A NEW PERSPECTIVE TO THE MAZUR-ULAM PROBLEM IN 2-FUZZY 2-NORMED LINEAR SPACES

C. ALACA

Abstract. In this paper, we introduce the concepts of 2-isometry, collinearity, 2-Lipschitz mapping in 2-fuzzy 2-normed linear spaces. Also, we give a new generalization of the Mazur-Ulam theorem when X is a 2-fuzzy 2-normed linear space or $\Im(X)$ is a fuzzy 2-normed linear space, that is, the Mazur-Ulam theorem holds, when the 2-isometry mapped to a 2-fuzzy 2-normed linear space is affine.

1. Introduction

The theory of fuzzy sets was introduced by Zadeh [25]. A satisfactory theory of 2-norms and n-norms on a linear space has been introduced and developed by Gähler in [9, 10]. Different authors introduced various definitions of fuzzy norms on a linear space. For reference, one may see [8, 11, 13, 14, 21, 23]. Following Cheng and Mordeson [3], Bag and Samanta [1] introduced a concept of fuzzy norm on a linear space.

Recently, Somasundaram and Beaula [20] introduced a concept of 2-fuzzy 2-normed linear space or fuzzy 2-normed linear space of the set of all fuzzy sets of a set. The authors gave the notion of α-2-norm on a linear space corresponding to the 2-fuzzy 2-norm by using some ideas of [1] and also gave some fundamental properties of this space.

In 1932, Mazur and Ulam [15] proved the following theorem.

Mazur-Ulam Theorem. *Every isometry of a real normed linear space onto a real normed linear space is a linear mapping up to translation.*

Baker [2] showed an isometry from a real normed linear space into a strictly convex real normed linear space is affine. Also, Jian [12] investigated the generalizations of the Mazur–Ulam theorem in F^*-spaces. Rassias and Wagner [19] described all volume preserving mappings from a real finite dimensional vector space into itself and Väisälä [22] gave a short and simple proof of the Mazur–Ulam theorem. Chu [6] proved that the Mazur–Ulam theorem holds when X is a linear 2-normed space. Chu et al. [7] generalized the Mazur–Ulam theorem when X is a linear n-normed space, that is, the Mazur–Ulam theorem holds, when the n-isometry mapped to a linear n-normed space is affine. In addition, Moslehian and Sadeghi [16] investigated the Mazur-Ulam theorem in non-archimedean spaces. Chu et al. [7] also

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obtained extensions of Rassias and Šemrl's theorem [18]. Cho et al. [5] investigated the Mazur–Ulam theorem on probabilistic 2-normed spaces. The Mazur–Ulam theorem has been extensively studied by many authors (see [17, 19, 24]).

In the present paper, we introduce the concepts of 2-isometry, collinearity, 2-Lipschitz mapping in 2-fuzzy 2-normed linear spaces. Also, we give a new generalization of the Mazur-Ulam theorem when X is a 2-fuzzy 2-normed linear space or $\Im(X)$ is a fuzzy 2-normed linear space, that is, the Mazur-Ulam theorem holds, when the 2-isometry mapped to a 2-fuzzy 2-normed linear space is affine.

2. On 2-Fuzzy 2-Normed Linear Spaces

In this section at first we give a concept of linear 2-normed space and later a concept of 2-fuzzy 2-normed linear space and it's fundamental properties by using some ideas of [20]. For more details we refer the readers to [1, 4, 20].

Definition 2.1. [4] Let X be a real vector space of dimension greater than 1 and let $\|\bullet,\bullet\|$ be a real valued function on $X\times X$ satisfying the following four properties:

(1) $\|x,y\|=0$ if and only if x and y are linearly dependent,
(2) $\|x,y\|=\|y,x\|$,
(3) $\|x,\alpha y\|=|\alpha|\,\|x,y\|$ for any $\alpha\in\mathbb{R}$,
(4) $\|x,y+z\|\le\|x,y\|+\|x,z\|$,

$\|\bullet,\bullet\|$ is called a 2-norm on X and the pair $(X,\|\bullet,\bullet\|)$ is called a linear 2-normed space.

Definition 2.2. [1] Let X be a linear space over S (field of real or complex numbers). A fuzzy subset N of $X\times\mathbb{R}$ ($\mathbb{R}$, the set of real numbers) is called a fuzzy norm on X if and only if :

(N1) For all $t\in\mathbb{R}$ with $t\le 0$, $N(x,t)=0$,
(N2) For all $t\in\mathbb{R}$ with $t>0$, $N(x,t)=1$ if and only if $x=0$,
(N3) For all $t\in\mathbb{R}$ with $t>0$, $N(\lambda x,t)=N(x,\frac{t}{|\lambda|})$, if $\lambda\ne 0$, $\lambda\in S$,
(N4) For all $s,t\in\mathbb{R}$, $x,y\in X$, $N(x+y,s+t)\ge\min\{N(x,s),N(y,t)\}$,
(N5) $N(x,\cdot)$ is a non-decreasing function of $t\in\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$.

Then (X,N) is called a fuzzy normed linear space or in short f-NLS.

Theorem 2.3. [1] *Let (X,N) be a f-NLS. Assume the condition that*

(N6) $N(x,t)>0$ for all $t>0$ implies $x=0$.

Define $\|x\|_\alpha=\inf\{t:N(x,t)\ge\alpha\}$, $\alpha\in(0,1)$. Then $\{\|\bullet\|_\alpha:\alpha\in(0,1)\}$ is an ascending family of norms on X. We call these norms as α-norms on X corresponding to the fuzzy norm on X.

Definition 2.4. Let X be any non-empty set and $\Im(X)$ be the set of all fuzzy sets on X. For $U,V\in\Im(X)$ and $\lambda\in S$ the field of real numbers, define

$$U+V=\{(x+y,\nu\wedge\mu):(x,\nu)\in U,(y,\mu)\in V\}$$

and $\lambda U=\{(\lambda x,\nu):(x,\nu)\in U\}$.

Definition 2.5. A fuzzy linear space $\widehat{X} = X \times (0,1]$ over the number field S where the addition and scalar multiplication operation on X are defined by $(x,\nu)+(y,\mu) = (x+y, \nu \wedge \mu)$, $\lambda(x,\nu) = (\lambda x, \nu)$ is a fuzzy normed space if to every $(x,\nu) \in \widehat{X}$ there is associated a non-negative real number, $\|(x,\nu)\|$, called the fuzzy norm of (x,ν), in such away that

(i) $\|(x,\nu)\| = 0$ iff $x = 0$ the zero element of X, $\nu \in (0,1]$,
(ii) $\|\lambda(x,\nu)\| = |\lambda| \, \|(x,\nu)\|$ for all $(x,\nu) \in \widehat{X}$ and all $\lambda \in S$,
(iii) $\|(x,\nu)+(y,\mu)\| \le \|(x,\nu\wedge\mu)\| + \|(y,\nu\wedge\mu)\|$ for all $(x,\nu),(y,\mu) \in \widehat{X}$,
(iv) $\|(x, \vee_t \nu_t)\| = \wedge_t \, \|(x,\nu_t)\|$ for all $\nu_t \in (0,1]$.

Definition 2.6. [20] Let X be a non-empty set and $\Im(X)$ be the set of all fuzzy sets in X. If $f \in \Im(X)$ then $f = \{(x,\mu) : x \in X \text{ and } \mu \in (0,1]\}$. Clearly f is a bounded function, since $|f(x)| \le 1$. Let S be the space of real numbers, then $\Im(X)$ is a linear space over the field S where the addition and scalar multiplication are defined by

$$f + g = \{(x,\mu) + (y,\eta)\} = \{(x+y, \mu\wedge\eta) : (x,\mu) \in f \text{ and } (y,\eta) \in g\}$$

and

$$\lambda f = \{(\lambda x, \mu) : (x,\mu) \in f\}$$

where $\lambda \in S$.

The linear space $\Im(X)$ is said to be normed linear space if, for every $f \in \Im(X)$, there exists an associated non-negative real number $\|f\|$ (called the norm of f) that satisfies

(i) $\|f\| = 0$ if and only if $f = 0$. For

$$\begin{aligned} \|f\| &= 0 \\ &\Longleftrightarrow \{\|(x,\mu)\| : (x,\mu) \in f\} = 0 \\ &\Longleftrightarrow x = 0,\, \mu \in (0,1] \Longleftrightarrow f = 0. \end{aligned}$$

(ii) $\|\lambda f\| = |\lambda| \, \|f\|$, $\lambda \in S$. For

$$\begin{aligned} \|\lambda f\| &= \{\|\lambda(x,\mu)\| : (x,\mu) \in f,\, \lambda \in S\} \\ &= \{|\lambda| \, \|(x,\mu)\| : (x,\mu) \in f\} = |\lambda| \, \|f\| . \end{aligned}$$

(iii) $\|f+g\| \le \|f\| + \|g\|$ for every $f,\, g \in \Im(X)$. For

$$\begin{aligned} \|f+g\| &= \{\|(x,\mu)+(y,\eta)\| : x,y \in X,\, \mu,\eta \in (0,1]\} \\ &= \{\|(x+y),(\mu\wedge\eta)\| : x,y \in X,\, \mu,\eta \in (0,1]\} \\ &= \{\|(x,\mu\wedge\eta)\| + \|(y,\mu\wedge\eta)\| : (x,\mu) \in f,\, (y,\eta) \in g\} \\ &= \|f\| + \|g\| . \end{aligned}$$

Then $(\Im(X), \|\bullet\|)$ is a normed linear space.

Definition 2.7. [20] A 2-fuzzy set on X is a fuzzy set on $\Im(X)$.

Definition 2.8. [20] Let $\Im(X)$ be a linear space over the real field S. A fuzzy subset N of $\Im(X) \times \Im(X) \times \mathbb{R}$ ($\mathbb{R}$, set of real numbers) is called a 2-fuzzy 2-norm on X (or fuzzy 2-norm on $\Im(X)$) if and only if,

(2-N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(f_1, f_2, t) = 0$,
(2-N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(f_1, f_2, t) = 1$ if and only if f_1 and f_2 are linearly dependent,
(2-N3) $N(f_1, f_2, t)$ is invariant under any permutation of f_1, f_2,
(2-N4) for all $t \in \mathbb{R}$ with $t > 0$, $N(f_1, \lambda f_2, t) = N(f_1, f_2, \frac{t}{|\lambda|})$, if $\lambda \neq 0$, $\lambda \in S$,
(2-N5) for all s, $t \in \mathbb{R}$,

$$N(f_1, f_2 + f_3, s + t) \geq \min\{N(f_1, f_2, s), N(f_1, f_3, t)\},$$

(2-N6) $N(f_1, f_2, \cdot) : (0, \infty) \to [0, 1]$ is continuous,
(2-N7) $\lim_{t \to \infty} N(f_1, f_2, t) = 1$.

Then $(\Im(X), N)$ is a fuzzy 2-normed linear space or (X, N) is a 2-fuzzy 2-normed linear space.

Remark 2.9. In a 2-fuzzy 2-normed linear space (X, N), $N(f_1, f_2, \cdot)$ is a non-decreasing function of $\mathbb{R}$ for all f_1, $f_2 \in \Im(X)$.

Theorem 2.10. [20] *Let $(\Im(X), N)$ be a fuzzy 2-normed linear space. Assume that*

(2-N8) $N(f_1, f_2, t) > 0$ for all $t > 0$ implies that f_1 and f_2 are linearly dependent.

Define $\|f_1, f_2\|_\alpha = \inf\{t : N(f_1, f_2, t) \geq \alpha, \alpha \in (0, 1)\}$.

Then $\{\|\bullet, \bullet\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of 2-norms on $\Im(X)$. These 2-norms are called α–2-norms on $\Im(X)$ corresponding to the 2-fuzzy 2-norm on X.

3. On the Mazur-Ulam Problem

In this section, we give a new generalization of the Mazur-Ulam theorem when X is a 2-fuzzy 2-normed linear space or $\Im(X)$ is a fuzzy 2-normed linear space. Hereafter we use the notion of fuzzy 2-normed linear space on $\Im(X)$ instead of 2-fuzzy 2-normed linear space on X.

Lemma 3.1. *For all f, $h \in \Im(X)$, $\alpha \in (0, 1)$ and $\lambda \in \mathbb{R}$. Then*

$$\|f, h\|_\alpha = \|f, h + \lambda f\|_\alpha .$$

Proof. The proof of Lemma is clear from [4, Theorem 2.1.6.]. □

As an immediate consequence of Lemma 3.1, we have the following.

Remark 3.2. For all f, g, $h \in \Im(X)$, $\alpha \in (0, 1)$,

$$\|f - h, f - g\|_\alpha = \|f - h, g - h\|_\alpha .$$

Lemma 3.3. *For g, $h \in \Im(X)$, if g and h are linearly dependent with the same direction, that is, $h = \lambda g$ for some $\lambda > 0$, then*

$$\|f, g + h\|_\alpha = \|f, g\|_\alpha + \|f, h\|_\alpha$$

for all $f \in \Im(X)$, $\alpha \in (0, 1)$.

Proof. For all $f \in \Im(X)$, $\|f, g + h\|_\alpha = \|f, g + \lambda g\|_\alpha = \|f, (1 + \lambda)g\|_\alpha = (1 + \lambda) \|f, g\|_\alpha = \|f, g\|_\alpha + \lambda \|f, g\|_\alpha = \|f, g\|_\alpha + \|f, h\|_\alpha$. □

Definition 3.4. Let $\Im(X)$ and $\Im(Y)$ be fuzzy 2-normed linear spaces and Ψ : $\Im(X) \to \Im(Y)$ a mapping. We call Ψ a 2-isometry if

$$\|f-h, g-h\|_{\alpha} = \|\Psi(f) - \Psi(h), \Psi(g) - \Psi(h)\|_{\beta}$$

for all f, g, $h \in \Im(X)$ and α, $\beta \in (0,1)$.

For a map Ψ, consider the following condition which is called the Area One Preserving Property (AOPP).

$$\text{(AOPP) Let } f,\ g,\ h \in \Im(X) \text{ with } \|f-h, g-h\|_{\alpha} = 1.$$

Then $\|\Psi(f) - \Psi(h), \Psi(g) - \Psi(h)\|_{\beta} = 1$.

Definition 3.5. The elements f, g and h are said to be collinear if and only if $g - h = r(f - h)$ for some real number r.

Now we define the concept of 2-Lipschitz mapping.

Definition 3.6. We call Ψ a 2-Lipschitz mapping if there is a $\kappa \geq 0$ such that

$$\|\Psi(f) - \Psi(h), \Psi(g) - \Psi(h)\|_{\beta} \leq \kappa \|f-h, g-h\|_{\alpha}$$

for all f, g, $h \in \Im(X)$ and α, $\beta \in (0,1)$. The constant κ is called the 2-Lipschitz constant.

Lemma 3.7. *Assume that if f, g and h are collinear, then $\Psi(f)$, $\Psi(g)$ and $\Psi(h)$ are collinear, and that Ψ satisfies (AOPP). Then Ψ preserves the area k for each $k \in \mathbb{N}$.*

Proof. Suppose that there exist f, $g \in \Im(X)$ with $f \neq g$ such that $\Psi(f) = \Psi(g)$. Since $\dim \Im(X) \geq 2$, there is $h' \in \Im(X)$ such that $g - f$ and $h' - f$ are linearly independent. Since $\|h' - f, g - f\|_{\alpha} \neq 0$, we can set

$$h = f + \frac{1}{\|h' - f, g - f\|_{\alpha}}(h' - f).$$

Then we have

$$\|h - f, g - f\|_{\alpha} = \left\| \frac{1}{\|h' - f, g - f\|_{\alpha}}(h' - f), g - f \right\|_{\alpha} = 1.$$

Since Ψ preserves the unit distance, $\|\Psi(h) - \Psi(f), \Psi(g) - \Psi(f)\|_{\beta} = 1$. But it follows from $\Psi(f) = \Psi(g)$ that

$$\|\Psi(h) - \Psi(f), \Psi(g) - \Psi(f)\|_{\beta} = 0,$$

which is a contradiction. Thus Ψ is injective.

Let f, g and h be elements of $\Im(X)$ and $k \in \mathbb{N}$ and $\|h - f, g - f\|_{\alpha} = k$. We put

$$f_i = f + \frac{i}{k}(g - f), \quad i = 0, 1, ..., k.$$

Thus

$$\begin{aligned} & \|h-f, f_{i+1}-f_i\|_\alpha \\ = & \left\| h-f, f+\frac{i+1}{k}(g-f)-\left(f+\frac{i}{k}(g-f)\right)\right\|_\alpha \\ = & \left\| h-f, \frac{1}{k}(g-f)\right\|_\alpha = \frac{1}{k}\|h-f, g-f\|_\alpha = \frac{k}{k} = 1 \end{aligned}$$

for all $i = 0, 1, ..., k$. Since Ψ satisfies (AOPP),

$$\|\Psi(h)-\Psi(f), \Psi(f_{i+1})-\Psi(f_i)\|_\beta = 1$$

for all $i = 0, 1, ..., k$. Since f_0, f_1 and f_2, are collinear, $\Psi(f_0)$, $\Psi(f_1)$ and $\Psi(f_2)$ are also collinear. Thus there is a real number r_0 such that $\Psi(f_2)-\Psi(f_1) = r_0(\Psi(f_1)-\Psi(f_0))$. Since

$$\begin{aligned} & \|\Psi(h)-\Psi(f), \Psi(f_1)-\Psi(f_0)\|_\beta = \|\Psi(h)-\Psi(f), \Psi(f_2)-\Psi(f_1)\|_\beta \\ = & \|(\Psi(h)-\Psi(f)), r_0(\Psi(f_1)-\Psi(f_0))\|_\beta = |r_0|\,\|\Psi(h)-\Psi(f), \Psi(f_1)-\Psi(f_0)\|_\beta, \end{aligned}$$

we have $r_0 = 1$ or -1. If $r_0 = -1$, $\Psi(f_2)-\Psi(f_1) = -\Psi(f_1)+\Psi(f_0)$, that is, $\Psi(f_2) = \Psi(f_0)$. Since Ψ is injective, $f_2 = f_0$, which is a contradiction. Thus $r_0 = 1$. Then we have $\Psi(f_2)-\Psi(f_1) = \Psi(f_1)-\Psi(f_0)$. Similarly, one can obtain that $\Psi(f_{i+1})-\Psi(f_i) = \Psi(f_i)-\Psi(f_{i-1})$ for all $i = 0, 1, ..., k-1$. Thus $\Psi(f_{i+1})-\Psi(f_i) = \Psi(f_1)-\Psi(f_0)$ for all $i = 0, 1, ..., k-1$. Hence

$$\begin{aligned} \Psi(g)-\Psi(f) &= \Psi(f_k)-\Psi(f_0) \\ &= \Psi(f_k)-\Psi(f_{k-1})+\Psi(f_{k-1})-\Psi(f_0)+...+\Psi(f_1)-\Psi(f_0) \\ &= k(\Psi(f_1)-\Psi(f_0)). \end{aligned}$$

Hence we obtain

$$\begin{aligned} & \|\Psi(h)-\Psi(f), \Psi(g)-\Psi(f)\|_\beta = \|\Psi(h)-\Psi(f), k(\Psi(f_1)-\Psi(f_0))\|_\beta \\ = & k\|\Psi(h)-\Psi(f), \Psi(f_1)-\Psi(f_0)\|_\beta = k. \end{aligned}$$

This completes the proof. □

Theorem 3.8. *Let Ψ be a 2-Lipschitz mapping with the 2-Lipschitz constant $\kappa \le 1$. Assume that if f, g and h are collinear, then $\Psi(f)$, $\Psi(g)$ and $\Psi(h)$ are collinear, and that Ψ satisfies (AOPP). Then Ψ is a 2-isometry.*

Proof. From Lemma 3.7, Ψ preserves distances k for all $k \in \mathbb{N}$. For f, g, $h \in \Im(X)$, there are two cases depending on whether $\|h-f, g-f\|_\alpha = 0$ or not.

In the first case $\|h-f, g-f\|_\alpha = 0$, $h-f$ and $g-f$ are linearly dependent. So f, g and h are collinear. Thus $\Psi(f)$, $\Psi(g)$ and $\Psi(h)$ are collinear, that is, $\Psi(h)-\Psi(f)$ and $\Psi(g)-\Psi(f)$ are linearly dependent. Hence $\|\Psi(h)-\Psi(f), \Psi(g)-\Psi(f)\|_\beta = 0$.

In the case $\|h-f, g-f\|_\alpha > 0$, there exists an $n_0 \in \mathbb{N}$ such that $n_0 > \|h-f, g-f\|_\alpha$. Assume that

$$\|\Psi(h)-\Psi(f), \Psi(g)-\Psi(f)\|_\beta < \|h-f, g-f\|_\alpha.$$

We can set

$$w = f + \frac{n_0}{\|h-f, g-f\|_\alpha}(g-f).$$

Then we get

$$\|h-f, w-f\|_\alpha = \left\| h-f, f + \frac{n_0}{\|h-f, g-f\|_\alpha}(g-f) - f \right\|_\alpha$$
$$= \frac{n_0}{\|h-f, g-f\|_\alpha}\|h-f, g-h\|_\alpha = n_0.$$

Thus,

$$\|\Psi(h) - \Psi(f), \Psi(w) - \Psi(f)\|_\beta = n_0.$$

By the definition of w,

$$w - g = \left(\frac{n_0}{\|h-f, g-f\|_\alpha} - 1\right)(g-f).$$

Since

$$\frac{n_0}{\|h-f, g-f\|_\alpha} > 1,$$

$h - f_1$ and $f_1 - f_0$ have the same direction. From Lemma 3.3,

$$\|h-f, w-f\|_\alpha = \|h-f, w-g\|_\alpha + \|h-f, g-f\|_\alpha.$$

Thus we have

$$\begin{aligned} & \|\Psi(h) - \Psi(f), \Psi(w) - \Psi(g)\|_\beta \\ \leq\ & \|h-f, w-g\|_\alpha \\ =\ & n_0 - \|h-f, g-f\|_\alpha. \end{aligned}$$

By the assumption,

$$\begin{aligned} n_0 &= \|\Psi(h) - \Psi(f), \Psi(w) - \Psi(f)\|_\beta \\ &\leq \|\Psi(h) - \Psi(f), \Psi(w) - \Psi(g)\|_\beta + \|\Psi(h) - \Psi(f), \Psi(g) - \Psi(f)\|_\beta \\ &< n_0 - \|h-f, g-f\|_\alpha + \|h-f, g-f\|_\alpha = n_0, \end{aligned}$$

which is a contradiction. Hence Ψ is a 2-isometry. This completes the proof. □

Lemma 3.9. *Let f, g be elements of $\Im(X)$. Then $v = \frac{f+g}{2}$ is the unique element of $\Im(X)$ satisfying*

$$\|f-h, f-v\|_\alpha = \|g-v, g-h\|_\alpha = \frac{1}{2}\|f-h, g-h\|_\alpha$$

for some $h \in \Im(X)$ with $\|f-h, g-h\|_\alpha \neq 0$ and v, f, g are collinear.

Proof. Let $\|f-h, g-h\|_\alpha \neq 0$ and $v = \frac{f+g}{2}$. Then v, f, g are 2-collinear. From Lemma 3.1, v satisfies

$$\|f-h, f-v\|_\alpha = \|g-v, g-h\|_\alpha = \frac{1}{2}\|f-h, g-h\|_\alpha$$

for all $h \in \Im(X)$ with $\|f-h, g-h\|_\alpha \neq 0$.

Now we prove the uniqueness.

Let u be an element of $\Im(X)$ satisfying the above properties. That is,

$$\|f-h,f-u\|_\alpha = \|g-u,g-h\|_\alpha = \frac{1}{2}\|f-h,g-h\|_\alpha$$

for some $h \in \Im(X)$ with $\|f-h,g-h\|_\alpha \neq 0$ and u, f, g are collinear. Since u, f, g are collinear, there exists a real number t such that $u = tf+(1-t)g$. From Lemma 3.1, we get

$$\begin{aligned}
& \frac{1}{2}\|f-h,g-h\|_\alpha = \|f-h,f-u\|_\alpha \\
= \;& \|f-h,f-(tf+(1-t)g)\|_\alpha \\
= \;& |1-t|\,\|f-h,f-g\|_\alpha \\
= \;& |1-t|\,\|f-h,g-h\|_\alpha
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{2}\|f-h,g-h\|_\alpha = \|g-u,g-h\|_\alpha \\
= \;& \|g-(tf+(1-t)g)\,,g-h\|_\alpha \\
= \;& \|-tf+tg,g-h\|_\alpha \\
= \;& |t|\,\|f-g,g-h\|_\alpha \\
= \;& |t|\,\|f-h,g-h\|_\alpha\,.
\end{aligned}$$

Since $\|f-h,g-h\|_\alpha \neq 0$, thus we have $\frac{1}{2} = |1-t| = |t|$. Therefore, we get $t = \frac{1}{2}$ and hence $v = u$. This completes the proof. □

Theorem 3.10. *Assume that $\Psi(f)$, $\Psi(g)$ and $\Psi(h)$ are collinear when f, g and h are collinear. If Ψ is a 2-isometry, then Ψ is affine.*

Proof. Let Ψ be a 2-isometry and $\Phi(f) = \Psi(f) - \Psi(0)$. Then Φ is a 2-isometry and $\Phi(0) = 0$. Thus we may assume that $\Psi(0) = 0$. Hence it suffices to show that Ψ is linear.

Let f, $g \in \Im(X)$ with $f \neq g$. Since $\dim\Im(X) > 1$, there exist an element $h \in \Im(X)$ such that

$$\|f-h,g-h\|_\alpha \neq 0.$$

Since Ψ is a 2-isometry, we have

$$\begin{aligned}
& \left\|\Psi(f)-\Psi(h),\Psi(f)-\Psi\left(\frac{f+g}{2}\right)\right\|_\beta \\
= \;& \left\|f-h,f-\frac{f+g}{2}\right\|_\alpha \\
= \;& \left\|f-h,\frac{f-g}{2}\right\|_\alpha \\
= \;& \frac{1}{2}\|f-h,f-g\|_\alpha \\
= \;& \frac{1}{2}\|f-h,g-h\|_\alpha = \frac{1}{2}\|\Psi(f)-\Psi(h),\Psi(g)-\Psi(h)\|_\beta
\end{aligned}$$

from Remark 3.2. Similarly, we can obtain

$$\left\|\Psi(g)-\Psi\left(\frac{f+g}{2}\right),\Psi(g)-\Psi(h)\right\|_\beta=\frac{1}{2}\left\|\Psi(f)-\Psi(h),\Psi(g)-\Psi(h)\right\|_\beta.$$

Since $\frac{f+g}{2}$, f and g are collinear, $\Psi\left(\frac{f+g}{2}\right)$, $\Psi(f)$ and $\Psi(g)$ are also collinear. By Lemma 3.9 we have

$$\Psi\left(\frac{f+g}{2}\right)=\frac{\Psi(f)+\Psi(g)}{2}$$

for all $f,\ g\in\Im(X)$, $\alpha,\ \beta\in(0,1)$. Since $\Psi(0)=0$, we can easily show that Ψ is additive. It follows that Ψ is $\mathbb{Q}$-linear.

Let $r\in\mathbb{R}^+$ with $r\neq 1$ and $f\in\Im(X)$. Since 0, f and rf are collinear, $\Psi(0)$, $\Psi(f)$ and $\Psi(rf)$ are also collinear. Since $\Psi(0)=0$, there exists a real number k such that $\Psi(rf)=k\Psi(f)$. Since $\dim\Im(X)>1$, there exist an element g of $\Im(X)$ such that $\|f,g\|_\alpha\neq 0$. Then we get

$$\begin{aligned}
r\|f,g\|_\alpha &= \|rf,g\|_\alpha=\|rf-0,g-0\|_\alpha\\
&= \|\Psi(rf)-\Psi(0),\Psi(g)-\Psi(0)\|_\beta\\
&= \|\Psi(rf),\Psi(g)\|_\beta=\|k\Psi(f),\Psi(g)\|_\beta\\
&= |k|\,\|\Psi(f),\Psi(g)\|_\beta=k\,\|\Psi(f)-\Psi(0),\Psi(g)-\Psi(0)\|_\beta\\
&= |k|\,\|f-0,g-0\|_\alpha=|k|\,\|f,g\|_\alpha.
\end{aligned}$$

Since $\|f,g\|_\alpha\neq 0$, $|k|=r$. Then $\Psi(rf)=r\Psi(f)$ or $\Psi(rf)=-r\Psi(f)$. Firstly, assume that $k=-r$, that is, $\Psi(rf)=-r\Psi(f)$. Then there exist positive rational numbers q_1, q_2 such that $0<q_1<r<q_2$. Since $\dim\Im(X)>1$, there exist an element $h\in\Im(X)$ such that $\|rf-q_2f,h-q_2f\|_\alpha\neq 0$. Then we have

$$\begin{aligned}
&(q_2+r)\,\|\Psi(f),\Psi(h)-\Psi(q_2f)\|_\beta\\
=\ &\|r\Psi(f)+q_2\Psi(f),\Psi(h)-\Psi(q_2f)\|_\beta\\
=\ &\|-\Psi(rf)+\Psi(q_2f),\Psi(h)-\Psi(q_2f)\|_\beta\\
=\ &\|\Psi(rf)-\Psi(q_2f),\Psi(h)-\Psi(q_2f)\|_\beta\\
=\ &\|rf-q_2f,h-q_2f\|_\alpha\\
=\ &|r-q_2|\,\|f,h-q_2f\|_\alpha\\
=\ &|q_2-r|\,\|f,h-q_2f\|_\alpha\\
\leq\ &(q_2-q_1)\,\|f,h-q_2f\|_\alpha\\
=\ &\|q_1f-q_2f,h-q_2f\|_\alpha\\
=\ &\|\Psi(q_1f)-\Psi(q_2f),\Psi(h)-\Psi(q_2f)\|_\beta\\
=\ &\|q_1\Psi(f)-q_2\Psi(f),\Psi(h)-\Psi(q_2f)\|_\beta\\
=\ &|q_1-q_2|\,\|\Psi(f),\Psi(h)-\Psi(q_2f)\|_\beta\\
=\ &(q_2-q_1)\,\|\Psi(f),\Psi(h)-\Psi(q_2f)\|_\beta.
\end{aligned}$$

Since $\|rf-q_2f,h-q_2f\|_\alpha\neq 0$,

$$\|\Psi(rf)-\Psi(q_2f),\Psi(h)-\Psi(q_2f)\|_\beta.\neq 0.$$

Thus we have $r + q_2 \leq q_2 - q_1$, which is a contradiction. Hence $k = r$, that is, $\Psi(rf) = r\Psi(f)$ for all positive real number r. Thus for every real number r, $\Psi(rf) = r\Psi(f)$. This completes the proof. $\square$

We get the following corollary from Theorem 3.8 and Theorem 3.10.

Corollary 3.11. *Let Ψ be a 2-Lipschitz mapping with the 2-Lipschitz constant $\kappa \leq 1$. Suppose that $\Psi(f)$, $\Psi(g)$ and $\Psi(h)$ are collinear when f, g and h are collinear. If Ψ satisfies (AOPP), then Ψ is an affine 2-isometry.*

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Cihangir Alaca, Department of Mathematics, Faculty of Science and Arts, Sinop University, 57000 Sinop, Turkey
E-mail address: cihangiralaca@yahoo.com.tr