FURTHER STUDY ON (L, M)-FUZZY TOPOLOGIES AND (L, M)-FUZZY NEIGHBORHOOD SYSTEMS

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Abstract. Following the idea of L-fuzzy neighborhood system as introduced by Fu-Gui Shi, and its generalization to (L, M)-fuzzy neighborhood system, the relationship between (L, M)-fuzzy topology and (L, M)-fuzzy neighborhood system will be further studied. As an application of the obtained results, we will describe the initial structures of (L, M)-fuzzy neighborhood subspaces and (L, M)-fuzzy topological product spaces.

1. **Introduction and Preliminaries**

In this paper, based on the idea of (L, M)-fuzzy topological space introduced by T. Kubiak and A. Šostak [6, 7], and the notion of (L, M)-fuzzy neighborhood system as a generalization of L-fuzzy neighborhood system of Fu-Gui Shi [10], the relationship between (L, M)-fuzzy topology and (L, M)-fuzzy neighborhood system will be further studied. As an application of the obtained results, we will describe the initial structures of (L, M)-fuzzy neighborhood subspaces and (L, M)-fuzzy topological product spaces.

The following preliminaries will be used throughout this paper, which can be found in [3, 8].

A complete lattice L is called completely distributive, if one of the following conditions hold (the second then follows as a consequence [3]):

(CD1)

$$\bigwedge_{i\in I}\left(\bigvee_{i\in J_i} a_{i,j}\right) = \bigvee_{f\in\prod J_i}\left(\bigwedge_{i\in I} a_{i,f(i)}\right),$$

(CD2)

$$\bigvee_{i\in I}\left(\bigwedge_{i\in J_i} a_{i,j}\right) = \bigwedge_{f\in\prod J_i}\left(\bigvee_{i\in I} a_{i,f(i)}\right),$$

where for each $i \in I$ and $j \in J_i, a_{i,j} \in L$ and $f \in \prod J_i$ means that f is a mapping $f : I \to \bigcup J_i$ such that $f(i) \in J_i$ for each $i \in I$.

An element $a \neq 0$ in a lattice is called coprime if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$ for all $b, c \in L$. Further, a is said to be join irreducible if $a = b \vee c$

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implies $a = b$ or $a = c$ for all $b, c \in L$. The set of all coprime elements (resp. join irreducible elements) of L is denoted by $\mathrm{Copr}(L)$ (resp. $J(L)$). It can be verified that if L is distributive, then $a \in L$ is coprime iff it is join irreducible, which means $\mathrm{Copr}(L) = J(L)$. So, for convenience, we usually use $J(L)$ to stand for the set of all coprime elements of L if L is distributive. If L is a completely distributive lattice and $x \lhd \bigvee_{t\in T} y_t$, then there must be $t^\star \in T$ such that $x \lhd y_{t^\star}$ (here $x \lhd a$ means: $K \subset L, a \leq \bigvee K \Rightarrow \exists y \in K$ such that $x \leq y$). Some more properties of $\lhd$ can be found in [8].

In the rest of the paper, L and M always denote Hutton algebras. A Hutton algebra L, is a completely distributive lattice with order-reversing involution with the least element 0 and the greatest element 1. Recall that an order-reversing involution $'$ on L is a map $(-)' : L \longrightarrow L$ such that for any $a, b \in L$, the following conditions hold: (1) $a \leq b$ implies $b' \leq a'$. (2) $a'' = a$. The following properties hold for any subset $\{b_i : i \in I\} \in L$: (1) $(\bigvee_{i\in I} b_i)' = \bigwedge_{i\in I} b_i'$; (2) $(\bigwedge_{i\in I} b_i)' = \bigvee_{i\in I} b_i'$. We notice that L^X, the set of all L-subsets of X, is also a Hutton algebra with pointwise order. Its smallest element and the largest element are denoted 0_X and 1_X, respectively. For each $A \in L^X$, the L-subset A' is defined $A'(x) = (A(x))'$ for each $x \in X$. Clearly, $J(L^X) = \{x_\lambda : x \in X, \lambda \in J(L)\}$, where x_λ is defined by $x_\lambda(y) = \lambda$ if $y = x$ and $x_\lambda(y) = 0$ otherwise.

Definition 1.1. (Kubiak and Šostak [6, 7]) An (L, M)-fuzzy topology on a set X is a map $\mathcal{T} : L^X \longrightarrow M$ such that

(LMFT1)
$$\mathcal{T}(1_X) = \mathcal{T}(0_X) = 1;$$

(LMFT2)
$$\forall\, U, V \in L^X, \mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V);$$

(LMFT3)
$$\forall \{U_j : j \in J\} \subseteq L^X, \mathcal{T}\left(\bigvee_{j\in J} U_j\right) \geqslant \bigwedge_{j\in J} \mathcal{T}(U_j).$$

$\mathcal{T}(U)$ can be interpreted as the degree to which U is an open L-set, $\mathcal{T}^\star(U) = \mathcal{T}(U')$ will be called the degree of closedness. The pair $(X, \mathcal{T})$ is called (L, M)-fuzzy topological space. A mapping $f : X \longrightarrow Y$ from an (L, M)-fuzzy topological space $(X, \mathcal{T}_1)$ to another (L, M)-fuzzy topological space $(Y, \mathcal{T}_2)$ is said to be continuous if $\mathcal{T}_1(f^{\leftarrow}(B)) \geq \mathcal{T}_2(B)$ for each $B \in L^Y$. The category of all (L, M)-fuzzy topological spaces and their continuous mappings is denoted by (L, M)-**FTOP**.

The following Definition 1.2 and Lemma 1.3 were introduced by Shi [10] for an L-fuzzy topology and can be easily transformed to an (L, M)-fuzzy topology as follows.

Definition 1.2. An (L, M)-fuzzy neighborhood system on a set X is a map $\mathcal{N} : L^X \longrightarrow M^{J(L^X)}$ satisfying the following conditions:

(LMFN1)

$$\mathcal{N}(1_X)(x_\lambda) = 1,\ \mathcal{N}(0_X)(x_\lambda) = 0 \quad (\forall\ x_\lambda \in J(L^X));$$

(LMFN2)

$$\mathcal{N}(U)(x_\lambda) = 0 \quad (\forall\ U \in L^X, \forall\ x_\lambda \in J(L^X), x_\lambda \not\leq U);$$

(LMFN3)

$$\mathcal{N}(U \wedge V)(x_\lambda) = \mathcal{N}(U)(x_\lambda) \wedge \mathcal{N}(V)(x_\lambda) \quad (\forall\ U, V \in L^X, \forall\ x_\lambda \in J(L^X));$$

(LMFN4)

$$\mathcal{N}(U)(x_\lambda) = \bigvee_{x_\lambda \leq V \leq U} \bigwedge_{y_\mu \lhd V} \mathcal{N}(V)(y_\mu) \text{ (where } \forall U \in L^X, x_\lambda, y_\mu \in J(L^X)).$$

$\mathcal{N}(U)(x_\lambda)$ is called the degree to which x_λ belongs to the neighborhood of U. The pair $(X, \mathcal{N})$ is called (L, M)-fuzzy neighborhood space. A mapping $f : X \longrightarrow Y$ from an (L, M)-fuzzy neighborhood space $(X, \mathcal{N}_1)$ to another (L, M)-fuzzy neighborhood space $(Y, \mathcal{N}_2)$ is said to be continuous if $\mathcal{N}_2(U)(f^{\rightarrow}(x_\lambda)) \leq \mathcal{N}_1(f^{\leftarrow}(U))(x_\lambda)$ for each $U \in L^Y$ and each $x_\lambda \in J(L^X)$. The category of all (L, M)-fuzzy neighborhood spaces and their continuous mappings is denoted by (L, M)-**FNS**.

Lemma 1.3. *(L, M)-**FTOP** is isomorphic to (L, M)-**FNS**.*

Proof. **Step 1:** Define $\mathcal{N}_{\mathcal{T}} : L^X \longrightarrow M^{J(L^X)}$ by

$$\mathcal{N}_{\mathcal{T}}(U)(x_\lambda) = \bigvee_{x_\lambda \leq V \leq U} \mathcal{T}(V) \quad (\forall U \in L^X, \forall x_\lambda \in J(L^X)).$$

Then $\mathcal{N}_{\mathcal{T}}$ is an (L, M)-fuzzy neighborhood system induced by $\mathcal{T}$.

In fact, (LMFN1) and (LMFN2) are easily obtained.

(LMFN3) If $A \leq B$, then by the definition of $\mathcal{N}_{\mathcal{T}}$, we have

$$\mathcal{N}_{\mathcal{T}}(A)(x_\lambda) \leq \mathcal{N}_{\mathcal{T}}(B)(x_\lambda) \ (\forall\ A, B \in L^X, \forall x_\lambda \in J(L^X)).$$

Hence

$$\mathcal{N}_{\mathcal{T}}(U \wedge V)(x_\lambda) \leq \mathcal{N}_{\mathcal{T}}(U)(x_\lambda) \wedge \mathcal{N}_{\mathcal{T}}(V)(x_\lambda) \ (\forall\ U, V \in L^X, \forall x_\lambda \in J(L^X)).$$

On the other hand, if $a \lhd \mathcal{N}_{\mathcal{T}}(U)(x_\lambda) \wedge \mathcal{N}_{\mathcal{T}}(V)(x_\lambda)$, then

$$a \lhd \mathcal{N}_{\mathcal{T}}(U)(x_\lambda) = \bigvee_{x_\lambda \leq E \leq U} \mathcal{T}(E), \text{ and } a \lhd \mathcal{N}_{\mathcal{T}}(V)(x_\lambda) = \bigvee_{x_\lambda \leq G \leq V} \mathcal{T}(G).$$

Further, there exist E and G such that

$$x_\lambda \leq E \leq U, x_\lambda \leq G \leq V, \text{and } a \leq \mathcal{T}(E),\ a \leq \mathcal{T}(G).$$

So

$$x_\lambda \leq E \wedge G \leq U \wedge V, \text{ and } a \leq \mathcal{T}(E) \wedge \mathcal{T}(G) \leq \mathcal{T}(E \wedge G).$$

Hence

$$a \leq \mathcal{T}(E \wedge G) \leq \bigvee_{x_\lambda \leq M \leq U \wedge V} \mathcal{T}(M) = \mathcal{N}_{\mathcal{T}}(U \wedge V)(x_\lambda).$$

This shows

$$\mathcal{N}_{\mathcal{T}}(U \wedge V)(x_\lambda) \geq \mathcal{N}_{\mathcal{T}}(U)(x_\lambda) \wedge \mathcal{N}_{\mathcal{T}}(V)(x_\lambda).$$

(LMFN4) We first show that

$$\mathcal{N}_{\mathcal{T}}(U)(x_\lambda) = \bigwedge_{\mu \lhd \lambda} \mathcal{N}_{\mathcal{T}}(U)(x_\mu). \tag{1}$$

By the definition of $\mathcal{N}_{\mathcal{T}}$, we can easily obtain

$$\mathcal{N}_{\mathcal{T}}(U)(x_\lambda) \leq \bigwedge_{\mu \lhd \lambda} \mathcal{N}_{\mathcal{T}}(U)(x_\mu).$$

On the other hand, if $a \lhd \bigwedge_{\mu \lhd \lambda} \mathcal{N}_{\mathcal{T}}(U)(x_\mu)$, then $a \lhd \mathcal{N}_{\mathcal{T}}(U)(x_\mu) = \bigvee_{x_\mu \leq G \leq U} \mathcal{T}(G)$ for each $\mu \lhd \lambda$. Further, there exists $G_{x_\mu} \in L^X$ such that $x_\mu \leq G_{x_\mu} \leq U$ and $a \leq \mathcal{T}(G_{x_\mu})$. Assuming $E = \bigvee_{\mu \lhd \lambda} G_{x_\mu}$, we have $x_\lambda \leq E \leq U$ and

$$a \leq \bigwedge_{\mu \lhd \lambda} \mathcal{T}(G_{x_\mu}) \leq \mathcal{T}(\bigvee_{\mu \lhd \lambda} G_{x_\mu}) = \mathcal{T}(E) \leq \bigvee_{x_\lambda \leq V \leq U} \mathcal{T}(V) = \mathcal{N}_{\mathcal{T}}(U)(x_\lambda).$$

This shows

$$\mathcal{N}_{\mathcal{T}}(U)(x_\lambda) \geq \bigwedge_{\mu \lhd \lambda} \mathcal{N}_{\mathcal{T}}(U)(x_\mu).$$

Now, let $x_\lambda \leq V \leq U$ and $\mu \lhd \lambda$, then we have

$$\mathcal{T}(V) \leq \bigwedge_{y_\mu \lhd V} \mathcal{N}_{\mathcal{T}}(V)(y_\mu) \leq \mathcal{N}_{\mathcal{T}}(V)(x_\mu) \leq \mathcal{N}_{\mathcal{T}}(U)(x_\mu).$$

So

$$\mathcal{N}_{\mathcal{T}}(U)(x_\lambda) = \bigvee_{x_\lambda \leq V \leq U} \mathcal{T}(V) \leq \bigvee_{x_\lambda \leq V \leq U} \bigwedge_{y_\mu \lhd V} \mathcal{N}_{\mathcal{T}}(V)(y_\mu) \leq \mathcal{N}_{\mathcal{T}}(U)(x_\mu).$$

Hence

$$\mathcal{N}_{\mathcal{T}}(U)(x_\lambda) \leq \bigvee_{x_\lambda \leq V \leq U} \bigwedge_{y_\mu \lhd V} \mathcal{N}_{\mathcal{T}}(V)(y_\mu) \leq \bigwedge_{\mu \lhd \lambda} \mathcal{N}_{\mathcal{T}}(U)(x_\mu) = \mathcal{N}_{\mathcal{T}}(U)(x_\lambda).$$

Therefore,

$$\mathcal{N}_{\mathcal{T}}(U)(x_\lambda) = \bigvee_{x_\lambda \leq V \leq U} \bigwedge_{y_\mu \lhd V} \mathcal{N}_{\mathcal{T}}(V)(y_\mu).$$

Step 2: Define $\mathcal{T}_{\mathcal{N}} : L^X \longrightarrow M$ by

$$\mathcal{T}_{\mathcal{N}}(U) = \bigwedge_{x_\lambda \lhd U} \mathcal{N}(U)(x_\lambda) \ (\forall U \in L^X).$$

Then $\mathcal{T}_{\mathcal{N}}$ is an (L, M)-fuzzy topology induced by $\mathcal{N}$.

In fact, (LMFT1) is easily obtained from (LMFN1).

(LMFT2) $\forall U, V \in L^X$,

$$\mathcal{T}_{\mathcal{N}}(U \wedge V) = \bigwedge_{x_\lambda \lhd U \wedge V} \mathcal{N}(U \wedge V)(x_\lambda) = \bigwedge_{x_\lambda \lhd U \wedge V} [\mathcal{N}(U)(x_\lambda) \wedge \mathcal{N}(V)(x_\lambda)]$$

$$\geq \left(\bigwedge_{x_\lambda \lhd U} \mathcal{N}(U)(x_\lambda)\right) \wedge \left(\bigwedge_{x_\lambda \lhd V} \mathcal{N}(V)(x_\lambda)\right) = \mathcal{T}_{\mathcal{N}}(U) \wedge \mathcal{T}_{\mathcal{N}}(V).$$

(LMFT3) $\forall \{E_j : j \in J\} \subseteq L^X$,

$$\mathcal{T}_{\mathcal{N}}\left(\bigvee_{j \in J} E_j\right) = \bigwedge_{x_\lambda \lhd \bigvee_{j \in J} E_j} \mathcal{N}\left(\bigvee_{j \in J} E_j\right)(x_\lambda) \geq \bigwedge_{j \in J} \bigwedge_{x_\lambda \lhd E_j} \mathcal{N}(E_j)(x_\lambda) = \bigwedge_{j \in J} \mathcal{T}_{\mathcal{N}}(E_j).$$

Step 3: We show that

$$\mathcal{N}_{\mathcal{T}_{\mathcal{N}}} = \mathcal{N}.$$

In fact, $\forall U \in L^X, \forall x_\lambda \in J(L^X)$, by (LMFN4), we have

$$\mathcal{N}_{\mathcal{T}_{\mathcal{N}}}(U)(x_\lambda) = \bigvee_{x_\lambda \leq V \leq U} \mathcal{T}_{\mathcal{N}}(V) = \bigvee_{x_\lambda \leq V \leq U} \bigwedge_{y_\mu \lhd V} \mathcal{N}(V)(y_\mu) = \mathcal{N}(U)(x_\lambda).$$

Hence $\mathcal{N}_{\mathcal{T}_{\mathcal{N}}} = \mathcal{N}$.

Step 4: We show that

$$\mathcal{T}(U) = \bigwedge_{x_\lambda \lhd U} \mathcal{N}_{\mathcal{T}}(U)(x_\lambda) \ (\forall U \in L^X) \text{ and } \mathcal{T}_{\mathcal{N}_{\mathcal{T}}} = \mathcal{T}.$$

In fact, for each $x_\lambda \lhd U$,

$$\mathcal{N}_{\mathcal{T}}(U)(x_\lambda) = \bigvee_{x_\lambda \leq V \leq U} \mathcal{T}(V) \geq \mathcal{T}(U).$$

Hence,

$$\bigwedge_{x_\lambda \lhd U} \mathcal{N}_{\mathcal{T}}(U)(x_\lambda) \geq \mathcal{T}(U).$$

On the other hand, if $a \lhd \bigwedge_{x_\lambda \lhd U} \mathcal{N}_{\mathcal{T}}(U)(x_\lambda)$, then $a \lhd \mathcal{N}_{\mathcal{T}}(U)(x_\lambda)$ for each $x_\lambda \lhd U$. Further, there exists $V_{x_\lambda} \in L^X$ such that $x_\lambda \leq V_{x_\lambda} \leq U$ and $a \leq \mathcal{T}(V_{x_\lambda})$. Obviously, $U = \bigvee_{x_\lambda \lhd U} V_{x_\lambda}$. So

$$\mathcal{T}(U) = \mathcal{T}\left(\bigvee_{x_\lambda \lhd U} V_{x_\lambda}\right) \geq \bigwedge_{x_\lambda \lhd U} \mathcal{T}(V_{x_\lambda}) \geq a.$$

This shows

$$\bigwedge_{x_\lambda \lhd U} \mathcal{N}_{\mathcal{T}}(U)(x_\lambda) \leq \mathcal{T}(U).$$

Hence

$$\mathcal{T}(U) = \bigwedge_{x_\lambda \lhd U} \mathcal{N}_{\mathcal{T}}(U)(x_\lambda) \ (\forall U \in L^X).$$

Now, by the definition of $\mathcal{T}_{\mathcal{N}}$, we have

$$\mathcal{T}_{\mathcal{N}_{\mathcal{T}}}(U) = \bigwedge_{x_\lambda \lhd U} \mathcal{N}_{\mathcal{T}}(U)(x_\lambda) = \mathcal{T}(U) \ (\forall U \in L^X).$$

Therefore, $\mathcal{T}_{\mathcal{N}_{\mathcal{T}}} = \mathcal{T}$.

Step 5: If $f:(X,\mathcal{T}_1)\longrightarrow(Y,\mathcal{T}_2)$ is continuous with respect to (L,M)-fuzzy topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, then

$$\mathcal{T}_1(f^{\leftarrow}(U))\geq\mathcal{T}_2(U)\ (\forall U\in L^Y).$$

Hence

$$\mathcal{N}_{\mathcal{T}_2}(U)\,(f^{\rightarrow}(x_\lambda))=\bigvee_{f^{\rightarrow}(x_\lambda)\leq V\leq U}\mathcal{T}_2(V)\leq\bigvee_{x_\lambda\leq f^{\leftarrow}(V)\leq f^{\leftarrow}(U)}\mathcal{T}_1(f^{\leftarrow}(V))$$
$$\leq\mathcal{N}_{\mathcal{T}_1}(f^{\leftarrow}(U))(x_\lambda).$$

Therefore $f:(X,\mathcal{N}_{\mathcal{T}_1})\longrightarrow(Y,\mathcal{N}_{\mathcal{T}_2})$ is continuous with respect to (L,M)-fuzzy neighborhood systems $\mathcal{N}_{\mathcal{T}_1}$ and $\mathcal{N}_{\mathcal{T}_2}$.

Step 6: If $f:(X,\mathcal{N}_1)\longrightarrow(Y,\mathcal{N}_2)$ is continuous with respect to (L,M)-fuzzy neighborhood systems $\mathcal{N}_1$ and $\mathcal{N}_1$, then

$$\mathcal{N}_2(V)\,(f^{\rightarrow}(x_\lambda))\leq\mathcal{N}_1(f^{\leftarrow}(V))(x_\lambda)\ (\forall V\in L^Y,\forall x_\lambda\in J(L^X)).$$

Hence

$$\mathcal{T}_{\mathcal{N}_2}(V)=\bigwedge_{y_\mu\lhd V}\mathcal{N}_2(V)(y_\mu)\leq\bigwedge_{f^{\rightarrow}(x_\lambda)\lhd V}\mathcal{N}_2(V)(f^{\rightarrow}(x_\lambda))=\bigwedge_{x_\lambda\lhd f^{\leftarrow}(V)}\mathcal{N}_2(V)(f^{\rightarrow}(x_\lambda)$$
$$\leq\bigwedge_{x_\lambda\lhd f^{\leftarrow}(V)}\mathcal{N}_1(f^{\leftarrow}(V))(x_\lambda)=\mathcal{T}_{\mathcal{N}_1}(f^{\leftarrow}(V)).$$

Therefore $f:(X,\mathcal{T}_{\mathcal{N}_1})\longrightarrow(Y,\mathcal{T}_{\mathcal{N}_2})$ is continuous with respect to (L,M)-fuzzy topologies $\mathcal{T}_{\mathcal{N}_1}$ and $\mathcal{T}_{\mathcal{N}_2}$.

□

2. Further Study on (L,M)-fuzzy Topologies and (L,M)-fuzzy Neighborhood Systems

Theorem 2.1. *Let X be a nonempty set, let $(Y,\mathcal{T}_Y)$ be an (L,M)-fuzzy topological space, and let $f:X\longrightarrow Y$ be a mapping. Define $\mathcal{N}:L^X\longrightarrow M^{J(L^X)}$ as follows:*

$$\mathcal{N}(A)(x_\lambda)=\mathcal{N}_{\mathcal{T}_Y}([f^{\rightarrow}(A')]')\,(f^{\rightarrow}(x_\lambda)).$$

Then $\mathcal{N}$ is an (L,M)-fuzzy neighborhood system on X.

Proof. (LMFN1–LMFN2). $\mathcal{N}(1_X)(x_\lambda)=\mathcal{N}_{\mathcal{T}_Y}(1_Y)(f^{\rightarrow}(x_\lambda))=1$. $x_\lambda\not\leq A$, then $f^{\rightarrow}(x_\lambda)\not\leq[f^{\rightarrow}(A')]'$. In fact, if we have $f^{\rightarrow}(x_\lambda)\leq[f^{\rightarrow}(A')]'$, thus

$$x_\lambda\leq f^{\leftarrow}[f^{\rightarrow}(x_\lambda)]\leq f^{\leftarrow}([f^{\rightarrow}(A')]')=[f^{\leftarrow}f^{\rightarrow}(A')]',$$

so $(x_\lambda)'\geq f^{\leftarrow}f^{\rightarrow}(A')\geq A'$. Hence $x_\lambda\leq A$, which is a contradiction. Therefore,

$$\mathcal{N}(0_X)(x_\lambda)=\mathcal{N}_{\mathcal{T}_Y}([f^{\rightarrow}(1_X)]')(f^{\rightarrow}(x_\lambda))=0$$

and

$$\mathcal{N}(A)(x_\lambda)=\mathcal{N}_{\mathcal{T}_Y}([f^{\rightarrow}(A')]')(f^{\rightarrow}(x_\lambda))=0\ (\forall\ x_\lambda\not\leq A).$$

(LMFN3) For each $A=A_1\wedge A_2$, we have

$$f^{\rightarrow}(A')=f^{\rightarrow}(A_1'\vee A_2')=f^{\rightarrow}(A_1')\vee f^{\rightarrow}(A_2').$$

Hence

$$\begin{aligned}\mathcal{N}(A_1 \wedge A_2)(x_\lambda) &= \mathcal{N}_{\mathcal{T}_Y}([f^{\rightarrow}((A_1 \wedge A_2)')]')(f^{\rightarrow}(x_\lambda))\\ &= \mathcal{N}_{\mathcal{T}_Y}([f^{\rightarrow}(A_1') \vee f^{\rightarrow}(A_2')]')(f^{\rightarrow}(x_\lambda))\\ &= \mathcal{N}_{\mathcal{T}_Y}([f^{\rightarrow}(A_1')]')(f^{\rightarrow}(x_\lambda)) \wedge \mathcal{N}_{\mathcal{T}_Y}([f^{\rightarrow}(A_2')]')(f^{\rightarrow}(x_\lambda))\\ &= \mathcal{N}(A_1)(x_\lambda) \wedge \mathcal{N}(A_2)(x_\lambda).\end{aligned}$$

Therefore, $\mathcal{N}(A_1 \wedge A_2)(x_\lambda) = \mathcal{N}(A_1)(x_\lambda) \wedge \mathcal{N}(A_2)(x_\lambda)$.

(LMFN4) **Step 1:** We show that

$$\mathcal{N}(A)(x_\lambda) = \bigvee_{B \in L^Y} \{\mathcal{N}_{\mathcal{T}_Y}(B)(f^{\rightarrow}(x_\lambda)) \mid f^{\leftarrow}(B) \leq A\}.$$

If $f^{\leftarrow}(B) \leq A$, then $A' \leq f^{\leftarrow}(B')$ and $f^{\rightarrow}(A') \leq B'$, so $A' \leq f^{\leftarrow}(B')$ and $B \leq (f^{\rightarrow}(A'))'$. Hence

$$\mathcal{N}(A)(x_\lambda) = \mathcal{N}_{\mathcal{T}_Y}([f^{\rightarrow}(A')]')(f^{\rightarrow}(x_\lambda)) \geq \mathcal{N}_{\mathcal{T}_Y}(B)(f^{\rightarrow}(x_\lambda)).$$

Therefore,

$$\mathcal{N}(A)(x_\lambda) \geq \bigvee_{B \in L^Y} \{\mathcal{N}_{\mathcal{T}_Y}(B)(f^{\rightarrow}(x_\lambda)) \mid f^{\leftarrow}(B) \leq A\}.$$

On the other hand, let $B = (f^{\rightarrow}(A'))'$, we have $f^{\leftarrow}(B) = (f^{\leftarrow}f^{\rightarrow}(A'))'$, thus

$$(f^{\leftarrow}(B))' = f^{\leftarrow}f^{\rightarrow}(A') \geq A',$$

so $f^{\leftarrow}(B) \leq A$. Hence

$$\begin{aligned}\mathcal{N}(A)(x_\lambda) &= \mathcal{N}_{\mathcal{T}_Y}([f^{\rightarrow}(A')]')(f^{\rightarrow}(x_\lambda))\\ &= \mathcal{N}_{\mathcal{T}_Y}(B)(f^{\rightarrow}(x_\lambda)) \leq \bigvee_{B \in L^Y} \{\mathcal{N}_{\mathcal{T}_Y}(B)(f^{\rightarrow}(x_\lambda)) \mid f^{\leftarrow}(B) \leq A\}.\end{aligned}$$

Step 2: We show that

$$\mathcal{N}(A)(x_\lambda) = \bigvee_{x_\lambda \leq V \leq A} \bigwedge_{y_\mu \triangleleft V} \mathcal{N}(V)(y_\mu).$$

By Step 1, let $a \triangleleft \mathcal{N}(A)(x_\lambda)$. Then there exists $B \in L^Y$ satisfying $f^{\leftarrow}(B) \leq A$ such that $a \triangleleft \mathcal{N}_{\mathcal{T}_Y}(B)(f^{\rightarrow}(x_\lambda))$, since

$$\mathcal{N}_{\mathcal{T}_Y}(B)(f^{\rightarrow}(x_\lambda)) = \bigvee_{f^{\rightarrow}(x_\lambda) \leq V \leq B} \bigwedge_{z_t \triangleleft V} \mathcal{N}_{\mathcal{T}_Y}(V)(z_t).$$

So there exists $V \in L^Y$ satisfying $f^{\rightarrow}(x_\lambda) \leq V \leq B$ such that $a \triangleleft \mathcal{N}_{\mathcal{T}_Y}(V)(z_t)$ for each $z_t \triangleleft V$. Let $U = f^{\leftarrow}(V)$, then $x_\lambda \leq U \leq A$ for all $y_\mu \triangleleft U$. By Step 1, we have

$$a \triangleleft \mathcal{N}_{\mathcal{T}_Y}(V)(f^{\rightarrow}(y_\mu)) \leq \mathcal{N}(U)(y_\mu).$$

Hence $\mathcal{N}(A)(x_\lambda) \leq \bigvee_{x_\lambda \leq V \leq A} \bigwedge_{y_\mu \triangleleft V} \mathcal{N}(V)(y_\mu)$.

On the other hand, let $b \in M$ and $\bigvee_{x_\lambda \leq V \leq A} \bigwedge_{y_\mu \triangleleft V} \mathcal{N}(V)(y_\mu) \not\leq b$. Then there exists $a \in \alpha(b)$ (where $\alpha(b)$ is the largest maximal set of b (see [12])) such that $\bigvee_{x_\lambda \leq V \leq A} \bigwedge_{y_\mu \triangleleft V} \mathcal{N}(V)(y_\mu) \not\leq a$. Further, there exists $V \in L^Y$ such that $x_\lambda \leq V \leq$

A and $\bigwedge_{y_\mu \lhd V} \mathcal{N}(V)(y_\mu) \nleq a$, thus $\mathcal{N}(V)(y_\mu) \nleq a$ $(\forall y_\mu \lhd V)$, and, in particular, $\mathcal{N}(V)(x_\gamma) \nleq a$ $(\forall \gamma \lhd \lambda)$.

By Step 1, we have

$$\mathcal{N}(V)(x_\gamma) = \bigvee_{D \in L^Y} \{\mathcal{N}_{\mathcal{T}_Y}(D)(f^{\rightarrow}(x_\gamma)) \mid f^{\leftarrow}(D) \leq V\}.$$

There exists $D \in L^Y$ such that $f^{\leftarrow}(D) \leq V$ and $\mathcal{N}_{\mathcal{T}_Y}(D)(f^{\rightarrow}(x_\gamma)) \nleq a$, and therefore $f^{\leftarrow}(D) \leq A$. By Lemma 1.3 , we have

$$\mathcal{N}_{\mathcal{T}_Y}(D)(f^{\rightarrow}(x_\lambda)) = \bigwedge_{f^{\rightarrow}(x_\gamma) \lhd f^{\rightarrow}(x_\lambda)} \mathcal{N}_{\mathcal{T}_Y}(D)(f^{\rightarrow}(x_\gamma)) \nleq b.$$

By Step 1, we have $\mathcal{N}(A)(x_\lambda) \nleq b$. Hence $\mathcal{N}(A)(x_\lambda) \geq \bigvee_{x_\lambda \leq V \leq A} \bigwedge_{h \lhd V} \mathcal{N}(V)(h)$. □

Theorem 2.2. *Let $\mathcal{N}, \mathcal{T}_Y$ and f be defined as in Theorem 2.1 and define a mapping $f^{\leftarrow}(\mathcal{T}_Y) : L^X \longrightarrow M$ by*

$$f^{\leftarrow}(\mathcal{T}_Y)(A) = \bigwedge_{x_\lambda \lhd A} \mathcal{N}(A)(x_\lambda) \ (\forall A \in L^X).$$

Then

(1) $f^{\leftarrow}(\mathcal{T}_Y)$ is the weakest (L, M)-fuzzy topology on X such that f is continuous.

(2) If $(Z, \mathcal{T}_Z)$ is an (L, M)-fuzzy topological space and $g : (Z, \mathcal{T}_Z) \longrightarrow (X, f^{\leftarrow}(\mathcal{T}_Y))$ is a map, then g is continuous iff $f \circ g$ is continuous.

Proof. (1) First, by Lemma 1.3 , we know that $f^{\leftarrow}(\mathcal{T}_Y) = \mathcal{T}_{\mathcal{N}}$ is an (L, M)-fuzzy topology on X. Second, we show that f is continuous, i.e., $\mathcal{T}_{\mathcal{N}}(f^{\leftarrow}(A)) \geq \mathcal{T}_Y(A)$ for each $A \in L^Y$. In fact, by Lemma 1.3, we can obtain

$$\begin{aligned}\mathcal{T}_{\mathcal{N}}(f^{\leftarrow}(A)) &= \bigwedge_{x_\lambda \lhd f^{\leftarrow}(A)} \mathcal{N}(f^{\leftarrow}(A))(x_\lambda) = \bigwedge_{x_\lambda \lhd f^{\leftarrow}(A)} \mathcal{N}_{\mathcal{T}_Y}([f^{\rightarrow}(f^{\leftarrow}(A'))]')(f^{\rightarrow}(x_\lambda)) \\ &\geq \bigwedge_{x_\lambda \lhd f^{\leftarrow}(A)} \mathcal{N}_{\mathcal{T}_Y}(A)(f^{\rightarrow}(x_\lambda)) = \bigwedge_{f^{\rightarrow}(x_\lambda) \lhd f^{\rightarrow} f^{\leftarrow}(A)} \mathcal{N}_{\mathcal{T}_Y}(A)(f^{\rightarrow}(x_\lambda)) \\ &\geq \bigwedge_{f^{\rightarrow}(x_\lambda) \lhd A} \mathcal{N}_{\mathcal{T}_Y}(A)(f^{\rightarrow}(x_\lambda)) = \mathcal{T}_Y(A).\end{aligned}$$

Hence f is continuous.

Now, let $\mathcal{T}_X$ be an (L, M)-fuzzy topology on X such that f is continuous, and let $A \in L^X$. If $B = (f^{\rightarrow}(A'))'$, then $f^{\leftarrow}(B) \leq A$. We only need to show that $\mathcal{T}_X(A) \geq \mathcal{T}_{\mathcal{N}}(A)$ $(\forall A \in L^X)$. In fact, since $f : (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ is continuous, we have that $f : (X, \mathcal{N}_{\mathcal{T}_X}) \longrightarrow (Y, \mathcal{N}_{\mathcal{T}_Y})$ is continuous, and then for all $A \in L^X$, we have

$$\mathcal{N}_{\mathcal{T}_X}(A)(x_\lambda) \geq \mathcal{N}_{\mathcal{T}_X}(f^{\leftarrow}(B))(x_\lambda) \geq \mathcal{N}_{\mathcal{T}_Y}(B)(f^{\rightarrow}(x_\lambda) = \mathcal{N}(A)(x_\lambda).$$

For any $A \in L^X$, we have

$$\mathcal{T}_X(A) = \mathcal{T}_{\mathcal{N}_{\mathcal{T}_X}}(A) = \bigwedge_{x_\lambda \lhd A} \mathcal{N}_{\mathcal{T}_X}(A)(x_\lambda) \geq \bigwedge_{x_\lambda \lhd A} \mathcal{N}(A)(x_\lambda) = \mathcal{T}_{\mathcal{N}}(A).$$

So $\mathcal{T}_X \geq \mathcal{T}_\mathcal{N}$. Hence $\mathcal{T}_\mathcal{N}$ is the weakest (L, M)-fuzzy topology on X such that f is continuous.

(2) If g is continuous, then $f \circ g$ is continuous. Now, suppose $f \circ g$ is continuous. we need to show that $\mathcal{T}_Z(g^{\leftarrow}(A)) \geq \mathcal{T}_\mathcal{N}(A)$ $(\forall A \in L^X)$. By Lemma 1.3, we only need to show that

$$\mathcal{N}_{\mathcal{T}_Z}(g^{\leftarrow}(A))(z_\lambda) \geq \mathcal{N}_{\mathcal{T}_\mathcal{N}}(A)(g^{\rightarrow}(z_\lambda)) = \mathcal{N}(A)(g^{\rightarrow}(z_\lambda)) \ (\forall z_\lambda \in J(L^Z), \forall A \in L^X).$$

In fact, for $a \lhd \mathcal{N}(A)(g^{\rightarrow}(z_\lambda))$, there exists $f^{\leftarrow}(B) \leq A$ such that

$$a \lhd \mathcal{N}_{\mathcal{T}_Y}(B)(f^{\rightarrow}(g^{\rightarrow}(z_\lambda))).$$

Hence

$$a \lhd \mathcal{N}_{\mathcal{T}_Y}(B)(f^{\rightarrow}(g^{\rightarrow}(z_\lambda))) \leq \mathcal{N}_{\mathcal{T}_Z}(g^{\leftarrow}(f^{\leftarrow}(B)))(z_\lambda) \leq \mathcal{N}_{\mathcal{T}_Z}(g^{\leftarrow}(A))(z_\lambda).$$

Therefore, $\mathcal{N}_{\mathcal{T}_Z}(g^{\leftarrow}(A))(z_\lambda) \geq \mathcal{N}(A)(g^{\rightarrow}(z_\lambda))$. □

Theorem 2.3. *Let X be a nonempty set, let $\{(X_i, \mathcal{T}_i)\}_{i \in I}$ be a collection of (L, M)-fuzzy topological space and let $f_j : X \longrightarrow X_j$ be a mapping for each $j \in I$. Define $\mathcal{N} : L^X \longrightarrow M^{J(L^X)}$ by*

$$\mathcal{N}(A)(x_\lambda) = \bigvee_{J \subseteq I finite} \left\{ \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}_j}(A_j)(f_j^{\rightarrow}(x_\lambda)) \mid \bigwedge_{j \in J} f_j^{\leftarrow}(A_j) \leq A \right\},$$

where I is an index set. Then

(1) $\mathcal{N}$ is an (L, M)-fuzzy neighborhood system on X.

(2) Define a mapping $\mathcal{T}_\mathcal{N} : L^X \longrightarrow M$ as follows:

$$\mathcal{T}_\mathcal{N}(A) = \bigwedge_{x_\lambda \lhd A} \mathcal{N}(A)(x_\lambda).$$

Then $\mathcal{T}_\mathcal{N}$ is the weakest (L, M)-fuzzy topology on X such that each f_j is continuous for each $j \in I$, and $\mathcal{T}_\mathcal{N} = \bigvee_{j \in I} f_j^{\leftarrow}(\mathcal{T}_i)$.

(3) If $(Z, \mathcal{T}_Z)$ is an (L, M)-fuzzy topological space and $g : (Z, \mathcal{T}_Z) \longrightarrow (X, \mathcal{T}_\mathcal{N})$ a function, then g is continuous if and only if $f_j \circ g$ $(j \in I)$ is continuous.

Proof. (1) (LMFN1)–(LMFN2) are easily obtained.

(LMFN3) If $A \leq B$, then we can easily obtain $\mathcal{N}(A)(x_\lambda) \leq \mathcal{N}(B)(x_\lambda)$. Hence

$$\mathcal{N}(A \wedge B)(x_\lambda) \leq \mathcal{N}(A)(x_\lambda) \wedge \mathcal{N}(B)(x_\lambda).$$

On the other hand, suppose that $a \lhd \mathcal{N}(A)(x_\lambda) \wedge \mathcal{N}(B)(x_\lambda)$. There exist finite subsets J_1, J_2 of I, $A_j \in L^{X_j}$ $(\forall j \in J_1)$, $B_j \in L^{X_j}$ $(\forall j \in J_2)$ such that

$$\bigwedge_{j \in J_1} f_j^{\leftarrow}(A_j) \leq A, \quad \bigwedge_{j \in J_2} f_j^{\leftarrow}(B_j) \leq B,$$

$$a \lhd \bigwedge_{j \in J_1} \mathcal{N}_{\mathcal{T}_j}(A_j)(f_j^{\rightarrow}(x_\lambda)), \text{and } a \lhd \bigwedge_{j \in J_2} \mathcal{N}_{\mathcal{T}_j}(B_j)(f_j^{\rightarrow}(x_\lambda)).$$

Let $J = J_1 \cup J_2$. Taking $A_j = 1$ ($\forall j \in J - J_1$), we may suppose that $J = J_1$, Taking $B_j = 1$ ($\forall j \in J - J_2$), we may suppose that $J = J_2$. Let $C_j = A_j \wedge B_j$ for every $j \in J$. Then $\bigwedge_{j\in J} f_j^{\leftarrow}(C_j) \le A \wedge B$ and $a \le \bigwedge_{j\in J} \mathcal{N}(C_j)(f_j^{\rightarrow}(x_\lambda))$. Therefore

$$\mathcal{N}(A \wedge B)(x_\lambda) \ge \mathcal{N}(A)(x_\lambda) \wedge \mathcal{N}(B)(x_\lambda).$$

(LMFN4) Suppose that $a \lhd \mathcal{N}(A)(x_\lambda)$. Then there exists a finite subset J of I and $A_j \in L^{X_j}$ ($\forall j \in J$) such that

$$\bigwedge_{j\in J} f_j^{\leftarrow}(A_j) \le A, \quad a \lhd \mathcal{N}_{\mathcal{T}_j}(A_j)(f_j^{\rightarrow}(x_\lambda)) \ (\forall j \in J).$$

Since

$$\mathcal{N}_{\mathcal{T}_j}(A_j)(f_j^{\rightarrow}(x_\lambda)) = \bigvee_{f_j^{\rightarrow}(x_\lambda) \le B_j \le A_j} \ \bigwedge_{y_{\mu_j} \lhd B_j} \mathcal{N}_{\mathcal{T}_j}(B_j)(y_{\mu_j}),$$

there exists $f_j^{\rightarrow}(x_\lambda) \le B_j \le A_j$ such that $a \lhd \bigwedge_{y_{\mu_j} \lhd B_j} \mathcal{N}_{\mathcal{T}_j}(B_j)(y_{\mu_j})$. Let

$$B = \bigwedge_{j\in J} f_j^{\leftarrow}(B_j),$$

then $x_\lambda \le B \le A$. For all $y_\mu \lhd B$, we have

$$a \lhd \bigwedge_{y_{\mu_j} \lhd B_j} \mathcal{N}_{\mathcal{T}_j}(B_j)(y_{\mu_j}) \le \bigwedge_{j\in J} \mathcal{N}_{\mathcal{T}_j}(B_j)(f_j^{\rightarrow}(y_\mu)) \le \mathcal{N}(B)(y_\mu).$$

Hence

$$\mathcal{N}(A)(x_\lambda) \le \bigvee_{x_\lambda \le B \le A} \ \bigwedge_{y_\mu \lhd B} \mathcal{N}(B)(y_\mu).$$

On the other hand, suppose that $b \in M$ and

$$\bigvee_{x_\lambda \le B \le A} \ \bigwedge_{y_\mu \lhd B} \mathcal{N}(B)(y_\mu) \not\le b.$$

Then there exists $a \in \alpha(b)$ such that

$$\bigvee_{x_\lambda \le B \le A} \ \bigwedge_{y_\mu \lhd B} \mathcal{N}(B)(y_\mu) \not\le a.$$

Further, there exists $B \in L^X$ such that $x_\lambda \le B \le A$ and $\bigwedge_{y_\mu \lhd B} \mathcal{N}(B)(y_\mu) \not\le a$. Hence $\mathcal{N}(B)(y_\mu) \not\le a$ for any $y_\mu \lhd B$. In particular, $\mathcal{N}(B)(x_\gamma) \not\le a$ for each $\gamma \lhd \lambda$ (this is because $x_\gamma \lhd x_\lambda \le B \Longrightarrow x_\gamma \lhd B$). By the definition of $\mathcal{N}$, there exist finite subsets J_1 of I, $B_j \in L^{X_j}$ ($\forall j \in J_1$) such that $\bigwedge_{j\in J_1} f_j^{\leftarrow}(B_j) \le B$ (thus we have $\bigwedge_{j\in J_1} f_j^{\leftarrow}(B_j) \le A$) and $\bigwedge_{j\in J_1} \mathcal{N}_{\mathcal{T}_j}(B_j)(f_j^{\rightarrow}(x_\gamma)) \not\le a$. By (1), we can obtain

$$\bigwedge_{j\in J_1} \mathcal{N}_{\mathcal{T}_j}(B_j)(f_j^{\rightarrow}(x_\lambda)) = \bigwedge_{j\in J_1} \bigwedge_{\gamma \lhd \lambda} \mathcal{N}_{\mathcal{T}_j}(B_j)(f_j^{\rightarrow}(x_\gamma)) = \bigwedge_{\gamma \lhd \lambda} \bigwedge_{j\in J_1} \mathcal{N}_{\mathcal{T}_j}(B_j)(f_j^{\rightarrow}(x_\gamma)) \not\le b.$$

By the definition of $\mathcal{N}$, since $\bigwedge_{j\in J_1} f_j^{\leftarrow}(B_j) \leq A$, we have $\mathcal{N}(A)(x_\lambda) \nleq b$. This shows

$$\bigvee_{x_\lambda \leq B \leq A} \bigwedge_{g \triangleleft B} \mathcal{N}(B)(y_\mu) \leq \mathcal{N}(A)(x_\lambda).$$

(2) By Lemma 1.3, it is obvious that $\mathcal{T}_\mathcal{N}$ is an $(L, M)-$fuzzy topology on X. In order to prove that $f_j : (X, \mathcal{T}_\mathcal{N}) \longrightarrow (X_j, \mathcal{T}_j)$ is continuous, i.e.,

$$\mathcal{T}_\mathcal{N}(f_j^{\leftarrow}(A_j)) \geq \mathcal{T}_j(A_j) = \mathcal{T}_{\mathcal{N}_{\mathcal{T}_j}}(A_j)\ (\forall A_j \in L^{X_j}, \forall j \in I),$$

we need to prove that $f_j : (X, \mathcal{N}) \longrightarrow (X_j, \mathcal{N}_{\mathcal{T}_j})$ is continuous. In fact, $\forall x_\lambda \in J(L^X)$, $A_j \in L^{X_j}$, by the definition of $\mathcal{N}$,

$$\mathcal{N}(f_j^{\leftarrow}(A_j))(x_\lambda) \geq \mathcal{N}_{\mathcal{T}_j}(A_j)(f_j^{\rightarrow}(x_\lambda)).$$

By Theorem 2.2, we have $\mathcal{T}_\mathcal{N} \geq f_j^{\leftarrow}(\mathcal{T}_j)$ $(\forall j \in I)$.
Hence

$$\mathcal{T}_\mathcal{N} \geq \mathcal{T}^\star = \bigvee_{j\in I} f_j^{\leftarrow}(\mathcal{T}_j).$$

On the other hand, suppose that for every $x_\lambda \in J(L^X)$, $A_j \in L^{X_j}$ and every finite subset $J \subseteq I$ and $\bigwedge_{j\in J} f_j^{\leftarrow}(A_j) \leq A$. We have that

$$\begin{aligned}\mathcal{N}_{\mathcal{T}^\star}(A)(x_\lambda) &\geq \mathcal{N}_{\mathcal{T}^\star}(\bigwedge_{j\in J} f_j^{\leftarrow}(A_j))(x_\lambda)\\ &= \bigwedge_{j\in J} \mathcal{N}_{\mathcal{T}^\star}(f_j^{\leftarrow}(A_j))(x_\lambda)\\ &\geq \bigwedge_{j\in J} \mathcal{N}_{\mathcal{T}_j}(A_j)(f_j^{\rightarrow}(x_\lambda)).\end{aligned}$$

By the definition of $\mathcal{N}$, we have $\mathcal{N}_{\mathcal{T}^\star} \geq \mathcal{N}$. Further, by Lemma 1.3, we have

$$\mathcal{T}^\star = \mathcal{T}_{\mathcal{N}_{\mathcal{T}^\star}} \geq \mathcal{T}_\mathcal{N}.$$

Therefore $\mathcal{T}_\mathcal{N} = \bigvee_{j\in I} f_j^{\leftarrow}(\mathcal{T}_j)$.

Now, since $f_j : (X, \mathcal{T}_\mathcal{N}) \longrightarrow (X_j, \mathcal{T}_j)$ is continuous, suppose that δ is an $(L, M)-$fuzzy topology on X such that $f_j : (X, \delta) \longrightarrow (X_j, \mathcal{T}_j)$ is continuous for each $j \in I$. By Theorem 2.2, we have $\delta \geq f_j^{\leftarrow}(\mathcal{T}_j)$ for each $j \in I$, and therefore $\delta \geq \mathcal{T}^\star = \bigvee_{j\in I} f_j^{\leftarrow}(\mathcal{T}_j)$.

(3) Necessity is straightforward. Suppose that $f_j \circ g$ is continuous for each $j \in I$. We show that $g : (Z, \mathcal{N}_{\mathcal{T}_Z}) \longrightarrow (X, \mathcal{N})$ is continuous i.e.

$$\mathcal{N}_{\mathcal{T}_Z}(g^{\leftarrow}(A))(x_\lambda) \geq \mathcal{N}(A)(g^{\rightarrow}(x_\lambda))\ (\forall x_\lambda \in J(L^X), \forall A \in L^X).$$

In fact, suppose that $a \triangleleft \mathcal{N}(A)(g^{\rightarrow}(x_\lambda))$. Then there exists a finite subset J of I such that $\bigwedge_{j\in J} f_j^{\leftarrow}(A_j) \leq A$ and $a \triangleleft \bigwedge_{j\in J} \mathcal{N}_{\mathcal{T}_j}(A_j)((f_j \circ g)^{\rightarrow}(x_\lambda))$. If $B = \bigwedge_{j\in J} f_j^{\leftarrow}(A_j)$, then $g^{\leftarrow}(B) \le g^{\leftarrow}(A)$ and

$$\begin{aligned} a \lhd \bigwedge_{j\in J} \mathcal{N}_{\mathcal{T}_j}(A_j)(f_j\circ g)^{\rightarrow}(x_\lambda)) \\ \le \bigwedge_{j\in J} \mathcal{N}_{\mathcal{T}_Z}((f_j\circ g)^{\leftarrow}(A_j))(x_\lambda) \\ = \mathcal{N}_{\mathcal{T}_Z}(g^{\leftarrow}(\bigwedge_{j\in J} f_j^{\leftarrow}(A_j))(x_\lambda) \\ = \mathcal{N}_{\mathcal{T}_Z}(g^{\leftarrow}(B))(x_\lambda) \le \mathcal{N}_{\mathcal{T}_Z}(g^{\leftarrow}(A))(x_\lambda). \end{aligned}$$

□

3. Subspaces and Product Spaces

Theorem 3.1. *Let $(Y, \mathcal{N}_Y)$ be an (L, M)-fuzzy neighborhood system, let X be a subset of Y, and let $id_Y|_X : X \longrightarrow Y$ be its respective embedding. Define $\mathcal{N}|_X : L^X \longrightarrow M^{J(L^X)}$ as follows:*

$$\begin{aligned} \mathcal{N}|_X(A)(x_\lambda) &= \mathcal{N}_{\mathcal{T}_{\mathcal{N}_Y}}\left([(id_Y|_X)^{\rightarrow}(A')]'\right)\left((id_Y|_X)^{\rightarrow}(x_\lambda)\right) \\ &= \mathcal{N}_Y\left([(id_Y|_X)^{\rightarrow}(A')]'\right)(x_\lambda). \end{aligned}$$

Then $\mathcal{N}|_X$ is an (L, M)-fuzzy neighborhood system on X.

Proof. The proof of Theorem 3.1 is easily obtained from Theorem 2.1. □

Definition 3.2. If $\mathcal{N}|_X$ be defined as in Theorem 3.1, then the pair $(X, \mathcal{N}|_X)$ is called a subspace of $(Y, \mathcal{N}_Y)$.

Theorem 3.3. $\mathcal{N}|_X(A) = \bigvee\{\mathcal{N}_Y(D)(x_\lambda) \mid D|_X = A\}$ $(\forall A \in L^X)$.

Proof. Let $[(id_Y|_X)^{\rightarrow}(A')]' = C$, we have $C|_X = A$. By Theorem 3.1,

$$\begin{aligned} \mathcal{N}|_X(A)(x_\lambda) &= \mathcal{N}_Y\left([(id_Y|_X)^{\rightarrow}(A')]'\right)(x_\lambda) = \mathcal{N}_Y(C)(x_\lambda) \\ &\le \bigvee\{\mathcal{N}_Y(D)(x_\lambda) \mid D|_X = A\}. \end{aligned}$$

On the other hand, by the proof of Theorem 2.1(see (LMFN4)) and Theorem 3.1,

$$\begin{aligned} \mathcal{N}|_X(A)(x_\lambda) &= \bigvee\{\mathcal{N}_{\mathcal{T}_{\mathcal{N}_Y}}(B)(x_\lambda) \mid (id_Y|_X)^{\leftarrow}(B) \le A\} \\ &= \bigvee\{\mathcal{N}_Y(B)(x_\lambda) \mid (id_Y|_X)^{\leftarrow}(B) \le A\} \ge \bigvee\{\mathcal{N}_Y(D)(x_\lambda) \mid D|_X = A\}. \end{aligned}$$

□

Definition 3.4. For any set X, let $\{(X_j, \mathcal{T}_j)\}_{j\in I}$ be a family of (L, M)-**FTOP**-objects, let $X = \prod_{j\in I} X_j$, and let $p_j : X \longrightarrow X_j$ be the j-th projection. The product (L, M)-fuzzy topology on X, denoted by $\prod_{j\in I} \mathcal{T}_j$, is the weakest (L, M)-fuzzy topology on X such that p_j is continuous. The pair $(X, \prod_{j\in I} \mathcal{T}_j)$ is called the product space of $\{(X_j, \mathcal{T}_j)\}_{j\in I}$.

Theorem 3.5. *(1) If* $\mathcal{T} = \prod_{j\in I} \mathcal{T}_j$, *then* $\mathcal{T} = \bigvee_{j\in I} p_j^{\leftarrow}(\mathcal{T}_j)$.

(2) If $(Y, \mathcal{T}_Y)$ *is an* (L, M)*-fuzzy topological space, then a mapping* $g : Y \longrightarrow X$ *is continuous if and only if* $p_j \circ g$ $(\forall j \in I)$ *is continuous.*

(3) $\forall x_\lambda \in J(L^X)$, $\forall A \in L^X$ *and every index set* I, *we have*

$$\mathcal{N}_{\mathcal{T}}(A)(x_\lambda) = \bigvee_{J\subseteq I finite} \left\{ \bigwedge_{j\in J} \mathcal{N}_{\mathcal{T}_j}(A_j)(p_j^{\rightarrow}(x_\lambda)) \mid \bigwedge_{j\in J} p_j^{\leftarrow}(A_j) \leq A \right\}.$$

(4) If J *is a finite subset of* I *and* $A = \prod_{j\in I} A_j$, *and* $A_j = 1$ *when* $j \notin J$, *then*

$$\mathcal{N}_{\mathcal{T}}(A)(x_\lambda) = \bigwedge_{j\in I} \mathcal{N}_{\mathcal{T}}(A_j)(p_j^{\rightarrow}(x_\lambda)), \quad \mathcal{T}(A) = \bigwedge_{j\in J} \mathcal{T}_j(A_j).$$

Proof. By $\mathcal{N} = \mathcal{N}_{\mathcal{T}_{\mathcal{N}}}$ and Theorem 2.3, we can easily obtain (1)–(3).

(4) We first show that

$$\mathcal{N}_{\mathcal{T}}(A)(x_\lambda) = \bigwedge_{j\in I} \mathcal{N}_{\mathcal{T}}(A_j)(p_j^{\rightarrow}(x_\lambda)).$$

It is obvious when $A = 1_X$ or $A = 0_X$. Without loss of generality, we assume $A \neq 1_X$ and $A \neq 0_X$. We also assume that $A_j \neq 1$ for each $j \in J$ (if not , then we have $\mathcal{N}_{\mathcal{T}}(A_j)(p_j^{\rightarrow}(x_\lambda)) = 1$). By the definition of $\mathcal{N}_{\mathcal{T}}$, it is obvious that

$$\mathcal{N}_{\mathcal{T}}(A)(x_\lambda) \geq \bigwedge_{j\in J} \mathcal{N}_{\mathcal{T}_j}(A_j)(p_j^{\rightarrow}(x_\lambda)).$$

On the other hand, let J_1 be a finite subset of I, and let $B_j \in L^{X_j}$ $(\forall j \in J_1)$ be such that $B = \bigwedge_{j\in J_1} p_j^{\leftarrow}(B_j) \leq A$. By $A = \prod_{j\in I} A_j = \bigwedge_{j\in J} p_j^{\leftarrow}(A_j)$, we have $J \subseteq J_1$ and $B_j \leq A_j$ $(\forall j \in J)$. Hence,

$$\begin{aligned}
\bigwedge_{j\in J_1} \mathcal{N}_{\mathcal{T}_j}(B_j)(p_j^{\rightarrow}(x_\lambda)) &\leq \bigwedge_{j\in J} \mathcal{N}_{\mathcal{T}_j}(B_j)(p_j^{\rightarrow}(x_\lambda)) \\
&\leq \bigwedge_{j\in J} \mathcal{N}_{\mathcal{T}_j}(A_j)(p_j^{\rightarrow}(x_\lambda)) \\
\text{(by the definition of } \mathcal{N}_{\mathcal{T}}) &\leq \mathcal{N}_{\mathcal{T}}(A)(x_\lambda).
\end{aligned}$$

Therefore, $\mathcal{N}_{\mathcal{T}}(A)(x_\lambda) = \bigwedge_{j\in J} \mathcal{N}_{\mathcal{T}_j}(A_j)(p_j^{\rightarrow}(x_\lambda))$. (2)

Now, since $p_j : (X, \mathcal{T}) \longrightarrow (X_j, \mathcal{T}_j)$ $(\forall j \in J)$ is continuous, we have

$$\mathcal{T}(A) = \mathcal{T}(\bigwedge_{j\in J} p_j^{\leftarrow}(A_j)) \geq \bigwedge_{j\in J} \mathcal{T}(p_j^{\leftarrow}(A_j)) \geq \bigwedge_{j\in J} \mathcal{T}_j(A_j).$$

In order to prove $\mathcal{T}(A) = \bigwedge_{j\in J} \mathcal{T}_j(A_j)$, we need to show that $\mathcal{T}(A) \leq \bigwedge_{j\in J} \mathcal{T}_j(A_j)$. If $\mathcal{T}(A) \not\leq \bigwedge_{j\in J} \mathcal{T}_j(A_j)$, then there exists $j_0 \in J$ such that $\mathcal{T}(A) \not\leq \mathcal{T}_{j_0}(A_{j_0})$. By

Lemma 1.3, we can obtain

$$\mathcal{T}(A) = \bigwedge_{x_\lambda \lhd A} \mathcal{N}_{\mathcal{T}}(A)(x_\lambda) \quad \text{and} \quad \mathcal{T}_{j_0}(A_{j_0}) = \bigwedge_{y_{\mu_{j_0}} \lhd A_{j_0}} \mathcal{N}_{\mathcal{T}_{j_0}}(A_{j_0})(y_{\mu_{j_0}}).$$

Hence $\mathcal{N}_{\mathcal{T}}(A)(x_\lambda) \not\leq \mathcal{T}_{j_0}(A_{j_0})$ for each $x_\lambda \lhd A$. Further, there exists $y_{\mu_{j_0}} \lhd A_{j_0}$ such that $\mathcal{N}_{\mathcal{T}}(A)(x_\lambda) \not\leq \mathcal{N}_{\mathcal{T}_{j_0}}(A_{j_0})(y_{\mu_{j_0}})$. However, by (2), we have

$$\mathcal{N}_{\mathcal{T}}(A)(x_\lambda) = \bigwedge_{j \in J} \mathcal{N}_{\mathcal{T}_j}(A_j)(p_j^{\rightarrow}(x_\lambda)) \leq \mathcal{N}_{\mathcal{T}_{j_0}}(A_{j_0})(y_{\mu_{j_0}}),$$

which is a contradiction. □

4. Conclusions

In this paper, the relationship between (L, M)-fuzzy topology and (L, M)-fuzzy neighborhood system is further studied, and the initial structures of (L, M)-fuzzy neighborhood subspaces and (L, M)-fuzzy topological product spaces are given. Similarly, we can also give the initial structures of (L, M)-fuzzy topological subspaces and (L, M)-fuzzy neighborhood product spaces.

The construction of initial structures in the category of (L, M)-fuzzy topological spaces through those in the category of (L, M)-fuzzy neighborhood systems really looks rather interesting; the fact that the two categories are isomorphic, however, enables researchers to substitute one of them with the other, to find a solution of a complicated problem.

The related topic of (L, M)-fuzzy topological spaces will be studied further in our subsequent papers (e.g. (L, M)-fuzzy topological groups and (L, M)-fuzzy topological vector spaces), involving, possibly, product of the latter.

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