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CLASSIFYING FUZZY NORMAL SUBGROUPS OF FINITE GROUPS

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Abstract. In this paper a first step in classifying the fuzzy normal subgroups of a finite group is made. Explicit formulas for the number of distinct fuzzy normal subgroups are obtained in the particular cases of symmetric groups and dihedral groups.

1. **Introduction**

One of the most important problems of fuzzy group theory is to classify the fuzzy subgroups of a finite group. This topic has enjoyed a rapid development in the last few years. Several papers have treated the particular case of finite abelian groups. Thus, in [9] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined, while [10, 11, 12, 17] deal with this number for cyclic groups of order $p^n q^m$ (p, q primes). Recall here the paper [16], where a recurrence relation is indicated which can successfully be used to count the number of distinct fuzzy subgroups for two classes of finite abelian groups: (arbitrary) finite cyclic groups and finite elementary abelian p-groups. Also, the above study has been extended to some remarkable classes of non-abelian groups: dihedral groups, symmetric groups, finite p-groups having a cyclic maximal subgroup, and hamiltonian groups.

In the present paper we will focus on classifying the fuzzy normal subgroups of a finite group G. Clearly, if G is abelian or hamiltonian, then its fuzzy normal subgroups coincide with its (classical) fuzzy subgroups and therefore the study is reduced to the above one. We will use a natural equivalence relation which is similar to that introduced in [16] and we will determine the number and nature of fuzzy normal subgroups of G with respect to this equivalence. For a different approach for classification see [4, 5]. In our case the corresponding equivalence classes of fuzzy normal subgroups are closely connected to the chains of normal subgroups in G. As a guiding principle in determining the number of these classes, we first find the number of maximal chains of normal subgroups in G. Another important ingredient is the expression of this number obtained in [15] for finite cyclic groups. Our method will be exemplified for the well-known classes of symmetric groups and dihedral groups. These groups are very important in abstract group theory, in combinatorics, as well as in geometry.

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The paper is organized as follows. In Section 2 we present some preliminary results on fuzzy (normal) subgroups and recall the main theorems of [15] and [16]. Section 3 deals with the normal subgroup structure of symmetric groups and dihedral groups, and with the way to compute the numbers of maximal chains of normal subgroups and distinct fuzzy normal subgroups of these groups. Explicit formulas for the above numbers will be obtained in Section 4. In the final section some conclusions and further research directions are indicated.

Most of our notation is standard and will usually not be repeated here. Basic notions and results on lattices (respectively on groups) can be found in [2] (respectively in [14]). For subgroup lattice concepts we refer the reader to [13].

2. **Preliminaries**

Let $(G, \cdot, e)$ be a group (where e denotes the identity of G) and $\mathcal{F}(G)$ be the collection of all fuzzy subsets of G. An element μ of $\mathcal{F}(G)$ is said to be a *fuzzy subgroup* of G if it satisfies the following two conditions:

a) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in G$;

b) $\mu(x^{-1}) \geq \mu(x)$, for any $x \in G$.

In this situation we have $\mu(x^{-1}) = \mu(x)$, for any $x \in G$, and $\mu(e) = \sup \mu(G)$. If μ satisfies the supplementary condition

$$\mu(xy) = \mu(yx), \text{ for all } x, y \in G,$$

then it is called a *fuzzy normal subgroup* of G. As in the case of subgroups, the sets $FL(G)$ and $FN(G)$ consisting of all fuzzy subgroups and all fuzzy normal subgroups of G form lattices with respect to fuzzy set inclusion (more precisely, $FN(G)$ is a sublattice of $FL(G)$), called the *fuzzy subgroup lattice* and the *fuzzy normal subgroup lattice* of G, respectively. Their binary operations $\wedge, \vee$ are defined by

$$\mu \wedge \eta = \mu \cap \eta, \ \mu \vee \eta = \langle \mu \cup \eta \rangle, \text{ for all } \mu, \eta \in FL(G),$$

where $\langle \mu \cup \eta \rangle$ denotes the fuzzy subgroup of G generated by $\mu \cup \eta$ (that is, the intersection of all fuzzy subgroups of G containing both μ and η). Note that the lattice $FN(G)$ is modular, as show [1] and [3].

For each $\alpha \in [0, 1]$, we define the level subset

$$_{\mu}G_{\alpha} = \{x \in G \mid \mu(x) \geq \alpha\}.$$

These subsets allow us to characterize the fuzzy (normal) subgroups of G, in the following manner: μ is a fuzzy (normal) subgroup of G if and only if its level subsets are (normal) subgroups in G (see e.g. Lemma 1.2.6 and Theorem 1.3.3 of [6]). This well-known theorem gives a link between $FL(G)$ and the classical subgroup lattice $L(G)$ of G, respectively between $FN(G)$ and the classical normal subgroup lattice $N(G)$ of G. It is important in studying many basic properties of $FL(G)$ and $FN(G)$, such as the distributivity and the fully ordering. We also recall that the fuzzy subgroup $\mu^* = \langle \mu \rangle$ generated by $\mu \in \mathcal{F}(G)$ can be described by using the level subsets $_{\mu}G_{\alpha}$, $\alpha \in [0, 1]$:

$$\mu^*(x) = \sup\{\alpha \mid x \in \langle _{\mu}G_{\alpha} \rangle\}, \text{ for any } x \in G.$$

The fuzzy (normal) subgroups of G can be classified up to some natural equivalence relations on $\mathcal{F}(G)$. One of them (used in [16] and [17], too) is defined by

$$\mu \sim \eta \text{ iff } (\mu(x) > \mu(y) \Longleftrightarrow \eta(x) > \eta(y), \text{ for all } x, y \in G)$$

and two fuzzy (normal) subgroups μ, η of G are said to be *distinct* if $\mu \not\sim \eta$. This equivalence relation generalizes that used in Murali's papers [7]–[12]. Also, it can be connected to the concept of level subgroup. In this way, suppose that the group G is finite and let $\mu : G \to [0,1]$ be a fuzzy (normal) subgroup of G. Put $\mu(G)$=$\{\alpha_1, \alpha_2, ..., \alpha_r\}$ and assume that $\alpha_1 > \alpha_2 > ... > \alpha_r$. Then μ determines the following chain of (normal) subgroups of G which ends in G:

$$_\mu G_{\alpha_1} \subset {_\mu G_{\alpha_2}} \subset ... \subset {_\mu G_{\alpha_r}} = G. \qquad (*)$$

Moreover, for any $x \in G$ and $i = \overline{1, r}$, we have

$$\mu(x) = \alpha_i \Longleftrightarrow i = \max\left\{j \mid x \in {_\mu G_{\alpha_j}}\right\} \Longleftrightarrow x \in {_\mu G_{\alpha_i}} \setminus {_\mu G_{\alpha_{i-1}}},$$

where, by convention, we set $_\mu G_{\alpha_0} = \emptyset$.

A necessary and sufficient condition for two fuzzy (normal) subgroups μ, η of G to be equivalent with respect to $\sim$ has been identified in [17]: $\mu \sim \eta$ if and only if μ and η have the same set of level subgroups, that is they determine the same chain of (normal) subgroups of type $(*)$. This result shows that *there exists a bijection between the equivalence classes of fuzzy* (*normal*) *subgroups of* G *and the set of chains of* (*normal*)*subgroups of* G *which end in* G. So, the problem of counting all distinct fuzzy (normal) subgroups of G can be translated into a combinatorial problem on the subgroup lattice $L(G)$ (or on the normal subgroup lattice $N(G)$) of G: finding the number of all chains of (normal) subgroups of G that terminate in G. Recall also that in our previous papers we have denoted by $g(G)$ (respectively by $h(G)$) the number of all maximal chains of subgroups of G (respectively the number of all chains of subgroups of G that terminate in G).

Even for some particular classes of finite groups, as finite abelian groups, this problem is very difficult. The largest class of groups for which it was completely solved is constituted by finite cyclic groups (see Corollary 4 of [16]). If $\mathbb{Z}_n$ is the finite cyclic group of order n and $n = p_1^{m_1} p_2^{m_2} ... p_s^{m_s}$ is the decomposition of n as a product of prime factors, then the number $f(n)$ of all distinct fuzzy (normal) subgroups of $\mathbb{Z}_n$ is given by the equality

$$f(n) = 2^{\sum\limits_{\alpha=1}^{s} m_\alpha} \sum_{i_2=0}^{m_2} \sum_{i_3=0}^{m_3} ... \sum_{i_s=0}^{m_s} \left(-\frac{1}{2}\right)^{\sum\limits_{\alpha=2}^{s} i_\alpha} \prod_{\alpha=2}^{s} \binom{m_\alpha}{i_\alpha} \binom{m_1 + \sum\limits_{\beta=2}^{\alpha} (m_\beta - i_\beta)}{m_\alpha},$$

where the above iterated sums are equal to 1 for $s = 1$. The number $g(n)$ of all maximal chains of (normal) subgroups of $\mathbb{Z}_n$ has been also determined in [15]:

$$g(n) = \binom{m_1 + m_2 + ... + m_s}{m_1, m_2, ..., m_s} = \frac{(m_1 + m_2 + ... + m_s)!}{m_1! m_2! ... m_s!}.$$

The above two equalities will play an essential role in the next section to solve our counting problems for finite symmetric groups and finite dihedral groups. This is due to the fact that the normal subgroup structure of these groups is closely connected to the subgroup structure of a certain finite cyclic group.

3. The Normal Subgroup Structure of Symmetric Groups and Dihedral Groups

For a finite group G, let us denote by $g'(G)$ and $h'(G)$ the number of all maximal chains of normal subgroups of G and the number of all distinct fuzzy normal subgroups of G, respectively. In the following our main goal is to compute explicitly these numbers for symmetric groups S_n and dihedral groups D_{2n}, $n \geq 3$. First of all, we must describe the normal subgroup lattices of these groups.

The symmetric group S_n consists of all permutations of the set $\{1, 2, ..., n\}$. A well-known group theory result states that it possesses only one proper normal subgroup (namely the alternating group A_n) if $n \neq 4$, while S_4 possesses two proper normal subgroups (namely $K = \{e, (12)(34), (13)(24),$
$(14)(23)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and A_4). In other words, we have:

$$N(S_n) = \begin{cases} \{\{e\}, A_n, S_n\}, & n \neq 4 \\ \\ \{\{e\}, K, A_4, S_4\}, & n = 4. \end{cases}$$

Remark that in both cases $N(S_n)$ is a chain of length 2 or 3, which shows that the following lattice isomorphisms hold

$$N(S_n) \cong L(\mathbb{Z}_{p^2}) \text{ for } n \neq 4, \text{ and } N(S_4) \cong L(\mathbb{Z}_{p^3}), \tag{1}$$

where p is an arbitrary prime.

Remark 3.1. The structure of finite groups whose lattices of subgroups are chains is well-known: these groups are cyclic of prime power orders. The same problem for lattices of normal subgroups is very difficult and unsolved yet. Note that there are known several classes of finite groups G with $N(G)$ fully ordered: cyclic groups of prime power orders, dihedral groups of type D_{2p^k} with p an odd prime, simple groups and symmetric groups.

Next, we shall focus on the dihedral group D_{2n}. Remind that it is the symmetry group of a regular polygon with n sides and has the order $2n$. The most convenient abstract description of D_{2n} is obtained by using its generators: a rotation x of order n and a reflection y of order 2. Under these notations, we have:

$$D_{2n} = \langle x, y \mid x^n = y^2 = 1,\ yxy = x^{-1} \rangle.$$

The structure of the subgroup lattice of D_{2n} is the following: if $n = p_1^{m_1} p_2^{m_2} ... p_s^{m_s}$ is the decomposition of n as a product of prime factors, then, for all divisors r of n, D_{2n} possesses a unique cyclic subgroup of order r (namely $\langle x^{\frac{n}{r}} \rangle \cong \mathbb{Z}_r$) and $\frac{n}{r}$ subgroups isomorphic to D_{2r} (namely $\langle x^{\frac{n}{r}}, x^i y \rangle, i = 0, 1, ..., \frac{n}{r} - 1$). Remark that D_{2n} has always a maximal cyclic normal subgroup $M = \langle x \rangle \cong \mathbb{Z}_n$ of index 2, which plays an important role in describing the normal subgroup lattice $N(D_{2n})$. Clearly, all subgroups of M are also normal in D_{2n}. On the other hand, if n is even, then

D_{2n} has another two maximal normal subgroups of order 2, namely $M_1 = \langle x^2, y\rangle$ and $M_2 = \langle x^2, xy\rangle$, both isomorphic to $D_{2\frac{n}{2}}$. In this way, one easily obtains:

$$N(D_{2n}) = \begin{cases} L(M) \cup \{D_{2n}\}, & n \equiv 1 \pmod 2 \\ \\ L(M) \cup \{D_{2n}, M_1, M_2\}, & n \equiv 0 \pmod 2. \end{cases} \tag{2}$$

Since both maximal chains and arbitrary chains ended in M of the subgroup lattice $L(M)$ of the cyclic subgroup M have been computed in Section 2 (their number is $g(n)$ and $f(n)$, respectively), the equality (2) will allow us to compute the corresponding chains of $N(D_{2n})$.

4. **Counting Fuzzy Normal Subgroups of Symmetric Groups and Dihedral Groups**

In the previous section we have seen that the lattice of normal subgroups of S_n is a chain in all cases. Obviously, this leads to the following result.

Theorem 4.1. *The number $g'(S_n)$ of all maximal chains of normal subgroups of the symmetric group S_n is equal to* 1*, for all $n \geq 2$.*

Moreover, from the lattice isomorphisms (1), we easily find the number $h'(S_n)$ of all distinct fuzzy normal subgroups of S_n

$$h'(S_n) = \begin{cases} f(p^2), & n \neq 4 \\ \\ f(p^3), & n = 4, \end{cases}$$

where p is an arbitrary prime. By using the explicit expression of $f(n)$ established in Section 2, one obtains that $f(p^2) = 4$ and $f(p^3) = 8$ (in fact, we have $f(p^k) = 2^k$, for all positive integers k). Hence we have proved the following theorem.

Theorem 4.2. *The number $h'(S_n)$ of all distinct fuzzy normal subgroups of the symmetric group S_n, $n \geq 3$, is given by the equality:*

$$h'(S_n) = \begin{cases} 4, & n \neq 4 \\ \\ 8, & n = 4. \end{cases}$$

According to (2), a maximal chain of normal subgroups of D_{2n} is of type

$$H_0 = \{e\} \subset H_1 \subset ... \subset H_{r-1} = M \subset H_r = D_{2n}$$

if n is odd and of one of the following types

$$H_0 = \{e\} \subset H_1 \subset ... \subset H_{r-1} = M \subset H_r = D_{2n},$$

$$H_0 = \{e\} \subset H_1 \subset ... \subset H_{r-1} = M_i \subset H_r = D_{2n},\ i = 1, 2,$$

if n is even. Clearly, if n is odd, then $g'(D_{2n})$ will coincide with the number of all maximal chains in $L(M)$, that is:

$$g'(D_{2n}) = g(n).$$

For n even, we have $M \cap M_1 = M \cap M_2 = \langle x^2\rangle$, and so a maximal chain in $N(D_{2n})$ can be obtained either by completing a maximal chain of $L(M)$ with D_{2n}, or by adding to a maximal chain of $L(\langle x^2\rangle)$ one of the subgroups M_i, $i = 1, 2$, and D_{2n}. Because the subgroup $\langle x^2\rangle$ is cyclic of order $\frac{n}{2}$, it possesses $g(\frac{n}{2})$ maximal chains of subgroups. We infer that:

$$g'(D_{2n}) = g(n) + 2g(\frac{n}{2}).$$

Now, the above two equalities and the explicit formula for $g(n)$ in Section 2 lead us to the next result.

Theorem 4.3. *Let $n = 2^k m \geq 3$ be a positive integer, where $k \in \mathbb{N}$ is arbitrary and $m \in \mathbb{N}^*$ is odd. Suppose that $m = p_1^{m_1} p_2^{m_2} ... p_q^{m_s}$ is the decomposition of m as a product of (odd) prime factors, where, by convention, we put $m_1 = m_2 = ... = m_s = 0$ for $m = 1$. Then the number $g'(D_{2n})$ of all maximal chains of normal subgroups of the dihedral group D_{2n} is given by the equality:*

$$g'(D_{2n}) = \begin{cases} \binom{m_1 + m_2 + ... + m_s}{m_1, m_2, ..., m_s}, & k = 0 \\ \\ \frac{3k + m_1 + m_2 + ... + m_s}{k} \binom{k - 1 + m_1 + m_2 + ... + m_s}{k - 1, m_1, m_2, ..., m_s}, & k \geq 1. \end{cases}$$

The number $h'(D_{2n})$ of all distinct fuzzy normal subgroups of D_{2n} can be easily computed if n is odd. In this case, every chain of normal subgroups of D_{2n} ended in D_{2n} is obtained by adding the term D_{2n} to an arbitrary chain of subgroups

$$\{e\} = H_0 \subset H_1 \subset ... \subset H_q$$

of M (we also include here the "void chain" of M, that is the chain whose set of terms is empty). Since, for a fixed $H_q \in L(M)$, the number of such chains is $f(|H_q|)$, it follows that:

$$h'(D_{2n}) = 1 + \sum_{d|n} f(d).$$

In view of (2), if n is even, then a chain of normal subgroups of D_{2n} ended in D_{2n} can be obtained in one of the following two ways: by adding the term D_{2n} to an arbitrary chain of subgroups of M or, for any $i \in \{1, 2\}$, by adding the terms M_i and D_{2n} to an arbitrary chain of subgroups of $\langle x^2\rangle$ (again, in both cases, we also include the void chain of M or $\langle x^2\rangle$). So, we infer that:

$$h'(D_{2n}) = 3 + \sum_{d|n} f(d) + 2\sum_{d|\frac{n}{2}} f(d).$$

Finally, we put together the above two explicit formulas of $h'(D_{2n})$ obtained for n odd and n even, respectively.

Theorem 4.4. *The number $h'(D_{2n})$ of all distinct fuzzy normal subgroups of the dihedral group D_{2n}, $n \geq 3$, is given by the equality:*

$$h'(D_{2n}) = \begin{cases} 1 + \sum_{d|n} f(d), & n \equiv 1 \pmod 2 \\ \\ 3 + \sum_{d|n} f(d) + 2\sum_{d|\frac{n}{2}} f(d), & n \equiv 0 \pmod 2. \end{cases}$$

Remark 4.5. 1) Set $n = 2^k m$, where $k \in \mathbb{N}^*$ is arbitrary and $m \in \mathbb{N}^*$ is odd. Then, for any divisor d of n, we have either $d \mid \frac{n}{2}$ or d is of the form $2^k m_1$, where m_1 is a divisor of m. This shows that

$$\sum_{d|n} f(d) = \sum_{d|\frac{n}{2}} f(d) + \sum_{m_1|m} f(2^k m_1),$$

and therefore the equality obtained in Theorem 4.4 for $n \equiv 0 \pmod 2$ can be rewritten in the following manner

$$h'(D_{2n}) = 3 + 3\sum_{d|\frac{n}{2}} f(d) + \sum_{m_1|m} f(2^k m_1).$$

One account that $f(2^q) = 2^q$ for all positive integers q, when $m = 1$ (that is, when $n = 2^k$), we get:

$$h'(D_{2^{k+1}}) = 3 + 3\sum_{i=0}^{k-1} f(2^i) + f(2^k) = 3 + 3(2^k - 1) + 2^k = 2^{k+2}.$$

In particular, the number of all distinct fuzzy normal subgroups of the dihedral group D_8 is:

$$h'(D_8) = 16.$$

2) A well-known arithmetic technique is to associate to a function $f : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ its sum-function $F : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ defined by

$$F(n) = \sum_{d|n} f(d), \text{ for all } n \in \mathbb{N},$$

and to compute the values of f by using the Möbius's inversion formula

$$f(n) = \sum_{d|n} F(d)\mu(\frac{n}{d}), \text{ for all } n \in \mathbb{N},$$

where μ denotes the Möbius's function. Theorem 4.4 shows that the number $h'(D_{2n})$ depends in fact only on the sum-function associated to our function f (defined in Section 2):

$$h'(D_{2n}) = \begin{cases} 1 + F(n), & n \equiv 1 \pmod 2 \\ 3 + F(n) + 2F(\frac{n}{2}), & n \equiv 0 \pmod 2. \end{cases}$$

In this case, a precise expression of $F(n)$ for n arbitrary is difficult to be obtained. Nevertheless, we remark that the above equality allows us to connect the numbers of all distinct fuzzy normal subgroups of dihedral groups D_{2n}, $n \in \mathbb{N}$, with the number of all distinct fuzzy subgroups of finite cyclic groups. Thus, if the numbers $h'(D_{2n})$, $n \in \mathbb{N}$, are known, then we can find the values $F(n)$, $n \in \mathbb{N}$, and then we can compute the numbers $f(n)$, $n \in \mathbb{N}$, according to the Möbius's inversion formula.

5. **Conclusions and Further Research**

The study concerning to classifying fuzzy subgroups, started in [8]-[12], [16] and [17] for finite abelian groups, and continued for finite nonabelian groups, is a significant aspect of fuzzy group theory. Since the fuzzy normal subgroups of a group constitute a particular class of fuzzy subgroups (recall again that for abelian groups and hamiltonian groups the notions of "fuzzy subgroup" and "fuzzy normal subgroup" are the same), their classification is also very important. Clearly, the study started in the present paper can successfully be extended to other remarkable classes of finite nonabelian groups. This will surely constitute the subject of some further research.

Three open problems with respect to this topic are the following.

Problem 5.1. In all the above mentioned papers, the fuzzy (normal) subgroups of a finite group G are classified up to the equivalence relation $\sim$ on $\mathcal{F}(G)$, defined in Section 2. Classify the fuzzy (normal) subgroups of G under other equivalence relations (different from $\sim$) on $\mathcal{F}(G)$.

Problem 5.2. For a finite group G, let $h(G)$ (respectively $h'(G)$) be the number of all distinct fuzzy subgroups (respectively fuzzy normal subgroups) of G, studied above. We consider the quantity

$$fndeg(G) = \frac{h'(G)}{h(G)},$$

which will be called the *fuzzy normality degree* of G. It measures the probability of a random fuzzy subgroup of G to be fuzzy normal and can constitute an interesting probabilistic aspect of finite groups. We formulate several problems related to the fuzzy normality degree:

a) Compute explicitly $fndeg(G)$ for some finite groups G.

b) Calculate

$$\lim_{n\to\infty} fndeg(G_n)$$

for some remarkable classes of finite groups $(G_n)_{n\in\mathbb{N}}$.

c) Obviously, there is no finite group with the fuzzy normality degree equal to 0 and, for a finite group G, we have $fndeg(G) = 1$ if and only if G is abelian or hamiltonian. So, the following qvestion is natural: given a fixed $a \in (0, 1)$, describe the structure of finite groups G satisfying

$$fndeg(G) = (\leq,\ \geq)\ a.$$

d) Study the properties of the map $fndeg$ from the class of finite groups to [0,1]. What can be said about two finite groups having the same fuzzy normality degree?

Problem 5.3. The fuzzy normal subgroups of a finite group G can be seen as the fixed points of the action of G on the set $FL(G)$ by conjugacy:

$$(g, \mu) \longmapsto g \circ \mu,\ (g \circ \mu)(x) = \mu(g^{-1}xg),\ \text{for all } x \in G.$$

Give a detailed description of the above action of G on $FL(G)$ and adjust the well-known Burnside's lemma to obtain the explicit formulas for the number of distinct fuzzy normal subgroups of symmetric groups and dihedral groups, established in Section 4. Also, introduce and study the dual concept of representation of G by permutations of the set $FL(G)$.

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