SOME FIXED POINT THEOREMS FOR SINGLE AND MULTI VALUED MAPPINGS ON ORDERED NON-ARCHIMEDEAN FUZZY METRIC SPACES

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Abstract. In the present paper, a partial order on a non- Archimedean fuzzy metric space under the Łukasiewicz t-norm is introduced and fixed point theorems for single and multivalued mappings are proved.

1. Introduction and Preliminaries

Research in the field of fixed point theory on fuzzy metric spaces ([1], [2], [3], [6], [7], [8], [16]) has been developed following the definition such spaces [5], [10], [20], the Generally, this theory is concerned with contractive or contractive type mappings ([9], [11], [12], [13], [14], [17], [19]).

In this paper we introduce a partial order on a non-Archimedean fuzzy metric space (in the sense of Kramosil and Michalek) under the Łukasiewicz t-norm and prove a fixed point theorem for single-valued nondecreasing mappings. Similar results are obtained for multivalued mappings.

For the sake of completeness, we first recall some notions from the theory of fuzzy metric spaces.

Definition 1.1. [18] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an Abelian topological monoid with the unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Definition 1.2. [10] A fuzzy metric space (in the sense of Kramosil and Michalek) is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$, satisfying the following properties:

(KM-1) $M(x, y, 0) = 0, \forall x \in X$
(KM-2) $M(x, y, t) = 1, \forall t > 0$ iff $x = y$
(KM-3) $M(x, y, t) = M(y, x, t), \forall x, y \in X$ and $t > 0$
(KM-4) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous, $\forall x, y \in X$
(KM-5) $M(x, z, t + s) \ge M(x, y, t) * M(y, z, s), \forall x, y, z \in X, \forall t, s > 0.$

We will refer to such spaces as FM-spaces.

If in the above definition, the triangular inequality (KM-5) is replaced by

$$(NA)\ M(x, z, \max\{t, s\}) \ge M(x, y, t) * M(y, z, s), \forall x, y, z \in X, \forall t, s > 0$$

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or, equivalently,

$$M(x,z,t) \geq M(x,y,t) * M(y,z,t), \forall x,y,z \in X, \forall t > 0$$

then the triple $(X, M, *)$ is called a *non- Archimedean fuzzy metric space.* It is easy to check that the triangular inequality (NA) implies (KM-5), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

Definition 1.3. [5], [18] Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is called an M-Cauchy sequence, if for each $\varepsilon \in (0,1)$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1-\varepsilon$ for all $m, n \geq n_0$. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be convergent to $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$. An FM space $(X, M, *)$ is called M- complete if every M-Cauchy sequence is convergent.

2. Fixed point Theory for Single-valued Mappings

We first prove the following lemma.

Lemma 2.1. *Let $(X, M, *)$ be a non-Archimedean fuzzy metric space with $a * b \geq \max\{a+b-1, 0\}$ and $\phi : X \times [0,\infty) \to \mathbb{R}$. Define the relation "$\preceq$" on X as follows:*

$$x \preceq y \Longleftrightarrow M(x,y,t) \geq 1 + \phi(x,t) - \phi(y,t), \forall t > 0.$$

Then "$\preceq$" is a (partial) order on X, called the partial order induced by ϕ.

Proof. For all $x \in X$ and $t > 0$, $M(x,x,t) = 1 = 1 + \phi(x,t) - \phi(x,t)$, then $x \preceq x$, that is, "$\preceq$" is reflexive. Again, if $x, y \in X$, be such that $x \preceq y$ and $y \preceq x$, then for all $t > 0$,

$$M(x,y,t) \geq 1 + \phi(x,t) - \phi(y,t)$$

and

$$M(y,x,t) \geq 1 + \phi(y,t) - \phi(x,t).$$

This shows that $M(x,y,t) = 1$ for all $t > 0$, that is, $x = y$. Thus "$\preceq$" is antisymmetric. Now for $x, y, z \in X$, let $x \preceq y$ and $y \preceq z$, then for all $t > 0$,

$$M(x,y,t) \geq 1 + \phi(x,t) - \phi(y,t) \tag{1}$$

and

$$M(y,z,t) \geq 1 + \phi(y,t) - \phi(z,t). \tag{2}$$

By (1) and (2) we have,

$$\begin{aligned} M(x,z,t) &\geq M(x,y,t) * M(y,z,t) \\ &= \max\{M(x,y,t) + M(y,z,t) - 1, 0\} \\ &\geq M(x,y,t) + M(y,z,t) - 1 \\ &\geq 1 + \phi(x,t) - \phi(z,t), \forall t > 0. \end{aligned}$$

This shows that $x \preceq z$. □

Example 2.2. Let $X = (0, \infty)$, $a * b = ab$ and

$$M(x, y, t) = \frac{\min\{x, y\}}{\max\{x, y\}}, \forall t > 0.$$

Then $(X, M, *)$ is an M-complete non-Archimedean fuzzy metric space (see [15]) Let $\phi : X \times [0, \infty) \to \mathbb{R}$, $\phi(x, t) = \dfrac{1}{x}$. Then for $x, y \in X$,

$$\begin{aligned} x \preceq y &\iff M(x, y, t) \geq 1 + \phi(x, t) - \phi(y, t) \\ &\iff M(x, y, t) \geq 1 + \frac{1}{x} - \frac{1}{y} \\ &\iff 1 - M(x, y, t) \leq \frac{1}{y} - \frac{1}{x}. \\ &\iff 1 - \tfrac{\min\{x,y\}}{\max\{x,y\}} \leq \frac{1}{y} - \frac{1}{x}. \end{aligned}$$

It follows that $2 \preceq \frac{1}{2}, 3 \preceq 1, 1 \preceq \frac{1}{3}$ but $3 \npreceq 5$ and $5 \npreceq 3$. Therefore X is a partially ordered space.

Example 2.3. Let $X = \mathbb{N} = \{1, 2, \cdots\}$, $a * b = ab$ and

$$M(x, y, t) = \begin{cases} \dfrac{x}{y} & \text{if} \quad x \leq y \\ & \\ \dfrac{y}{x} & \text{if} \quad y \leq x \end{cases}, \forall t > 0.$$

Then $(X, M, *)$ is a non-Archimedean fuzzy metric space ([5]). Also this space is M-complete. Let $\phi : X \times [0, \infty) \to \mathbb{R}$, $\phi(x, t) = x$.

Then it is obvious that $x \preceq y \iff x \leq y$. Therefore X is a partially ordered space. Also X is a totally ordered space. If we define $\phi : X \times [0, \infty) \to \mathbb{R}$, $\phi(x, t) = x - \dfrac{1}{x}$, then it is again obvious that $x \preceq y \iff x \leq y$. Now, if $\phi : X \times [0, \infty) \to \mathbb{R}$, $\phi(x, t) = 1 - \dfrac{5}{x}$, then $1 \preceq 2 \preceq 3 \preceq 4 \preceq 5 \preceq 6$ but $6 \npreceq 7$ and $7 \npreceq 6$. Therefore X is a partially ordered space.

Definition 2.4. Let $(X, M, *)$ be a fuzzy metric space. If "$\preceq$" is an order on X, then the fuzzy metric space is called an ordered fuzzy metric space. Let $(X, M, *)$ be an ordered fuzzy metric space and let $f : X \to X$ be a mapping. If $x \preceq y$ implies that $fx \preceq fy$, then f is called a nondecreasing mapping.

Theorem 2.5. *Let $(X, M, *)$ be an M-complete non- Archimedean fuzzy metric space with $a * b \geq \max\{a + b - 1, 0\}$, $\phi : X \times [0, \infty) \to \mathbb{R}$ be a function bounded from above and "$\preceq$" the partial order induced by ϕ. If $f : X \to X$ is a continuous nondecreasing function with $x_0 \preceq fx_0$ for some $x_0 \in X$, then f has a fixed point in X.*

Proof. Consider a point $x_0 \in X$ satisfying $x_0 \preceq fx_0$. We define a sequence $\{x_n\}$ in X such that $x_n = fx_{n-1}$ for $n = 1, 2, \cdots$. Then, since f is nondecreasing we have $x_0 \preceq x_1 \preceq x_2 \preceq \cdots$, that is the sequence $\{x_n\}$ is nondecreasing. By the definition of "$\preceq$" we have, $\forall t > 0$, $\phi(x_0, t) \le \phi(x_1, t) \le \phi(x_2, t) \le \cdots$. In other words, for all $t > 0$, the sequence $\{\phi(x_n, t)\}$ is nondecreasing in $\mathbb{R}$. Since ϕ is bounded from above, $\{\phi(x_n, t)\}$ is convergent and hence it is Cauchy. So, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ and $t > 0$ we have $|\phi(x_m, t) - \phi(x_n, t)| = \phi(x_m, t) - \phi(x_n, t) < \varepsilon$. Since $x_n \preceq x_m$, it follows that

$$\begin{aligned} M(x_n, x_m, t) &\ge 1 + \phi(x_n, t) - \phi(x_m, t) \\ &= 1 - [\phi(x_m, t) - \phi(x_n, t)] \\ &> 1 - \varepsilon. \end{aligned}$$

This shows that the sequence $\{x_n\}$ is Cauchy in X and since X is M-complete, it converges to a point $z \in X$. Consequently, by the continuity of f, we have $fz = z$. □

Example 2.6. Let $(X, M, *)$ be as in Example 2.3 and $\phi : X \times [0, \infty) \to \mathbb{R}$, $\phi(x, t) = 1 - \dfrac{5}{x}$. Define $A = \{1, 2, \cdots, 5\}$ and $B = \{6, 7, \cdots\}$. Now if $x, y \in A$ and $x \le y$, then $x \preceq y$. If $x \in A$ and $y \in B$ then $x \preceq y$. If $x, y \in B$, then x and y are not comparable. Now define $f : X \to X$,

$$fx = \begin{cases} x + 1 & \text{if} \quad x \le 5 \\ \\ 6 & \text{if} \quad x > 5 \end{cases}.$$

It is clear that f is continuous and nondecreasing. Also $1 \preceq 2 = f1$ and all the conditions of Theorem 2.5 are satisfied. Therefore, f has a fixed point.

3. Fixed point Theory for Multi-valued Mappings

In the following we provide multivalued versions of the preceding theorem. The results are related to those in [4].

Let X be a topological space and $\preceq$ be a partial order on X. Let 2^X denote the family of all nonempty subsets of X.

Definition 3.1. [4] Let A and B be two nonempty subsets of X. Then

(R-1) If for every $a \in A$, there exists $b \in B$ such that $a \preceq b$, then $A \prec_1 B$.
(R-2) If for every $b \in B$, there exists $a \in A$ such that $a \preceq b$, then $A \prec_2 B$.
(R-3) If $A \prec_1 B$ and $A \prec_2 B$, then $A \prec B$.

Remark 3.2. [4] The relations $\prec_1$ and $\prec_2$ are different. For example, let $X = \mathbb{R}$, $A = [\frac{1}{2}, 1]$, $B = [0, 1]$, $\preceq$ be usual order on X, then $A \prec_1 B$ but $A \not\prec_2 B$; if $A = [0, 1]$, $B = [0, \frac{1}{2}]$, then $A \prec_2 B$ while $A \not\prec_1 B$.

Remark 3.3. [4] $\prec_1$, $\prec_2$and $\prec$ are reflexive and transitive, but are not antisymmetric. For instance, let $X = \mathbb{R}$, $A = [0, 3]$, $B = [0, 1] \cup [2, 3]$, $\preceq$ be the usual order on X, then $A \prec B$ and $B \prec A$, but $A \ne B$. Hence, they are not partial orders on 2^X.

Definition 3.4. [4] A multi-valued operator $T : X \to 2^X$ is called order closed if, for monotone sequences $\{u_n\}$ and $\{v_n\}$ in X, $u_n \to u_0, v_n \to v_0$ and $v_n \in Tu_n$ imply $v_0 \in Tu_0$.

Theorem 3.5. *Let $(X, M, *)$ be an M-complete non- Archimedean fuzzy metric space with $a * b \geq \max\{a + b - 1, 0\}$, $\phi : X \times [0, \infty) \to \mathbb{R}$ a function bounded from above and "$\preceq$" a partial order induced by ϕ. Suppose $F : X \to 2^X$ is an order closed operator with $\{x_0\} \prec_1 Fx_0$ for some $x_0 \in X$. If $\forall x, y \in X$, $x \preceq y \Longrightarrow Fx \prec_1 Fy$ (that is, F is nondecreasing with respect to $\prec_1$), then F has a fixed point in X.*

Proof. Since Fx is nonempty for all $x \in X$, there exists $x_1 \in Fx_0$ such that $x_0 \preceq x_1$. Now since $Fx_0 \prec_1 Fx_1$, there exists $x_2 \in Fx_1$ such that $x_1 \preceq x_2$. Continuing this process, we get an increasing sequence $\{x_n\}$, which satisfies $x_{n+1} \in Fx_n$. By the definition of "$\preceq$", we have $\phi(x_0, t) \leq \phi(x_1, t) \leq \phi(x_2, t) \leq \cdots \ \forall t > 0$. In other words, for all $t > 0$ the sequence $\{\phi(x_n, t)\}$ is nondecreasing in $\mathbb{R}$. Since ϕ is bounded from above, $\{\phi(x_n, t)\}$ is convergent and hence Cauchy. So, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ and $t > 0$ we have $|\phi(x_m, t) - \phi(x_n, t)| = \phi(x_m, t) - \phi(x_n, t) < \varepsilon$. Therefore, since $x_n \preceq x_m$,

$$\begin{aligned} M(x_n, x_m, t) &\geq 1 + \phi(x_n, t) - \phi(x_m, t) \\ &= 1 - [\phi(x_m, t) - \phi(x_n, t)] \\ &> 1 - \varepsilon. \end{aligned}$$

This shows that the sequence $\{x_n\}$ is Cauchy in X and since X is M-complete, it converges to a point $z \in X$. Consequently, since F is order closed, we have $z \in Fz$ and $x_{n+1} \in Fx_n$. $\square$

We can simlarly prove the following theorem.

Theorem 3.6. *Let $(X, M, *)$ be an M-complete non- Archimedean fuzzy metric space with $a * b \geq \max\{a + b - 1, 0\}$, $\phi : X \times [0, \infty) \to \mathbb{R}$ a function bounded from below and "$\preceq$" the partial order induced by ϕ. Suppose $F : X \to 2^X$ is an order closed operator with $Fx_0 \prec_2 \{x_0\}$ for some $x_0 \in X$. If $\forall x, y \in X$, $x \preceq y \Longrightarrow Fx \prec_2 Fy$ (F is nondecreasing with respect to $\prec_2$), then F has a fixed point in X.*

Example 3.7. Let $(X, M, *)$ be as in Example 2.3 and $\phi : X \times [0, \infty) \to \mathbb{R}$, $\phi(x, t) = 1 - \dfrac{5}{x}$. Define $A = \{1, 2, \cdots, 5\}$ and $B = \{6, 7, \cdots\}$. Now if $x, y \in A$ and $x \leq y$, then $x \preceq y$. If $x \in A$ and $y \in B$ then $x \preceq y$. If $x, y \in B$, then x and y are not comparable. Now define $F : X \to 2^X$,

$$Fx = \{6, x + 1\}.$$

It is clear that F is order closed and $\{1\} \prec_1 \{2, 6\} = F1$. Also, if $x \preceq y$ then $Fx \prec_1 Fy$, and all the conditions of Theorem 3.5 are satisfied. Therefore, F has a fixed point.

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