

## A NEW STOCK MODEL FOR OPTION PRICING IN UNCERTAIN ENVIRONMENT

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**ABSTRACT.** The option-pricing problem is always an important part in modern finance. Assuming that the stock diffusion is a constant, some literature has introduced many stock models and given corresponding option pricing formulas within the framework of the uncertainty theory. In this paper, we propose a new stock model with uncertain stock diffusion for uncertain markets. Some option pricing formulas on the proposed uncertain stock model are derived and a numerical calculation is illustrated.

### 1. Introduction

An option is a contract that gives the holder the right to buy or sell a stock in the time permitted. As the core of the option trade, option-pricing is one of the most complex problems in modern finance. In 1900, Bachelier [1] first used advanced mathematics to study finance and the option pricing problem. More than half a century after the seminal work of Bachelier, in 1973, Black and Scholes [2] and Merton [18], independently, applied the geometric Wiener process to stock markets and proposed the famous Black-Scholes stock model. After that, Merton [19] examined the option pricing on a stock that has a stochastic volatility. Cox et al [6] presented a binomial model and gave a numerical method to price option. Hull and White [9] and Stein and Stein [23] investigated a special type of stock model with stochastic volatility and gave some option pricing formulas, respectively. Today, the stochastic finance theory based on the stock model has become a vital tool in financial markets and been widely used in option pricing and financial derivative.

Fuzzy set was initiated by Zadeh [25] in 1965. In a fuzzy environment, some researchers applied fuzzy theory to study the stock models in financial markets. In 2004, Liu [10] founded credibility theory which is a branch of mathematics to deal with the phenomena with subjective information. In 2008, Liu [12] applied credibility theory to finance and gave a stock model for fuzzy stock markets. Meanwhile, Qin and Li [22] obtained the European option pricing formulas based on this stock model. Afterward Gao and Gao [8] presented the mean reversion model in fuzzy stock market and discussed the problem of pricing zero-coupon when short rate followed this mean reversion model. Peng [20] extended above two types of fuzzy stock models and derived some option pricing formulas.

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The probability theory and the fuzzy theory are two important tools to deal with indeterminacy phenomena. However, in real life, there is some subjective information, especially those lacking or even those without historical records, behave neither like randomness nor like fuzziness. In order to deal with this subjective information, Liu [11] founded uncertainty theory which was refined by Liu [15] based on normality, duality, subadditivity and product axioms in 2010. Uncertainty theory is a branch of mathematics to study the uncertainty in human systems and has gained considerable achievement. Gao [7] discussed some properties of continuous uncertain measure. You [24] made an investigation into the mathematical properties of uncertain sequences. Liu [14] built uncertain programming based on uncertainty theory. Liu and Ha [17] gave the important formula of the expected value of function of uncertain variables. Chen and Dai [4] studied the maximum entropy principle of uncertainty distribution for uncertain variables. More recent developments of uncertainty theory can be found in Liu [15].

In order to study the evolution of uncertain phenomena with time, uncertain process was introduced by Liu [12], which is essentially a sequence of uncertain variables indexed by time. After that, a significant uncertain process called canonical process was designed by Liu [13]. The canonical process is a stationary independent increment process with Lipschitz continuous sample paths. Liu [12, 13] proposed uncertain calculus and uncertain differential equation with respect to canonical process. In 2009, Liu [13] first applied uncertain differential equation to finance and produced a new topic of uncertain finance. Based on the assumption that stock price follows a geometric canonical process, Liu [13] presented the uncertain counterpart of the Black-Scholes model and derived the European option pricing formulas in uncertain financial market. Afterward Chen [3] gave the America option pricing formulas. Peng and Yao [21] presented a mean reversion uncertain stock model and deduced the corresponding option pricing formulas. Besides, in order to apply the uncertain differential equation well, Chen and Liu [5] proved an existence and uniqueness theorem of solution under global Lipschitz condition. Liu [16] proved the extreme value theorems of uncertain process.

Well known is that, the stock diffusion changes over the time in the financial market, and how to describe the stock diffusion exactly is always an important problem in pricing options. When the estimated stock diffusion curve from sample data is close enough to the real varying regularity, it is applicable to represent stock diffusion process by stochastic process. However, we often lack observed stock diffusion data, for example, a new stock. In such a case, we need to invite some domain experts to evaluate the change of stock diffusion. Since human beings usually overestimate unlikely events, the evaluated stock diffusion curve may have much a larger variance than the real varying regularity and then probability theory is no longer valid. Thus we have no choice but to use uncertainty theory. This provides motivation to study the uncertain stock model with uncertain stock diffusion.

In this paper, we propose a new uncertain stock model based on the assumption that stock diffusion follows some uncertain process and derive some corresponding option pricing formulas. The rest of the paper is organized as follows. Some preliminary concepts of uncertainty theory are recalled in Section 2. Section 3 reviews

some uncertain stock models. A new uncertain stock model for uncertain market is presented in Section 4. Some option pricing formulas for the proposed uncertain stock model are derived in Section 5. A numerical example is given in Section 6. The last section contains a brief summary.

## 2. Preliminaries

As a branch of mathematics to deal with human uncertainty, uncertainty theory, founded by Liu [11], is an axiomatic system. In this section, uncertainty theory is introduced and some basic concepts are given.

Let  $\Gamma$  be a nonempty set and  $L$  a  $\sigma$ -algebra over  $\Gamma$ . Each element  $\Lambda$  in  $L$  is called an event. A set function  $M$  from  $L$  to  $[0, 1]$  is called an uncertain measure if it satisfies normality, duality, subadditivity and product axioms.

An uncertain variable is defined as a measurable function from an uncertainty space  $(\Gamma, L, M)$  to the set of real numbers, i.e., for any Borel set of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

The uncertainty distribution  $\Phi : \mathfrak{R} \rightarrow [0, 1]$  of an uncertain variable  $\xi$  is defined by Liu [11] as

$$\Phi(x) = M \{ \gamma \in \Gamma \mid \xi(\gamma) \leq x \}.$$

and the inverse function  $\Phi^{-1}$  is called the inverse uncertainty distribution of  $\xi$ .

An uncertain variable  $\xi$  is called normal if it has a normal uncertainty distribution

$$\Phi(x) = \left( 1 + \exp \left( \frac{\pi(e-x)}{\sqrt{3}\sigma} \right) \right)^{-1}, \quad x \in \mathfrak{R},$$

denoted by  $N(e, \sigma)$ , where  $e$  and  $\sigma$  are real numbers with  $\sigma > 0$ . and the inverse uncertainty distribution is

$$\Phi^{-1}(\alpha) = e + \frac{\sqrt{3}\sigma}{\pi} \ln \frac{\alpha}{1-\alpha}, \quad (0 < \alpha < 1).$$

The expected value of uncertain variable  $\xi$  is defined by Liu [11] as

$$E[\xi] = \int_0^{+\infty} M\{\xi \geq r\} dr - \int_{-\infty}^0 M\{\xi \leq r\} dr$$

provided that at least one of the two integrals is finite.

The operational law of independent uncertain variables was given by Liu [15] for calculating the uncertainty distribution of monotone function of uncertain variables. Let  $\xi_1, \xi_2, \dots, \xi_n$  be independent uncertain variables with uncertainty distributions  $\Phi_1, \Phi_2, \dots, \Phi_n$ , respectively. If  $f(x_1, x_2, \dots, x_n)$  is a strictly increasing function with respect to  $x_1, x_2, \dots, x_m$  and strictly decreasing with respect to  $x_{m+1}, x_{m+2}, \dots, x_n$ , then  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$  is an uncertain variable with inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \dots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \dots, \Phi_n^{-1}(1-\alpha)).$$

Furthermore, the expected value of uncertain variable  $\xi = f(\xi_1, \xi_2, \dots, \xi_n)$  was obtained by Liu and Ha [17] as follows,

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \dots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \dots, \Phi_n^{-1}(1-\alpha))d\alpha.$$

**Definition 2.1.** [12] Let  $T$  be an index set and let  $(\Gamma, L, M)$  be an uncertain space. An uncertain process is a measurable function from  $T \times (\Gamma, L, M)$  to the set of real numbers, i.e., for each  $t \in T$  and any Borel set  $B$  of real numbers, the set

$$\{X_t \in B\} = \{\gamma \in \Gamma \mid X_t(\gamma) \in B\}$$

is an event.

An uncertain process  $X_t$  is said to have independent increments if

$$X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables where  $t_0$  is the initial time and  $t_1, t_2, \dots, t_k$  are any times with  $t_0 < t_1 < \dots < t_k$ . For this case,  $X_t$  is said to be an independent increment process.

**Theorem 2.2.** ([16], *Extreme Value Theorem*) Let  $X_t$  be a sample continuous independent increment process and have a continuous uncertainty distribution  $\Phi_t(x)$  at each time  $t$ . Then the supremum

$$\sup_{0 \leq t \leq s} X_t$$

has an uncertainty distribution

$$\Psi(x) = \inf_{0 \leq t \leq s} \Phi_t(x).$$

**Definition 2.3.** [13] An uncertain process  $C_t$  is said to be a canonical process if  
(i)  $C_0 = 0$  and almost all sample paths are Lipschitz continuous;  
(ii)  $C_t$  has stationary and independent increments;  
(iii) every increment  $C_{s+t} - C_t$  is a normal uncertain variable with expected value 0 and variance  $t^2$ , whose uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3t}}\right)\right)^{-1}, \quad x \in \mathfrak{R}.$$

The canonical process  $C_t$  is one of the most significant uncertain processes, which is a normal uncertain variable  $N(0, t)$ .

**Definition 2.4.** [13] Let  $X_t$  be an uncertain process and  $C_t$  be a canonical process. For any partition of closed interval  $[a, b]$  with  $a = t_1 < t_2 < \dots < t_{k+1} = b$ , the mesh is written as

$$\Delta = \max_{1 \leq i \leq k} |t_{i+1} - t_i|.$$

Then uncertain integral of  $X_t$  with respect to  $C_t$  is

$$\int_a^b X_t dC_t = \lim_{\Delta \rightarrow 0} \sum_{i=1}^k X_{t_i} (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite.

**Example 2.5.** Let  $f(t)$  be a continuous function with respect to  $t$ . Then the uncertain integral

$$\int_a^b f(t)dC_t$$

is a normal uncertain variable at each time  $s$ , and

$$\int_a^b f(t)dC_t \sim N\left(0, \int_a^b |f(t)| dt\right).$$

**Definition 2.6.** [12] Suppose  $C_t$  is a canonical process, and  $f, g$  are two given functions. Then

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t$$

is called an uncertain differential equation.

### 3. Some Uncertain Stock Models

In this section, we will review two types of uncertain stock models for uncertain financial markets.

Let  $X_t$  be the bond price, and  $Y_t$  the stock price. Assume that the stock price  $Y_t$  follows a geometric canonical process. Then Liu's uncertain stock model is written as follows,

$$\begin{cases} dX_t = rX_t dt \\ dY_t = eY_t dt + \sigma Y_t dC_t \end{cases}$$

where  $r$  is the riskless interest rate,  $e$  is the stock drift,  $\sigma$  is the stock diffusion, and  $C_t$  is a canonical process.

This stock model is an uncertain counterpart of Black-Scholes model [2]. Based on this model, Liu [13] and Chen [3] derived the European and America option pricing formulas, respectively.

Let  $X_t$  be the bond price, and  $Y_t$  the stock price. Then Peng-Yao's uncertain stock model is written as follows,

$$\begin{cases} dX_t = rX_t dt \\ dY_t = (m - \alpha Y_t)dt + \sigma dC_t \end{cases}$$

where  $r, m, \alpha$  are some given positive constants,  $\sigma$  is the stock diffusion, and  $C_t$  is a canonical process.

This stock model is an uncertain counterpart of Black-Karasinski model and incorporates a general economic behavior called mean reversion. In addition, Peng and Yao [21] derived the European and America option pricing formulas.

### 4. A New Uncertain Stock Model

Liu's and Peng-Yao's uncertain stock models all assumed that the stock diffusion  $\sigma$  is a constant. In this section, we use Peng-Yao's uncertain stock model to characterize the stock diffusion process and obtain a new uncertain stock model. Let  $X_t$  be the bond price, and  $Y_t$  the stock price. Then we obtain the uncertain stock model as follows,

$$\begin{cases} dX_t = rX_t dt \\ dY_t = eY_t dt + \sigma_t Y_t dC_t. \end{cases} \quad (1)$$

Here, the stock diffusion process  $\sigma_t$  satisfies Peng-Yao's uncertain stock model, i.e.,  $\sigma_t$  follows the uncertain differential equation

$$d\sigma_t = -\delta(\sigma_t - \theta)dt + p d\tilde{C}_t \quad (2)$$

where  $\delta, \theta, p$  are positive constants, and  $\tilde{C}_t$  is a canonical process independent on  $C_t$ .

This model is an uncertain counterpart of the Stein-Stein model [23]. It means that the stock diffusions appear to be pulled back to some long-run average level over time.

Next, we derive the analytical solution for uncertain differential equation (2). It follows from the chain rule that

$$\begin{aligned} d(\exp(\delta t)\sigma_t) &= \delta \exp(\delta t)\sigma_t dt + \exp(\delta t)d\sigma_t \\ &= \delta \exp(\delta t)\sigma_t dt + \exp(\delta t)(-\delta\sigma_t + \theta\delta)dt + p \exp(\delta t)d\tilde{C}_t \\ &= \theta\delta \exp(\delta t)dt + p \exp(\delta t)d\tilde{C}_t. \end{aligned}$$

Integration on both sides of above equation yields

$$\exp(\delta t)\sigma_t - \sigma_0 = \theta\delta \int_0^t \exp(\delta s)ds + p \int_0^t \exp(\delta s)d\tilde{C}_s$$

with initial value  $\sigma_0$ .

Then we have

$$\sigma_t = \theta + (\sigma_0 - \theta) \exp(-\delta t) + p \exp(-\delta t) \int_0^t \exp(\delta s)d\tilde{C}_s.$$

It follows from Example 2.5 that  $\int_0^t \exp(\delta s)d\tilde{C}_s$  is a normal uncertain variable  $N(0, (\exp(\delta t) - 1)/\delta)$  at each time  $t$ . Using the operational law of independent uncertain variables, we obtain that  $\sigma_t$  is also a normal uncertain variable and

$$\sigma_t \sim N\left(\theta + (\sigma_0 - \theta) \exp(-\delta t), \frac{p}{\delta} - \frac{p}{\delta} \exp(-\delta t)\right).$$

That is,  $\sigma_t$  has the expected value  $E[\sigma_t] = \theta + (\sigma_0 - \theta) \exp(-\delta t)$  and  $\sqrt{V[\sigma_t]} = \frac{p}{\delta} - \frac{p}{\delta} \exp(-\delta t)$ . Obviously,  $\sigma_t$  has initial value  $\sigma_0$  at  $t = 0$ ; and  $E[\sigma_t] \rightarrow \theta$  when  $t \rightarrow +\infty$ , which shows that when the stock diffusion  $\sigma_t$  is high, the mean reversion tends to cause it to have a negative drift; when the stock diffusion  $\sigma_t$  is low, the mean reversion tends to cause it to have a positive drift. Here, the stock diffusion is pulled to a long-run average level  $\theta$  (see Figure 1).

Moreover,  $\sqrt{V[\sigma_t]}$  is an increasing function of  $t$  and  $\sqrt{V[\sigma_t]} \rightarrow p/\delta$  when  $t \rightarrow \infty$ , which shows the stock diffusion becomes more and more wide around its expected

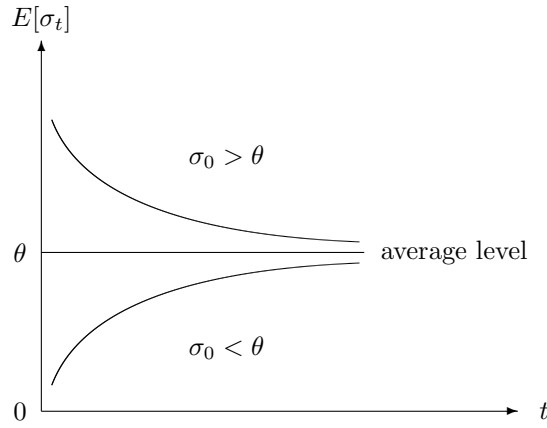


FIGURE 1. Mean Reversion

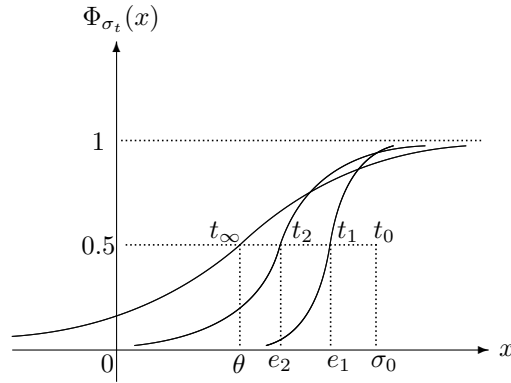


FIGURE 2. The Uncertainty Distribution of Stock Diffusion

value. When  $\sigma_0 > \theta$ , the uncertainty distribution of stock diffusion can be illustrated in Figure 2 where  $t_0 = 0$ ,  $t_\infty = \infty$ ,  $t_1 < t_2$ ,  $E[\sigma_{t_1}] = e_1$  and  $E[\sigma_{t_2}] = e_2$ .

Although we derive the analytic solution of stock diffusion process  $\sigma_t$ , it is difficult to represent the stock price  $Y_t$  because of the complexity of uncertain differential equation in model (1). Considering the expected value is the average value of uncertain stock diffusion and represents the size of uncertain stock diffusion, we can employ expected value  $E[\sigma_t]$  to calculate stock price  $Y_t$  instead of  $\sigma_t$  in real life. Hence the uncertain stock model (1) can be approximated by the following model,

$$\begin{cases} dX_t = rX_t dt \\ dY_t = eY_t dt + E[\sigma_t]Y_t dC_t. \end{cases} \quad (3)$$

We can find that the stock price is

$$Y_s = Y_0 \exp(es + \int_0^s E[\sigma_t] dC_t)$$

where  $\int_0^s E[\sigma_t]dC_t$  is a normal uncertain variable at each time  $s$ , and

$$\int_0^s E[\sigma_t]dC_t \sim N(0, \int_0^s |E[\sigma_t]| dt).$$

Note that  $\int_0^s |E[\sigma_t]| dt = \theta s + (\sigma_0 - \theta)(1 - \exp(-\delta s))/\delta$ , then  $\int_0^s E[\sigma_t]dC_t$  has the uncertainty distribution

$$\Upsilon_s(x) = \left(1 + \exp\left(\frac{-\pi\delta x}{\sqrt{3}\beta(s)}\right)\right)^{-1} \quad (4)$$

where  $\beta(s) = \theta\delta s + \sigma_0 - \theta - (\sigma_0 - \theta)\exp(-\delta s)$ .

When  $\sigma_t \equiv \sigma$ , the expected value  $E[\sigma_t] \equiv \sigma$  and consequently uncertain stock model (1) and (3) degenerate to the Liu's uncertain stock model, respectively. Next, we will derive some option pricing formulas from uncertain stock model (3).

### 5. Some Option Pricing Formulas Based on New Stock Model

The European option is the most classic and useful option. A European option gives one the right, but not the obligation, to buy or sell a stock at a specified time for a specified price. In this subsection, we will give the European option pricing formulas based on the modified stock model.

A European call (put) option is a contract that gives the holder the right to buy (sell) a stock at an expiration time  $s$  for a strike price  $K$ . If  $Y_s$  is the final price of the underlying stock, then the payoff from a European call (put) option is  $(Y_s - K)^+$  ( $(K - Y_s)^+$ ). Considering the time value of money resulted from the bond, the present value of this payoff is  $\exp(-rs)(Y_s - K)^+$  ( $\exp(-rs)(K - Y_s)^+$ ). Hence the European call (put) option price should be the expected present value of the payoff  $\exp(-rs)E[(Y_s - K)^+]$  ( $\exp(-rs)E[(K - Y_s)^+]$ ).

**Theorem 5.1.** (*European Call Option Pricing Formula*) Assume a European put option for the stock model (3) has a strike price  $K$  and an expiration time  $s$ . Then the European call option price is

$$f_c = \exp(-rs)Y_0 \int_{K/Y_0}^{+\infty} \left(1 + \exp\left(\frac{\pi\delta(\ln y - es)}{\sqrt{3}\beta(s)}\right)\right)^{-1} dy.$$

*Proof.* It follows from the stock price  $Y_s = Y_0 \exp(es + \int_0^s E[\sigma_t]dC_t)$  and the definition of  $f_c$  that

$$\begin{aligned} f_c &= \exp(-rs)E[(Y_0 \exp(es + \int_0^s E[\sigma_t]dC_t) - K)^+] \\ &= \exp(-rs) \int_0^{+\infty} M\{Y_0 \exp(es + \int_0^s E[\sigma_t]dC_t) - K \geq x\} dx \\ &= \exp(-rs)Y_0 \int_{K/Y_0}^{+\infty} M\{\int_0^s E[\sigma_t]dC_t \geq (\ln y - es)\} dy \\ &= \exp(-rs) \int_{K/Y_0}^{+\infty} 1 - \Upsilon_s(\ln y - es) dy \\ &= \exp(-rs)Y_0 \int_{K/Y_0}^{+\infty} \left(1 + \exp\left(\frac{\pi\delta(\ln y - es)}{\sqrt{3}\beta(s)}\right)\right)^{-1} dy. \end{aligned}$$



The European call option pricing formula is verified.  $\square$

**Theorem 5.2.** *European call option price formula  $f_c$  has the following properties:*

- (1)  $f_c$  is a decreasing function of interest rate  $r$ ;
- (2)  $f_c$  is a decreasing and convex function of strike price  $K$ ;
- (3)  $f_c$  is an increasing and convex function of stock's initial price  $Y_0$ .

*Proof.* (1) It directly follows from the definition  $f_c = \exp(-rs)E[(Y_s - K)^+]$  that the option price is a decreasing function of  $r$ . That is, the European call option will devalue if the interest rate is raised; and will appreciate in value if the interest rate is reduced.

(2) This follows from the fact that  $\exp(-rs)(Y_s - K)^+$  is decreasing and convex in  $K$ . It means that European call option price is a decreasing and convex function of the stock's strike price when the other variables remain unchanged.

(3) For any positive constant  $c$ ,  $\exp(-rs)(Y_0c - K)^+$  is an increasing and convex function of  $Y_0$ . Then,  $\exp(-rs)(Y_0 \exp(es + \int_0^s E[\sigma_t]dC_t) - K)^+$  is an increasing and convex function of  $Y_0$ . Because the uncertainty distribution of  $\exp(es + \int_0^s E[\sigma_t]dC_t)$  does not depend on  $Y_0$ , the desired result is verified. This property means that if the other variables remain unchanged, then the option price is an increasing and convex function of the stock's initial price.  $\square$

**Theorem 5.3.** *(European Put Option Pricing Formula) Assume a European put option for the stock model (3) has a strike price  $K$  and an expiration time  $s$ . Then the European put option price is*

$$f_p = \exp(-rs)Y_0 \int_0^{K/Y_0} \left(1 + \exp\left(\frac{\pi\delta(es - \ln y)}{\sqrt{3}\beta(s)}\right)\right)^{-1} dy.$$

*Proof.* It follows from the stock price  $Y_s = Y_0 \exp(es + \int_0^s E[\sigma_t]dC_t)$  and the definition of  $f_p$  that

$$\begin{aligned} f_p &= \exp(-rs)E[(K - Y_0 \exp(es + \int_0^s E[\sigma_t]dC_t))^+] \\ &= \exp(-rs) \int_0^{+\infty} M\{K - Y_0 \exp(es + \int_0^s E[\sigma_t]dC_t) \geq x\}dx \\ &= \exp(-rs)Y_0 \int_0^{K/Y_0} M\left\{\int_0^s E[\sigma_t]dC_t \leq (\ln y - es)\right\}dy \\ &= \exp(-rs)Y_0 \int_0^{K/Y_0} \Upsilon_s(\ln y - es)dy \\ &= \exp(-rs)Y_0 \int_0^{K/Y_0} \left(1 + \exp\left(\frac{\pi\delta(es - \ln y)}{\sqrt{3}\beta(s)}\right)\right)^{-1} dy. \end{aligned}$$

The European put option pricing formula is verified.  $\square$

**Theorem 5.4.** *European put option price formula  $f_p$  has the following properties:*

- (1)  $f_p$  is a decreasing function of interest rate  $r$ ;
- (2)  $f_p$  is an increasing and convex function of strike price  $K$ ;
- (3)  $f_p$  is a decreasing and convex function of stock's initial price  $Y_0$ .

*Proof.* These properties can be obtained by the similar analysis to Theorem 5.2.  $\square$

An American option is a contract that gives the holder the right to buy or sell a stock at any time prior to an expiration time  $s$  for a strike price  $K$ . American option is widely accepted by investors for its flexibility of exercising time. Next, we will derive the American option pricing formulas from our uncertain stock model.

An American call (put) option is a contract that gives the holder the right to buy (sell) a stock at any time prior to an expiration time  $s$  for a strike price  $K$ . If  $Y_s$  is the final price of the underlying stock, then the payoff from an American call (put) option is the supremum of  $(Y_u - K)^+$  ( $(K - Y_u)^+$ ) over the time interval  $(0, s]$ . Considering the time value of money resulted from the bond, the present value of this payoff is

$$\sup_{0 \leq u \leq s} \exp(-ru)(Y_u - K)^+ \quad \left( \sup_{0 \leq u \leq s} \exp(-ru)(K - Y_u)^+ \right).$$

Hence the American call (put) option price should be the expected present value of the payoff

$$E\left[ \sup_{0 < u \leq s} \exp(-ru)(Y_u - K)^+ \right] \quad \left( E\left[ \sup_{0 < u \leq s} \exp(-ru)(K - Y_u)^+ \right] \right).$$

**Theorem 5.5.** *(American Call Option Pricing Formula) Assume an American call option for the stock model (3) has a strike price  $K$  and an expiration time  $s$ . Then the American call option price is*

$$f_c = \int_0^{+\infty} \sup_{0 < u \leq s} \left( 1 + \exp\left( -\frac{\pi\delta eu}{\sqrt{3}\beta(u)} + \frac{\pi\delta}{\sqrt{3}\beta(u)} \ln \frac{y \exp(ru) + K}{Y_0} \right) \right)^{-1} dy.$$

*Proof.* We first solve the uncertainty distribution  $\Phi_u(x)$  of  $\exp(-ru)(Y_u - K)^+$ . For each  $u \in (0, s]$ , it is obvious that  $\Phi_u(x) = 0$  when  $x < 0$ . When  $x \geq 0$ , we have

$$\begin{aligned} \Phi_u(x) &= M\{\exp(-ru)(Y_u - K)^+ \leq x\} \\ &= M\left\{ \int_0^u E[\sigma_t] dC_t \leq -eu + \ln \frac{x \exp(ru) + K}{Y_0} \right\} \\ &= \Upsilon_u\left(-eu + \ln \frac{x \exp(ru) + K}{Y_0}\right) \\ &= \left( 1 + \exp\left( \frac{\pi\delta eu}{\sqrt{3}\beta(u)} + \frac{\pi\delta}{\sqrt{3}\beta(u)} \ln \frac{Y_0}{x \exp(ru) + K} \right) \right)^{-1}. \end{aligned}$$

It is obvious that  $\Phi_u(x)$  is continuous for each fixed  $u \in (0, s]$ . Furthermore,  $\exp(-ru)(Y_u - K)^+$  is a sample continuous independent increment process since  $C_t$  is a Lipschitz continuous uncertain process with stationary and independent

increments. It follows from the extreme value theorem that  $\sup_{0 < u \leq s} \exp(-ru)(Y_u - K)^+$  has an uncertainty distribution

$$\Psi(x) = \inf_{0 < u \leq s} \Phi_u(x) = \inf_{0 < u \leq s} \Upsilon(-eu + \ln \frac{x \exp(ru) + K}{Y_0}).$$

Then

$$\begin{aligned} f_c &= E[\sup_{0 < u \leq s} \exp(-ru)(Y_u - K)^+] \\ &= \int_0^{+\infty} M\{\sup_{0 < u \leq s} \exp(-ru)(Y_u - K)^+ \geq y\} dy \\ &= \int_0^{+\infty} 1 - \Psi(y) dy \\ &= \int_0^{+\infty} \sup_{0 < u \leq s} \left( 1 + \exp\left(-\frac{\pi \delta eu}{\sqrt{3}\beta(u)} + \frac{\pi \delta}{\sqrt{3}\beta(u)} \ln \frac{y \exp(ru) + K}{Y_0}\right) \right)^{-1} dy. \end{aligned}$$

The American call option pricing formula is verified.  $\square$

**Theorem 5.6.** *America call option price formula  $f_c$  has the following properties:*

- (1)  $f_c$  is a decreasing function of interest rate  $r$ ;
- (2)  $f_c$  is a decreasing and convex function of strike price  $K$ ;
- (3)  $f_c$  is an increasing and convex function of stock's initial price  $Y_0$ .

*Proof.* (1) It directly follows from the definition  $f_c = E[\sup_{0 < u < s} \exp(-rs)(Y_s - K)^+]$  that the option price is a decreasing function of  $r$ . That is, the American call option will devalue if the interest rate is raised; and will appreciate in value if the interest rate is reduced.

(2) This follows from the fact that  $\sup_{0 < u < s} \exp(-ru)(Y_u - K)^+$  is decreasing and convex in  $K$ . It means that American call option price is a decreasing and convex function of the stock's strike price when the other variables remain unchanged.

(3) For any positive constant  $c$ ,  $\exp(-ru)(Y_0 c - K)^+$  is an increasing and convex function of  $Y_0$ . Then,  $\sup_{0 < u < s} \exp(-ru)(Y_0 \exp(eu + \int_0^u E[\sigma_t] dC_t) - K)^+$  is an increasing and convex function of  $Y_0$ . Because the uncertainty distribution of  $\exp(eu + \int_0^u E[\sigma_t] dC_t)$  does not depend on  $Y_0$ , the desired result is verified. This property means that if the other variables remain unchanged, then the option price is an increasing and convex function of the stock's initial price.  $\square$

**Theorem 5.7.** *(American Put Option Pricing Formula) Assume an American put option for the stock model (3) has a strike price  $K$  and an expiration time  $s$ . Then the American put option price is*

$$f_p = \int_0^{+\infty} \sup_{0 < u \leq s} \left( 1 + \exp\left(\frac{\pi \delta eu}{\sqrt{3}\beta(u)} + \frac{\pi \delta}{\sqrt{3}\beta(u)} \ln \frac{Y_0}{K - y \exp(ru)}\right) \right)^{-1} dy.$$

*Proof.* We first solve the uncertainty distribution  $\Phi_u(x)$  of  $\exp(-ru)(K - Y_u)^+$ . For each  $u \in (0, s]$ , it is obvious that  $\Phi_u(x) = 0$  when  $x < 0$ . When  $x \geq 0$ , we have

$$\begin{aligned}\Phi_u(x) &= M\{\exp(-ru)(K - Y_u)^+ \leq x\} \\ &= M\{\exp(-ru)(K - Y_0 \exp(eu + k \int_0^u E[\sigma_t]dC_t)) \leq x\} \\ &= M\{Y_0 \exp(eu + \int_0^u E[\sigma_t]dC_t) \geq K - x \exp(ru)\} \\ &= M\{\int_0^u E[\sigma_t]dC_t \geq -eu + \ln \frac{K - x \exp(ru)}{Y_0}\} \\ &= 1 - \Upsilon_u(-eu + \ln \frac{K - x \exp(ru)}{Y_0}).\end{aligned}$$

It follows from the extreme value theorem that  $\sup_{0 < u \leq s} \exp(-ru)(K - Y_u)^+$  has an uncertainty distribution

$$\Psi(x) = \inf_{0 < u \leq s} \Phi_u(x) = 1 - \sup_{0 < u \leq s} \Upsilon_u(-eu + \ln \frac{K - x \exp(ru)}{Y_0}).$$

Then

$$\begin{aligned}f_p &= E[\sup_{0 < u \leq s} \exp(-ru)(K - Y_u)^+] \\ &= \int_0^{+\infty} M\{\sup_{0 < u \leq s} \exp(-ru)(K - Y_u)^+ \geq y\}dy \\ &= \int_0^{+\infty} 1 - \Psi(y)dy \\ &= \int_0^{+\infty} \sup_{0 < u \leq s} \Upsilon_u(-eu + \ln \frac{K - y \exp(ru)}{Y_0})dy \\ &= \int_0^{+\infty} \sup_{0 < u \leq s} \left(1 + \exp\left(\frac{\pi\delta eu}{\sqrt{3}\beta(u)} + \frac{\pi\delta}{\sqrt{3}\beta(u)} \ln \frac{Y_0}{K - y \exp(ru)}\right)\right)^{-1} dy.\end{aligned}$$

The American put option pricing formula is verified.  $\square$

**Theorem 5.8.** *American put option price formula  $f_p$  has the following properties:*

- (1)  $f_p$  is a decreasing function of interest rate  $r$ ;
- (2)  $f_p$  is an increasing and convex function of strike price  $K$ ;
- (3)  $f_p$  is a decreasing and convex function of stock's initial price  $Y_0$ .

*Proof.* These properties can be obtained by the similar analysis to Theorem 5.6.  $\square$

## 6. A Numerical Example

As an illustration, in this section, we calculate the European call option prices based on new uncertain stock model (3) and make a comparison with Liu's uncertain stock model.

$K$	Liu( $s=1$ )	New( $s=1$ )	Liu( $s=1.5$ )	New( $s=1.5$ )	Liu( $s=2$ )	New( $s=2$ )
15	7.0676	7.2441	9.1266	9.4504	11.9077	12.4131
16	6.2774	6.4657	8.4293	8.7622	11.2745	11.7866
17	5.5329	5.7329	7.7695	8.1108	10.6712	11.1895
18	4.8410	5.0518	7.1492	7.4978	10.0983	10.6219
19	4.2078	4.4274	6.5695	6.9241	9.5558	10.0836
20	3.6369	3.8623	6.0309	6.3898	9.0432	9.5743
21	3.1292	3.3572	5.5330	5.8943	8.5599	9.0931
22	2.6836	2.9107	5.0746	5.4367	8.1050	8.6394
23	2.2968	2.5198	4.6544	5.0153	7.6775	8.2120
24	1.9639	2.1801	4.2702	4.6284	7.2763	7.8098
25	1.6795	1.8867	3.9198	4.2739	6.9000	7.4316

TABLE 1. The Comparison of Liu and New European Call Option Prices

$K$	Liu( $s=1$ )	Mod( $s=1$ )	Liu( $s=1.5$ )	Mod( $s=1.5$ )	Liu( $s=2$ )	Mod( $s=2$ )
15	7.0676	7.2165	9.1266	9.2468	11.9077	11.9686
16	6.2774	6.4363	8.4293	8.5530	11.2745	11.3362
17	5.5329	5.7017	7.7695	7.8964	10.6712	10.7338
18	4.8410	5.0191	7.1492	7.2790	10.0983	10.1616
19	4.2078	4.3934	6.5695	6.7016	9.5558	9.6196
20	3.6369	3.8274	6.0309	6.1647	9.0432	9.1074
21	3.1292	3.3220	5.5330	5.6677	8.5599	8.6243
22	2.6836	2.8756	5.0746	5.2097	8.1050	8.1696
23	2.2968	2.4853	4.6544	4.7890	7.6775	7.7422
24	1.9639	2.1466	4.2702	4.4037	7.2763	7.3408
25	1.6795	1.8545	3.9198	4.0517	6.9000	6.9643

TABLE 2. The Comparison of Liu and Modified European Call Option Prices

**Example 6.1.** Assume the riskless interest rate  $r = 0.08$ , the stock drift  $e = 0.06$ , the initial stock price  $Y_0 = 20$ , the stock diffusion  $\sigma = 0.32$  in Liu's uncertain stock model. In the new stock model (3), taking  $\sigma_0 = 0.35$ ,  $\theta = \sigma = 0.32$ ,  $\delta = 1$ . At three expiration times  $s = 1$ ,  $s = 1.5$  and  $s = 2$ , the Matlab Uncertainty Toolbox (<http://orsc.edu.cn/liu/resources.htm>) yields the European call option prices based on Liu's uncertain stock model and the model (3), respectively (see Table 1).

It is observed that the uncertain stock diffusion exerts an upward influence on all European call option prices when initial value  $\sigma_0 > \theta$ . Whenever average level  $\theta = \sigma$ , the new price exceeds Liu price for the same  $\sigma$ . Moreover, for each strike price  $K$ , the gap between Liu and new prices tend to become big when  $s$  changes from 1, 1.5 to 2.

We use Peng-Yao's uncertain stock model to characterize the stock diffusion process and obtain a new uncertain stock model. All the option pricing formulas derived from this model are based on expected value  $E[\sigma_t]$  of uncertain stock diffusion. Since  $\sqrt{V[\sigma_t]} = \frac{p}{\delta} - \frac{p}{\delta} \exp(-\delta t)$  represents the spread around expected value  $E[\sigma_t]$ , we usually need to modify the option prices according to circumstances. In Example 6.1, the stock diffusion tends to have a negative drift because of  $\sigma_0 > \theta$ . Taking  $p = 0.02$ , we can calculate the modified European call option prices by  $\widehat{E[\sigma_t]} = E[\sigma_t] - \sqrt{V[\sigma_t]}$ . The Table 2 shows the Liu and modified European call option prices.

It is easy to find that the gap between Liu and modified prices tend to become small when  $s$  changes from 1, 1.5 to 2 for each strike price  $K$ .

## 7. Conclusions

In this paper, a new uncertain stock model for uncertain market is presented via uncertainty theory. Based on this new model, some option pricing formulas are proved. As an illustration, we calculate the European call option prices and make a comparison with Liu's uncertain stock model. Comparison shows the uncertain stock diffusion exerts an upward influence on all European call option prices when initial value is larger than average level. Besides, considering the diffusion  $p$  of uncertain stock diffusion process  $\sigma_t$ , we also calculate the modified European call option prices. It should be emphasized that there exist many stock models in uncertain financial market. Every model has its own advantages and disadvantages. It will always be a challenge to find and choose a suitable stock model on the option-pricing problem.

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