NUMERICAL METHODS FOR FUZZY LINEAR PARTIAL DIFFERENTIAL EQUATIONS UNDER NEW DEFINITION FOR DERIVATIVE

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Abstract. In this paper difference methods to solve "fuzzy partial differential equations" (FPDE) such as fuzzy hyperbolic and fuzzy parabolic equations are considered. The existence of the solution and stability of the method are examined in detail. Finally examples are presented to show that the Hausdorff distance between the exact solution and approximate solution tends to zero.

1. Introduction

The topic of numerical methods for solving fuzzy differential equations has been rapidly growing in recent years. The concept of fuzzy derivative was first introduced by S.L. Chang, L.A. Zadeh in [7]. It was followed up by D. Dubois, H. Prade in [8], who defined any used the extension principle. Other methods have been discussed by M. L Puri, D. A. ralescu in [14] and R. Goetschel, W. Voxman in [9]. The fuzzy differential equations and fuzzy initial value problem were treated in a standard way by O. Kaleva in [10] and [11], by S. Seikkala in [15], ... The numerical methods for solving fuzzy differential equations were introduced by M. Ma, M. Friedman, A. Kandel in [12] by the standard Euler method.

In [4], J. Buckley and T. Feuring proposed a procedure to examine solutions of fuzzy partial differential equations. They checked to see if the Buckley-Feuring solution exist or not. If the Buckley-Feuring solution fails to exist they check if the Seikkala solution exists. Their proposed method only works for elementary partial differential in their sense of elementary. They assumed solution of FPDE is not defined series and so Bessel functions and Legendre function are not used in the solution. In [1] T. Allahviranloo used a numerical method to solve FPDE, that was based on the Seikala derivative.

In this paper we use a difference method for solving the fuzzy hyperbolic equations and the fuzzy parabolic equations. The paper is organized as follows:

In section 2 we noted some basic definitions of fuzzy numbers and fuzzy derivative which have been discussed by B. Bede, SG. Gal and which are used in the paper. In section 3 we define two FPDEs, in particular, the fuzzy heat equation and the fuzzy wave equation and also use difference methods for them. The necessary conditions for stability of proposed method are discussed in section 4. The difference methods

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are illustrated by solving some examples in section 5 and conclusions are drown in section 6.

2. Preliminaries

We begin this section by defining the notation that we will use in the paper. We place a $\sim$ sign over a letter to denote a fuzzy subset of the real numbers. We write $\widetilde{A}(x)$, a number in $[0,1]$, for the membership function of $\widetilde{A}$ evaluated at x. An $\alpha-$cut of $\widetilde{A}$, written $\widetilde{A}[\alpha]$, is defined as $\{x|\widetilde{A}(x) \geq \alpha\}$, for $0 < \alpha \leq 1$.

We represent an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$, which satisfies the following requirements:

(1) $\underline{u}(r)$ is a bounded left continuous non decreasing function over $[0,1]$.
(2) $\overline{u}(r)$ is a bounded left continuous non increasing function over $[0,1]$.
(3) $\underline{u}(r) \leq \overline{u}(r),\ 0 \leq r \leq 1$.

A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha, \quad 0 \leq r \leq 1$.
Let F be the set of all upper semicontinuous normal convex fuzzy numbers with bounded $\alpha-$level sets. Since the $\alpha-$cuts of fuzzy numbers are always closed and bounded, we denote the intervals by $\widetilde{N}[\alpha] = [\underline{N}(\alpha), \overline{N}(\alpha)]$, for all α. The Hausdorff metric is defined on F as

$$d_\infty(\widetilde{u}, \widetilde{v}) = \sup\{d_H(\widetilde{u}[\alpha], \widetilde{v}[\alpha]) : 0 \leq r \leq 1\}, \quad \widetilde{u}, \widetilde{v} \in E.$$

where d_H is the Hausdorff on classic sets.
Puri and Ralescu [13] introduced H-derivative that is based in the H-difference of sets, as follows.

Definition 2.1. Let $u, v \in F^n$. If there exists $w \in F^n$ such that $u = v + w$, then w is called the H-difference of u and v and it is denoted by $u - v$.

In [3] the authors introduce a more general definition of derivative for fuzzy mappings enlarging the class of differentiable fuzzy mappings by considering a lateral type of H-derivatives.

Definition 2.2. Let $F : (a,b) \longrightarrow F^n$ and $t_0 \in (a,b)$. We say that F is differentiable at t_0 if:
(1) There exists an element $F'(t_0) \in F^n$ such that, for all $h > 0$ sufficiently near to 0, there are $F(t_0 + h) - F(t_0)$, $F(t_0) - F(t_0 - h)$ and

$$\lim_{h \to o^+} \frac{F(t_0 + h) - F(t_0)}{h} = \lim_{h \to o^+} \frac{F(t_0) - F(t_0 - h)}{h} = F'(t_0). \tag{1}$$

or
(2) There exists an element $F'(t_0) \in F^n$ such that, for all $h < 0$ sufficiently near to 0, there are $F(t_0 + h) - F(t_0)$, $F(t_0) - F(t_0 - h)$ and

$$\lim_{h \to o^-} \frac{F(t_0 + h) - F(t_0)}{h} = \lim_{h \to o^-} \frac{F(t_0) - F(t_0 - h)}{h} = F'(t_0). \tag{2}$$

Theorem 2.3. *Let $F : T \longrightarrow \mathrm{F}$ be a function and denote $[F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$, for $\alpha \in [0,1]$. Then* *(i) If F is differentiable in the first form* (1)*, then f_α and g_α are differentiable functions and*

$$[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]. \tag{3}$$

(ii) If F is differentiable in the second form (2)*, then f_α and g_α are differentiable functions and*

$$[F'(t)]^\alpha = [g'_\alpha(t), f'_\alpha(t)]. \tag{4}$$

Proof. [6] □

Consider the FPDE

$$\varphi(D_x, D_y)\widetilde{U}(x,y) = \widetilde{F}(x,y,\widetilde{K}), \tag{5}$$

subject to certain boundary conditions where the operator $\varphi(D_x, D_y)$ is a polynomial, with constant coefficient, in D_x and D_y, where $D_x(D_y)$ stands for the partial differential with respect to $x(y)$. The boundary conditions can be of the form $\widetilde{U}(0,y) = \widetilde{C}_1$, $\widetilde{U}(x,0) = \widetilde{C}_2$, $\widetilde{U}(M_1,y) = \widetilde{C}_3$, $\ldots$, $\widetilde{U}(0,y) = \widetilde{C}_1$, $\widetilde{U}(0,y) = \widetilde{g}_1(y;\widetilde{C}_4)$, Where $\widetilde{U}(x,0) = \widetilde{f}_1(x;\widetilde{C}_5)$, $\ldots$. $\widetilde{F}(x,y,\widetilde{K})$ is the fuzzy function with $\widetilde{K} = (\widetilde{k}_1,\ldots,\widetilde{k}_n)$, where $\widetilde{k}_i$ is a triangular fuzzy number in J_i, $1 \le i \le n$. Let $I_1 = [0, M_1]$, $I_2 = [0, M_2]$. The fuzzy function $\widetilde{U}$ maps $I_1 \times I_2$ into fuzzy numbers. Also let $\widetilde{C} = (\widetilde{c}_1,\ldots,\widetilde{c}_m)$ with $\widetilde{c}_i$ being triangular fuzzy number in the interval L_i, $1 \le i \le m$. Let

$$\widetilde{K}[\alpha] = \prod_{i=1}^{n} \widetilde{k}_i[\alpha], \quad \widetilde{C}[\alpha] = \prod_{i=1}^{m} \widetilde{c}_i[\alpha].$$

Let $\widetilde{U}(x,y)[\alpha] = [\underline{U}(x,y;\alpha), \overline{U}(x,y;\alpha)]$. We assume that the $\underline{U}(x,y;\alpha)$ and $\overline{U}(x,y;\alpha)$ have partial continuous so that $\varphi(D_x,D_y)\underline{U}(x,y;\alpha)$ and $\varphi(D_x,D_y)\overline{U}(x,y;\alpha)$ are continuous for all $(x,y) \in I_1 \times I_2$, all α. Then, we have the following alternatives for solving the problem (5).

Case 1: If we consider $\varphi(D_x,D_y)\widetilde{U}(x,y)$ by using the derivative in the first form (1), then we have

$$[\varphi(D_x,D_y)\widetilde{U}(x,y)]^\alpha = [\varphi(D_x,D_y)\underline{U}(x,y,\alpha), \varphi(D_x,D_y)\overline{U}(x,y,\alpha)]$$

and we have to solve the system of partial differential equations

$$\varphi(D_x,D_y)\underline{U}(x,y;\alpha) = \underline{F}(x,y;\alpha) = min\{F(x,y,k)|k \in \widetilde{K}[\alpha]\}, \tag{6}$$

$$\varphi(D_x,D_y)\overline{U}(x,y;\alpha) = \overline{F}(x,y;\alpha) = max\{F(x,y,k)|k \in \widetilde{K}[\alpha]\}, \tag{7}$$

for all $(x,y) \in I_1 \times I_2$ and all $\alpha \in [0,1]$. We append equations (6) and (7) any boundary conditions, for example, if they were $\widetilde{U}(0,y) = \widetilde{C}_1$ and $\widetilde{U}(M_1,y) = \widetilde{C}_2$, then we add

$$\underline{U}(0,y;\alpha) = \underline{C}_1(\alpha), \quad \underline{U}(M_1,y;\alpha) = \underline{C}_2(\alpha) \tag{8}$$

to equation (6) and

$$\overline{U}(0,y;\alpha) = \overline{C}_1(\alpha), \quad \overline{U}(M_1,y;\alpha) = \overline{C}_2(\alpha) \tag{9}$$

to equation (7) where $\widetilde{C}_i[\alpha] = [\underline{C}_i(\alpha), \overline{C}_i(\alpha)],\ i = 1, 2$. Let $\underline{U}(x, y; \alpha)$ and $\overline{U}(x, y; \alpha)$ solves equations (6) and (7), add the boundary equations, respectively.

Case 2: If we consider $\varphi(D_x, D_y)\widetilde{U}(x, y)$by using the derivative in the second form (2), then we have

$$[\varphi(D_x, D_y)\widetilde{U}(x, y)]^{\alpha} = [\varphi(D_x, D_y)\overline{U}(x, y, \alpha), \varphi(D_x, D_y)\underline{U}(x, y, \alpha)]$$

and, we should solve the system of partial differential equations

$$\varphi(D_x, D_y)\underline{U}(x, y; \alpha) = \overline{F}(x, y; \alpha) = max\{F(x, y, k)|k \in \widetilde{K}[\alpha]\}, \tag{10}$$

$$\varphi(D_x, D_y)\overline{U}(x, y; \alpha) = \underline{F}(x, y; \alpha) = min\{F(x, y, k)|k \in \widetilde{K}[\alpha]\}, \tag{11}$$

for all $(x, y) \in I_1 \times I_2$ and all $\alpha \in [0, 1]$. We append equations (10) and (11) any boundary conditions, for example, if they were $\widetilde{U}(0, y) = \widetilde{C}_1$ and $\widetilde{U}(M_1, y) = \widetilde{C}_2$, then we add

$$\underline{U}(0, y; \alpha) = \underline{C}_1(\alpha), \quad \underline{U}(M_1, y; \alpha) = \underline{C}_2(\alpha) \tag{12}$$

to equation (10) and

$$\overline{U}(0, y; \alpha) = \overline{C}_1(\alpha), \quad \overline{U}(M_1, y; \alpha) = \overline{C}_2(\alpha) \tag{13}$$

to equation (11) where $\widetilde{C}_i[\alpha] = [\underline{C}_i(\alpha), \overline{C}_i(\alpha)],\ i = 1, 2$. Let $\underline{U}(x, y; \alpha)$ and $\overline{U}(x, y; \alpha)$ solves equations (10) and (11), add the boundary equations, respectively.

3. A Fuzzy Partial Differential Equation

In this section we consider $\varphi(D_x, D_y)\widetilde{U}(x, y)$ by using the derivative in the first form (1) or second form (2) also we solve two types of FPDE as numerically.

(1) **Fuzzy Parabolic Equation**

Consider the fuzzy heat equation which is illustrated below with the parabolic equation:

$$(D_t - \beta^2 D_x D_x)\widetilde{U}(x, t) = \widetilde{0}, \quad 0 < x < l, \quad t > 0, \tag{14}$$

where

$$\widetilde{U}(0, t) = \widetilde{\widetilde{K_1}}, \quad \widetilde{U}(l, t) = \widetilde{\widetilde{K_2}}, \qquad t > 0, \qquad \widetilde{U}(x, 0) = \widetilde{f}(x), \qquad 0 < x < l$$

We have four different cases:

(a) If both $(\frac{\partial^2 \widetilde{U}}{\partial x^2})$ and $(\frac{\partial \widetilde{U}}{\partial t})$ are differentiable in the first form (1) then by (6) and (7) we have

$$(D_t)\underline{U}(x, t; \alpha) - (\beta^2 D_x D_x)\overline{U}(x, t; \alpha) = \underline{0},$$

$$(D_t)\overline{U}(x, t; \alpha) - (\beta^2 D_x D_x)\underline{U}(x, t; \alpha) = \overline{0},$$

$$0 < x < l, \quad t > 0, \quad \alpha \in [0, 1] \tag{15}$$

where

$$\underline{U}(0,t;\alpha) = \underline{K_1}(\alpha), \quad \underline{U}(l,t;\alpha) = \underline{K_2}(\alpha), \qquad t > 0, \quad \alpha \in [0,1],$$

$$\overline{U}(0,t;\alpha) = \overline{K_1}(\alpha), \quad \overline{U}(l,t;\alpha) = \overline{K_2}(\alpha), \qquad t > 0, \quad \alpha \in [0,1],$$

$$\underline{U}(x,0;\alpha) = \underline{f}(x;\alpha), \overline{U}(x,0;\alpha) = \overline{f}(x;\alpha) \quad 0 < x < l, \quad \alpha \in [0,1].$$

(b) if both $(\frac{\partial^2 \widetilde{U}}{\partial x^2})$ and $(\frac{\partial \widetilde{U}}{\partial t})$ are differentiable in the second form (2), then by (10) and (11) we have

$$(D_t)\underline{U}(x,t;\alpha) - (\beta^2 D_x D_x)\overline{U}(x,t;\alpha) = \overline{0},$$
$$(D_t)\overline{U}(x,t;\alpha) - (\beta^2 D_x D_x)\underline{U}(x,t;\alpha) = \underline{0},$$
$$0 < x < l, \quad t > 0, \quad \alpha \in [0,1] \tag{16}$$

The boundary conditions are the same as the first case.

(c) if $(\frac{\partial^2 \widetilde{U}}{\partial x^2})$ is differentiable in the first form (1) and $(\frac{\partial \widetilde{U}}{\partial t})$ is differentiable in the second form (2), then by (6), (7), (10) and (11) we have

$$(D_t)\overline{U}(x,t;\alpha) - (\beta^2 D_x D_x)\overline{U}(x,t;\alpha) = \underline{0},$$
$$(D_t)\underline{U}(x,t;\alpha) - (\beta^2 D_x D_x)\underline{U}(x,t;\alpha) = \overline{0},$$
$$0 < x < l, \quad t > 0, \quad \alpha \in [0,1] \tag{17}$$

The boundary conditions are the same as the first case.
(d) if $(\frac{\partial^2 \widetilde{U}}{\partial x^2})$ is differentiable in the second form (2) and $(\frac{\partial \widetilde{U}}{\partial t})$ is differentiable in the first form (1) then by (6), (7), (10) and (11) we have

$$(D_t)\underline{U}(x,t;\alpha) - (\beta^2 D_x D_x)\underline{U}(x,t;\alpha) = \underline{0},$$
$$(D_t)\overline{U}(x,t;\alpha) - (\beta^2 D_x D_x)\overline{U}(x,t;\alpha) = \overline{0},$$
$$0 < x < l, \quad t > 0, \quad \alpha \in [0,1] \tag{18}$$

The boundary conditions are the same as the first case.
Assume $\widetilde{U}$ is a fuzzy function of the independent crisp variables x and t. Subdivide the x-t plane into sets of equal rectangles of sides $\delta x = h$, $\delta t = k$, by equally spaced grid lines parallel to O_y, defined by $x_i = ih$, $i = 0, 1, 2, \ldots$ and equally spaced grid lines parallel to O_x, defined by $y_j = jk$, $j = 0, 1, 2, \ldots$.

Denote the value of $\widetilde{U}$ at the representative mesh point $p(ih, jk)$ by

$$\widetilde{U}_p = \widetilde{U}(ih, jk) = \widetilde{U}_{i,j} \tag{19}$$

and also denote the parametric form of fuzzy number, $\widetilde{U}_{i,j}$ as follow

$$\widetilde{U}_{i,j} = (\underline{u}_{i,j}, \overline{u}_{i,j}). \tag{20}$$

If we consider $(D_xD_x)\widetilde{U}_{i,j}$ by using the derivative in the first form (1) then by Taylor's theorem and definition of standard difference

$$(D_xD_x)\widetilde{U}_{i,j} = (\underline{(D_xD_x)\widetilde{U}_{i,j}}, \overline{(D_xD_x)\widetilde{U}_{i,j}}),$$

where

$$\underline{(D_xD_x)\widetilde{U}_{i,j}} \simeq \frac{\underline{u}\{(i+1)h, jk\} - 2\overline{u}\{ih, jk\} + \underline{u}\{(i-1)h, jk\}}{h^2},$$

$$\overline{(D_xD_x)\widetilde{U}_{i,j}} \simeq \frac{\overline{u}\{(i+1)h, jk\} - 2\underline{u}\{ih, jk\} + \overline{u}\{(i-1)h, jk\}}{h^2}.$$

By (19) and (20) we have

$$\underline{(D_xD_x)\widetilde{U}_{i,j}} \simeq \frac{\underline{u}_{i+1,j} - 2\overline{u}_{i,j} + \underline{u}_{i-1,j}}{h^2}, \quad \overline{(D_xD_x)\widetilde{U}_{i,j}} \simeq \frac{\overline{u}_{i+1,j} - 2\underline{u}_{i,j} + \overline{u}_{i-1,j}}{h^2} \tag{21}$$

also if we consider, $(D_xD_x)\tilde{U}_{i,j}$ by using the derivative in the second form (2) then by Taylor's theorem and definition of standard difference we yield equations as follows

$$\underline{(D_xD_x)\widetilde{U}_{i,j}} \simeq \frac{\overline{u}_{i+1,j} - 2\underline{u}_{i,j} + \overline{u}_{i-1,j}}{h^2}, \quad \overline{(D_xD_x)\widetilde{U}_{i,j}} \simeq \frac{\underline{u}_{i+1,j} - 2\overline{u}_{i,j} + \underline{u}_{i-1,j}}{h^2} \tag{22}$$

with a leading error of order h^2. Similarly if $(D_tD_t)\tilde{U}_{i,j}$ comes from the first form (1),

$$\underline{(D_tD_t)\widetilde{U}_{i,j}} \simeq \frac{\underline{u}_{i,j+1} - 2\overline{u}_{i,j} + \underline{u}_{i,j-1}}{k^2}, \quad \overline{(D_tD_t)\widetilde{U}_{i,j}} \simeq \frac{\overline{u}_{i,j+1} - 2\underline{u}_{i,j} + \overline{u}_{i,j-1}}{k^2} \tag{23}$$

and if consider the second form (2) we have,

$$\underline{(D_tD_t)\widetilde{U}_{i,j}} \simeq \frac{\overline{u}_{i,j+1} - 2\underline{u}_{i,j} + \overline{u}_{i,j-1}}{k^2}, \quad \overline{(D_tD_t)\widetilde{U}_{i,j}} \simeq \frac{\underline{u}_{i,j+1} - 2\overline{u}_{i,j} + \underline{u}_{i,j-1}}{k^2} \tag{24}$$

with a leading error of order k^2. With this notation the forward - difference approximation for $(D_t)\widetilde{U}$ at P is

$$\underline{(D_t)\widetilde{U}_{i,j}} \simeq \frac{\underline{u}_{i,j+1} - \overline{u}_{i,j}}{k}, \qquad \overline{(D_t)\widetilde{U}_{i,j}} \simeq \frac{\overline{u}_{i,j+1} - \underline{u}_{i,j}}{k} \tag{25}$$

if we use the first form (1) and if we use the second form (2) is

$$\underline{(D_t)\widetilde{U}_{i,j}} \simeq \frac{\overline{u}_{i,j+1} - \underline{u}_{i,j}}{k}, \qquad \overline{(D_t)\widetilde{U}_{i,j}} \simeq \frac{\underline{u}_{i,j+1} - \overline{u}_{i,j}}{k} \tag{26}$$

with a leading error of o(k).
One finite-difference approximation to

$$(D_t)\widetilde{U} - \beta^2(D_xD_x)\widetilde{U} = \widetilde{0} = (\underline{0}, \overline{0}) \tag{27}$$

is

$$\underline{(D_t)\widetilde{U}} - \beta^2\overline{(D_xD_x)\widetilde{U}} = \underline{0} = \varepsilon(\alpha - 1),$$

$$\overline{(D_t)\widetilde{U}} - \beta^2\underline{(D_xD_x)\widetilde{U}} = \overline{0} = \varepsilon(1 - \alpha)$$

Now there are two cases, the **first** one by using (21), (25) or (22), (26) and definition of standard difference and $\tilde{0} = (\underline{0}, \overline{0}) = (0, 0)$ following equations must be hold:

$$\frac{\underline{u}_{i,j+1} - \overline{u}_{i,j}}{k} = \frac{\beta^2(\overline{u}_{i+1,j} - 2\underline{u}_{i,j} + \overline{u}_{i-1,j})}{h^2} \tag{28}$$

$$\frac{\overline{u}_{i,j+1} - \underline{u}_{i,j}}{k} = \frac{\beta^2(\underline{u}_{i+1,j} - 2\overline{u}_{i,j} + \underline{u}_{i-1,j})}{h^2} \tag{29}$$

where $\alpha \in [0,1]$ and $\widetilde{U} = (\underline{u}, \overline{u})$ is the exact solution of the approximating difference equations, $x_i = ih, \quad (i = 0, 1, 2, \ldots)$ and $t_j = jk, \quad (j = 0, 1, 2, \ldots)$. This can be written as

$$\underline{u}_{i,j+1} = r\overline{u}_{i-1,j} + (-2r)\underline{u}_{i,j} + r\overline{u}_{i+1,j} + \overline{u}_{i,j} \tag{30}$$

$$\overline{u}_{i,j+1} = r\underline{u}_{i-1,j} + (-2r)\overline{u}_{i,j} + r\underline{u}_{i+1,j} + \underline{u}_{i,j} \tag{31}$$

where $r = \frac{\beta^2 k}{h^2}$ and $\alpha \in [0,1]$.

The **second** case by using (21), (26) or (22), (25) and definition of standard difference the following equations must be hold:

$$\frac{\overline{u}_{i,j+1} - \underline{u}_{i,j}}{k} = \frac{\beta^2(\overline{u}_{i+1,j} - 2\underline{u}_{i,j} + \overline{u}_{i-1,j})}{h^2} \tag{32}$$

$$\frac{\underline{u}_{i,j+1} - \overline{u}_{i,j}}{k} = \frac{\beta^2(\underline{u}_{i+1,j} - 2\overline{u}_{i,j} + \underline{u}_{i-1,j})}{h^2} \tag{33}$$

This can be written as

$$\underline{u}_{i,j+1} = r\underline{u}_{i+1,j} + (1 - 2r)\overline{u}_{i,j} + r\underline{u}_{i-1,j} \tag{34}$$

$$\overline{u}_{i,j+1} = r\overline{u}_{i+1,j} + (1 - 2r)\underline{u}_{i,j} + r\overline{u}_{i-1,j} \tag{35}$$

Hence we can calculate the unknown pivotal values of u along the first time-row, $t = k$, in terms of known boundary and initial values along $t = 0$, then the unknown pivotal values along the second time-row in terms of the calculated pivotal values along the first, and so on.

(2) **Fuzzy Hyperbolic Equation**

Consider the fuzzy wave equation which is illustrated below with the hyperbolic equation:

$$(D_t D_t - \beta^2 D_x D_x)\widetilde{U}(x,t) = \widetilde{0}, \quad 0 < x < l, \quad t > 0, \tag{36}$$

where

$$\widetilde{U}(0,t) = \widetilde{U}(l,t) = \widetilde{0}, \quad t > 0, \quad \widetilde{U}(x,0) = \widetilde{f}(x), \quad 0 \leq x \leq l.$$

and

$$\frac{\partial \widetilde{U}}{\partial t}(x,0) = \widetilde{g}(x), \quad 0 \leq x \leq l.$$

All conditions are similar to heat equation. One finite-difference approximation to

$$(D_t D_t)\widetilde{U} - \beta^2 (D_x D_x)\widetilde{U} = \widetilde{0} = (\underline{0}, \overline{0}) \tag{37}$$

is

$$\underline{(D_tD_t)\widetilde{U}} - \beta^2\overline{(D_xD_x)\widetilde{U}} = \underline{0} = \varepsilon(\alpha-1),$$

$$\overline{(D_tD_t)\widetilde{U}} - \beta^2\underline{(D_xD_x)\widetilde{U}} = \overline{0} = \varepsilon(1-\alpha)$$

Now there are four cases that we can classify them to two cases, the **first** case by (21), (23) or (22), (24) and definition of standard difference and $\tilde{0} = (\underline{0}, \overline{0}) = (0,0)$ that the following equations must be hold:

$$\frac{\underline{u}_{i,j+1} - 2\overline{u}_{i,j} + \underline{u}_{i,j-1}}{k^2} = \frac{\beta^2(\overline{u}_{i+1,j} - 2\underline{u}_{i,j} + \overline{u}_{i-1,j})}{h^2} \tag{38}$$

$$\frac{\overline{u}_{i,j+1} - 2\underline{u}_{i,j} + \overline{u}_{i,j-1}}{k^2} = \frac{\beta^2(\underline{u}_{i+1,j} - 2\overline{u}_{i,j} + \underline{u}_{i-1,j})}{h^2} \tag{39}$$

where $\widetilde{U} = (\underline{u}, \overline{u})$ is the exact solution of the approximating difference equations, $x_i = ih, \quad (i = 0,1,2,\ldots)$ and $t_j = jk, \quad (j = 0,1,2,\ldots)$. This can be written as

$$\underline{u}_{i,j+1} = (-2\lambda^2)\underline{u}_{i,j} + \lambda^2(\overline{u}_{i+1,j} + \overline{u}_{i-1,j}) - \underline{u}_{i,j-1} + 2\overline{u}_{i,j} \tag{40}$$

$$\overline{u}_{i,j+1} = (-2\lambda^2)\overline{u}_{i,j} + \lambda^2(\underline{u}_{i+1,j} + \underline{u}_{i-1,j}) - \overline{u}_{i,j-1} + 2\underline{u}_{i,j} \tag{41}$$

The **second** case by using (21), (24) or (22), (23) and definition of standard difference the following equations yield by similar calculations:

$$\underline{u}_{i,j+1} = (-2\lambda^2+2)\overline{u}_{i,j} + \lambda^2(\underline{u}_{i+1,j} + \underline{u}_{i-1,j}) - \underline{u}_{i,j-1} \tag{42}$$

$$\overline{u}_{i,j+1} = (-2\lambda^2+2)\underline{u}_{i,j} + \lambda^2(\overline{u}_{i+1,j} + \overline{u}_{i-1,j}) - \overline{u}_{i,j-1} \tag{43}$$

where $\lambda = \frac{\beta k}{h}$ and $\alpha \in [0,1]$. Hence we can calculate the unknown pivotal values of u along the first time-row, $t = k$, in terms of known boundary and initial values along $t = 0$, then the unknown pivotal values along the second time-row in terms of the calculated pivotal values along the first, and so on.

4. A Necessary Condition for Stability

Now we are going to consider the stability of the classical explicit equations (30), (31). If the boundary values at $i = 0$ and N, $j > 0$ are known, then $2(N-1)$ equations can be written in the following matrix

$$\begin{bmatrix}
-2r & & & & & 1 & r & & & \\
& \cdot & & & & r & 1 & r & & \\
& & \cdot & & & & \ddots & \ddots & \ddots & \\
& & & \cdot & & & & r & 1 & r \\
& & & & -2r & & & & r & 1 \\
1 & r & 0 & & & -2r & & & & \\
r & 1 & r & & & & \cdot & & & \\
& \ddots & \ddots & \ddots & & & & \cdot & & \\
& & r & 1 & r & & & & \cdot & \\
& & & r & 1 & & & & & -2r
\end{bmatrix}
\begin{bmatrix}
\underline{u}_{1,j} \\ \underline{u}_{2,j} \\ \vdots \\ \underline{u}_{N-1,j} \\ \overline{u}_{1,j} \\ \overline{u}_{2,j} \\ \vdots \\ \overline{u}_{N-1,j}
\end{bmatrix}$$

$$= (\underline{u}_{1,j+1}, \underline{u}_{2,j+1}, \ldots, \underline{u}_{N-1,j+1}, \overline{u}_{1,j+1}, \overline{u}_{2,j+1}, \ldots, \overline{u}_{N-1,j+1})^t \tag{44}$$

i.e.

$$\begin{bmatrix} \underline{w}_{j+1} \\ \overline{w}_{j+1} \end{bmatrix} = P \begin{bmatrix} \underline{w}_j \\ \overline{w}_j \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{45}$$

where

$$P = \begin{bmatrix} A & B \\ B & A \end{bmatrix}, \tag{46}$$

$$A = (-2r)I, \qquad B = I + r \begin{bmatrix} 0 & 1 & & \\ 1 & . & \ddots & \\ & \ddots & . & 1 \\ & & 1 & 0 \end{bmatrix}.$$

Lemma 4.1. *The eigenvalues of a common tridiagonal matrix the eigenvalue of the $N \times N$ matrix*

$$\begin{bmatrix} a & b & & & \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ & & & c & a \end{bmatrix}$$

are

$$\lambda_k = a + 2\{\sqrt{bc}\} \cos \frac{k\pi}{N+1} \qquad k = 1, 2, \ldots, N$$

where a, b and c may be real or complex.

Theorem 4.2. *Let matrix P be of the follow form*

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}.$$

Then the eigenvalues of P are union of eigenvalues of $A + B$ and eigenvalues of $A - B$. [2]

Now we prove the stability of this method in the following theorem.

Theorem 4.3. *If $r = \frac{k}{h^2} < \frac{1}{2}$ then the difference of equations (30) and (31) are stable.*

Proof. It is sufficient to show in (45) that $\rho(P) < 1$, thus by Theorem 4.2 it is sufficient to find eigenvalues of

$$A + B = \begin{bmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & 1-2r & r \\ & & & r & 1-2r \end{bmatrix}$$

and

$$A-B=\begin{bmatrix} -2r-1 & -r & & & \\ -r & -2r-1 & -r & & \\ & \ddots & \ddots & \ddots & \\ & & -r & -2r-1 & -r \\ & & & -r & -2r-1 \end{bmatrix}.$$

Let matrices $(N-1)\times(N-1)$, T and T' be as follows

$$T=\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, T'=\begin{bmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ & & & 1 & 2 \end{bmatrix}$$

thus

$$A+B=I-rT,$$
$$A-B=-rT'-I,$$

where I is the unit matrix of order $2(N-1)$ and T a $(N-1)\times(N-1)$ matrix whose eigenvalues λ_T are given by

$$\lambda_T=\lambda_{T'}=4\cos^2\frac{k\pi}{2(N+1)} \qquad k=1,2,\ldots,N-1.$$

By using Lemma 4.1 the eigenvalues of $A-B$ and $A+B$ are obtained as follows:

$$\lambda_{A-B}=-4r\cos^2\frac{k\pi}{2(N+1)}-1, \qquad \lambda_{A+B}=1-4r\cos^2\frac{k\pi}{2(N+1)}.$$

Therefore the equations will be stable when

$$\rho(A-B)=\max_k|-4r\cos^2\frac{k\pi}{2(N+1)}-1|<1 \quad k=1,2,\ldots,N-1$$

$$\rho(A+B)=\max_k|1-4r\cos^2\frac{k\pi}{2(N+1)}|<1 \quad k=1,2,\ldots,N-1$$

i.e.

$$-1<-1-4r\cos^2\frac{k\pi}{2(N+1)}<1 \quad k=1,2,\ldots,N-1,$$

$$-1<1-4r\cos^2\frac{k\pi}{2(N+1)}<1 \quad k=1,2,\ldots,N-1.$$

As $h\to 0$, $N\to\infty$ and $\cos^2\frac{k\pi}{2(N+1)}\to 1$ hence

$$-1<\pm 1-4r\cos^2\frac{k\pi}{2(N+1)}<1$$

so we have $|r|<\frac{1}{2}$. □

This is the necessary and sufficient condition for the difference equations to be stable when the solution of the FPDE dose not increase as t increases.

Corollary 4.4. *If $r = \frac{k}{h^2} < \frac{1}{2}$ than the difference equations (34) and (35) are stable.*

Proof. The stability can be deal with in a similar manner where

$$P = \begin{bmatrix} A & B \\ B & A \end{bmatrix}, \tag{47}$$

$$B = (1-2r)I, \quad A = r\begin{bmatrix} 0 & 1 & & \\ 1 & . & \ddots & \\ & \ddots & . & 1 \\ & & 1 & 0 \end{bmatrix}. \tag{48}$$

and

$$A+B = \begin{bmatrix} 1-2r & r & & & \\ r & 1-2r & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & 1-2r & r \\ & & & r & 1-2r \end{bmatrix},$$

$$A-B = \begin{bmatrix} 2r-1 & r & & & \\ r & 2r-1 & r & & \\ & \ddots & \ddots & \ddots & \\ & & r & 2r-1 & r \\ & & & r & 2r-1 \end{bmatrix} \tag{49}$$

thus

$$A+B = I - rT, \qquad A-B = rT^{'} - I$$

then the condition for stability will yield the same as last case. □

The stability of (40), (41) (wave equations) and (42), (43) can be deal with in a similar manner. If the boundary values at $i = 0$ and M, $j > 0$ are known, then $2(M-1)$ equations can be written in matrix as

$$(\underline{v}_{1,j+1},\underline{v}_{2,j+1},\ldots,\underline{v}_{M-1,j+1},\overline{v}_{1,j+1},\overline{v}_{2,j+1},\ldots,\overline{v}_{M-1,j+1})^t$$

$$= \begin{bmatrix} -2\lambda^2 & & & & & 2 & \lambda^2 & & & \\ & . & & & & \lambda^2 & 2 & \lambda^2 & & \\ & & . & & & & \ddots & \ddots & \ddots & \\ & & & . & & & & \lambda^2 & 2 & \lambda^2 \\ & & & & -2\lambda^2 & & & & \lambda^2 & 2 \\ 2 & \lambda^2 & 0 & & & -2\lambda^2 & & & & \\ \lambda^2 & 2 & \lambda^2 & & & & . & & & \\ & \ddots & \ddots & \ddots & & & & . & & \\ & & \lambda^2 & 2 & \lambda^2 & & & & . & \\ & & & \lambda^2 & 2 & & & & & -2\lambda^2 \end{bmatrix} \begin{bmatrix} \underline{v}_{1,j} \\ \underline{v}_{2,j} \\ \vdots \\ \underline{v}_{M-1,j} \\ \overline{v}_{1,j} \\ \overline{v}_{2,j} \\ \vdots \\ \overline{v}_{M-1,j} \end{bmatrix}$$

$$+ \begin{bmatrix} -1 & & 0 & 0 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & -1 & 0 & & 0 \\ 0 & & 0 & -1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & 0 & 0 & & -1 \end{bmatrix} \begin{bmatrix} \underline{v}_{1,j-1} \\ \underline{v}_{2,j-1} \\ \vdots \\ \underline{v}_{M-1,j-1} \\ \overline{v}_{1,j-1} \\ \overline{v}_{2,j-1} \\ \vdots \\ \overline{v}_{M-1,j-1} \end{bmatrix} + \begin{bmatrix} \varepsilon(\alpha-1) \\ \\ \vdots \\ \varepsilon(\alpha-1) \\ \varepsilon(1-\alpha) \\ \\ \vdots \\ \varepsilon(1-\alpha) \end{bmatrix} \quad (50)$$

we have

$$V_{j+1} = P.V_j - IV_{j-1},$$

where

$$P = \begin{bmatrix} A & B \\ B & A \end{bmatrix},$$

$$B = 2I + \lambda^2 \begin{bmatrix} 0 & 1 & & \\ 1 & . & \ddots & \\ & \ddots & . & 1 \\ & & 1 & 0 \end{bmatrix}, \qquad A = -2\lambda^2 I.$$

Now let

$$W_{j+1} = \begin{bmatrix} V_{j+1} \\ V_j \end{bmatrix}, \ \ W_j = \begin{bmatrix} V_j \\ V_{j-1} \end{bmatrix} \ \ and \ \ W_{j-1} = \begin{bmatrix} V_{j-1} \\ V_{j-2} \end{bmatrix}.$$

Then the iterative method yields as follow

$$\begin{bmatrix} V_{j+1} \\ V_j \end{bmatrix} = Q \begin{bmatrix} V_j \\ V_{j-1} \end{bmatrix} + E \quad (51)$$

where

$$Q = \begin{bmatrix} P & -I \\ I & 0 \end{bmatrix}$$

Now we prove the stability of this method in the following theorem.

Theorem 4.5. *If $\lambda = \frac{k}{h} < 1$ then the difference equations (40) and (41) are stable.*

Proof. It is sufficient to show that in (51) that $\rho(Q) < 1$, so we should find the value of μ such that it satisfies in the following equation

$$\det(Q - \mu I) = 0.$$

It is easy to show that

$$\det(Q - \mu I) = 0$$

if and only if

$$\det(\mu^2 I - \mu P + I) = 0$$

or

$$\det(P - \frac{\mu^2 + 1}{\mu} I) = 0$$

so the eigenvalues of Q are related to P as follow

$$\lambda_P = \frac{\mu^2 + 1}{\mu} = \frac{\lambda_Q^2 + 1}{\lambda_Q}. \tag{52}$$

On the other hands by Theorem 4.2 eigenvalues of P are the union of eigenvalues of $A + B$ and $A - B$, so we have

$$A + B = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & & & \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & & \\ & \ddots & \ddots & \ddots & \\ & & \lambda^2 & 2(1-\lambda^2) & \lambda^2 \\ & & & \lambda^2 & 2(1-\lambda^2) \end{bmatrix}$$

and

$$A - B = \begin{bmatrix} -2(\lambda^2+1) & -\lambda^2 & & & \\ -\lambda^2 & -2(\lambda^2+1) & -\lambda^2 & & \\ & \ddots & \ddots & \ddots & \\ & & -\lambda^2 & -2(\lambda^2+1) & -\lambda^2 \\ & & & -\lambda^2 & -2(\lambda^2+1) \end{bmatrix}.$$

Let matrices $(M-1) \times (M-1)$, T and T' as follows

$$T = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, T' = \begin{bmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ & & & 1 & 2 \end{bmatrix}$$

thus

$$A + B = 2I - \lambda^2 T, \qquad A - B = -\lambda^2 T' - 2I,$$

where $I_{2(M-1)}$ is the unit matrix of order $2(M-1)$ and T , T' are $(M-1) \times (M-1)$ matrices whose eigenvalues $\lambda_T$and $\lambda_{T'}$ are given by

$$\lambda_T = \lambda_{T'} = 4\cos^2 \frac{k\pi}{2(M+1)} \qquad k = 1, 2, \dots, M-1.$$

By using Lemma 4.1, the eigenvalues of $A - B$ and $A + B$ are obtained as follows:

$$\lambda_{A-B} = -4\lambda^2 \cos^2 \tfrac{k\pi}{2(M+1)} - 2, \quad \lambda_{A+B} = 2 - 4\lambda^2 \cos^2 \tfrac{k\pi}{2(M+1)}. \tag{53}$$

thus

$$\lambda_P = (\pm 2 - 4\lambda^2 \cos^2 \frac{k\pi}{2(M+1)}) \tag{54}$$

Now by (52) and (54) we have

$$\frac{\mu^2 + 1}{\mu} = (\pm 2 - 4\lambda^2 \cos^2 \frac{k\pi}{2(M+1)}).$$

Then

$$\mu^2 - (\pm 2 - 4\lambda^2 \cos^2 \frac{k\pi}{2(M+1)})\mu + 1 = 0$$

by solving quadratic equation

$$\Delta = (\pm 1 - 2\lambda^2 \cos^2 \frac{k\pi}{2(M+1)})^2 - 1.$$

Thus

$$\mu = (\pm 1 - 2\lambda^2 \cos^2 \frac{k\pi}{2(M+1)}) \pm \sqrt{(\pm 1 - 2\lambda^2 \cos^2 \frac{K\pi}{2(M+1)})^2 - 1}.$$

or

$$\mu = l \pm \sqrt{l^2 - 1},$$

where

$$l = (\pm 1 - 2\lambda^2 \cos^2 \frac{k\pi}{2(M+1)}). \tag{55}$$

A necessary condition for stability is

$$|\mu| < 1.$$

Since λ , k and M are real, $l \leq 1$ by (55).
When $l \leq -1$ or $l \geq 1$, $|\mu| \geq 1$, giving instability.
When

$$-1 < l < 1, \qquad l^2 < 1, \qquad \mu = l \pm i\sqrt{l^2 - 1},$$

hence

$$|\mu| = \sqrt{l^2 + (l^2 - 1)} = \sqrt{2l^2 - 1} < 1,$$

showing that a necessary condition for stability is $-1 < l < 1$. By (55),

$$-1 < (1 - 2\lambda^2 \cos^2 \frac{k\pi}{2(M+1)}) < 1.$$

The only useful inequality is

$$-1 < (\pm 1 - 2\lambda^2 \cos^2 \frac{k\pi}{2(M+1)})$$

giving

$$\lambda = \frac{k}{h} < 1.$$

□

Corollary 4.6. *If* $\lambda = \frac{k}{h} < 1$ *then the difference equations (42) and (43) are stable.*

Proof. The stability can be deal with in a similar manner, where

$$P = \begin{bmatrix} A & B \\ B & A \end{bmatrix}, \qquad A = \lambda^2 \begin{bmatrix} 0 & 1 & & \\ 1 & \cdot & \ddots & \\ & \ddots & \cdot & 1 \\ & & 1 & 0 \end{bmatrix}, \quad B = (2\lambda^2 + 2)I \tag{56}$$

and

$$\begin{aligned} A + B &= \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & & & \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & & \\ & \ddots & \ddots & \ddots & \\ & & \lambda^2 & 2(1-\lambda^2) & \lambda^2 \\ & & & \lambda^2 & 2(1-\lambda^2) \end{bmatrix}, \\ A - B &= \begin{bmatrix} 2(\lambda^2-1) & \lambda^2 & & & \\ \lambda^2 & 2(\lambda^2-1) & \lambda^2 & & \\ & \ddots & \ddots & \ddots & \\ & & \lambda^2 & 2(\lambda^2-1) & \lambda^2 \\ & & & \lambda^2 & 2(\lambda^2-1) \end{bmatrix}. \end{aligned} \tag{57}$$

Thus

$$A + B = 2I - \lambda^2 T, \qquad A - B = \lambda^2 T' - 2I.$$

Then the condition for stability will be yield as in the last case. □

5. **Examples**

Example 5.1. Consider the fuzzy parabolic equation

$$\frac{\partial \widetilde{U}}{\partial t}(x,t) - \frac{\partial^2 \widetilde{U}}{\partial x^2}(x,t) = 0, \quad 0 < x < l, \quad t > 0,$$

with the boundary conditions $\widetilde{U}(0,t) = \widetilde{U}(l,t) = 0,\ t > 0$, and $\widetilde{U}(x,0) = \widetilde{f}(x) = \widetilde{k}cos(\pi x - \pi/2),\ 0 \le x \le 1$. and $\widetilde{k}[\alpha] = [\underline{k}(\alpha), \overline{k}(\alpha)] = [\alpha - 1, 1 - \alpha]$.

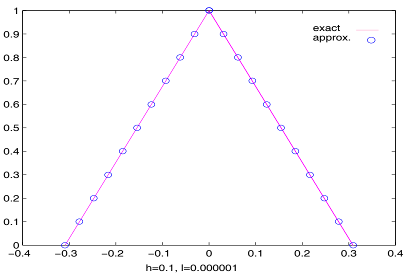

FIGURE 1.

The exact solution for

$$\frac{\partial \underline{U}}{\partial t}(x,t;\alpha) - \frac{\partial^2 \underline{U}}{\partial x^2}(x,t;\alpha) = 0,$$

$$\frac{\partial \overline{U}}{\partial t}(x,t;\alpha) - \frac{\partial^2 \overline{U}}{\partial x^2}(x,t;\alpha) = 0,$$

with $0 < x < l, \quad t > 0$ are $\underline{U}(x,y;\alpha) = \underline{k}(\alpha)e^{-\pi^2 t}cos(\pi x - \pi/2)$ and $\overline{U}(x,y;\alpha) = \overline{k}(\alpha)e^{-\pi^2 t}cos(\pi x - \pi/2)$. It is clear that the partial derivative of $\frac{\partial \widetilde{U}}{\partial t}$ and $\frac{\partial^2 \widetilde{U}}{\partial x^2}$ exist as the second form (2) of Definition 2.3. We use the equations (28) and (29) in to approximate the exact solution with $h = 0.1$ and $k = 0.00001$, therefore, $r = 0.001$. Figure 1 shows the exact and approximate solution at the point $(0.1, 0.000001)$ for each $\alpha \in (0,1]$. The *Hausdorff distance* between the solutions is $\mathbf{1.2e-003}$.

Example 5.2. Consider the fuzzy hyperbolic problem

$$\frac{\partial^2 \widetilde{U}}{\partial t^2}(x,t) - 4\frac{\partial^2 \widetilde{U}}{\partial x^2}(x,t) = 0, \ 0 < x < l, \ t > 0,$$

with the boundary and initial conditions

$$\widetilde{U}(0,t) = \widetilde{U}(l,t) = 0, \ t > 0, \ \widetilde{U}(x,0) = \widetilde{k}sin(\pi x), \ 0 \leq x \leq 1,$$

and the initial conditions

$$\frac{\partial \widetilde{U}}{\partial t}(x,0) = 0, \ 0 \leq x \leq 1,$$

where $\widetilde{k}[\alpha] = [\underline{k}(\alpha), \overline{k}(\alpha)] = [0.75 + 0.25\alpha, 1.25 - 0.25\alpha]$.

The exact solutions are

$$\underline{U}(x,t;\alpha) = \underline{k}(\alpha)sin(\pi x)cos(2\pi t),$$
$$\overline{U}(x,t;\alpha) = \overline{k}(\alpha)sin(\pi x)cos(2\pi t),$$

for $\alpha \in (0,1]$. It is clear that the partial derivative of $\frac{\partial^2 \widetilde{U}}{\partial t^2}$ and $\frac{\partial^2 \widetilde{U}}{\partial x^2}$ exist as in the second form (2) of Definition 2.3. We use the equations (38) and (39) to approximate the exact solution with $h = 0.1$ and $k = 0.001$, therefore $\lambda = 0.02$. Figure 2 shows the exact and approximate solution at the point $(0.1, 0.001)$, for each $\alpha \in (0,1]$. The *Hausdorff distance* between the solutions is $7.6247e - 006$.

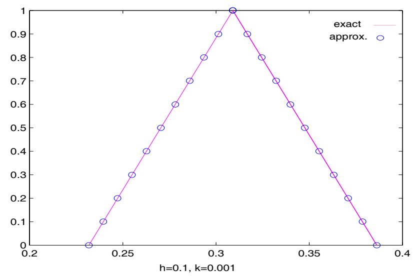

FIGURE 2.

6. Conclusions

We presented different methods for solving fuzzy partial differential equations. These numerical methods based on the definition of derivative that considered by *B. Bede, SG. Gal* [3]. If all terms of FPDE are differentiable in the sense of the first form (1) or the second form (2) of Definition 2.3 then the solutions of FPDE could be concluded from the numerical values.We presented necessary conditions for stability of this method.

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