As mentioned above, the tensor product of two modules is an abelian group. We will study this problem based on the above situation. So, the tensor product of two fuzzy modules should be a fuzzy abelian group.

**Definition 4.1.** [6] Let  $\mu_A \in \text{Mod-}FR$ ,  $\nu_B \in FR\text{-Mod}$  and let  $\eta_C$  be a fuzzy abelian group, a map  $\tilde{f} : \mu_A \times \nu_B \rightarrow \eta_C$  is called fuzzy biadditive if,  $f$  is biadditive and

for all  $\sum(a_i, b_i) \in A \times B$ ,

$$\eta\tilde{f}(\sum(a_i, b_i)) \geq (\mu \times \nu)(\sum(a_i, b_i)).$$

**Definition 4.2.** [6] Let  $\mu_A \in \text{Mod-}FR$  and  $\nu_B \in FR\text{-Mod}$ . A pair  $(\chi_G, \tilde{\varphi})$  is called a tensor product of  $\mu_A$  and  $\nu_B$  if, for every fuzzy biadditive  $f : \mu_A \times \nu_B \rightarrow \rho_H$ , where  $\rho_H$  is a fuzzy abelian group, there is a unique fuzzy map  $\tilde{\theta} \in \text{Hom}(\chi_G, \rho_H)$  such that  $\tilde{\theta}\tilde{\varphi} = f$ .

**Theorem 4.3.** Let  $\mu_A \in \text{Mod-}FR$  and  $\nu_B \in FR\text{-Mod}$ . The tensor product of the two fuzzy  $R$ -modules  $\mu_A$  and  $\nu_B$  exists and it is unique up to isomorphism.

*Proof.* Let  $\tilde{\varphi} : A \times B \rightarrow A \otimes B$  be the tensor product of the right  $R$ -module  $A$  and the left  $R$ -module  $B$ . We can define a map  $\mu \otimes \nu : A \otimes B \rightarrow [0, 1]$  by putting

$$\mu \otimes \nu(\sum(a_i \otimes b_i)) = \bigvee \{ \mu \times \nu(\sum(a'_i, b'_i)) \mid \sum(a'_i \otimes b'_i) = \sum(a_i \otimes b_i) \}.$$

It is easily check that  $\tilde{\varphi}$  is fuzzy biadditive. Let  $\eta_H$  be a fuzzy abelian group and  $\tilde{\psi} : (\mu \times \nu)_{A \times B} \rightarrow \eta_H$  be fuzzy biadditive. By the definition of tensor product in module category, there is a unique homomorphism  $\tilde{\theta} : A \otimes B \rightarrow H$  such that  $\tilde{\theta}\tilde{\varphi} = \tilde{\psi}$ . We now have to show that for every  $\sum(a_i \otimes b_i) \in A \otimes B$ ,

$$\eta\tilde{\theta}(\sum(a_i \otimes b_i)) \geq (\mu \otimes \nu)(\sum(a_i \otimes b_i)).$$

Suppose  $\sum(a'_i \otimes b'_i) = \sum(a_i \otimes b_i) \in A \otimes B$ . We have

$$\begin{aligned} \eta\tilde{\theta}(\sum(a'_i \otimes b'_i)) &= \eta(\sum \tilde{\theta}(a'_i \otimes b'_i)) \geq \bigwedge \eta\tilde{\theta}(a'_i \otimes b'_i) \\ &= \bigwedge \eta\tilde{\theta}\tilde{\varphi}(a'_i, b'_i) = \bigwedge \eta\tilde{\psi}(a'_i, b'_i) \\ &\geq \bigwedge \mu \times \nu(a'_i, b'_i) = \mu \times \nu(\sum(a'_i, b'_i)). \end{aligned}$$

This concludes that  $\eta\tilde{\theta}(\sum(a_i \otimes b_i)) \geq (\mu \otimes \nu)(\sum(a_i \otimes b_i))$  and hence  $((\mu \otimes \nu)_{A \otimes B}, \tilde{\varphi})$  is the tensor product of  $\mu_A$  and  $\nu_B$ .

It is obvious that the tensor product is unique up to isomorphism.  $\square$

For simplicity, we shall write  $\mu_A \otimes \nu_B$  for the tensor product of  $\mu_A$  and  $\nu_B$  instead of  $(\mu \otimes \nu)_{A \otimes B}$ .

If  $A \in R\text{-Mod}$ , we have  $R \otimes_R A \cong A$ . Similarly, if  $A \in \text{Mod-}R$ , then we have  $A \otimes_R R \cong A$ . Using this fact, we can get the following statements.

**Proposition 4.4.** Let  $\mu_A \in FR\text{-Mod}$ . We have  $0_R \otimes \mu_A \cong \mu_A$ .

**Proposition 4.5.** Let  $\mu_A \in \text{Mod-}FR$ . We have  $\mu_A \otimes 0_R \cong \mu_A$ .

For  $M \in R\text{-Mod}$  ( $M \in \text{Mod-}R$ ), it is well-known that the functor  $- \otimes_R M$  ( $M_R \otimes -$ ) is right exact. In  $FR\text{-Mod}$ , we can easily get the following results.

**Proposition 4.6.** Let  $\xi_M \in FR\text{-Mod}$ . Then  $\xi_M \otimes -$  preserves epimorphisms in  $FR\text{-Mod}$ .

*Proof.* Let  $\mu_A \xrightarrow{\tilde{g}} \nu_B \rightarrow \bar{s}$  be an epic. Since  $M \otimes -$  is right exact, we have an epimorphism

$$\xi_M \otimes \mu_A \xrightarrow{1_M \otimes \tilde{g}} \xi_M \otimes \nu_B \rightarrow \bar{s}.$$

Now, we only need to show that  $1_M \otimes \tilde{g}$  is a fuzzy homomorphism. For every  $\sum m_i \otimes a_i \in \xi_M \otimes \mu_A$ , we have

$$\begin{aligned} & (\xi \otimes \nu)(1_M \otimes \tilde{g})(\sum m_i \otimes a_i) \\ &= (\xi \otimes \nu)(\sum m_i \otimes \tilde{g}(a_i)) \\ &= \bigvee \{ \xi \times \nu(\sum (m'_i, b'_i)) \mid \sum m'_i \otimes b'_i = \sum m_i \otimes \tilde{g}(a_i) \} \\ &\geq \bigvee \{ \xi \times \mu(\sum (m'_i, a'_i)) \mid \tilde{g}(a'_i) = b'_i \text{ and } \sum m'_i \otimes b'_i = \sum m_i \otimes \tilde{g}(a_i) \} \\ &\geq \bigvee \{ \xi \times \mu(\sum (m''_i, a''_i)) \mid \sum m''_i \otimes a''_i = \sum m_i \otimes a_i \} \\ &= \xi \otimes \mu(\sum m_i \otimes a_i). \end{aligned}$$

Hence, we get the desired results.  $\square$

**Proposition 4.7.** *Let  $\xi_M \in FR\text{-Mod}$ . Then  $- \otimes \xi_M$  preserves epimorphisms in  $\text{Mod-}FR$ .*

*Proof.* The proof is similar to that of Proposition 4.6.  $\square$

Now, we study the relationships between Hom functor and tensor functor.

**Theorem 4.8.** *Let  $R$  be a commutative ring. Then there are quasi-isomorphisms*

$$\begin{aligned} \tau : \gamma \text{Hom}_{(\nu_B \otimes \mu_A, \eta_C)} &\cong^Q \alpha \text{Hom}_{(\mu_A, \beta \text{Hom}_{(\nu_B, \eta_C)})}; \\ \tau' : \gamma \text{Hom}_{(\mu_A \otimes \nu_B, \eta_C)} &\cong^Q \alpha \text{Hom}_{(\mu_A, \beta \text{Hom}_{(\nu_B, \eta_C)})}. \end{aligned}$$

*Proof.* We only prove the first quasi-isomorphism. Firstly, we recall the definition of  $\tau$  in Theorem 2.11 [13]. Suppose  $f \in \text{Hom}(\nu_B \otimes \mu_A, \eta_C)$ ,  $a \in A$  and  $b \in B$ , define  $f_a : \nu_B \rightarrow \eta_C$  by  $f_a(b) = f(b \otimes a)$  and define  $\bar{f} : \mu_A \rightarrow \text{Hom}(\nu_B, \eta_C)$  by  $\bar{f}(b) = f_a$ . Then  $\tau : \gamma \text{Hom}_{(\nu_B \otimes \mu_A, \eta_C)} \rightarrow \alpha \text{Hom}_{(\mu_A, \beta \text{Hom}_{(\nu_B, \eta_C)})}$  is defined by  $\tau(f) = \bar{f}$ .

So, we have to show that  $f_a$ ,  $\bar{f}$  and  $\tau$  are fuzzy morphisms.

1)  $f_a$  is a fuzzy homomorphism. For  $b \in B$ , we have

$$\eta(f_a(b)) = \eta(f(b \otimes a)) \geq \nu \otimes \mu(b \otimes a) \geq \nu \times \mu(b, a) = \bigvee \{ \nu(b), \mu(a) \} \geq \nu(b).$$

2)  $\bar{f}$  is a fuzzy homomorphism. For  $a \in A$ , we have

$$\begin{aligned} \beta(\bar{f}(a)) &= \beta(f_a) = \bigwedge \{ \eta(f_a(b)) \mid b \in B \} = \bigwedge \{ \eta(f(b \otimes a)) \mid b \in B \} \\ &\geq \bigwedge \{ \bigvee \{ \nu(b), \mu(a) \} \mid b \in B \} \geq \mu(a). \end{aligned}$$

3)  $\tau$  is a fuzzy homomorphism. For  $f \in \text{Hom}_R(B \otimes A, C)$  we have

$$\begin{aligned} \alpha(\tau(f)) &= \alpha(\bar{f}) = \bigwedge \{ \beta(\bar{f}(a)) \mid a \in A \} = \bigwedge \{ \beta(f_a) \mid a \in A \} \\ &= \bigwedge \{ \bigwedge \{ \eta(f_a(b)) \mid b \in B \} \mid a \in A \} \\ &= \bigwedge \{ \bigwedge \{ \eta f(b \otimes a) \mid b \in B \} \mid a \in A \} \\ &= \bigwedge \{ \eta f(b \otimes a) \mid b \in B, a \in A \}. \end{aligned}$$

While,  $\gamma(f) = \bigwedge \{\eta f(\sum b_i \otimes a_i) | b_i \in B, a_i \in A\}$ . So, we get  $\alpha(\tau(f)) \geq \gamma(f)$ .  $\square$

## 5. Conclusion

In this paper, we make  $\text{Hom}(\mu_A, \nu_B)$  into fuzzy  $R$ -module by redefining a function  $\alpha : \text{Hom}(\mu_A, \nu_B) \rightarrow [0, 1]$ . Then we get some related properties of Hom functor. We also obtain that  $\text{Hom}(\xi_P, -)$  is exact if and only if  $\xi_P$  is a fuzzy projective  $R$ -module. In addition, we prove that the tensor product of two fuzzy  $R$ -modules exists. We also get some properties of tensor product. Finally, we study the adjointness of Hom functor and tensor functor. The obtained results generalized the related theory in  $R\text{-Mod}$ .

To conclude, we would like to notice that there are several open questions on the topic of the paper, some of which we provide below. 1) Is the functor  $\text{Hom}(-, \xi_M)$  left exact? 2) What are the relationships between a fuzzy injective  $R$ -module  $\xi_M$  and the functor  $\text{Hom}(-, \xi_M)$ ? 3) Can we define fuzzy module structures in such a way that these functors do become left exact? 4) If  $R$  is not a commutative semiperfect ring, can we get Theorem 3.9? 5) Is the tensor product in fuzzy module category associative? 6) Are  $\tau$  and  $\tau'$  in Theorem 4.8 fuzzy isomorphisms? Our future work on this topic will focus on these questions. Our obtained results may help in the study of the homological properties of  $FR\text{-Mod}$  and equivalences of fuzzy module categories.

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